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INTRODUCTION

Ce mémoire présente une synthèse de mes travaux de recherche depuis ma thèse. Ces travaux font partie du domaine des probabilités intégrables. L'adjectif « intégrable » reflète l'apport d'idées issus des systèmes intégrables classiques ou quantiques. Pour moi, les probabilités intégrables désignent plus largement l'étude des modèles aléatoires exactement résolubles, qui peuvent être analysés au moyen de formules exactes. L'intérêt scientifique de cette étude provient principalement de l'universalité des résultats découverts au moyen de modèles intégrables.

L'exemple fondateur de l'entrelacement entre les notions d'intégrabilité et d'universalité est le théorème de Moivre–Laplace (1733), selon lequel si X_n est la somme de n variables aléatoires de Bernoulli indépendantes et de probabilité de succès $1/2$, qu'on retranche l'espérance $n/2$, et qu'on divise par la racine de la variance $\sqrt{n/4}$, alors la loi de cette somme renormalisée $(X_n - n/2)/\sqrt{n/4}$ converge vers la loi Gaussienne. L'intérêt de ce résultat est qu'il est universel, c'est-à-dire vrai pour un grand ensemble de lois initiales. C'est la raison pour laquelle la gaussienne modélise bien les phénomènes aléatoires résultants d'une somme d'aléas indépendants.

Un théorème de Baik, Deift et Johansson (1999) dit que si λ_1 est la plus grande composante d'une partition d'entier λ tirée au hasard selon la mesure de Plancherel associée au groupe symétrique $\mathcal{S}_n$, qu'on retranche $2\sqrt{n}$, ce qui n'est pas tout à fait l'espérance mais presque, et qu'on divise par $n^{1/6}$, alors la quantité renormalisée converge vers la loi de Tracy et Widom (1994), qui décrit les fluctuations de la plus grande valeur propre de matrices aléatoires hermitiennes en grande dimension. Plus simplement, λ_1 est aussi la plus longue sous-suite croissante d'une permutation aléatoire tirée uniformément dans $\mathcal{S}_n$. Ce résultat sert de théorème de Moivre–Laplace pour les modèles de croissance de la classe de Kardar–Parisi–Zhang (KPZ). Cette classe d'universalité est un domaine de la physique statistique assez vaste regroupant des modèles de croissance d'interface, des modèles de transport (systèmes de particules en interaction), de polymères dirigés, etc. décrits soit par des modèles combinatoires, soit par des équations aux dérivées partielles stochastiques. D'autres types de modèles physiques se sont ajoutés à la liste au fil des années. Le domaine a des liens étroits avec la théorie des matrices aléatoires, en termes de méthodes et de résultats, et rejoint par certains aspects la mécanique statistique planaire comme les modèles de dimères.

L'analyse des modèles intégrables dans la classe KPZ a connu une activité intense pendant les vingt-cinq dernières années. Au début du vingt-et-unième siècle, les modèles aléatoires exactement résolubles de la classe KPZ étaient toujours liés à des processus déterminantaux, comme la mesure de Schur introduite par Okounkov (2001), qui généralise la mesure de Plancherel du groupe symétrique. La première percée au sujet de modèles non déterminantaux est la série d'articles de Tracy et Widom (2008, 2009) qui ont calculé les probabilités de transition du processus d'exclusion asymétrique, au moyen d'une méthode de diagonalisation d'opérateurs venant de la physique théorique : l'ansatz de Bethe. A peine plus tard, O'Connell (2011) reliait certains modèles de polymères dirigés à une généralisation de la mesure de Schur, la mesure de Whittaker. Les fonctions de Whittaker ne s'écrivent pas comme des déterminants, contrairement aux polynômes de Schur, mais de nombreux arguments basés sur la théorie des fonctions symétriques fonctionnent toujours. Ces avancées ont motivé des aller-retours fructueux entre d'une part la théorie des fonctions symétriques et d'autre part les méthodes provenant de l'intégrabilité quantique, et en particulier de l'étude des modèles sur réseau satisfaisant à l'équation de Yang-Baxter.

En quelques mots, l'état de l'art est le suivant. En dimension spatiale égale à un, pour les modèles sur un domaine infini, on sait maintenant complètement caractériser leur limite d'échelle commune *via* le point fixe KPZ de Matetski–Quastel–Remenik (2021) ou le paysage dirigé de Dauvergne–Ortmann-Virág (2022). En revanche, bien que l'on puisse démontrer que l'équation KPZ apparaît à la limite de certaines classes de modèles de manière universelle, l'universalité du point fixe KPZ est un problème complètement ouvert. Par ailleurs, en dimensions spatiales supérieures, la classe KPZ est également un domaine méconnu, en partie car il y a peu de modèles intégrables véritablement tri-dimensionnels.

Je présente ici le contenu de ce mémoire, qui se focalise sur ma propre contribution au domaine.

Le premier chapitre porte sur l'analyse de modèles de la classe KPZ sur un domaine semi-infini à l'aide des processus de Macdonald. Ces processus sont des mesures de probabilité sur les partitions d'entiers, introduites par Borodin et Corwin (2013), basées sur des identités remarquables satisfaites par les fonctions de Macdonald. La variante en demi-espace que nous introduisons permet d'étudier des modèles de la classe KPZ définis sur une moitié de quadrant, ou bien des modèles où le désordre est symétrique par rapport à un axe, ou encore des systèmes de particules sur un réseau semi-infini, relié à un réservoir. Si je devais choisir un résultat emblématique de ce chapitre, je choisirais le Théorème 1.16, qui exprime la transformée de Laplace de la solution de l'équation KPZ sur $\mathbb{R}_+$ comme une fonctionnelle multiplicative très simple d'un processus ponctuel Pfaffien qui décrit la limite d'échelle des valeurs propres du GOE. Ce théorème est démontré en prenant la limite d'une relation analogue entre, d'un côté, la transformée de Laplace q-déformée d'un modèle discret, le modèle à six sommets stochastique, et de l'autre, une fonctionnelle multiplicative de la mesure de Schur pfaffienne. Cette identité discrète est elle-même la traduction probabiliste d'une remarquable identité satisfaite par les polynômes de Macdonald (1.5.7).

Le deuxième chapitre porte sur des modèles de la classe KPZ sur un domaine borné, particulièrement dans le cas de conditions au bord dites ouvertes. Dans ce cadre, il est beaucoup plus difficile d'obtenir des formules exactes. La loi limite universelle des fluctuations de la fonction de hauteur de tels modèles n'est notamment pas connue. Le résultat principal de ce chapitre est une caractérisation probabiliste de la mesure stationnaire de l'équation KPZ sur un intervalle, donnée par le Théorème 2.6. Cette dernière est associée à une mesure de Gibbs sur un couple de processus stochastiques continus, appelé ensemble de lignes gibbsien. Ces objets sont intimement liés à l'intégrabilité. En fait, nous avons étudié avec Ivan Corwin et Zongrui Yang des analogues discrets de ces mesures de Gibbs, qui peuvent être vus comme des variantes de processus de Schur ou de Whittaker.

Enfin, pour décrire le troisième chapitre, revenons au théorème de Moivre–Laplace. Il est souvent illustré par l'expérience de la planche de Galton (voir Figure 7) : des clous sont plantés de manière régulière en quinconce sur une planche verticale, de sorte qu'une bille lâchée à travers le réseau de clous passera de manière aléatoire à gauche ou à droite de chaque clou avec probabilité 1/2, simulant ainsi une marche aléatoire simple. Que se passe t-il si les clous ne sont pas arrangés de manière régulière ? Ou que se passe-t-il si on remplace la bille par un photon, et les clous par des miroirs semi-réfléchissants qui transforment la fonction d'onde du photon de manière unitaire et aléatoire ? Étonnamment, les deux questions sont équivalentes comme expliqué à la toute fin de ce manuscrit (voir la Section 3.7.2). Plus généralement, l'influence du désordre sur les phénomènes de diffusion, et en particulier sur la marche aléatoire simple, est étudiée depuis bien longtemps. A grande échelle, une trajectoire typique est peu influencée par le désordre. Einstein l'avait déjà compris dans son article fondateur de 1905 sur la diffusion. En revanche, le comportement extrême de diffusions en environnement aléatoire n'est pas bien décrit par la théorie classique des valeurs extrêmes mais obéit plutôt aux lois de la classe KPZ. Ce phénomène, que nous avons étudié avec Ivan Corwin (2015) à la fin de ma thèse, à travers l'étude d'un modèle exactement résoluble, a débouché sur de nombreux développements dont une partie est décrite dans le troisième chapitre.

INTRODUCTION (IN ENGLISH)

This memoir presents a synthesis of my research work since my PhD thesis. This work falls within the field of integrable probability. The notion of integrability arises from the fact that ideas originating from the study of classical or quantum integrable systems are useful in this context. For me, integrable probability more broadly refers to the study of exactly solvable random models, which can be analyzed using exact formulas. The primary scientific interest of this study lies in the universality of the results discovered through integrable models.

The foundational example of the interplay between integrability and universality is the Moivre–Laplace theorem (1733), which states that if X_n is the sum of n independent Bernoulli random variables with success probability $1/2$, and if we subtract the expectation $n/2$ and divide by the square root of the variance $\sqrt{n/4}$, then the law of this renormalized sum $(X_n - n/2)/\sqrt{n/4}$ converges to the Gaussian distribution. The significance of this result lies in its universality, meaning it holds for a wide class of initial distributions. This is why the Gaussian distribution effectively models random phenomena resulting from the sum of independent random variables.

A theorem by Baik, Deift, and Johansson (1999) states that if λ_1 is the largest part of an integer partition λ drawn according to the Plancherel measure associated with the symmetric group $\mathcal{S}_n$, and if we subtract $2\sqrt{n}$ (which is not exactly the expectation but close) and divide by $n^{1/6}$, then the renormalized quantity converges to the Tracy–Widom distribution (1994), which describes the fluctuations of the largest eigenvalue of large-dimensional Hermitian random matrices. More simply, λ_1 is also the length of the longest increasing subsequence of a random permutation drawn uniformly from $\mathcal{S}_n$. This result serves as a Moivre–Laplace theorem for growth models in the Kardar–Parisi–Zhang (KPZ) universality class. This class encompasses a broad area of statistical physics, including interface growth models, transport models (interacting particle systems), directed polymers, and more, described either by combinatorial models or stochastic partial differential equations. Over the years, other types of physical models have been added to this list. The field has strong connections with random matrix theory, both in terms of methods and results, and overlaps in certain aspects with planar statistical mechanics, such as dimer models.

The analysis of integrable models in the KPZ class has seen intense activity over the past twenty-five years. At the beginning of the twenty-first century, exactly solvable random models in the KPZ class were still linked to determinantal processes, such as the Schur measure introduced by Okounkov (2001), which generalizes the Plancherel measure of the symmetric group. The first breakthrough regarding non-determinantal models was the series of papers by Tracy and Widom (2008, 2009), who computed the transition probabilities of the asymmetric simple exclusion process using a method of operator diagonalization from theoretical physics: the Bethe ansatz. Shortly thereafter, O'Connell (2011) connected certain directed polymer models to a generalization of the Schur measure, the Whittaker measure. While Whittaker functions are not written as determinants—unlike Schur polynomials—many arguments based on symmetric function theory still apply. These advances motivated fruitful exchanges between the theory of symmetric functions and methods from quantum integrability, particularly the study of lattice models satisfying the Yang–Baxter equation.

In summary, the state of the art is as follows. In one spatial dimension, for models on an infinite domain, we now have a complete characterization of their common scaling limit via the KPZ fixed point of Matetski–Quastel–Remenik (2021) or the directed landscape of Dauvergne–Ortmann–Virág (2022). However, while it can be shown that the KPZ equation universally emerges as the limit of certain classes of models under special scaling, the universality of the KPZ fixed point remains a completely open problem. Furthermore, in higher spatial dimensions, the KPZ class is also poorly understood, partly because there are few genuinely three-dimensional integrable models.

I now present the content of this memoir, which focuses solely on my own contributions to the field.

The first chapter addresses the analysis of KPZ class models on a semi-infinite domain using Macdonald processes. These processes are probability measures on integer partitions, introduced by Borodin and Corwin (2013), based on remarkable identities satisfied by Macdonald functions. The half-space variant we introduce allows us to study KPZ class models defined on a half-quadrant, or models where the disorder is symmetric with respect to an axis, or particle systems on a semi-infinite lattice connected to a reservoir. If I were to choose an emblematic result from this chapter, it would be Theorem 1.16, which expresses the Laplace transform of the solution to the KPZ equation on $\mathbb{R}_+$ as a very simple multiplicative functional of a Pfaffian point process describing the scaling limit of the eigenvalues of the GOE. This theorem is proven by taking the limit of an analogous relation between, on one hand, the q-deformed Laplace transform of a discrete model—the stochastic six-vertex model—and, on the other hand, a multiplicative functional of the Pfaffian Schur measure. This discrete identity is itself the probabilistic translation of a remarkable identity satisfied by Macdonald polynomials, see Equation (1.5.7).

The second chapter focuses on KPZ class models on a bounded domain, particularly in the case of so-called open boundary conditions. In this context, it is much more difficult to obtain exact formulas. The universal limiting law of the height function fluctuations for such models is notably unknown. The main result of this chapter is a probabilistic characterization of the stationary measure of the KPZ equation on an interval, given by Theorem 2.6. This measure is associated with a Gibbs measure on a pair of continuous stochastic processes, called two-layer Gibbs line ensemble. These objects are intimately linked to integrability. In fact, with Ivan Corwin and Zongrui Yang, we studied discrete analogs of these Gibbs measures, which can be seen as variants of Schur or Whittaker processes.

Finally, to describe the third chapter, let us return to the Moivre–Laplace theorem. It is often illustrated by the Galton board experiment (see Figure 7): nails are arranged in a regular quincunx pattern on a vertical board, so that a ball dropped through the array of nails will randomly pass to the left or right of each nail with probability $1/2$, thus simulating a simple random walk. What happens if the nails are not arranged regularly? Or what happens if we replace the ball with a photon and the nails with beam-splitters that transform the photon's wave function in a unitary and random manner? Surprisingly, these two questions are equivalent, as explained at the very end of this manuscript (see Section 3.7.2). More generally, the influence of disorder on diffusion phenomena, and in particular on the simple random walk, has been studied for a long time. On a large scale, a typical trajectory is little influenced by disorder. Einstein already understood this in his foundational 1905 paper on diffusion. On the other hand, the extreme behavior of diffusions in a random environment is not well described by classical extreme value theory but rather obeys the laws of the KPZ class. This phenomenon, which we studied with Ivan Corwin (2015) at the end of my PhD thesis through the analysis of an exactly solvable model, has led to numerous developments, some of which are described in the third chapter.

PUBLICATIONS

[1] G. Barraquand. « A short proof of a symmetry identity for the q-Hahn distribution ». Electron. Commun. in Probab. 19.50 (2014).

[2] G. Barraquand. « A phase transition for q-TASEP with a few slower particles ». Stochastic Process. Appl. 125.7 (2015). arXiv: 1404.7409.

[3] G. Barraquand and I. Corwin. « The q-Hahn asymmetric exclusion process ». Ann. Appl. Probab. 26.4 (2016). arXiv: 1501.03445.

[4] G. Barraquand and I. Corwin. « Random-walk in beta-distributed random environment ». Probab. Theory Rel. Fields 167.3-4 (2017). arXiv: 1503.04117.

[5] J. Baik, G. Barraquand, I. Corwin, and T. Suidan. « Facilitated exclusion process ». Computation and combinatorics in dynamics, stochastics and control. The Abel symposium 2016. springer, 2018. arXiv: 1707.01923.

[6] J. Baik, G. Barraquand, I. Corwin, and T. Suidan. « Pfaffian Schur processes and last passage percolation in a half-quadrant ». Ann. Probab. 46.6 (2018). arXiv: 1606.00525.

[7] G. Barraquand, A. Borodin, I. Corwin, and M. Wheeler. « Stochastic six-vertex model in a half-quadrant and half-line open asymmetric simple exclusion process ». Duke Math. J. 167.13 (2018).

[8] G. Barraquand and M. Rychnovsky. « Tracy-Widom asymptotics for a river delta model ». Stochastic dynamics out of equilibrium. Lecture notes from the IHP trimester, Institut Henri Poincaré (IHP), Paris, France, April – July, 2017. Cham: Springer, 2019. arXiv: 1807.01824.

[9] G. Barraquand, P. Le Doussal, and A. Rosso. « Stochastic growth in time-dependent environments ». Phys. Rev. E (rapid communications) 101.4 (2020). arXiv: 1909.11557.

[10] G. Barraquand, A. Borodin, and I. Corwin. « Half-space Macdonald processes ». Forum of Mathematics, Pi 8 (2020). arXiv: 1802.08210.

[11] G. Barraquand, A. Krajenbrink, and P. Le Doussal. « Half-space stationary Kardar-Parisi-Zhang equation ». J. Stat. Phys. 181.4 (2020). arXiv: 2003.03809.

[12] G. Barraquand and P. Le Doussal. « Moderate deviations for diffusion in time dependent random media ». J. Phys. A Math. Theor. 53.21 (2020). arXiv: 1912.11085.

[13] G. Barraquand and M. Rychnovsky. « Large deviations for sticky Brownian motions ». Electron. J. Probab. 25 (2020). arXiv: 1905.10280.

[14] G. Barraquand and P. Le Doussal. « Kardar-Parisi-Zhang equation in a half space with flat initial condition and the unbinding of a directed polymer from an attractive wall ». Phys. Rev. E 104.2 (2021). arXiv: 2104.08234.

[15] G. Barraquand, I. Corwin, and E. Dimitrov. « Fluctuations of the log-gamma polymer free energy with general parameters and slopes ». Probab. Theory Rel. Fields 181.1-3 (2021). arXiv: 2012.12316.

[16] G. Barraquand and P. Le Doussal. « Steady state of the KPZ equation on an interval and Liouville quantum mechanics ». Europhys. Lett. 137.6 (2022). arXiv: 2105.15178.

[17] G. Barraquand, A. Krajenbrink, and P. Le Doussal. « Half-space stationary Kardar-Parisi-Zhang equation beyond the Brownian case ». J. Phys. A:Math. Theor. 55.27 (2022). arXiv: 2202.10487.

[18] G. Barraquand and P. Le Doussal. « A stationary model of non-intersecting directed polymers ». J. Phys. A: Math. Theor. 56.4 (2023). arXiv: 2205.08023.

[19] G. Barraquand and I. Corwin. « Stationary measures for the log-gamma polymer and KPZ equation in half-space ». Ann. Probab. 51.5 (2023). arXiv: 2203.11037.

[20] G. Barraquand, I. Corwin, and E. Dimitrov. « Spatial tightness at the edge of Gibbsian line ensembles ». Commun. Math. Phys. 397.3 (2023). arXiv: 2101.03045.

[21] G. Barraquand and P. Le Doussal. « Stationary measures of the KPZ equation on an interval from Enaud–Derrida's matrix product ansatz representation ». J. Phys. Math. Theor. 56.14 (2023). arXiv: 2209.03131.

[22] G. Barraquand and M. Rychnovsky. « Random walk on nonnegative integers in beta distributed random environment ». Commun. Math. Phys. 398.2 (2023). arXiv: 2201.07270.

[23] G. Barraquand and S. Wang. « An identity in distribution between full-space and half-space log-gamma polymers ». Int. Math. Res. Notices 2023.14 (2023). arXiv: 2108.08737.

[24] G. Barraquand. « Integral formulas for two-layer Schur and Whittaker processes ». Bulletin de la Société Mathématique de France (2024). arXiv: 2409.08927.

[25] G. Barraquand and I. Corwin. « Markov duality and Bethe ansatz formula for half-line open ASEP ». Probab. Math. Phys. 5.1 (2024). arXiv: 2212.07349.

[26] G. Barraquand, I. Corwin, and Z. Yang. « Stationary measures for integrable polymers on a strip ». Invent. Math. 237.3 (2024). arXiv: 2306.05983.

[27] G. Barraquand and D. Bernard. « Introduction to quantum exclusion processes » (2025). arXiv: 2507.01570.

[28] G. Barraquand, O. Blondel, and M. Simon. « Weakly asymmetric facilitated exclusion process ». Electron. J. Probab. 30 (2025). arXiv: 2301.04689.

[29] G. Barraquand and F. Casini. « Convergence of the KMP model to the KPZ equation ». preprint (2025). arXiv: 2507.19222.

[30] G. Barraquand, I. Corwin, and S. Das. « KPZ exponents for the half-space log-gamma polymer ». Probab. Theory Rel. 192 (Oct. 2025). arXiv: 2310.10019.

[31] G. Barraquand, I. Corwin, and E. Dimitrov. « Maximal free energy of the log-gamma polymer ». Journal d'Analyse Mathématique (May 2025). arXiv: 2105.05283.

[32] G. Barraquand and P. Le Doussal. « Large time cumulants of the KPZ equation on an interval ». J. Stat. Phys. 192.111 (2025). arXiv: 2504.18292.

[33] G. Barraquand and Z. Ouyang. « Stationary inverse-Wishart polymers ». preprint (2025). arXiv: 2511.14375.

CHAPTER 1

HALF-SPACE MACDONALD PROCESSES

In this chapter, I present my work on the Pfaffian Schur measure [6] and its Macdonald analogue [10], and I review several applications to Kardar–Parisi–Zhang growth on semi-infinite domains. The bulk of this chapter presents the article [10], joint with Alexei Borodin and Ivan Corwin, where we introduced the "half-space Macdonald measure". Its application to the stochastic six-vertex model and the KPZ equation on $\mathbb{R}_+$ is treated in a separate article with Michael Wheeler [7]. I also present in this chapter various further results revolving around half-space KPZ growth, taken from a further work on the symmetrized log-gamma polymer model [23], as well as my physics articles [14; 11; 17] written with Pierre Le Doussal and Alexandre Krajenbrink.

1.1. Around the Schur measure

1.1.1. Cauchy identity. — Schur symmetric polynomials can be defined as the ratio of determinants

$$s_\lambda(x) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n}{\det\left(x_i^{n-j}\right)_{i,j=1}^n},$$

where $x = (x_1, \dots, x_n)$ is a set of variables and $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n) \in \mathbb{N}^n$ is an integer partition with at most n parts. They satisfy a remarkable summation identity:

$$\sum_{\lambda_1\geqslant\lambda_2\geqslant\dots\geqslant 0} s_\lambda(a_1,\dots,a_n) s_\lambda(b_1,\dots,b_n) = \prod_{i,j=1}^n \frac{1}{1-a_ib_j} =: H(a;b). \qquad (\star)$$

It can be proved using linear algebra as follows. By the well-known formula for Vandermonde determinants, $(\star)$ is equivalent to

$$\sum_{\lambda_1\geqslant\lambda_2\geqslant\dots\geqslant 0} \det\left(a_i^{\lambda_j+n-j}\right)_{i,j=1}^n \det\left(b_i^{\lambda_j+n-j}\right)_{i,j=1}^n = \frac{\prod_{i<j}(x_i-x_j)(b_i-b_j)}{\prod_{i,j=1}^n 1-x_ib_j}.$$

Using the Binet-Cauchy identity, the summation on the left can be rewritten as a single determinant, so that $(\star)$ becomes equivalent to a well-known determinant identity, often attributed to Cauchy [Cau41],

$$\det\left(\frac{1}{1-a_ib_j}\right)_{i,j=1}^n = \frac{\prod_{i<j}(a_i-a_j)(b_i-b_j)}{\prod_{i,j=1}^n 1-a_ib_j}.$$

This is the reason why summation identities of the form $(\star)$ are often called "Cauchy identity".

In order to understand probabilistic applications to such formulas, it is useful to consider the power series expansion of both sides of $(\star)$. Regarding the right hand side, as long as $|a_ib_j| < 1$, we may write

$$\prod_{i,j=1}^{n} \frac{1}{1-a_ib_j} = \prod_{i,j=1}^{n}\left(\sum_{w=0}^{\infty}(a_ib_j)^w\right) = \sum_{W=(w_{i,j})\in\mathbb{N}^{n\times n}} \prod_{i,j=1}^{n}(a_ib_j)^{w_{i,j}} \tag{1.1.1}$$

Up to normalization, $\prod_{i,j=1}^{n}(a_ib_j)^{w_{i,j}}$ may be interpreted as the probability that the $n\times n$ integer matrix W has entries $w_{i,j}$ distributed as independent geometric random variables with parameter a_ib_j.

Let us turn to the left hand side of $(\star)$. Schur functions satisfy a branching rule

$$s_\lambda(a_1,\dots,a_n) = \sum_{\mu} s_{\lambda/\mu}(a_1)s_\mu(a_2,\dots,a_n), \tag{1.1.2}$$

where for $a\in\mathbb{R}$,

$$s_{\lambda/\mu}(a) = \mathbb{1}_{\mu\prec\lambda}a^{|\lambda|-|\mu|} \tag{1.1.3}$$

with $|\lambda| = \sum_i \lambda_i$ and we say that μ interlaces with λ, denoted $\mu\prec\lambda$, when $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \dots$. The latter are the same inequalities as the interlacing satisfied by eigenvalues λ_i of a real symmetric matrix and eigenvalues μ_j of a principal sub-matrix (1). The branching rule (1.1.2) allows to expand the Schur polynomials in monomials and obtain what is called the combinatorial formula:

$$s_\lambda(x) = \sum_{\lambda^{(1)}\prec\lambda^{(2)}\prec\cdots\prec\lambda^{(n)}=\lambda} \prod_{i=1}^{n} a_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|}, \tag{1.1.4}$$

where $\lambda^{(1)}\prec\lambda^{(2)}\prec\cdots\prec\lambda^{(n)}$ is a sequence of interlacing partitions and we have denoted by $|\lambda|$ the size of a partition, that is the sum of the parts $\lambda_1+\lambda_2+\dots$. Back to the Cauchy identity $(\star)$, its left hand side can now be expanded as

$$\sum_{\substack{\lambda^{(1)}\prec\lambda^{(2)}\prec\cdots\prec\lambda^{(n)}=\lambda\\ \mu^{(1)}\prec\mu^{(2)}\prec\cdots\prec\mu^{(n)}=\lambda}} \prod_{i=1}^{n} a_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|} \prod_{j=1}^{n} b_j^{|\mu^{(j)}|-|\mu^{(j-1)}|}, \tag{1.1.5}$$

where the sum now runs over two sequences of integer partitions. Since there is an equality between (1.1.1) and (1.1.5) there must be a bijection between integer matrices and sequences of interlaced partitions which matches the monomials in the two sums. There are actually several such bijections, the most famous being called the Robinson-Schensted-Knuth (RSK) correspondence. This is an extension of the Robinson-Schensted correspondence, generally stated as a bijection between permutations (matrices) and couples of standard Young tableaux with the same shape, introduced in the context of representation theory of the symmetric group [Rob38].

1.1.2. Last passage percolation. — There are several ways to define the RSK correspondence. The most common way involves an algorithm for the insertion of integers into Young tableaux, but we will not give it here. The coordinates of the partitions $\lambda^{(1)},\lambda^{(2)},\dots,\mu^{(1)},\dots$ can also be expressed explicitly in terms of maximization problems involving the entries of $W\in\mathbb{N}^{n\times n}$. In particular, we have that $\lambda_1 = G(n,n)$ where $G(n,n)$ is the last passage time from $(1,1)$ to (n,n) in the array of weights $(w_{i,j})$. It is defined as the maximization problem

$$G(n,m) = \max_{\pi:(1,1)\to(n,m)} \prod_{(i,j)\in\pi} w_{i,j}, \tag{1.1.6}$$

1. The connection to random matrix theory actually goes much deeper, but here, our λ_i are integers and are not interpreted as the eigenvalues of random matrix or operator.

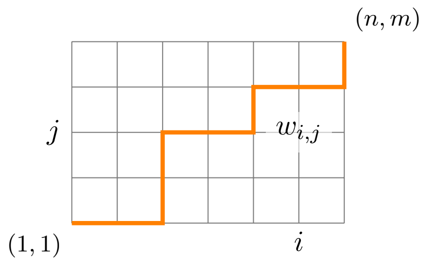


FIGURE 1. An up-right path $\pi : (1,1) \to (n,m)$ as in the definition of last passage percolation time (1.1.6).

where the maximum runs over all up-right lattice paths between $(1,1)$ and (n,m) in the grid in Figure 1.

Now, we may employ the RSK correspondence as a change of variable: if the weights $w_{i,j}$ are distributed as geometric random variables as above, we get

$$\begin{aligned} \mathbb{P}\left(G(n,n) \leqslant s\right) &= \sum_{\substack{W \in \mathbb{N}^{n\times n} \\ G(n,n)\leqslant s}} \prod_{i,j=1}^{n} (1-a_i b_j)(a_i b_j)^{w_{i,j}} \\ &= \frac{1}{H(a;b)} \sum_{\lambda\,:\,\lambda_1 \leqslant s} s_\lambda(a) s_\lambda(b). \end{aligned} \tag{1.1.7}$$

Indeed, in the first line, we sum the probability of observing matrices W with geometrically distributed entries, restricted on matrices such that $G(n,n) \leqslant s$. Via the RSK correspondence, this sum of monomials indexed by integer matrices can be written as a sum of monomials indexed by interlaced sequences of partitions, which can eventually be written as a sum of Schur functions. The constraint that $G(n,n) \leqslant s$ becomes $\lambda_1 \leqslant s$ after the change of variables. This exact formula for the probability distribution of $G(n,n)$ is essentially due to [Joh00], inspired by an earlier work on a closely related model of random permutations [BDJ99].

By using tools from random matrix theory, one can use (1.1.7) to prove that if parameters are all equal and such that $a_i b_j \equiv q$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{G(n,n) - \frac{\sqrt{q}}{1-\sqrt{q}} n}{\sigma(q) n^{1/3}} \leqslant s \right) = F_2(s),$$

where F_2 is the cumulative distribution function of the GUE Tracy-Widom distribution and

$$\sigma(q) = \left(\frac{\sqrt{q}(1+\sqrt{q})}{(1-\sqrt{q})^3} \right)^{1/3}.$$

1.1.3. Schur measure. — Given the form of the probability distribution of $G(n,m)$ in (1.1.7) it is natural to consider probability measures on random partitions of the form

$$\mathbb{P}^{\mathrm{Schur}}(\lambda) = \frac{1}{H(a;b)} s_\lambda(a_1,\dots,a_n) s_\lambda(b_1,\dots,b_m). \tag{1.1.8}$$

This point of view is adopted in [Oko01] — building on earlier works [Joh00; BOO00] — where it was proved that the set $\{\lambda_i - i\}_{i\geqslant 1}$ forms a determinantal point process. It is actually an example of a special class of determinantal point processes called biorthogonal ensembles, which first arose in the context of random matrix theory: for many matrix models, the probability distribution of eigenvalues of a $n \times n$ matrix can be written as a product of $n \times n$ determinants.

The Schur measure has proven to be an extremely powerful framework. By varying the choice of parameters a_i, b_j, potentially taking some limits, one can study many models including, besides last passage percolation:

- The longest increasing subsequences of random permutations;
- Many models of non-intersecting paths, encoding for various combinatorial structures such as random tilings;
- Interacting particle systems such as the totally asymmetric simple exclusion process (TASEP) or the PushTASEP.

In the theory of symmetric functions, there are many other families of functions, besides Schur polynomials, which satisfy a Cauchy identity. The corresponding probability measure on random partitions have many nice properties inherited by the symmetric functions properties. In particular, they are typically symmetric with respect to their parameters, which is a trace of integrability. Important examples, discussed below in more details, include the case of Whittaker functions pioneered in [OCo09] and the more general Macdonald measures [BC14].

1.1.4. Further remarks about the RSK correspondence. — A celebrated theorem of Pitman [Pit75] states that if X_t is a Brownian motion, and $M_t = \sup_{s\in[0,t]} X_t$, then the process

$$t \mapsto 2M_t - X_t$$

is a Brownian motion conditioned to stay positive for all $t \in \mathbb{R}_+$, i.e. it is a Bessel(3) process. For continuous functions $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}$, let us define

$$f_1 \otimes f_2(t) = \inf_{0\leqslant s\leqslant t}\{f_1(s) + f_2(t) - f_2(s)\}$$
$$f_1 \odot f_2(t) = \sup_{0\leqslant s\leqslant t}\{f_1(s) + f_2(t) - f_2(s)\}.$$

Then the Pitman transform of $f = (f_1, f_2) : \mathbb{R}_+ \to \mathbb{R}^2$ is

$$\mathcal{P}^{(2)} f = (f_2 \odot f_1, f_1 \otimes f_2). \tag{1.1.9}$$

Pitman's theorem can be reformulated as follows: if $B = (B_1, B_2)$ is a Brownian motion in $\mathbb{R}^2$, then $\mathcal{P}^{(2)}B$ is a Brownian motion constrained to stay in the Weyl chamber

$$\mathbb{W}_2 = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > x_2\}$$

Remark 1.1. — In the discrete setup, the Pitman transform has a nice interpretation in terms of so-called box-ball systems. A random walks with ± 1 steps encodes a configuration of particles on $\mathbb{N}$ as in Figure 2, where each up step is associated to an empty site, and each down step is associated to a site occupied by a particle. The configuration is updated as follows. Let us number the particles from left to right, and update their position sequentially. Particle number 1 is first moved to the next empty site on the right. Then particle number 2 is moved to the next empty site on the right, and so on. After one step of the dynamics, the new configuration is encoded by the Pitman transform of X_t [CKST18]:

$$(\mathcal{P}X)_t = 2 \max_{1\leqslant s\leqslant t}\{X_s\} - X_t$$

Indeed, the Pitman transform consists in reflecting the trajectory off the trajectory of the running maximum, which is a way to encode the box-ball dynamics, see Figure 2 (right).

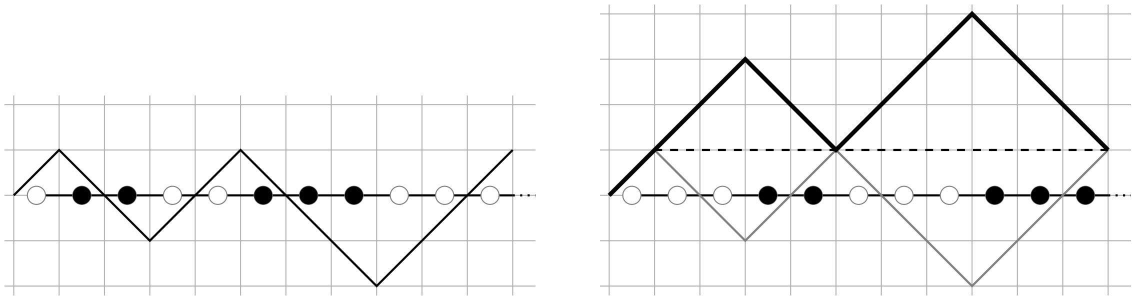


FIGURE 2. Left: the height function associated with a box-ball configuration. Right: Configuration after one step of the dynamics, given by the Pitman transform, that is reflecting the trajectory off the running maximum.

The reason why we introduced the compositions $f_1 \otimes f_2$ and $f_1 \odot f_2$ and the Pitman transform of a couple of functions is that it allows to generalize to a n-tuple of functions (or equivalently, a function from $\mathbb{R}_+$ to $\mathbb{R}^n$). Following [OY02], for $f = (f_1, \dots, f_n)$, we define $\mathcal{P}_i^{(n)} f$ to be the transformation that replaces (f_i, f_{i+1}) by $(f_{i+1} \odot f_i, f_i \otimes f_{i+1})$. Then, if

$$(n, n-1, \dots, 2, 1) = s_{i_1} s_{i_n} \dots s_{i_k}$$

is a minimal factorization of the reverse permutation in adjacent transpositions $s_i = (i, i+1)$, one defines the Pitman transform

$$\mathcal{P}^{(n)} = \mathcal{P}_{i_1}^{(n)} \dots \mathcal{P}_{i_k}^{(n)}.$$

Biane, Bougerol and O'Connell found that [BBO05] that the Pitman transforms $\mathcal{P}_i^{(n)}$ satisfy braid relations, ensuring that the above definition of $\mathcal{P}^{(n)}$ does not depend on the chosen factorization of the reverse permutation. The transformation $\mathcal{P}^{(n)}$ should be interpreted as some sort of sorting algorithm, and $\mathcal{P}_i^{(n)}$ is an elementary operation that sorts coordinates i and $i+1$. One then has a n-dimensional analogue of Pitman's theorem due to [OY02; BJ02]. If $B = (B_1, \dots, B_n)$ is a Brownian motion in $\mathbb{R}^n$, then $\mathcal{P}^{(n)} B$ is a Brownian motion conditioned (in Doob's sense) to stay in the Weyl chamber

$$\mathbb{W}_n = \{(x_1, \dots, x_n) \in \mathbb{R}^2; x_1 > \dots > x_n\}.$$

Pitman's transforms are intimately connected to the RSK correspondence and last passage percolation. Given an integer matrix of weights W, let us define discrete functions

$$R_i(m) = \sum_{j=1}^{m} w_{i,j}$$

and compositions

$$R_1 \otimes R_2(m) = \min_{1 \leqslant j \leqslant m} \{R_1(j-1) + R_2(m) - R_2(j)\}$$
$$R_1 \odot R_2(m) = \max_{1 \leqslant j \leqslant m} \{R_1(j) + R_2(m) - R_2(j-1)\}.$$

One can define the Pitman transform $\mathcal{P}^{(n)} R$ of $R = (R_1, \dots, R_n)$ exactly as in the continuous case. Since the first coordinate $(\mathcal{P}^{(n)} R)_1$ of the Pitman transform involves repeated application of the composition $\odot$, it can be shown that

$$G(m, n) = (\mathcal{P}^{(n)} R)_1(m).$$

More generally, the RSK correspondence

$$\begin{Bmatrix} \emptyset \prec \lambda^{(1)} \prec \dots \prec \lambda^{(n)} = \lambda \\ \emptyset \prec \mu^{(1)} \prec \dots \prec \mu^{(m)} = \lambda \end{Bmatrix} \quad \longleftrightarrow \quad W = (w_{i,j}) \in \mathbb{N}^{n \times m}$$

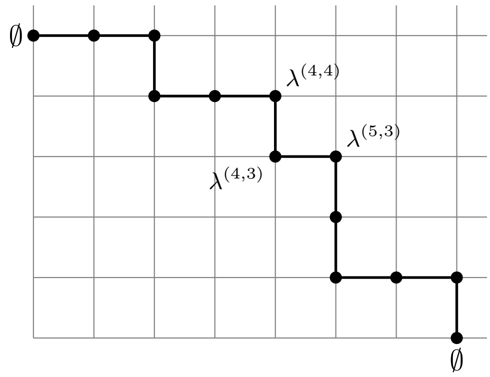


FIGURE 3. A downright path γ and random partitions associated to vertices on γ, illustrating the definition of Schur processes.

can be expressed in terms of discrete Pitman transforms: we have

$$(\mu^{(j)})_{1\leqslant j\leqslant m} = \left(\mathcal{P}^{(n)}R(j)\right)_{1\leqslant j\leqslant m}$$

The sequence of partitions $\lambda^{(i)}$ can also be written in terms of Pitman transforms after replacing $w_{i,j}$ by $w_{j,i}$.

This link between the RSK correspondence and last passage percolation, via Pitman transforms, provides a concrete probabilistic interpretation to the other coordinates of random partitions λ following the Schur measure, besides λ_1. It also gives hints about how to properly generalize the Schur measure to the Schur process, which is a measure on sequences of partitions that describes in particular the joint distribution of last passage times $G(n,m)$ at different points.

1.1.5. The Schur process. — The Schur process was introduced in [OR03] as a model of random plane partition, which may be encoded by a sequence of integer partitions. In their original paper, the Schur process is a probability measure on partitions

$$\emptyset = \mu^{(0)} \subset \lambda^{(1)} \supset \mu^{(1)} \subset \lambda^{(2)} \supset \mu^{(2)} \subset \cdots \subset \lambda^{(N)} \supset \mu^{(N+1)} = \emptyset \tag{1.1.10}$$

occurring with probability proportional to

$$\prod_{i=1}^{N} s_{\lambda^{(i)}/\mu^{(i-1)}}(\varrho_i^+) s_{\lambda^{(i)}/\mu^{(i)}}(\varrho_i^-)$$

where $\varrho_i^\pm$ are sets of variables (or, more generally, specializations of the ring of symmetric functions). By choosing some of the $\varrho_i^\pm$ to be zero, one may force some of the inclusions in (1.1.10) to be equalities, so that the Schur process may be viewed as a sequence of random integer partitions included into each other in an arbitrary way. The sequence of inclusions is encoded by a word on the alphabet $\{\subset, \supset\}$. Since we will be mostly interested in applications of the Schur measure to lattice models, it is convenient to view this word as a lattice path in the first quadrant of $\mathbb{Z}^2$, as depicted in Figure 3.

Thus, we will define Pfaffian Schur processes for any down right path γ (see Figure 3). Each edge e in the quadrant is decorated by a set ϱ_e (or a more general specialization). Each vertex v in the quadrant is associated to a partition $\lambda^{(v)}$. For each γ, the Pfaffian Schur process is a probability measure on the sequence of partitions $\Lambda_\gamma := (\lambda^{(v)})_{v\in\gamma}$ where

$$\mathbb{P}(\Lambda_\gamma) \propto \prod_{e\in\gamma} \text{weight}(e),$$

with

$$\mathrm{weight}\left(\begin{array}{c}\mu \qquad\quad \lambda\\ \bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\\ \varrho\end{array}\right) = s_{\lambda/\mu}(\varrho) \quad\text{and}\quad \mathrm{weight}\left(\begin{array}{cc} & \bullet\,\lambda\\ \varrho & |\\ & \bullet\,\mu\end{array}\right) = s_{\lambda/\mu}(\varrho).$$

An important property, which follows from branching and summation properties of Schur functions, is that the marginal distribution of any coordinate in the Schur process is a Schur measure.

The set of ϱ_e can be chosen, for instance, as being all single variable specializations and depend on two sets of parameters $a_1, a_2, \dots$ and $b_1, b_2, \dots$ as

$$\varrho_e = \begin{cases} a_i \text{ if } e = (i-1,j) \to (i,j) \text{ (a horizontal edge arriving on the } i\text{-th column)} \\ b_j \text{ if } e = (i,j-1) \to (i,j) \text{ (a vertical edge arriving on the } i\text{-th row).} \end{cases}$$

Then, for every down-right path γ, we have the joint distribution identity

$$\left(\lambda_1^{(v)}\right)_{v\in\gamma} = (G(v))_{v\in\gamma}$$

where $G(v)$ is the last passage percolation time from the vertex $(1,1)$ to v. This is a consequence of the RSK correspondence. Actually, the RSK correspondance provides an inpterpretation of all the coordinates in Λ_γ in terms of Pitman transforms or in terms of last passage percolation times associated to non-intersecting paths. This also suggests a coupling between all sequences Λ_γ for different paths γ, so that the identity holds simultaneously for all γ. This couplings may be described through Markov dynamics on the Schur process, where the dynamics are indexed by down-right paths instead of time. We note that there are several Markov dynamics on the Schur process that are such that their first coordinate marginal $\lambda_1^{(v)}$ has the same distribution as the last passage percolation time $G(v)$. One distinguished choice is the one corresponding to the RSK correspondence. There is another convenient choice, referred as "push-block dynamics", after [BF14; WW09; Bor11], which will play an important role in the sequel, especially in Chapter 2.

1.2. Pfaffian Schur process

As we have just seen, exchanging $w_{i,j}$ and $w_{j,i}$ has a simple effect on the RSK output, it simply exchanges the sequences of partitions $\lambda^{(i)}$ and $\mu^{(j)}$. This suggests the existence of some probabilistic model of last passage percolation having the symmetry $w_{i,j} = w_{j,i}$, which would be associated with a single sequence of random partitions.

1.2.1. Symmetrized last passage percolation. — Last passage percolation through an array of weights which is symmetric with respect to the main diagonal, i.e. $w_{i,j} = w_{j,i}$, was first considered by Baik and Rains in [BR01a; BR01b]. We illustrate the symmetry axis by a red line in Figure 4 and below. As we show in Figure 4, imposing a symmetry axis is equivalent to assuming that paths must stay below the diagonal, that is in the first octant (half-quadrant) of the lattice.

If the distribution of weights depends on a single sequence of parameters $(a_i)_{1\leqslant i\leqslant n}$ in $(0,1)$ and an extra parameter $c > 0$ as

$$\begin{aligned} w_{i,j} &\sim \mathrm{Geom}(a_i a_j) \text{ for } i > j \\ w_{i,i} &\sim \mathrm{Geom}(c a_i), \end{aligned}$$

then, the last passage percolation time

$$G^{\boxslash}(n,m) = \max_{\text{paths } (1,1)\to(n,m)} \Big\{ \sum_{(i,j)\in \text{ path}} w_{i,j} \Big\}.$$

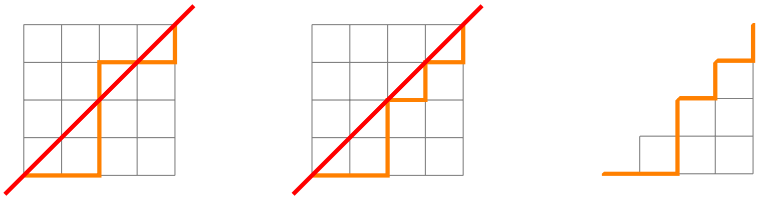


FIGURE 4. Since the weights are symmetric, the weight associated to any path (as on the left picture) is the same as the weight of a path that stay below the diagonal (as in the middle picture) so the last passage percolation time can be computed by maximizing only over such paths (as in the right picture).

is distributed as [BR01a]

$$\mathbb{P}\left(G^{\boxslash}(n,n) \leqslant r\right) = \frac{1}{H^{\boxslash}(a;c)} \sum_{\lambda:\lambda_1 \leqslant r} c^{\lambda_1-\lambda_2+\lambda_3-\lambda_4+\cdots} s_\lambda(a), \tag{1.2.1}$$

where

$$H^{\boxslash}(a;c) := \prod_i \frac{1}{1-a_i c} \prod_{i<j} \frac{1}{1-a_i a_j}.$$

As $n \to \infty$, $G^{\boxslash}(n,n)$ satisfies a phase transition, now referred to as the Baik-Rains transition.

$\underline{c<1}$ **:** It is shown in [BR01b] that if $a_i \equiv a \in (0,1)$ and $c<1$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{G^{\boxslash}(n,n) - \frac{a}{1-a}n}{\sigma(a^2)n^{1/3}} \leqslant s\right) = F_4(s)$$

where F_4 is now the GSE Tracy-Widom distribution.

$\underline{c>1}$ **:** When $c>1$, the weights along the diagonal become big enough so that the optimal path (whose weight is equal to $G^{\boxslash}(n,n)$) collects an order $O(n)$ weights along the diagonal. The fluctuations of the sum of these big weights yields a Gaussian central limit theorem with fluctuations on the scale $n^{1/2}$.

$\underline{c=1}$ **:** Exactly at the transition point, $c=1$, fluctuations are on the scale $n^{1/3}$ but with a slightly different distribution [BR01b]:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{G^{\boxslash}(n,n) - \frac{a}{1-a}n}{\sigma(a^2)n^{1/3}} \leqslant s\right) = F_1(s)$$

where F_1 is now the GOE Tracy-Widom distribution.

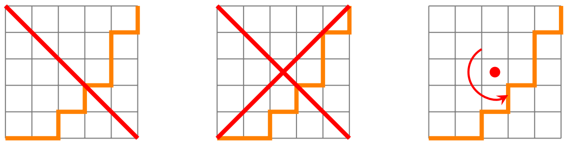
A heuristic explanation for the value of c at the transition is the following. The value $c=1$ is such that the following two quantities match:

1. The expectation of weights along the diagonal, which is $\mathbb{E}[w_{11}] = \frac{ac}{1-ac}$.
2. The typical size of weights along a geodesic which is given (when $c \leqslant 1$) by the limit

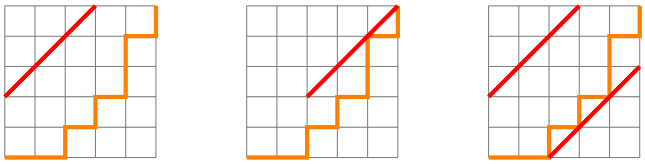
$$\lim_{n\to\infty} \frac{1}{2n} G^{\boxslash}(n,n) = \frac{a}{1-a}.$$

Open Problem 1.2. — *Characterize the geometry of the geodesic between* $(1,1)$ *and* (n,n) *at the critical point, i.e. when* $c=1$. *It is plausible that the geodesic touches the diagonal at* $O(n^{2/3})$ *points, but how these points are distributed is unknown.*

Remark 1.3. — Baik and Rains also computed the probability distribution of last passage times $\mathbb{P}\left(G^{\bullet}(n,n) \leqslant r\right)$ in terms of Schur functions for other symmetry types $\bullet$:



In the last picture, the semi-circular arrow means that weights are symmetric with respect to the central point. However, the computation of the last passage time distribution is an open problem when imposing other symmetries such as:



The last problem is equivalent to last passage percolation in a band, discussed in details in Chapter 2.

1.2.2. Pfaffian Schur measure. — The Pfaffian Schur measure was introduced in [BR05] (this is a slight generalization of the measure appearing in (1.2.1))

$$\mathbb{P}^{Schur}_{\boxslash}(\lambda) = \frac{1}{H^{\boxslash}(a;b,c)} \tau_\lambda(b_1, \dots, b_k, c) s_\lambda(a_1, \dots, a_n) \tag{1.2.2}$$

where

$$H^{\boxslash}(a;b,c) = \prod_{i=1}^{n} \frac{1}{1-a_i c} \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{1}{1-a_i b_k} \prod_{1 \leqslant i<j \leqslant n} \frac{1}{1-a_i a_j},$$

and

$$\tau_\lambda = \sum_{\mu' \text{even}} s_{\lambda/\mu}$$

and "μ' even" means that we sum over partition whose dual has even coordinates. It corresponds to imposing that the coordinates of the partition μ come in pairs $\mu_1 = \mu_2$, $\mu_3 = \mu_4$, etc. as the eigenvalues of a GSE matrix. When the function τ_λ is evaluated into a single variable c, the definition of Schur function readily imply

$$\tau_\lambda(c) = c^{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \dots}.$$

In particular, if $c = 0$, the Pfaffian Schur measure is supported on partitions such that λ' is even.

Borodin and Rains [BR05] actually introduced the more general Pfaffian Schur process and proved that they form Pfaffian point process. These are analogues of determinantal point processes where determinants are replaced by Pfaffians. We will not define the Pfaffian Schur process here, but we define a generalization of it in Section 1.3.4.

In [6], we exploited Borodin-Rains's result to study half-space last passage percolation. The connection between the Pfaffian Schur measure and last passage percolation is the following.

Proposition 1.4 **([6]).** — *Let λ de distributed according to $\mathbb{P}^{Schur}_{\boxslash}$ as in* (1.2.2)*. Then,*

$$\lambda_1 \overset{(dist)}{=} G^{\boxslash}(n+k, n)$$

when the weights are independent with

$$\begin{aligned} w_{i,j} &\sim \mathrm{Geom}(a_i a_j) \textit{ for } 1 \leqslant j < i \leqslant n \\ w_{i,i} &\sim \mathrm{Geom}(c a_i), \\ w_{n+k,j} &\sim \mathrm{Geom}(b_k a_j). \end{aligned}$$

We proved this result in [6] by constructing dynamics on sequences of interlaced partitions preserving the class of Pfaffian Schur processes. It generalizes one of the results of Baik and Rains from [BR01a] which corresponds to the case $k = 0$. Our method is very simple to implement. It was suggested to us by Alexei Borodin, and it also works in the case of Macdonald processes, which we discuss in Section 1.3.4. In the Schur case, we could however have used a more traditional approach via the RSK correspondence. The latter method was implemented shortly after in [BBNV18].

Then, using the results of [BR05] on the Pfaffian Schur process, and performing asymptotics of Fredholm Pfaffians, we were able to complement the asymptotic results of [BR01b]. We mostly worked with a model with exponentially distributed weights — this is a limit of the geometric case. We assume that

$$\begin{aligned} w_{i,j} &\sim \mathrm{Exp}(1) \text{ for } 1 \leqslant j < i \\ w_{i,i} &\sim \mathrm{Exp}(\alpha). \end{aligned} \tag{1.2.3}$$

Theorem 1.5 **([6]).** — *Denoting by $E^{\boxslash}(n,n)$ the last passage percolation time with exponential weights* (1.2.3)*, we have the following asymptotics: When $\alpha > \frac{1}{2}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{E^{\boxslash}(n,n) - 4n}{2^{4/3}n^{1/3}} \leqslant s\right) = F_4(s).$$

When $\alpha = \frac{1}{2}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{E^{\boxslash}(n,n) - 4n}{2^{4/3}n^{1/3}} \leqslant s\right) = F_1(s).$$

When $\alpha < \frac{1}{2}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\alpha(1-\alpha)E^{\boxslash}(n,n) - n}{\sqrt{(1-2\alpha)n}} \leqslant s\right) = G(s),$$

where G is the standard Gaussian distribution. Moreover, for any $\kappa > 1$ and $\alpha > \frac{1}{1+\sqrt{\kappa}}$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{E^{\boxslash}(\kappa n,n) - (1+\sqrt{\kappa})^2 n}{\left(\frac{(1+\sqrt{\kappa})^4}{\sqrt{\kappa}}n\right)^{1/3}} \leqslant s\right) = F_2(s).$$

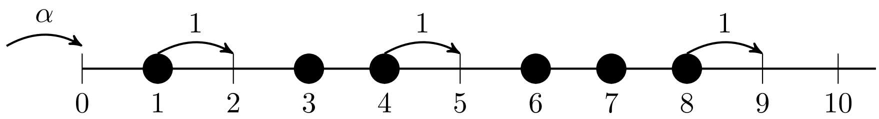
The last asymptotic results means that when $\kappa > 1$, $E^{\boxslash}(\kappa n,n)$ has the same fluctuations as $E(\kappa n,n)$ (the unsymmetrized model), as long as α is large enough. Intuitively, at large scale, the shape of the optimal path is not influenced by the boundary. Theorem 1.5 is a subset of the main results in [6]. In the paper, we also study the joint distribution of the vector

$$\left(E^{\boxslash}(n_i,m_i)\right)_{1\leqslant i\leqslant k},$$

where the points (n_i,m_i) form a down-right path, that is we have $n_1 \geqslant m_1 \geqslant \ldots \geqslant m_k$. The joint distribution is expressed in terms of an extended Pfaffian process. We can then study asymptotics when the points are at distance $O(n^{2/3})$ from the point (n,n) and the boundary parameter α is scaled closed to critical as $\alpha = 1/2 + 2^{-1/3}\varpi n^{-1/3}$. The resulting Pfaffian point process encodes the finite dimensional distribution of the half-space Airy line ensemble, recently defined [DY25]. The latter is an important tool for studying universal properties of half-space growth models in the KPZ class. It should be a key input in order to define appropriate half-space generalizations of the directed landscape and KPZ fixed point, which we discuss in more details in Section 1.8.4.

1.2.3. Application to interacting particle systems. — One of the motivations for establishing the asymptotic results in Theorem 1.5 was to study the fluctuations in some interacting particle system called the facilitated (totally) asymmetric simple exclusion process (FTASEP). In this system, particles on $\mathbb{Z}$ move at rate 1 to the right by one unit, provided the site on the right is empty, and the site on the left is occupied. One may show that starting from the step initial configuration, the state of the system is in bijection with configurations of the TASEP on a semi-infinite lattice $\mathbb{N}$.

In the TASEP on $\mathbb{N}$, particles jump independently at exponential rate 1 to the next site on the right provided this site is empty. Moreover there is a reservoir that inserts a particle at site 0 at rate α



Letting $N_x(t)$ denote the number of particles to the right of x at time t, starting from the empty initial configuration at $t = 0$, Theorem 1.5 implies the following.

Corollary 1.6 **([5]).** — *1. If $\alpha > 1/2$,*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0(t) - t/4}{2^{-4/3}t^{1/3}} \geqslant s\right) = F_4(-s).$$

2. If $\alpha = 1/2$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0(t) - t/4}{2^{-4/3}t^{1/3}} \geqslant s\right) = F_1(-s). \tag{1.2.4}$$

3. If $\alpha < 1/2$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0(t) - t\alpha(1-\alpha)}{\sqrt{\frac{(1-2\alpha)^2}{\alpha(1-\alpha)}t}} \geqslant s\right) = G(s).$$

This corollary is based on a well-known coupling between TASEP and last passage percolation such that

$$\{N_n(t) \leqslant y\} = \{E^{\square}(n+y, y) \geqslant t\}.$$

1.3. Half-space Macdonald measure

1.3.1. Macdonald functions. — Macdonald polynomials were introduced in [Mac88]. They are symmetric polynomials whose coefficients are rational functions in $\mathbb{Q}(q,t)$, and degenerate to Schur functions when $q = t$. The motivation for introducing them was that several families of symmetric functions [(2)] originating in representation theory and harmonic analysis, namely Schur, zonal, Jack and Hall-Littlewood polynomials, satisfied a very similar theory, calling for an interpolating family. Most properties of Schur polynomials, such as orthogonality, summation identities, combinatorial formulas, extend to Macdonald polynomials in a similar way, up to deformations based on the parameters q, t.

2. The term "symmetric function" refers to, informally speaking, a symmetric polynomial in infinitely many variables. This is the best formalism to work with symmetric polynomials, so as to forget the number n of variables. The algebra of symmetric functions can be defined as an inverse limit from the algebra of symmetric polynomials in n variables. We refer to [Mac95] for details.

Macdonald functions are usually defined as the unique family of symmetric functions $\{P_\lambda\}$, indexed by integer partitions λ, such that they are triangular in the basis of monomial symmetric functions, i.e.

$$P_\lambda = \sum_{\mu \leqslant \lambda} u_{\lambda,\mu} m_\mu, \text{ where } m_\lambda = \sum_{\alpha\,:\,\mathrm{sort}(\alpha)=\lambda} x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

and $u_{\lambda,\mu} \in \mathbb{Q}(q,t)$ with $u_{\lambda,\lambda=1}$, satisfying the orthogonality condition

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \text{ if } \lambda \neq \mu,$$

where the scalar product $\langle \cdot, \cdot \rangle_{q,t}$ is a deformation of the so-called Hall inner product. It is defined as

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \prod_{j \geqslant 1} j^{m_j} m_j! \mathbb{1}_{\lambda=\mu}$$

where $p_\lambda = \prod_i p_{\lambda_i}$ are power-sum symmetric polynomials, i.e. $p_k(x) = \sum_i x_i^k$, and we used the notation that for a partition λ, its multiplicities m_j are the number of times the integer j appears in the list of coordinates of λ. This inner product was proposed by Macdonald in order to interpolate between the cases of Schur, Jack and Hall-Littlewood functions. Letting $b_\lambda = \langle P_\lambda, P_\lambda \rangle_{q,t}$, it is customary to define a dual family $Q_\lambda = P_\lambda / b_\lambda$, so that $\langle P_\lambda, Q_\mu \rangle_{q,t} = \mathbb{1}_{\lambda=\mu}$.

1.3.2. Macdonald measures. — Borodin and Corwin generalized the Schur measure in [BC14] by replacing Schur polynomials by Macdonald functions. For $q,t \in [0,1)$, they defined the probability measure

$$\mathbb{P}^{\mathrm{Macdonald}}(\lambda) = \frac{1}{H(a;b|q,t)} P_\lambda(a_1,\dots,a_n|q,t) Q_\lambda(b_1,\dots,b_m|q,t), \tag{1.3.1}$$

where P_λ and Q_λ are Macdonald polynomials and the normalization constant is

$$H(a;b|q,t) = \prod_{i,j} \frac{(ta_ib_j;q)_\infty}{(a_ib_j;q)_\infty}, \quad \text{where } (a;q)_\infty = \prod_{k=0}^{\infty} (1-aq^k).$$

Macdonald measures and processes have many applications. Although the coordinates λ_i of the random partition no longer form a determinantal point process for general parameters q,t, it is still possible to express observables of interest in terms of integrals involving determinants, and eventually in terms of Fredholm determinants amenable for large scale asymptotic analysis. The study of Macdonald processes at $t=0$ led to q-deformations of integrable particle systems such as the q-TASEP and q-PushTASEP. As $q \to 1$, Macdonald symmetric functions become $\mathfrak{gl}_n$-Whittaker functions, which appeared previously in [OCo09] in the context solvable directed polymers. At $q=0$, Macdonald symmetric functions become Hall-Littlewood polynomials. The corresponding measures become related to the stochastic six-vertex model [Bor18], and after some scaling limit, the asymmetric simple exclusion process (ASEP). After yet another scaling limit of either ASEP or directed polymers, the exact formulas in the theory of Macdonald measures [BC14; BCGS16; BCFV15; BO17] allow to obtain information on the probability distribution of the KPZ equation.

1.3.3. Pfaffian or half-space variant. — In view of the usefulness of Macdonald measures to study growth processes beyond last passage percolation and TASEP, it is natural to look for a Pfaffian version of Macdonald measures, degenerating to the Pfaffian Schur measure (1.2.2) when $q=t$. We introduced and studied it with Alexei Borodin and Ivan

Corwin in [10], with a special case addressed in [7] with Michael Wheeler. Since Pfaffians do not really arise in the general case, we called the probability measures "half-space Macdonald measures" instead of "Pfaffian Macdonald measures". These are defined as

$$\mathbb{P}_{\boxslash}^{\text{Macdonald}}(\lambda) = \frac{1}{H^{\boxslash}(a;\varrho|q,t)} \mathcal{E}_\lambda(\varrho|q,t) P_\lambda(a_1,\dots,a_n|q,t) \tag{1.3.2}$$

where ϱ is a nonnegative specialization (3) of the ring of symmetric functions, and the symmetric function $\mathcal{E}_\lambda$ is defined as

$$\mathcal{E}_\lambda = \sum_{\mu' \text{even}} b_\mu^{\text{el}}(q,t) Q_{\lambda/\mu}.$$

The combinatorial coefficient $b_\mu^{\text{el}}(q,t)$ is defined in [Mac95] in such a way that the function $\mathcal{E}_\lambda$ is the appropriate analogue of the function τ_λ above and satisfies similar properties. The definition is chosen so that the normalization constant still has a factorized form:

$$H^{\boxslash}(a;\varrho|q,t) = \Phi(a|q,t) H(a;\varrho|q,t) \text{ with } \Phi(a|q,t) = \prod_{1\leqslant i<j\leqslant n} \frac{(a_i a_j;q)_\infty}{(a_i a_j;q)_\infty}.$$

The expression of the normalization means, when ϱ is the empty or zero specialization, that we have

$$\sum_{\lambda' \text{ even}} P_\lambda(a) = \Phi(a|q,t) \tag{1.3.3}$$

which is usually referred as the Littlewood summation identity for Macdonald functions.

In [10], we compute observables of various marginals of the Macdonald measures, typically allowing to determine the distribution of λ_1 or λ_n. Our method, similar to [BC14], is based on certain operators acting on symmetric functions and such that Macdonald polynomials are eigenfunctions. We also build dynamics preserving the class of half-space Macdonald processes (that is a probability measure on several partitions, which we will not define here). This allows to relate half-space Macdonald measures to q-deformed particle systems when $q = 0$, and to the log-gamma polymer model in a half-quadrant when q goes to 1. See Figure 5 for a chart illustrating the various limits and applications.

1.3.4. Definition of half-space Macdonald processes. — We define half-space Macdonald processes in a similar way as Schur process (Section 1.1.5). Schur functions are replaced by Macdonald functions, and the path γ now starts somewhere on the diagonal of the quadrant, and then progresses down-right to the bottom boundary – see Figure 6.

Slightly more formally, the half-space Macdonald process is a probability measure on the sequence of partitions $\Lambda_\gamma := (\lambda^v)_{v\in\gamma}$ such that

$$\mathbb{P}(\Lambda_\gamma) = \frac{1}{H^\gamma(q,t)} \mathcal{E}_{\lambda^\circ}(\varrho_\circ) \prod_{e\in\gamma} \text{weight}(e),$$

where $H^\gamma(q,t)$ is a normalization constant which is explicit, and the weight of off-diagonal edges are chosen as in the Macdonald process, i.e.

$$\text{weight}\left(\begin{tikzcd} \overset{\mu}{\bullet} \arrow[r, dash, "\varrho"'] & \overset{\lambda}{\bullet} \end{tikzcd}\right) = Q_{\lambda/\mu}(\varrho) \quad \text{and} \quad \text{weight}\left(\begin{tikzcd} \bullet\,\lambda \arrow[d, dash, "\varrho"] \\ \bullet\,\mu \end{tikzcd}\right) = P_{\lambda/\mu}(\varrho).$$

As in the Schur case, branching and summation properties of Macdonald functions imply that for any $v \in \gamma$, the law of λ^v is a half-space Macdonald measure. The joint distribution of coordinates λ_i^v is related in [10] to various models for various choices of q, t. This relation

3. The specialization ϱ needs to be such that $\mathcal{E}_\lambda(\varrho)$ and $P_\lambda(\varrho)$ are nonnegative. It is known that the set of such ϱ is exactly the same as for the Schur measure [Mat19].

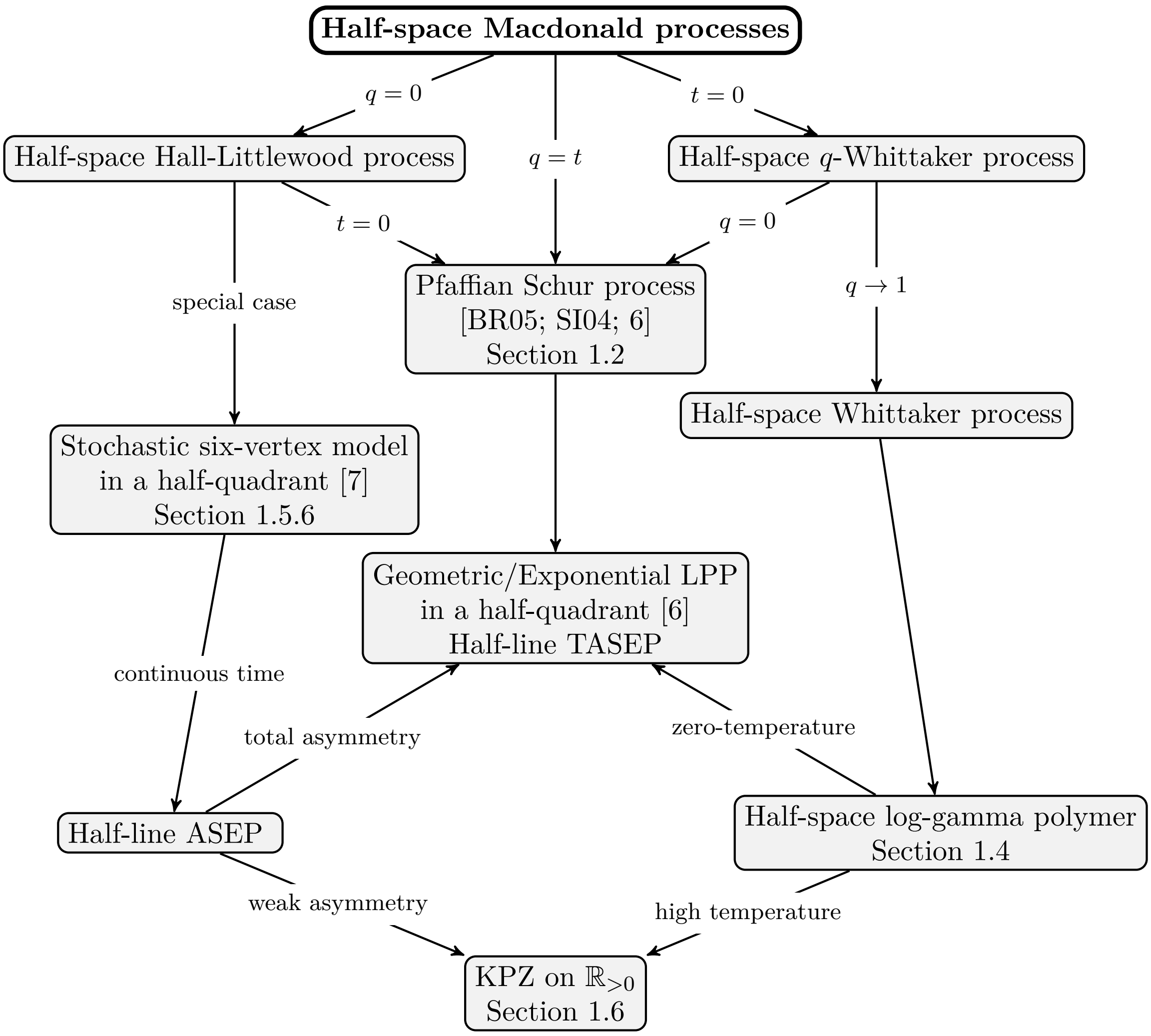


FIGURE 5. Hierarchy of half-space Macdonald processes and their degenerations. Arrows mean that one has to take a specialization of parameters or a scaling limit.

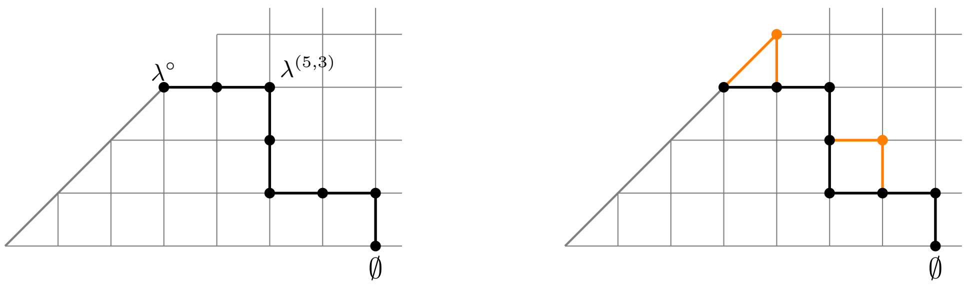
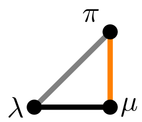


FIGURE 6. Left: A path γ used in the definition of half-space Macdonald processes. Right: When defining dynamics on half-space Macdonald measures, one considers elementary updates on down right paths. There are two kinds of updates, depicted in orange: bulk updates and boundary updates.

is based on the study of Markov dynamics on λ_γ, indexed by paths – see Figure 6 (right). There are, in principle, many dynamics which map the half-space Macdonald process Λ_γ on a path γ to the half-space Macdonald process $\Lambda_{\gamma'}$ on a path γ' obtained from γ after one of the updates depicted in Figure 6 (right). Regarding the bulk updates, dynamics were studied and classified in [MP17] in the context of the full-space Macdonald process. We can simply use the same bulk dynamics in the half-space setting. Regarding the boundary update, we need to define some dynamics. It is plausible that in some cases, such dynamics could be made through q-deformations or t-deformations of the RSK correspondence. We chose in [10] a simpler method: we use "push-block" type dynamics.

1.3.5. Push-block dynamics, first approach. — Consider a bulk update of the form:



We need to map the sequence $\Lambda_\gamma = (\lambda, \mu, \dots)$ to a random sequence $\Lambda_{\gamma'} = (\pi, \mu, \dots)$ by a Markov process with transition kernel denoted $U(\pi, \mu|\lambda, \mu)$. It is simpler to assume that during such an update of the path, we update the partition λ^v only for the vertex v which has been updated. The half-space Macdonald process on Λ_γ is weighted by $\mathcal{E}_\lambda(\varrho)P_{\mu/\lambda}(a)$. The half-space Macdonald process on Λ_γ is weighted by $\mathcal{E}_\pi(\varrho)Q_{\pi/\mu}(a)$. Hence, we need to find a kernel $U(\pi|\mu, \lambda)$ such that

$$\sum_\lambda U(\pi|\lambda, \mu)\mathcal{E}_\lambda(\varrho)P_{\mu/\lambda}(a) = \frac{H^\gamma(q,t)}{H^{\gamma'}(q,t)}\mathcal{E}_\pi(\varrho)Q_{\pi/\mu}(a).$$

A priori, there can exist many kernels $U(\pi|\mu, \lambda)$ satisfying such identity. However, if we assume that it does not depend on λ, i.e. $U(\pi|\mu, \lambda) = U(\pi|\lambda)$, then there is a unique choice, we must have

$$U(\pi|\lambda) = \frac{H^\gamma(q,t)}{H^{\gamma'}(q,t)}\frac{\mathcal{E}_\pi(\varrho)Q_{\pi/\mu}(a)}{\sum_\lambda \mathcal{E}_\lambda(\varrho)P_{\mu/\lambda}(a)}.$$

This assumption that dynamics do not depend on the partition that is updated is what characterizes the "push-block" dynamics – the term push-block refers to pushing and blocking mechanism arising in dynamics on Gelfand-Tsetlin patterns [WW09; BK08]. This choice is also related to a general method to couple together Markov chains satisfying some intertwining property [DF90].

Macdonald functions satisfy many properties, and one can show that

$$\sum_\lambda \mathcal{E}_\lambda(\varrho)P_{\mu/\lambda}(a) = \frac{H^\gamma(q,t)}{H^{\gamma'}(q,t)}\sum_\pi \mathcal{E}_\pi(\varrho)Q_{\pi/\mu}(a)$$

so that

$$\sum_\pi U(\pi|\mu) = 1,$$

which means that the kernel U is a well-defined Markov kernel.

These dynamics on half-space Macdonald processes are related to many models: when $q = t$, The coordinate largest λ_1 is related to last-passage percolation $G^{\boxslash}$ and symmetrized plane partitions. When $t = 0, q \in (0,1)$, both largest and smallest coordinates λ_1 and $\lambda_{\ell(\lambda)}$ are related to q-deformed interacting particle systems which are variants of the TASEP and the PushTASEP. When $t = 0$ and $q \to 1$, then one can show [10] that q^{λ_1} converges to the partition function of the half-space log-gamma polymer defined in Section 1.4. When $q = 0$, the length of partitions $\ell(\lambda)$ is related to the stochastic six-vertex model in a half-quadrant (Section 1.5.6).

1.3.6. Exact formulas. — In order to give a taste of the integral formulas that one can obtain in the general q, t case, we state the following:

Proposition 1.7 **([10]).** — *For any positive integer k,*

$$\mathbb{E}_{\square}^{\text{Macdonald}}\Big[\big(q^{\lambda_1}t^{n-1}+q^{\lambda_2}t^{n-2}+\cdots+q^{\lambda_n}\big)^k\Big]$$
$$=\frac{1}{(t-1)^k}\oint\frac{\mathrm{d}w_1}{2\mathbf{i}\pi}\cdots\oint\frac{\mathrm{d}w_k}{2\mathbf{i}\pi}\prod_{1\leqslant a<b\leqslant k}\frac{(tw_a-qw_b)(w_a-w_b)}{(w_a-qw_b)(tw_a-w_b)}\frac{(1-qw_aw_b)(1-tw_aw_b)}{(1-qtw_aw_b)(1-w_aw_b)}$$
$$\times\prod_{m=1}^{k}\left(\prod_{j=1}^{n}\frac{tw_m-a_j}{w_m-a_j}\,\frac{1-a_jw_m}{1-ta_jw_m}\right)\frac{H(qw_m;\varrho|q,t)(1-tw_m^2)}{H(w_m;\varrho|q,t)(1-w_m^2)}\,\frac{1}{w_m},\qquad(1.3.4)$$

where the contour for w_c, $1\leqslant c\leqslant k$, contains $\{a_1,\dots,a_n\}$ and the image of the $w_{c+1},\dots,w_k$ contours multiplied by q, and does not contain any other singularities of the integrand (this may restrict the choice of admissible specializations ϱ). All contours above are positively oriented.

Proof idea. — Define the shift operator $\mathbf{T}_{x_i}$ by

$$(\mathbf{T}_{x_i}F)(x_1,\dots,x_n)=F(x_1,\dots,qx_i,\dots,x_n),$$

and for any subset $I\subset\{1,\dots,n\}$ with r elements, define

$$A_I(x;t)=t^{\frac{r(r-1)}{2}}\prod_{i\in I,\,j\notin I}\frac{tx_i-x_j}{x_i-x_j}\,.$$

For $r=1,2,\dots,n$, the r^{th} Macdonald difference operator is

$$\mathbf{D}_r=\sum_{\substack{I\subset\{1,\dots,n\}\\|I|=r}}A_I(x;t)\prod_{i\in I}\mathbf{T}_{x_i}.\qquad(1.3.5)$$

It is such that

$$\mathbf{D}_rP_\lambda(a_1,\dots,a_n)=e_r(q^{\lambda_1}t^{n-1},\dots,q^{\lambda_n})P_\lambda(a_1,\dots,a_n),$$

where e_r is the rth elementary symmetric polynomial [(4)]. Now consider the identity

$$\sum_\lambda\mathcal{E}_\lambda(b)P_\lambda(a_1,\dots,a_n)=H^{\square}(a_1,\dots,a_n;\varrho).\qquad(1.3.6)$$

where we have omitted the dependency in q, t to simplify notations. If we apply $\mathbf{D}_1$ on both sides of (1.3.6), we get that

$$\mathbb{E}[q^{\lambda_1}t^{n-1}+\cdots+q^{\lambda_n}]=\frac{\mathbf{D}_1H^{\square}(a_1,\dots,a_n;\varrho)}{H^{\square}(a_1,\dots,a_n;\varrho)}$$

can be expressed as some explicit sum of products. If we express this sum as a contour integral and iterate the procedure k times, we obtain the claimed formula. □

It is not immediately clear how much asymptotic results can be obtained from a formula such as Proposition 1.7. Instead of using Macdonald difference operators $\mathcal{D}_r$ we may also use the operators

$$\mathbf{N}_z=\sum_{\eta_1,\dots,\eta_n=0}^{\infty}z^{|\eta|}\prod_{i<j}\frac{q^{\eta_j}x_j-q^{\eta_i}x_i}{x_j-x_i}\prod_{i,j}\frac{(tx_i/x_j)_{\eta_i}}{(qx_i/x_j)_{\eta_i}}\prod_{i=1}^{n}(\mathbf{T}_{x_i})^{\eta_i},$$

4. Actually, Macdonald polynomials can be constructed as eigenfunctions of the operator $\mathbf{D}_1$, which is self-adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle_{q,t}$, so that Macdonald functions are orthogonal. See [Mac95, Chap. VI, Section 4] for details.

such that

$$\mathbf{N}_z P_\lambda(a_1,\dots,a_n) = \prod_{i=1}^{n} \frac{(zq^{\lambda_i}t^{n-i+1};q)_\infty}{(zq^{\lambda_i}t^{n-i};q)_\infty} P_\lambda(a_1,\dots,a_n).$$

This eigenrelation is often attributed to Masatoshi Noumi (see [NS20] and references therein), hence the use of the letter $\mathbf{N}$. The eigenrelation can be deduced from the Pieri rules for Macdonald polynomials (see [BC14, Proposition 2.2.17]), by reversing the classical argument in Macdonald's book [Mac95] where the Pieri rules are proven from the eigenrelation involving Macdonald operators.

Employing the same method as in the proof of Proposition 1.7, now using Noumi's operator, we obtain explicit formulas for several $(q,t)-$analogues of Laplace transforms. For instance, we have the following:

Theorem 1.8 **([10]).** — *Let $z \in \mathbb{C}\setminus\mathbb{R}_{>0}$. Under mild assumptions on the specialization ϱ, we have*

$$\mathbb{E}^{\mathrm{Macdonald}}_{\square}\left[\prod_{i=1}^{n} \frac{(q^{-\lambda_i}t^{i}z;q)_\infty}{(q^{-\lambda_i}t^{i-1}z;q)_\infty}\right] = \sum_{k=0}^{n} \frac{1}{k!} \int_{R-\mathbf{i}\infty}^{R+\mathbf{i}\infty} \frac{\mathrm{d}s_1}{2\mathbf{i}\pi} \cdots \int_{R-\mathbf{i}\infty}^{R+\mathbf{i}\infty} \frac{\mathrm{d}s_k}{2\mathbf{i}\pi} \oint \frac{\mathrm{d}w_1}{2\mathbf{i}\pi} \cdots \oint \frac{\mathrm{d}w_k}{2\mathbf{i}\pi}$$
$$\mathcal{A}^{q,t}_{-\vec{s}}(\vec{w}) \prod_{i=1}^{k} \Gamma(-s_i)\Gamma(1+s_i) \frac{\overline{\mathcal{G}}^{q,t}(w_i)}{\overline{\mathcal{G}}^{q,t}(q^{-s_i}w_i)} \frac{\phi(w_i^2)(-z)^{s_i}}{\phi(q^{-s_i}w_i^2)(1-q^{s_i})w_i}, \quad (1.3.7)$$

where $R \in (0,1)$ is chosen such that $a_i < q^R < a_i/a_j$ for all i,j, the positively oriented contours for the variables w_j enclose all the a_i's and no other singularity, and we have used the shorthand notation

$$\mathcal{A}^{q,t}_{\vec{s}}(\vec{w}) := \prod_{1\leqslant i<j\leqslant k} \frac{(q^{s_j}w_j - q^{s_i}w_i)(w_i-w_j)\phi(q^{s_i+s_j}w_iw_j)\phi(w_iw_j)}{(q^{s_i}w_i - w_j)(q^{s_j}w_j - w_i)\phi(q^{s_i}w_iw_j)\phi(q^{s_j}w_jw_i)}$$

and

$$\overline{\mathcal{G}}^{q,t}(w) = \prod_{j=1}^{n} \frac{\phi(a_j/w)}{\phi(wa_j)} \frac{1}{H(w;\varrho)}, \quad \phi(z) = \frac{(tz;q)_\infty}{(z;q)_\infty}.$$

Remark 1.9. — Letting $(u_1,\dots,u_{2k}) := (w_1^{-1}, q^{\nu_1}w_1, \dots, w_k^{-1}, q^{\nu_k}w_k)$, we have

$$\mathcal{A}^{q,t}_{\nu}(\vec{w}) = \mathrm{Pf}\left[\frac{u_j-u_i}{1-u_iu_j}\right] \prod_{1\leqslant i<j\leqslant k} \frac{f(q^{\nu_i+\nu_j}w_iw_j)f(w_iw_j)}{f(q^{\nu_i}w_iw_j)f(q^{\nu_j}w_jw_i)} \prod_{i=1}^{k} \frac{1-q^{\nu_i}}{q^{\nu_i}w_i - w_i^{-1}}$$

where $f(u) = (tu;q)_\infty/(qu;q)_\infty$, and we have used Schur's Pfaffian identity

$$\mathrm{Pf}\left(\frac{x_i-x_j}{x_i+x_j}\right) = \prod_{i<j} \frac{x_i-x_j}{x_i+x_j}. \quad (1.3.8)$$

If $q=t$, the function f is constant, so that the whole integrand in (1.3.7) can be written as a Pfaffian. This is coherent with the fact that the Pfaffian Schur process determines a Pfaffian point process [BR05].

1.4. Application to the log-gamma polymer

1.4.1. The log-gamma polymer. — When $t=0$ and $q\to 1$, the half-space Macdonald measure on λ is such that $q^{-\lambda_1}$ converges the partition function of the log-gamma directed polymer in a half-quadrant. The log-gamma polymer was introduced by Seppäläinen in [Sep12] as the appropriate positive temperature generalization of exponential last passage percolation. Here, appropriate means that the model has a stationary structure described in terms of i.i.d. random variables, which allows to make explicit computations for the law

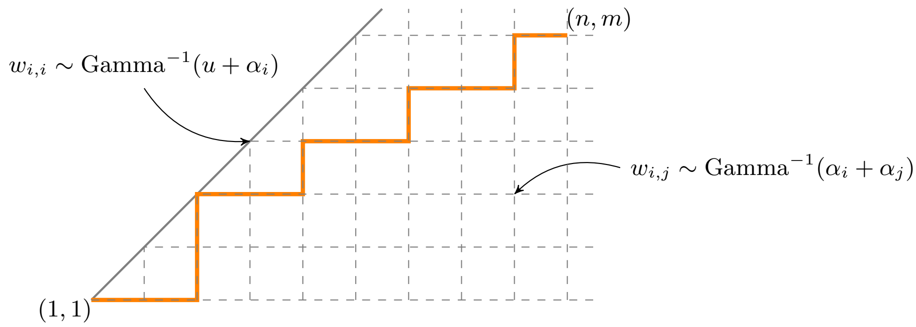


Figure 7. Illustration of the definition of the partition function of the half-space log-gamma polymer.

of large numbers of the free energy, and find fluctuation exponents: [Sep12] proved that the free energy of a polymer of length n has fluctuations of order $n^{1/3}$ and the polymer paths wander at distances $O(n^{2/3})$ around a straight line. For the model defined on $\mathbb{Z}^2$, it was then realized in [COSZ14] that the log-gamma polymer model possesses much more than a stationary structure. Via a variant of the RSK correspondence, the geometric RSK correspondence, the model is related to the Whittaker measure (the $t = 0, q \to 1$ limit of the Macdonald measure), and one can obtain exact formulas to characterize the free energy and prove Tracy-Widom asymptotics [BCR13].

In this section, we are interested in the log-gamma polymer model in a half-quadrant, that is when paths are confined below a wall. Let $\alpha_1, \alpha_2, \dots$ be positive parameters and $u \in \mathbb{R}$. Let $w_{i,j}$ be independent random variables such that $w_{i,j} \sim \mathrm{Gamma}^{-1}(\alpha_i + \alpha_j)$ for $i > j$ and $w_{i,i} \sim \mathrm{Gamma}^{-1}(u + \alpha_i)$. The partition function of the half-quadrant inverse-gamma polymer is

$$\mathcal{Z}^{\boxslash}(n, m) = \sum_{\pi:(1,1)\to(n,m)} \prod_{(i,j)\in\pi} w_{i,j}$$

where π is restricted to the lower half of the quadrant – see Figure 7. At zero temperature, that is when α_i and u go to zero, the free energy $\log(\mathcal{Z}^{\boxslash}(n, m))$ converges to passage times $E_{\boxslash}(n, m)$ of half-space exponential LPP. At high temperature, the free energy process, as n, m vary, weakly converges to the solution of the KPZ equation on $\mathbb{R}_+$ with Neumann boundary condition at 0, with a boundary parameter controlled by the parameter u (this convergence will be discussed in more details in Section 1.6.3 and in chapter 2).

Remark 1.10. — As explained in Figure 4, the last passage times are the same when weights are symmetric with respect to the diagonal, or when one constrains paths to stay below the diagonal. At positive temperature, this is no longer true. However, the partition function with symmetric weights is related to the partition function with constrained paths by multiplying or dividing diagonal weights by 2. In the text, we defined our partition function $\mathcal{Z}^{\boxslash}$ as the sum over constrained paths, but it is useful to relate it to a partition function in a symmetrized array of weights [OSZ14].

A limit of Proposition 1.7 yields the following moment formula

Proposition 1.11 **([10]).** —

$$\mathbb{E}[\mathcal{Z}^{\boxslash}(n,m)^k] = \oint \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \oint \frac{\mathrm{d}z_k}{2\mathbf{i}\pi} \prod_{1\leqslant a<b\leqslant k} \frac{z_a - z_b}{z_a - z_b - 1} \frac{z_a + z_b}{1 + z_a + z_b}$$
$$\times \prod_{i=1}^{k} \frac{2z_i}{z_i - u + 1/2} \prod_{j=1}^{n} \left(\frac{1}{\alpha_j - z_i - 1/2} \right) \prod_{j=1}^{m} \left(\frac{1}{z_i + \alpha_j - 1/2} \right), \quad (1.4.1)$$

where the contours are such that for all $1 \leqslant c \leqslant k$, the contour for z_c encloses $\{-\alpha_j + 1/2\}_{1\leqslant j\leqslant m}$ and $\{z_{c+1}+1, \dots, z_k+1\}$, and excludes the poles at $u - 1/2$, $\alpha_j - 1/2$ (for $1 \leqslant j \leqslant n$) and $-1 - z_j$ (for $j \neq c$). Because the integrand decays at least quadratically at infinity, one may chose the contours to be all vertical lines such that the contour for the variable z_i is $r_i + \mathbf{i}\mathbb{R}$ where

$$\max_j \{k - \alpha_j - 1/2\} < r_k + k - 1 < \cdots < r_2 + 2 < r_1 < \min_j \{u - 1/2, \alpha_j - 1/2, 0\}.$$

We note that if $k > \alpha_i + \alpha_j$ or $k > \alpha_i + u$ for some $i < j$, the k-th moment of $Z(n,m)$ fails to exist. Thus, it is not possible from this formula to compute the Laplace transform of $Z(n,m)$.

However, one can compute the Laplace transform by taking a limit of Theorem 1.8. We will not state here the result. In the large scale limit, the asymptotic analysis of such formulas yield the same asymptotics as in the Baik-Rains transition, as expected. However, there is a catch: the asymptotic analysis is not rigorous. The main obstruction is that the coefficient $\mathcal{A}^{q,t}_{-\vec{s}}(\vec{w})$ in the formula can be bounded at best by C^{k^2} but $\sum_{k=0}^{\infty} \frac{C^{k^2}}{k!}$ is not summable. If the coefficient was a Pfaffian (as it is the case when $q = t$, or in certain large scale limits), we could use Hadamard's bound, leading to a summable series. But it does not seem possible to write this term as a Pfaffian. In the case of the full-space Macdonald measure, there are similar looking formulas and somewhat surprisingly, the analogous coefficient can be written as a Cauchy determinant for all values of q, t. This illustrates the additional difficulties posed by the half-space case.

1.4.2. Whittaker functions and their Plancherel theory. — Using the Plancherel theorem associated with the Whittaker transform, [OSZ14] suggested another Laplace transform formula. Using the framework of [OSZ14], there are analytic difficulties that prevent to prove this formula, essentially that Plancherel theorem can only be applied to L^2 functions. However, using the flexibility afforded by the specialization ϱ in our definition of half-space Macdonald measure, we are able to find the appropriate regularization of the log-gamma polymer free energy and obtain the following (which is a slight variant of the formula proposed in [OSZ14]):

Theorem 1.12 **([10]).** — *For $m \geqslant n$ and any $s > 0$, we have*

$$\mathbb{E}\left[e^{-s\mathcal{Z}^{\boxslash}(m,n)}\right] =$$
$$\frac{1}{n!} \int_{r-\mathbf{i}\infty}^{r+\mathbf{i}\infty} \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \int_{r-\mathbf{i}\infty}^{r+\mathbf{i}\infty} \frac{\mathrm{d}z_n}{2\mathbf{i}\pi} \prod_{i\neq j} \frac{1}{\Gamma(z_i - z_j)} \prod_{1\leqslant i<j\leqslant n} \frac{\Gamma(z_i + z_j)}{\Gamma(\alpha_i + \alpha_j)} \prod_{i,j=1}^{n} \Gamma(z_i - \alpha_j)$$
$$\prod_{i=1}^{n} \left(s^{\alpha_i - z_i} \frac{\Gamma(u + z_i)}{\Gamma(u + \alpha_i)} \prod_{j=n+1}^{m} \frac{\Gamma(\alpha_j + z_i)}{\Gamma(\alpha_j + \alpha_i)} \right) \quad (1.4.2)$$

where the real part r of the contour is such that $r + u > 0$ and $r > \alpha_i$ for all $1 \leqslant i \leqslant n$.

Let us explain in more details how this formula was guessed in [OSZ14] when $m = 0$, and what is the obstacle to prove it. Using the geometric variant of RSK correspondence, applied to an array of weights symmetric with respect to the diagonal, [OSZ14] proved that the probability distribution of $\mathcal{Z}^{\boxslash}(n, n)$ is the same as that of $x_1 \in \mathbb{R}_+$ when the vector $x \in \mathbb{R}_+^n$ has density equal to

$$e^{-u(x_1-x_2+x_3-x_4+\dots)}\Psi_{\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(x)\frac{1}{\prod_{i<j}\Gamma(\alpha_i+\alpha_j)\prod_i\Gamma(u+\alpha_i)}$$

where $\Psi_{\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(x)$ denotes a $\mathfrak{gl}_n$-Whittaker function. The notion of Whittaker function was introduced in the context of automorphic forms by Jacquet [Jac67]. Whittaker functions also arise in works on the quantum Toda lattice [Kos78] and mirror symmetry [Giv97]. For more background, we refer to Reda Chhaibi's PhD thesis [Chh13] or the survey [Lam13]. Explicit integral representations of Whittaker functions exist for general classical groups [GLO07; GLO08] though we will only work with GL_n and SO_{2n+1} in this text.

From the symmetric functions point of view, $\mathfrak{gl}_n$-Whittaker functions may be seen as a limit of a Macdonald polynomial $P_\lambda(a)$, when $t = 0$ and $q \to 1$, where the partition λ becomes the variable x and the set of symmetric parameters a become α in the limit [GLO12]. As a consequence, Whittaker functions satisfy many properties which are analogous to Schur functions. For instance, they satisfy a Cauchy identity

$$\int_{\mathbb{R}_+^n} e^{-1/x_n}\Psi_{\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(x)\Psi_{\boldsymbol{\beta}}^{\mathfrak{gl}_n}(x)\frac{dx}{x} = \prod_{i,j=1}^{n}\Gamma(\alpha_i+\beta_j). \tag{1.4.3}$$

They also satisfy an analogue of the combinatorial formula for Schur functions (1.1.4), which is called Givental's integral representation. This is actually used as the definition of Whittaker functions in many references.

Consider a triangular array $\boldsymbol{z} = (z_{i,j} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant i)$ (Figure 8A). The $\mathfrak{gl}_n$-Whittaker function is defined as an integral on triangular arrays with fixed bottom row:

$$\Psi_{\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(\boldsymbol{x}) = \int_{\mathbb{R}_+^{n(n-1)/2}}\prod_{i=1}^{n}\left(\frac{\prod_{j=1}^{i-1}z_{i-1,j}}{\prod_{j=1}^{i}z_{i,j}}\right)^{\alpha_i}\exp\left(-\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{z_{i+1,j+1}+z_{i-1,j}}{z_{i,j}}\right)\prod_{\substack{1\leqslant i<n\\1\leqslant j\leqslant i}}\frac{dz_{i,j}}{z_{i,j}}, \tag{1.4.4}$$

where $z_{n,j} := x_j$ for $1 \leqslant j \leqslant n$ and by convention $z_{i,j} = 0$ if (i, j) is out of the range $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant i$. The second factor in the integral may be written informally as

$$\exp\left(-\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{z_{i+1,j+1}+z_{i-1,j}}{z_{i,j}}\right) = \exp\left(\sum_{z\to z'}\frac{z'}{z}\right)$$

where the sum runs over all the edges of Figure 8A.

Using this definition of the Whittaker function, along with the Cauchy identity and the geometric version of the RSK correspondence, the connection between the log-gamma polymer and Whittaker functions arises exactly as the connection between Schur functions and last passage percolation, as explained in Section 1.1.2. For the log-gamma polymer model in a quadrant, this was discovered in [COSZ14].

1.4.3. The argument of O'Connell, Seppäläinen and Zygouras. — Let us go back to the log-gamma polymer in a half quadrant. In order to compute the Laplace transform of $Z^{\boxslash}(n, n)$ in (1.4.2), one needs to compute the integral

$$\int_{\mathbb{R}_+^n} e^{-sx_1}e^{-u(x_1-x_2+x_3-x_4+\dots)}\Psi_{\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(x)\frac{1}{\prod_{i<j}\Gamma(\alpha_i+\alpha_j)\prod_i\Gamma(u+\alpha_i)}. \tag{1.4.5}$$

$$
\begin{array}{ccccccccccc}
 & & & & & z_{1,1} & & & & & \\
 & & & & & \nearrow\ \searrow & & & & & \\
 & & & & z_{2,2} & & z_{2,1} & & & & \\
 & & & & \nearrow\ \searrow & & \nearrow\ \searrow & & & & \\
 & & & z_{3,3} & & z_{3,2} & & z_{3,1} & & & \\
 & & & \nearrow\ \searrow & & \nearrow\ \searrow & & \nearrow\ \searrow & & & \\
 & & z_{4,4} & & z_{4,3} & & z_{4,2} & & z_{4,1} & & \\
 & & \nearrow\ \searrow & & \nearrow\ \searrow & & \nearrow\ \searrow & & \nearrow\ \searrow & & \\
 & z_{5,5} & & z_{5,4} & & z_{5,3} & & z_{5,2} & & z_{5,1} & \\
 & \nearrow\ \searrow & & \nearrow\ \searrow & & \nearrow\ \searrow & & \nearrow\ \searrow & & \nearrow\ \searrow & \\
z_{6,6} & & z_{6,5} & & z_{6,4} & & z_{6,3} & & z_{6,2} & & z_{6,1}
\end{array}
$$

(A) $\mathfrak{gl}_n$ case, $n = 6$.

$$
\begin{array}{ccccccc}
 & & & & & z_{1,1} & \\
 & & & & & \nearrow\ \searrow & \\
 & & & & z_{2,2} & & 1 \\
 & & & & \nearrow\ \searrow & & \\
 & & & z_{3,3} & & z_{3,2} & \\
 & & & \nearrow\ \searrow & & \nearrow\ \searrow & \\
 & & z_{4,4} & & z_{4,3} & & 1 \\
 & & \nearrow\ \searrow & & \nearrow\ \searrow & & \\
 & z_{5,5} & & z_{5,4} & & z_{5,3} & \\
 & \nearrow\ \searrow & & \nearrow\ \searrow & & \nearrow\ \searrow & \\
z_{6,6} & & z_{6,5} & & z_{6,4} & & 1
\end{array}
$$

(B) $\mathfrak{so}_{2n+1}$ case, $n = 3$.

FIGURE 8. Tableau of integral variables $z_{i,j}$ (including the fixed bottom row) for $\mathfrak{gl}_n$ Whittaker functions and $\mathfrak{so}_{2n+1}$ Whittaker functions; an arrow $a \to b$ means that the term $\frac{a}{b}$ appears in the exponential term in the corresponding integral representations.

This integral may be transformed – at least heuristically – using the Plancherel theorem associated with Whittaker functions. For a function $g \in L^2\left(\mathbb{R}^n_{>0}, \prod_{i=1}^n \; dx_i/x_i\right)$, we define its *Whittaker transform* as

$$\widehat{g}(\boldsymbol{\lambda}) := \int_{\mathbb{R}^n_{>0}} g(\boldsymbol{x}) \Psi^{\mathfrak{gl}_n}_{\boldsymbol{\lambda}}(\boldsymbol{x}) \prod_{i=1}^n \frac{dx_i}{x_i}.$$

Let $L^2_{\mathrm{sym}}\left((\mathrm{i}\mathbb{R})^n, \mu_n(\boldsymbol{\lambda}) d\boldsymbol{\lambda}\right)$ be the space of square integrable functions which are symmetric with respect to permuting their variables, equipped with the *Sklyanin measure*

$$\mu_n(\boldsymbol{\lambda}) = \frac{1}{(2\pi \mathrm{i})^n n!} \prod_{j \neq k} \Gamma\left(\lambda_j - \lambda_k\right)^{-1}. \tag{1.4.6}$$

It is known that the Whittaker transformation $g \mapsto \widehat{g}$ is an isometry [Sem94; KL01]

$$L^2(\mathbb{R}^n_{>0}, \prod_{i=1}^n dx_i/x_i) \to L^2_{\mathrm{sym}}\left((\mathrm{i}\mathbb{R})^n, \mu_n(\boldsymbol{\lambda}) d\boldsymbol{\lambda}\right).$$

In particular, the following Plancherel formula holds. For all $f_1, f_2 \in L^2\left(\mathbb{R}^n_{>0}, \prod_{i=1}^n dx_i/x_i\right)$, we have

$$\int_{\mathbb{R}^n_+} f_1(\boldsymbol{x}) \overline{f_2(\boldsymbol{x})} \prod_{i=1}^n \frac{dx_i}{x_i} = \int_{(\mathrm{i}\mathbb{R})^n} \widehat{f_1}(\boldsymbol{\lambda}) \overline{\widehat{f_2}(\boldsymbol{\lambda})} \mu_n(\boldsymbol{\lambda}) d\boldsymbol{\lambda}. \tag{1.4.7}$$

Now, in order to compute (1.4.5), we may let

$$f_1(x) = e^{-u(x_1 - x_2 + x_3 - x_4 + \dots)}, \quad f_2(x) = e^{-s x_1} \Psi^{\mathfrak{gl}_n}_{\boldsymbol{\alpha}}(x).$$

The Whittaker transform of f_2 is easily obtained using the Cauchy identity (1.4.3). The Whittaker transform of f_1 is obtained using another identity called the Littlewood identity – analogous to (1.3.3) – which yields

$$\widehat{f_1}(\boldsymbol{\lambda}) = \prod_{i<j} \Gamma(\lambda_i + \lambda_j) \prod_{i=1}^n \Gamma(u + \lambda_i),$$

so that the Laplace transform formula (1.4.2) is

$$\int_{(\mathrm{i}\mathbb{R})^n} \widehat{f_1}(\boldsymbol{\lambda}) \overline{\widehat{f_2}(\boldsymbol{\lambda})} \mu_n(\boldsymbol{\lambda}) d\boldsymbol{\lambda}.$$

The issue with this argument is that $\widehat{f_1}$ does not belong to $L^2_{\mathrm{sym}}\left((\mathrm{i}\mathbb{R})^n, \mu_n(\boldsymbol{\lambda}) d\boldsymbol{\lambda}\right)$, it is not integrable in L^2 against the Sklyanin measure, as noticed in [OSZ14]. By considering the partition function $Z^{\boxslash}(m,n)$ for $m > n$, as we do in [10], one is lead to consider a more complicated function f_2 depending on more parameters, which is in L^2 for certain ranges of parameters when m is large enough. This allows to establish a Laplace transform formula for large m, depending on many parameters, and at the end it becomes not difficult to deduce the formula for all $m \geqslant n$.

Of course, there is some technical work behind this regularization procedure. The main technical input that we need is a decay bound on some function, denoted $\mathcal{T}(x)$, which is the Whittaker limit of the symmetric function $\mathcal{E}_\lambda$ in (1.3.2). The function $\mathcal{T}(x)$ is defined in such a way that its Whittaker transform is

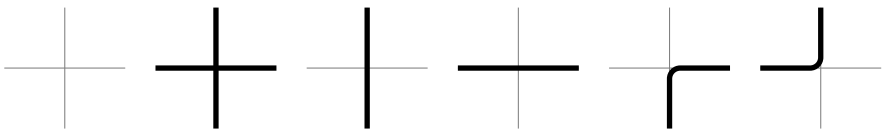
$$\prod_{1\leqslant i<j\leqslant n} \Gamma(\lambda_i+\lambda_j) \prod_{i=1}^{n} \left(e^{\tau \frac{\lambda_i^2}{2}} \prod_{j=1}^{m-n} \Gamma(\lambda_i+\alpha_{n+j}) \right),$$

where $\tau > 0$ is some regularization parameter. Our decay estimate [10, Proposition 6.16] is based on two ideas:

1. A similar idea as in Paley–Wiener type theorems, which roughly say that for a given function, the holomorphicity of its Fourier transform over a large enough domain in the complex plane can be translated into decay bounds at infinity for the function. It suffices to write the function as the Fourier inverse of its Fourier transform and analyze the integral using complex analysis. This idea works as well for the Whittaker transform.
2. An expansion of fundamental Whittaker functions as a linear combination of $\prod_{i=1}^n e^{\mathbf{i}\nu_i x_i}$, which we learnt from [OCo14], and comes from [Has82]. The coefficients arising in this decomposition were studied in [IS07]. The analysis of recurrence relations allows to bound them, and this eventually leads to bounds for the function $\mathcal{T}(x)$. We note that this decomposition of fundamental Whittaker functions is also very interesting from a probabilistic point of view, which is explored in [OCo22].

1.5. Application to the stochastic six-vertex model

1.5.1. Six-vertex model. — The six-vertex model is a probability measure on collections of interacting up-right paths in the lattice $\mathbb{Z}^2$. The random configuration is sampled according to a Gibbs measure: the probability to see a given configuration in some domain of $\mathbb{Z}^2$ is proportional to the product over all vertices of certain weights associated to each possible local configuration at a given vertex:



The weights are typically denoted by $a_1, a_2, b_1, b_2, c_1, c_2$ as in the Table 1. One may argue that one can take $c_1 = c_2$ without loss of generality (left turns and right turns must alternate). Then, it is convenient to write the weights in terms of three parameters a, b, c as well as electric and magnetic fields H, V [Bax82, Chap. 8]. The fields H, V will parametrize the density of vertical and horizontal paths. The parameters a, b, c however influence the shape of paths, and several phase diagrams are possible, based on the parameter

$$\Delta = \frac{a^2+b^2-c^2}{2ab}.$$

There are three cases: $\Delta > 1$ (ferroelectric phase), $-1 < \Delta < 1$ (disorder phase), and $\Delta < -1$ (anti-ferroelectric phase).

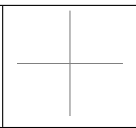
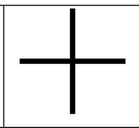
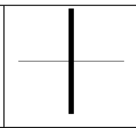
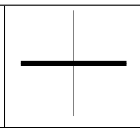
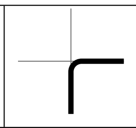
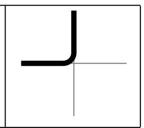
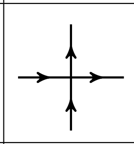
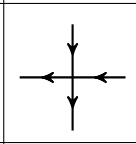
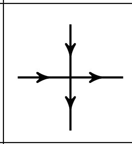
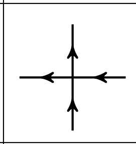
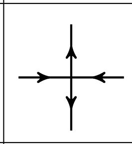
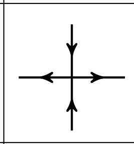
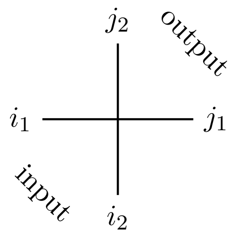
paths						
arrows						
General weights	a_1	a_2	b_1	b_2	c_1	c_2
Parametrization	$e^{-H-V}a$	$e^{H+V}a$	$e^{-H+V}b$	$e^{H-V}b$	c	c
Stochastic	1	1	b_1	b_2	$1-b_1$	$1-b_2$

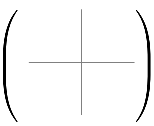
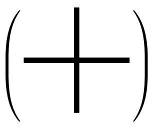
TABLE 1. Table of the six possible configurations and their weights.

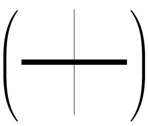
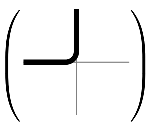
Our discussion will be restricted to the case where $\Delta > 1$, not for arbitrary weights parameters, but for a subset of parameters which are said stochastic, in the sense that certain weights will sum to 1 so that configurations will be sampled according to Markov chains. This special instance of the six-vertex model was first studied in [GS92].

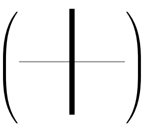
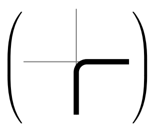
1.5.2. Stochastic six-vertex model. — We define a six-vertex configuration on $\mathbb{N}^2$ starting with a boundary condition such that there is an path arriving from the left on every row, and no path coming from the bottom on every column, see Figure 9. At every given vertex, we will denote by $i_1, i_2, j_1, j_2 \in \{0,1\}$ the number of paths on the south west north and east edges as follows:



The numbers i_1, i_2 will be regarded as inputs, and j_1, j_2 will be regarded as outputs. For any given input, the output will be sampled according to the probability weights:

$\mathbb{P}$ () $= 1$, $\mathbb{P}$ () $= 1$ (1.5.1a)

$\mathbb{P}$ () $= w_z$, $\mathbb{P}$ () $= 1 - w_z$ (1.5.1b)

$\mathbb{P}$ () $= qw_z$, $\mathbb{P}$ () $= 1 - qw_z$, (1.5.1c)

where we use the parametrization

$$w_z = \frac{1-z}{1-qz} \tag{1.5.2}$$

and the parameter z will have the form $z = a_x b_y$ when (x, y) is the position of the vertex. Hence, weights depend on two families of inhomogeneity parameters $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ which are often called rapidities or spectral parameters. It is also possible

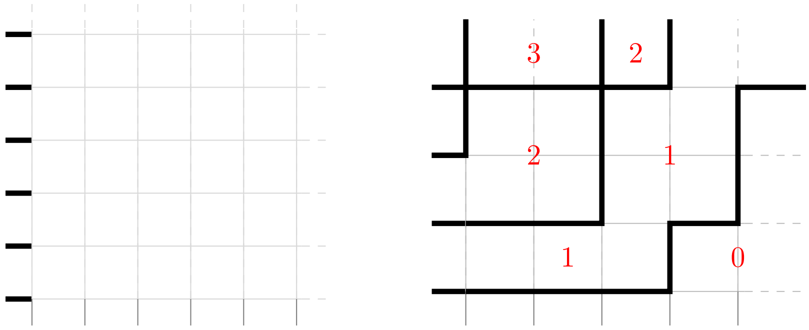


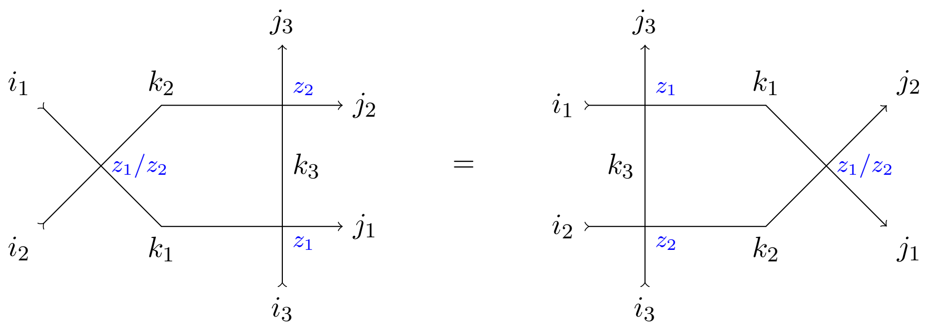
FIGURE 9. Left: boundary conditions for the configuration on $\mathbb{N}^2$. From the boundary on the left, a path at every row, while no path come from the bottom. Right: A possible configuration in the stochastic six-vertex model. We also indicate in red the value of the height function in different regions.

to make the weight depend on two extra families of inhomogeneity parameters [BP18], but we will not discuss this level of generality here.

To sample a configuration, given the boundary condition above, we may first sample the configuration of outgoing edges at the vertex $(0,0)$, according to the probability weights in (1.5.1b), since the input is determined by the boundary condition. Once the output is sampled, we may now sample the output of vertices $(0,1)$ and $(1,0)$, in any order, using the weights above. One can continue iteratively, whenever the output of the vertices (x,y) with $x+y=n$ is sampled, this determines the input configuration of the vertices $(x,y)$with $x+y=n+1$, so that one can sample their output and iterate. A possible configuration is shown in Figure 9.

In order to describe the configuration quantitatively, we define a height function $h(x,y)$ as the number of paths passing to the right of x, or at x, at row y – see Figure 9 – in such a way that paths are the level lines of the height function.

1.5.3. Yang-Baxter equation. — What is special with the parametrization of weights in (1.5.1) and (1.5.2) above? These weights satisfy the celebrated Yang-Baxter equation. It is often written in a graphical way. For all $i_1, i_2, i_3,\ j_1, j_2, j_3 \in \{0,1\}$,



(1.5.3)

We may view the pictures above as the intersection of three lines, labeled 1, 2 and 3. The lines are oriented as indicated in the picture, so that it is clear, for each vertex, which edges are the input, and which edges are the output. Each line $\ell \in \{1,2,3\}$ has input i_ℓ and output j_ℓ. Along each line, between the intersection with the other two, the state of the edge is denoted by k_ℓ (meaning that k_ℓ path is going through that segment). For each choice of state of internal edges $k_1, k_2, k_3 \in \{0,1\}$ we obtain a configuration, and we associate a weight to this configuration equal to the product of the three vertex weights, according to (1.5.1) and (1.5.2). At each vertex, the parameter z will be equal to either z_1, z_2 or z_1/z_2, according to the indications in blue in (1.5.3).

Then, the Yang-Baxter equation (1.5.3) means that after summing over all possible internal states k_1, k_2, k_3, the weights of the configurations is the same on both sides of (1.5.3).

We may restate this more explicitly. Let $w_z(i_1, i_2; j_1, j_2)$ denote the probability weight

$$w_z(i_1, i_2; j_1, j_2) = \mathbb{P}\left(\begin{array}{ccc} & j_2 & \\ i_1 & + & j_1 \\ & i_2 & \end{array} \right)$$

as in (1.5.1) and (1.5.2), and we use the convention that $w_z(i_1, i_2; j_1, j_2) = 0$ if $i_1 + i_2 \neq j_1 + j_2$. The Yang-Baxter equation (1.5.3) means that for all $i_1, i_2, i_3, j_1, j_2, j_3 \in \{0, 1\}$,

$$\begin{aligned} &\sum_{k_1,k_2,k_3 \in \{0,1\}} w_{z_1/z_2}(i_1, i_2 : k_1, k_2) w_{z_1}(k_1, i_3; j_1, k_3) w_{z_2}(k_2, k_3; j_2, j_3) \\ &\qquad = \sum_{k_1,k_2,k_3 \in \{0,1\}} w_{z_2}(i_2, i_3; k_2, k_3) w_{z_1}(i_1, k_3; k_1, j_3) w_{z_1/z_2}(k_1, k_2; j_1, j_2). \end{aligned} \tag{1.5.4}$$

It is natural to arrange the weights $w_z(i_1, i_2; j_1, j_2)$ in a matrix of size 4×4 that we may view as acting on the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$. To see how to do this naturally, let us assume that $(\mathsf{e}_0, \mathsf{e}_1)$ is a basis of $\mathbb{C}^2$. Then, $\mathbb{C}^2 \otimes \mathbb{C}^2$ is spanned by the $\mathsf{e}_{i_1} \otimes \mathsf{e}_{i_2}$, where $i_1, i_2 \in \{0, 1\}$. Hence, at each vertex, input and output states can be associated to the basis elements of $\mathbb{C}^2 \otimes \mathbb{C}^2$. We will denote by $R(z)$ the matrix

$$R(z) = (w_z(\mathsf{i}; \mathsf{j}))_{\mathsf{i},\mathsf{j} \in \{0,1\}^2},$$

which is acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$R(z)\ \mathsf{e}_{i_1} \otimes \mathsf{e}_{i_2} = \sum_{j_1,j_2} w_z(i_1, i_2; j_1, j_2) \mathsf{e}_{j_1} \otimes \mathsf{e}_{j_2}.$$

Explicitly, we have

$$R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q w_z & 1 - q w_z & 0 \\ 0 & 1 - w_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(1-z)}{1-qz} & \frac{1-q}{1-qz} & 0 \\ 0 & \frac{z(1-q)}{1-qz} & \frac{1-z}{1-qz} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, we denote by $R_{ij}(z)$ the endomorphism of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which is acting as $R(z)$ on the ith and jth tensor product, and acts trivially as identity on the third tensor. This means that, for example,

$$R_{13}(z)\ \mathsf{e}_{i_1} \otimes \mathsf{e}_{i_2} \otimes \mathsf{e}_{i_3} = \sum_{j_1,j_2,j_3} \mathbb{1}_{i_2=j_2} w_z(i_1, i_3; j_1, j_3) \mathsf{e}_{j_1} \otimes \mathsf{e}_{j_2} \otimes \mathsf{e}_{j_3}.$$

Then, the Yang-Baxter equation (1.5.3), reformulated as (1.5.4), is equivalent to the algebraic identity

$$R_{23}(z_2) R_{13}(z_1) R_{12}(z_1/z_2) = R_{12}(z_1/z_2) R_{13}(z_1) R_{23}(z_2). \tag{1.5.5}$$

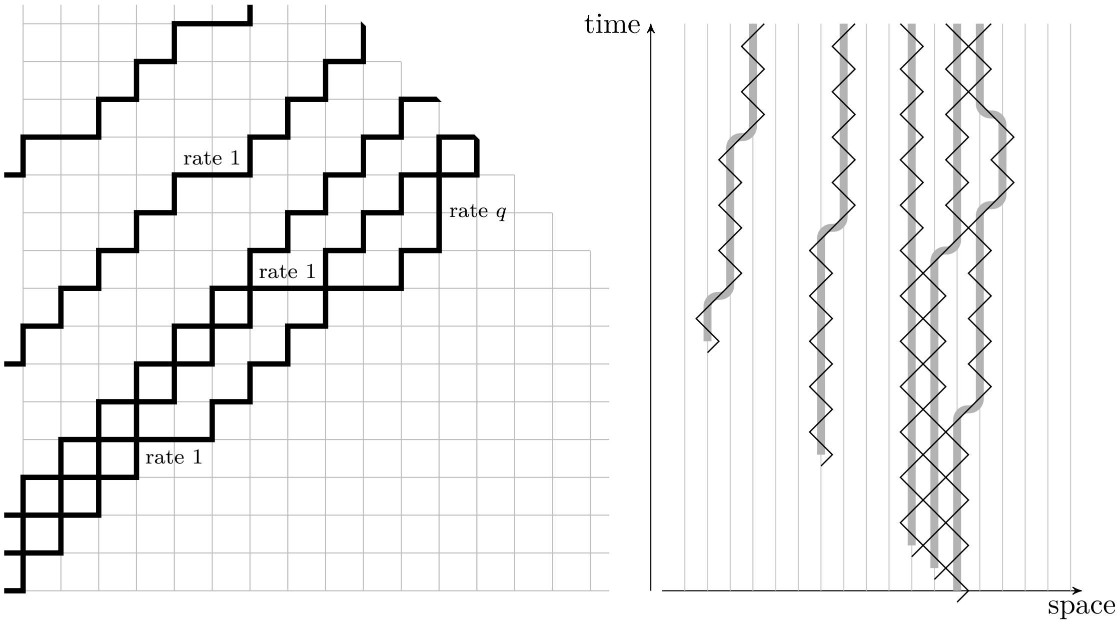


FIGURE 10. Illustration of the limit from the stochastic six-vertex model to ASEP in the limit where w_z goes to zero. Paths almost always zig-zag, except when they traverse a vertex straight, which happens with probability ε or $q\varepsilon$. After we rotate the picture by 45 degrees and rescale time by ε^{-1}, the two events occur at exponential rate 1 and q. Paths may be seen as the space-time trajectories of particles in ASEP (the gray thick paths on the right).

1.5.4. Connection to ASEP. — There are two important ways to see the stochastic six-vertex model. On one hand, it is a special case of the six-vertex model, which is some equilibrium system (a Gibbs measure) in 2d statistical mechanics. On the other hand, the stochasticity of weights allows to sample the configuration as a Markov chain, so that paths may be seen as space-time trajectories of particles in a $1+1$ dimensional out of equilibrium system [GS92]. We shall consider the second point of view in more details.

Consider the stochastic six-vertex model in the quadrant $\mathbb{N}^2$ and let us chose parameters in such a way that for all vertices, $w_z = \varepsilon$. If ε is very small, then the path will almost always zig zag — see Figure 10 — except at certain times when a vertex ┿ or ┿ will arise. After rotating by 45 degres, and scaling time by ε^{-1}, paths correspond to the space-time trajectory of a particle system. Figure 10 illustrates the fact in the $\varepsilon \to 0$ limit, particles jump at rate 1 to the right, and at rate q to the left, and cannot jump when the target site is occupied. These are exactly the rules of the asymmetric simple exclusion process (ASEP).

1.5.5. Comments on exact solvability. — Transition probabilities of ASEP on $\mathbb{Z}$ can be expressed as multiple contour integrals, using the (coordinate) Bethe ansatz [TW08b]. This work of Tracy and Widom has a huge impact in the field. It allowed them to prove that fluctuations of the current and tagged particles positions are asymptotically distributed as in the Kardar–Parisi–Zhang class, i.e. follow the GUE Tracy Widom distribution [TW08a; TW09]. It also led to characterizing the probability distribution of the Kardar–Parisi–Zhang equation [ACQ11; SS10] (through a limit from ASEP to the KPZ equation [BG97]).

The solvability of ASEP can be reinterpreted through that of the stochastic six-vertex model. Indeed, the stochastic six-vertex model is solvable via coordinate Bethe ansatz

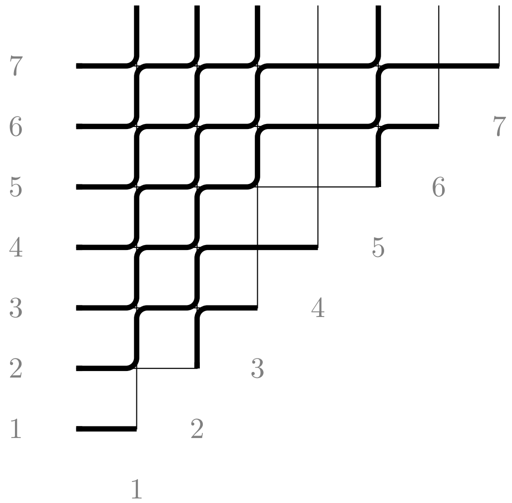


FIGURE 11. Sample configuration of the stochastic six-vertex model in a half-quadrant.

[GS92; BCG16], or algebraic Bethe ansatz [BP18], and related to Macdonald measures [Bor18]. In the following, we will see how the stochastic six vertex model can be analyzed using the framework of (half-space) Macdonald measures when $q = 0$ (when Macdonald symmetric functions become Hall-Littlewood polynomials). One may wonder why it is interesting to use such sophisticated methods if, as we have just mentioned, the model can be studied via standard methods such as the Bethe ansatz. The reason is that Bethe ansatz typically produces formulas which are sums over the symmetric group (or the hyperoctaedral group in the half-space case). Such formulas are not directly amenable for asymptotic analysis. In the free fermionic cases (e.g. models related to Schur processes), these sums over permutations become determinants or Pfaffians, and can be analyzed asymptotically. Using (half-space) Macdonald processes, we will see that we can relate models which are not free fermionic (ASEP, stochastic six-vertex model, the log-gamma polymer, the KPZ equation) to free fermionic models (typically the Schur measure).

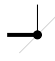
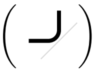
1.5.6. Stochastic six-vertex model in a half-quadrant. — In [7], we studied the stochastic six-vertex model in a half-quadrant – see Figure 11.

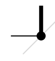
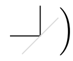
To define the model, one needs to prescribe boundary weights. The inhomogeneity parameters need to be the same on horizontal rows and columns with the same index, so as to satisfy a boundary Yang-Baxter equation (sometimes called reflection equation or KRKR relation). Hence the model depends now on a single family of parameters $a_1, a_2, \dots$ The boundary weights can be chosen quite arbitrarily. They are conveniently parametrized (following [He22]) by a density of the form $\varrho = \frac{1}{1+\nu^{-1}}$ and an extra parameter τ as

$$\mathbb{P}\Big(\quad\Big) = 1 - \mathbb{P}\Big(\quad\Big) \quad = B(a,\tau,\nu) := \frac{1-a^2}{(1-a\nu\tau)(1+a/\nu)} \tag{1.5.6a}$$

$$\mathbb{P}\Big(\quad\Big) = 1 - \mathbb{P}\Big(\quad\Big) \quad = \tau B(a,\tau,\nu) \tag{1.5.6b}$$

where a is the row/column parameter. In our work [7], we restricted to a very special case where $\nu = 1$, which corresponds to a density $1/2$ and $\tau = 1$, so that

$\mathbb{P}\Big($  $\Big) = 1 - \mathbb{P}\Big($  $\Big) \quad = 1$

$\mathbb{P}\Big($  $\Big) = 1 - \mathbb{P}\Big($  $\Big) \quad = 1$

It was recognized later [He24] that the general case can also be treated using half-space Macdonald processes.

Our first result is an identity in distribution relating the half-space Hall-Littlewood measure and the stochastic six-vertex model in a half-quadrant. It is a half-space analogue of a similar identity proved in [Bor18; BBW16].

Proposition 1.13 **([7]).** — *Let λ be a random partition sampled according to the half-space Macdonald measure* (1.3.2) *with ϱ being the empty specialization, in the case $q = 0$ and $t = \tilde{q}$. We denote by λ' the dual partition.*

Let $\mathfrak{h}(n,n)$ be the height function at point (n,n) of the stochastic six-vertex model on a half-quadrant with asymmetry parameter $\tilde{q}$, row/columns parameters $a_1, a_2, \dots$ The height function at (n,n) is defined as the number of paths exiting from the n-th row. For example, we have $\mathfrak{h}(7,7) = 4$ in Figure 11. Then,

$$\lambda'_1 \overset{(d)}{=} \mathfrak{h}(n,n).$$

The proof is based on a graphical argument adapted from [BBW16] and [WZ16], which combines the Yang-Baxter and reflection equations. Proposition 1.13 allows to use general results on half-space Macdonald measures to study the height function $\mathfrak{h}(n,n)$. For instance, one can obtain moment formulas of the random variable $q^{\mathfrak{h}(n,n)}$ using variants of Proposition 1.7. However, as we have already mentioned, expansions such as Theorem 1.8 are not convenient for asymptotic analysis.

1.5.7. q-independence trick. — In order to go further, we need another input, which is the following miracle formula conjectured by Betea, Wheeler and Zinn-Justin [BWZ15], and later proved by Rains [Rai18]. Let n be an even positive integer and $u \in \mathbb{C}$. Macdonald symmetric polynomials satisfy

$$\frac{1}{\Phi(x|q,t)} \sum_{\lambda' \text{ even}} \prod_{i \text{ even}} \left(1 - u q^{\lambda_i} t^{n-i}\right) \; b^{\mathrm{el}}_{\lambda}(q,t) P_{\lambda}(x_1, \dots, x_n | q,t)$$
$$= \frac{\operatorname{Pf}\left[\dfrac{x_i - x_j}{1 - x_i x_j} - u \dfrac{x_i - x_j}{1 - t x_i x_j}\right]}{\operatorname{Pf}\left[\dfrac{x_i - x_j}{1 - x_i x_j}\right]}, \quad (1.5.7)$$

where Pf denotes the Pfaffian (for an antisymmetric $2n \times 2n$ matrix A, the Pfaffian $\operatorname{Pf}(A)$ is such that $\operatorname{Pf}(A)^2 = \det(A)$. It can be computed as a sum over permutations in S_{2n} of products of entries in the matrix A). The most striking fact in the identity (1.5.7) is that the left hand side seem to depend on q in a highly intricate way, but the right-hand-side is obviously independent of q! This yields identities that relate half-space Macdonald measures for different values of q.

Consider a set of parameters $a_1, \dots, a_n \in (0,1)$ for n even, and let us denote by $\mathbb{P}^{\mathrm{Schur}}_{\boxslash}$ and $\mathbb{E}^{\mathrm{Schur}}_{\boxslash}$ the half-space Macdonald measure and expectation associated with parameters

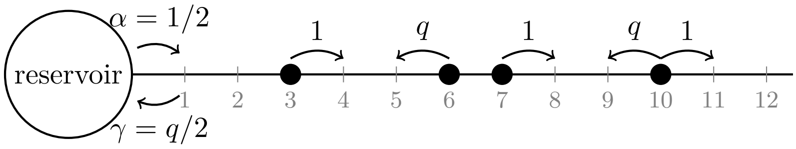


FIGURE 12. Jump rates in the half-line ASEP.

a_i in the Schur case $q = t$. We stress that $\mathbb{P}_{\llcorner}^{\mathrm{Schur}}$ and $\mathbb{E}_{\llcorner}^{\mathrm{Schur}}$ do not depend on the parameter $q = t$. The identity (1.5.7) implies that for any $q, t \in [0, 1)$ and $u \in \mathbb{C}$

$$\mathbb{E}_{\llcorner}^{\mathrm{Macdonald}}\left[\prod_{i \text{ even}}\left(1 - uq^{\lambda_i}t^{n-i}\right)\right] = \mathbb{E}_{\llcorner}^{\mathrm{Schur}}\left[\prod_{i \text{ even}}\left(1 - ut^{\lambda_i+n-i}\right)\right].$$

Letting $u = -t^x$, dividing both sides by $(-t^x; t^2)_\infty$, and with a bit of work, one arrives at the following:

Proposition 1.14 **([7]).** — *Let $\mathbb{E}_{\llcorner}^{HL}$ denote the expectation with respect to the half-space Macdonald measure $\mathbb{P}_{\llcorner}^{\mathrm{Macdonald}}$ when $q = 0$ and ϱ is trivial (HL stands for Hall-Littlewood, as Macdonald functions become Hall-Littlewood polynomials when $q = 0$). For any $x \in \mathbb{R}$, even positive integer n, and any parameters $(a_1, \dots, a_n) \in (0, 1)^n$,*

$$\begin{aligned}\mathbb{E}_{\llcorner}^{\mathrm{HL}}\left[\frac{1}{(-t^{x+n-\ell(\lambda)}, t^2)_\infty}\right] &= \mathbb{E}_{\llcorner}^{\mathrm{Schur}}\left[\prod_{j=1}^{n}\frac{1}{\mathsf{f}_x(\lambda_j + n - j)}\right] \\ &= \mathrm{Pf}\left[\mathsf{J} + (\mathsf{f}_x - 1)\cdot\mathsf{K}^{\mathsf{C}}\right]_{\ell^2(\mathbb{Z}_{\geqslant 0})},\end{aligned} \tag{1.5.8}$$

where

$$\mathsf{f}_x(j) = \frac{(-t^{x+j+1}; t^2)_\infty}{(-t^{x+j}; t^2)_\infty}.$$

and the kernel K^{C} is the kernel associated to the Pfaffian point process $\mathbb{Z}\setminus\{\lambda_i - i\}_{i\geqslant 1}$ when λ is sampled according to $\mathbb{P}_{\llcorner}^{\mathrm{Schur}}$.

The left hand side of (1.5.8) should be viewed as a deformation of a Laplace transform, characterizing the probability distribution of $\ell(\lambda)$. The right hand side is a Fredholm Pfaffian, which can be analyzed asymptotically with the same methods as for Fredholm determinants.

Taking a limit from the stochastic six-vertex model to the asymmetric simple exclusion process (ASEP) with injection rate $\alpha = 1/2$ and ejection rate $\gamma = q/2$ (5) — see Figure 12 — we obtain that the total number $N_0(t)$ of particles in the system at time t satisfies the following asymptotics, generalizing (1.2.4).

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{N_0\left(\frac{t}{1-q}\right) - t/4}{2^{-4/3}t^{1/3}} \geqslant s\right) = F_1(-s). \tag{1.5.9}$$

Subsequently, Jimmy He [He24] proved the full generalization of Corollary 1.6, for almost all values of injection α and ejection γ parameters. His proof uses a somewhat similar idea, reducing the appropriate Hall-Littlewood measure to some Pfaffian formula, but it is based on symmetric functions identities of Imamura, Mucciconi and Sasamoto [IMS21b; IMS22] instead of (1.5.7).

5. We do not really need here to introduce these variables α and γ, but these are the canonical notations for injection/ejection rates in the literature

It was definitely expected that ASEP behave at large scale as TASEP, by universality. Hence, the most interesting consequence of Proposition 1.13 is not the ASEP asymptotics (1.5.9) but rather an application to the half-space KPZ equation that we describe next.

1.6. Application to the KPZ equation

1.6.1. Boundary conditions for the KPZ equation. — The half-space KPZ equation on $\mathbb{R}_+$ with Neumann-type boundary with parameter $u \in \mathbb{R}$ is the stochastic PDE

$$\begin{cases} \partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x), & t \geqslant 0, x \in \mathbb{R}_+ \\ \partial_x h(t,x)\big|_{x=0} = u, \end{cases} \tag{KPZ$_u$}$$

where ξ is a space-time white noise. It is well known that the KPZ equation on $\mathbb{R}$ is ill-posed, but all solution theories, such as for instance the one involving regularity structures [Hai13], agree with the Cole-Hopf interpretation, whereby $h(t,x) = \log Z(t,x)$ and $Z(t,x)$ solves the multiplicative noise stochastic heat equation

$$\partial_t Z(t,x) = \frac{1}{2}\Delta Z(t,x) + Z(t,x)\xi(t,x). \tag{1.6.1}$$

For the equation on a domain with a boundary, such as $\mathbb{R}_+$, the definition of solutions is less canonical due to subtleties in the meaning of boundary parameters, see [GH19]. Nevertheless, the following definition is the physically correct one, based on the directed polymer interpretation, and based on the scaling limit of several discrete models.

Definition 1.15. — We say that $h(t,x) \in C(\mathbb{R}_+, C(\mathbb{R}_+,\mathbb{R}))$ is a solution to the half-space KPZ equation with parameter $u \in \mathbb{R}$ and initial condition $h_0 \in C(\mathbb{R}_+,\mathbb{R})$ if $h(t,x) = \log Z(t,x)$ and Z is a solution to the multiplicative noise stochastic heat equation (1.6.1) with initial condition $Z_0 = e^{h_0}$ and boundary condition

$$\partial_x Z(t,x)\big|_{x=0} = (u-1/2)Z(t,0).$$

As explained in [CS18], this means that

$$Z(t,x) = \int_{\mathbb{R}_+} dy\, p_t(x,y)Z_0(y) + \int_0^t ds \int_{\mathbb{R}_+} dy p_s(x,y)Z(s,y)\xi(s,y),$$

where $p_t(x,y)$ is the heat kernel on $\mathbb{R}_+$ with boundary condition

$$\partial_x p_t(x,y)\big|_{x=0} = (u-1/2)p_t(0,y).$$

When $u > 0$, it may be interpreted as the transition kernel for a Brownian motion killed at the boundary at rate u. When $u = 0$, this is the transition kernel associated to reflected Brownian motion.

We will also consider the solution starting from more general initial conditions, in particular the so-called narrow wedge solution, which is such that $Z_0(x) = \delta_0(x)$.

The shift by $1/2$ in the definition of the boundary parameter is somewhat puzzling. It is not simply due to some Itō correction when going from Z to $\log Z$ (otherwise the sign would be different [6]). The convention that we impose is justified by the fact that, with this convention, we have $\partial_x \mathbb{E}[h(t,x)]\big|_{x=0} = u$. The presence of this shift is related to nontrivial correlations between the increment $h(t,x) - h(t,0)$, which is locally a Brownian motion with drift u, and the value $h(t,0)$ at the boundary.

6. If the shift by $1/2$ was due to some Itō correction, the factor $(u-1/2)$ would be $(u+1/2)$. Indeed, if $B(x)$ denotes a standard Brownian motion and $Z(x) = e^{B(x)+ux}$, then Itō formula yields $dZ(x) = (u+1/2)Z(x)dx + Z(x)dB(x)$.

1.6.2. Probability distribution of the half-space KPZ equation. — The following result characterizes the probability distribution of $h(t,0)$. It is obtained as a scaling limit of Proposition 1.14.

Theorem 1.16. — *Let $h(t,x)$ be the solution to the half-space KPZ equation with parameter $u=0$ and narrow wedge initial condition. For any $z>0$,*

$$\mathbb{E}\left[e^{-ze^{\frac{t}{12}+h(2t,0)}}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{\sqrt{1+4ze^{t^{1/3}\mathfrak{a}_i^{\mathrm{GOE}}}}}\right] \tag{1.6.2}$$

where $\{\mathfrak{a}_i^{\mathrm{GOE}}\}_{i\geqslant 1}$ is the GOE point process, i.e. the limit of eigenvalues of a GOE matrix, rescaled near the edge in such a way that $\mathfrak{a}_1^{\mathrm{GOE}}$ has distribution F_1.

From Theorem 1.16, it is easy to deduce that

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h(t,0)+\frac{t}{24}}{(t/2)^{1/3}} \leqslant s\right) = \mathbb{P}\left(\mathfrak{a}_1 \leqslant s\right) = F_1(s).$$

Theorem 1.16 is a half-space version of a celebrated result about the full space KPZ equation, whose solution is temporarily denoted by $h_{\mathbb{R}}(t,x)$ below, to distinguish from the half-space solution:

$$\mathbb{E}\left[e^{-ze^{\frac{t}{12}+h_{\mathbb{R}}(2t,0)}}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{1+ze^{t^{1/3}\mathfrak{a}_i^{\mathrm{GUE}}}}\right] \tag{1.6.3}$$

where $\{\mathfrak{a}_i^{\mathrm{GUE}}\}_{i\geqslant 1}$ is now the GUE point process. This result is essentially due to the four independent works [ACQ11; SS10; CDR10; Dot10], though it was stated in this form only a few years later in [BG16].

Open Problem 1.17. — *Is there a generalization of* (1.6.2) *for arbitrary u and for arbitrary x? The natural point process to consider is the universal Pfaffian point process interpolating between GUE/GSE/GOE point processes that is defined in [6; 5].*

Remark 1.18. — In a series of works, [IMS21a; IMS21b; IMS22], Imamura, Mucciconi and Sasamoto related certain observables of the half-space Macdonald measure to the free boundary Schur measure [BBNV18]. Since the latter is a Pfaffian point process, this leads to Fredholm Pfaffian formulas for the half-space log-gamma polymer, and after a scaling limit, for the half-space KPZ equation. This provides a rigorous proof of conjectural formulas from [DKLT20; KL20] obtained using the conjectural completeness of Bethe ansatz wave functions of the delta-Bose gas Hamiltonian on $\mathbb{R}_+$. The work [IMS22] also leads to Baik-Rains transition asymptotics – i.e. an analogue of Corollary 1.6 – for the log-gamma polymer and the half-space KPZ equation.

The method is close in spirit to the argument presented in Section 1.5.7, with important differences. They do not rely on the refined Littlewood identity 1.5.7 but on some new symmetric function identity proved in [IMS21b]. That identity itself has further consequences about other models, such as the half-space ASEP [He24]. Moreover, in the special case where the boundary parameter $u=0$, it is not clear how to relate the formula of [KL20; IMS22] for the Laplace transform of $Z(t,0)$ to our Theorem 1.16.

Open Problem 1.19. — *Is there a multi-point generalization of* (1.6.3)*?*

1.6.3. KPZ equation as a limit of directed polymer models. — The half-space KPZ equation may also be studied as a limit of the half-space log-gamma polymer free energy. The formula from Theorem 1.12 is not particularly convenient as it would require to let n there go to infinity and in that limit the series is not very well controlled. However, the moment formula from Proposition 1.11 is quite powerful because it allows to vary the inhomogeneity parameters α_i. Before explaining how this is used, let us recall how the free energy of the log-gamma polymer converges to the KPZ equation. The convergence was first studied in [AKQ14]. The half-space case was first studied in [Wu20a; Par19a; 19].

The general idea is that up-right paths in Figure 7 become Brownian motions in the scaling limit, and we need to rescale the partition function in such a way that the product of weights along the path become the integral of a white noise along a Brownian paths. The simplest instance is to let $\alpha_i = \alpha$ scale to infinity and keep the boundary parameter u fixed. Let

$$\mathscr{Z}_n(t,x) = \frac{\mathscr{Z}(nt + x\sqrt{n}, nt - x\sqrt{n})}{(2\sqrt{n})^{nt}}, \quad \alpha = 1/2 + \sqrt{n}.$$

The reason for this choice of scaling is that with such α a random variable $w \sim \mathrm{Gamma}^{-1}(2\alpha)$ have expectation $\mathbb{E}[w] = \frac{1}{2\alpha-1} = \frac{1}{2\sqrt{n}}$. The scalings chosen ensure that $\mathbb{E}[\mathscr{Z}_n]$ converges to the heat kernel on $\mathbb{R}_+$ (with the correct boundary condition). Using essentially the same method as in [AKQ14], based on the chaos series expansion associated to the multiplicative noise stochastic heat equation, one shows that

$$\mathscr{Z}_n(t,x) \Longrightarrow Z(t,x),$$

where Z is the solution to the multiplicative noise stochastic heat equation with delta initial condition from Definition 1.15. In other terms, the free energy $\log \mathscr{Z}_n(t,x)$ converges to $h(t,x)$ the half-space KPZ equation. Such convergence is proven in [Wu20a] for fixed t,x, and we extended the result with Ivan Corwin in [19] to process level convergence and more general initial conditions, in order to solve a problem discussed in chapter 2.

1.6.4. "Stationary" KPZ equation in half-space. — In [11], we considered the solution to the half-space KPZ equation from Definition 1.15 with boundary parameter u and initial condition

$$h_0(x) = B(x) - vx,$$

where B is a standard Brownian motion. As we discuss in more details in Section 2.5, when $u = -v$, this initial condition is stationary in the sense that for all $t > 0$, the process $x \mapsto h(t,x) - h(t,0)$ remains distributed as a Brownian motion with drift $u = -v$.

It turns out that the moments of $e^{h(t,x)}$ can still be computed as a limit of Proposition 1.11, by choosing the first row parameter as $\alpha_1 = v$, and scaling the other parameters $\alpha_i, i \geqslant 2$ as in Section 1.6.3.

Let $Z(x,t) = e^{h(t,x)}$ and assume that $v > k - 1/2$, and $u + v > k$. Then, we have

$$\mathbb{E}\left[Z(x,t)^k\right] = 2^k \frac{\Gamma(u+v)}{\Gamma(u+v-k)} \int_{r_1 + \mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \int_{r_k + \mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_k}{2\mathbf{i}\pi} \prod_{1 \leqslant a < b \leqslant k} \frac{z_a - z_b}{z_a - z_b - 1} \frac{z_a + z_b}{z_a + z_b - 1}$$
$$\times \prod_{i=1}^{k} \frac{z_i}{z_i + u - 1/2} \frac{1}{(v-1/2)^2 - z_i^2} e^{t z_i^2 - x z_i}, \quad (1.6.4)$$

where the contours are chosen so that $v - 1/2 > r_1 > r_2 + 1 > \ldots, > r_k + k - 1 > \max\{k-1, k-u-1/2\}$.

For full-space models, this type of multiple integrals over contours nested into each other often arises in models related to Macdonald measures. Typically, it can be rewritten, after moving contours and rearranging residues, as a sum of contour integrals involving determinants. For half-space models, somewhat similar nested contour integral formulas arise, but the cross product is different and it is not known how to rearrange the residues in general. In some special case, there is a conjecture [BBC16, Conjecture 5.2] leading to integrals of Pfaffians. Applying that conjecture to the moment formula (1.6.4), we conjecture that for $u - 1/2 > k$ and $v - 1/2 > k$, we have

$$\mathbb{E}[Z(t,0)^k] = 2^k \frac{\Gamma(A+B+1)}{\Gamma(A+B+1-k)} \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \dots}} \frac{(-1)^{\ell(\lambda)}}{m_1! m_2! \dots} \int_{\mathbf{i}\mathbb{R}} \frac{dw_1}{2\mathbf{i}\pi} \cdots \int_{\mathbf{i}\mathbb{R}} \frac{dw_{\ell(\lambda)}}{2\mathbf{i}\pi}$$
$$\times \prod_{j=1}^{\ell(\lambda)} \frac{(w_j + 1/2)_{\lambda_j - 1}}{4(w_j)_{\lambda_j}} \mathrm{Pf}\left[\frac{u_i - u_j}{u_i + u_j}\right]_{i,j=1}^{2\ell(\lambda)}$$
$$\times E(w_1, w_1+1, \dots, w_1 + \lambda_1 - 1, w_2, \dots, w_2 + \lambda_2 - 1, \dots, w_{\ell(\lambda)}, \dots, w_{\ell(\lambda)} + \lambda_{\ell(\lambda)} - 1), \tag{1.6.5}$$

where we use the Pochhammer notation for rising factorials $(w)_\lambda = w(w+1)\dots(w+\lambda-1)$, we define variables u_i for $1 \leqslant i \leqslant 2\ell(\lambda)$ as

$$(u_1, \dots, u_{2\ell(\lambda)}) = (-w_1 + \tfrac{1}{2}, w_1 - \tfrac{1}{2} + \lambda_1, -w_2 + \tfrac{1}{2}, w_2 - \tfrac{1}{2} + \lambda_2, \dots, -w_{\ell(\lambda)} + \tfrac{1}{2}, w_{\ell(\lambda)} - \tfrac{1}{2} + \lambda_{\ell(\lambda)}), \tag{1.6.6}$$

and where

$$E(z_1, \dots, z_k) =$$
$$\prod_{i=1}^{k} \frac{e^{tz_i^2}}{(v-1/2)^2 - z_i^2} \sum_{\sigma \in BC_k} \sigma\left(\prod_{1 \leqslant j < i \leqslant k} \frac{z_i - z_j - 1}{z_i - z_j} \frac{z_i + z_j - 1}{z_i + z_j} \prod_{i=1}^{k} \frac{z_i}{z_i + u - 1/2} \right). \tag{1.6.7}$$

It turns out that the symmetrization can be performed using [BBC16, Equation (54)], relying on the theory of BC-symmetric Hall-Littlewood polynomials [Ven15], and one finds simply

$$E(z_1, \dots, z_k) = 2^k k! \prod_{i=1}^{k} \frac{e^{tz_i^2}}{(v-1/2)^2 - z_i^2} \frac{z_i^2}{z_i^2 - (u-1/2)^2}. \tag{1.6.8}$$

Assuming the validity of this conjecture, summing the moments to obtain a generating series (even though the series is divergent), and performing a saddle-point asymptotic analysis of the resulting Fredholm Pfaffian expansion, we conjecture precise formulas for the limiting fluctuations of $h(t,0)$. In particular, in the simplest case $u = v = 0$, we obtain an analogue of the celebrated Baik-Rains distribution (conjecturally, this limiting distribution also arises in [BFO20], through a limit of half-space last passage percolation, but our formulas are arguably simpler):

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{h(2t,0) + \frac{t}{12}}{t^{1/3}} \leqslant s \right) = F_{\mathrm{halfBR}}(s)$$

where F_{halfBR} is expressed in terms of the GOE and GUE Tracy-Widom distributions F_1 and F_2 respectively, as

$$F_{\mathrm{halfBR}}(s) = \partial_s \left[\frac{F_2(s)}{F_1(s)} \int_{-\infty}^{s} dt\, \frac{F_1(t)^4}{F_2(t)^2} \right]. \tag{1.6.9}$$

This formula is very reminiscent of the definition of the Baik-Rains distribution [BR00]. It can also be expressed in terms of the Hastings-McLeod solution $q(s)$ to the Painlevé II equation as

$$F_{\mathrm{halfBR}}(s) = \partial_s \left[e^{-\frac{1}{2}\int_s^{+\infty} dr [(r-s)q^2(r) - q(r)]} \int_{-\infty}^{s} dr\, e^{-2\int_r^{+\infty} dt\, q(t)} \right].$$

As we have stressed above, the computations rely on several conjectures. However, the probability distribution F_{halfBR} should also arise in other models in the KPZ universality class by universality. Among them, half-space last passage percolation is the most natural choice as its is related to Pfaffian Schur process, so that one can apply methods of asymptotic analysis of Fredholm Pfaffians. In fact, in a work in progress, Mattia Cafasso, Alessandra Occelli, Daniel Ofner and Harriet Walsh are computing the limiting distribution, and I hope that their formula will match with ours.

Remark 1.20. — As we will see in Section 2.5, the Brownian motion is not the only stationary solution of the half-space KPZ equation with drift u. There is an infinite family of stationary processes, parametrized by their drift at infinity, call it v. Surprisingly, such initial conditions can also be studied through the log-gamma polymer in a half-quadrant. We study the fluctuations of $h(t, 0)$ in that case in [17]. The methods are similar to the case of Brownian initial data, with additional technicalities.

1.7. Curious identities in distribution

1.7.1. Exchanging boundary and initial condition parameters. — Let $\mathcal{Z}_u^v(n, m)$ be the partition function of the half-space log-gamma polymer where we have set $\alpha_i \equiv \alpha$ for $i \geqslant 2$, the parameter along the first row is set to $\alpha_1 = v$ and the boundary parameter is u. Then, we have the identity in distribution [10, Proposition 8.1]

$$\mathcal{Z}_u^v(n, n) = \mathcal{Z}_v^u(n, n).$$

From the definition of the partition functions as sums over paths of products of weights, this identity is very puzzling. In the zero temperature limit, it degenerates to an identity from [BR01a, Corollary 7.6]. It is a special case of an identity in law satisfied by the half-space Macdonald measure (1.3.2) when the specialization ϱ is trivial, stated in [10, Proposition 2.6], generalizing the Schur case found in [BR01a, Corollary 7.6].

In the limit from the log-gamma polymer to the KPZ equation (see Section 1.6.3) we find that one can exchange the drift $-v$ of the initial condition and the boundary parameter u without changing the probability distribution of $h(t, 0)$. The law of $h(t, x)$ for $x > 0$ is not invariant, however, under the exchange of u and v. We note that the symmetry between u and v is not clear on the formula (1.6.4), which is valid for any $x \geqslant 0$, but it becomes obvious on (1.6.5) which is valid only for $x = 0$.

1.7.2. From half-space to full-space partition functions. — In their work on symmetrized variants of the Schur measure [BR01a], Baik and Rains also noticed identities in distributions between the last passage percolation times $G_{\boxtimes}(2n, 2n)$ and $G(n, n)$ when some parameters are appropriately chosen. In works with Pierre Le Doussal [14] and with Shouda Wang [23] we found generalizations and analogues of such identity in law for the log-gamma polymer and the KPZ equation.

Consider the following two solutions to the stochastic heat equation (1.6.1):

1. Let $Z_{\mathbb{R}_+}^{(u)}(t, x)$ be the solution to the multiplicative noise SHE (1.6.1) on $\mathbb{R}_+$ in the sense of Definition 1.15, with boundary parameter u, and delta initial condition.
2. Let $Z_{\mathbb{R}}^{\mathrm{half-Brownian}}(t, x)$ be the solution to the multiplicative noise SHE (1.6.1) on $\mathbb{R}$ with initial condition $Z(0, x) = \mathbb{1}_{x \geqslant 0} e^{B(x)+ux}$, that we call "half-Brownian" initial condition.

Proposition 1.21. — *We have the identity in distribution*

$$\int_x^{+\infty} Z_{\mathbb{R}_+}^{(u)}(t, y) dy \overset{(d)}{=} Z_{\mathbb{R}}^{\mathrm{half-Brownian}}(t, -x). \tag{1.7.1}$$

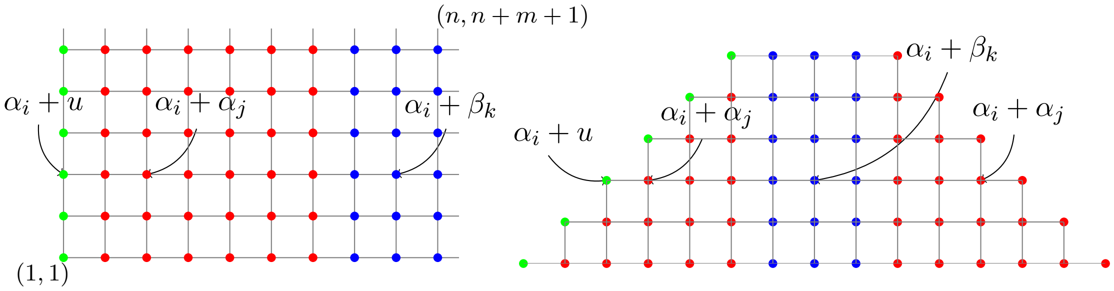


FIGURE 13. Parametrization of weights for the identity (1.7.2) to hold.

This identity was first guessed in a work with Pierre Le Doussal [14] where we computed the moments of the left hand side, that is

$$\int_x^{\infty} dy_1 \cdots \int_x^{\infty} dy_1 \mathbb{E}[Z_{\mathbb{R}_+}^{(u)}(t, y_1) \cdots Z_{\mathbb{R}_+}^{(u)}(t, y_k)]$$

using a variant of the moment formula (1.6.4). After playing with certain symmetrization identities, one notices a simplification that reduces to the well-known formula (coming essentially from [BC14]) for the moments of the right hand side. This identity is useful because the asymptotics of $\log Z_{\mathbb{R}}^{\text{half}-\text{Brownian}}$ are not very difficult to study, using the same methods as for the delta initial condition. Asymptotics depend on the drift of the initial condition and obey the Baik–Ben-Arous–Péché phase transition [BBP05]. On the other hand, the asymptotics of the left hand side of (1.7.1) are much more difficult a priori, as usual with half-space models. The identity (1.7.1) was eventually proved rigorously in [DS25, Proposition 8.1], by taking the scaling limit of a similar identity satisfied by the log-gamma polymer and that we described next.

Again, we define two type of partition functions for the log-gamma polymer:

1. Define the partition function for path in a trapezoidal domain (see Figure 13), defined as
$$\mathcal{Z}^{\frown}(n\|m) := \sum_{k=1}^{n} \mathcal{Z}^{\boxslash}(k, 2n-k+m+1),$$
where $\mathcal{Z}^{\boxslash}$ is the partition function of the log-gamma polymer in a half-quadrant defined in Section 1.4.
2. Let $\mathcal{Z}^{\square}(n, m)$ be the partition function of the log-gamma polymer in a rectangular grid of size $n \times m$, that is when paths are not constrained to stay below the diagonal.

We further assume that weights in both partition functions depend on two sets of parameters, α_i and β_j, as well as a boundary parameter u, as indicated in Figure 13.

Proposition 1.22. — *We have the identity in distribution*

$$\mathcal{Z}^{\frown}(n\|m) \overset{(d)}{=\!=} \mathcal{Z}^{\square}(n, n+m+1). \tag{1.7.2}$$

as long as parameters on both sides correspond as in Figure 13.

The identity (1.7.2) is proved by computing the probability distribution of both sides using the geometric RSK correspondence and variants of the half-space Whitttaker measure, drawing heavily on earlier results of Bisi and Zygouras [BZ19]. Actually, the $m = 0$ case of the identity can be guessed by applying formally the Plancherel theorem for Whittaker functions in some of the formulas in [BZ19]. Unfortunately, the application of the

Plancherel theorem there would not be valid, as one needs L^2 functions – a similar issue also arises in [OSZ14]. We find that the appropriate way to regularize both sides of the identity in order to prove it by an application of the Plancherel theorem is to have an extra parameter m and consider a partition function with m vertical layers in the middle – see the points in blue in Figure 13 (right).

Remark 1.23. — From a probabilistic perspective, the introduction of this parameter m is also interesting, as it provides information on the partition function $\mathcal{Z}^{\boxslash}(n, n+m)$. Actually, we exploit this information in [30] to prove that when the process

$$x \mapsto \frac{\log \mathcal{Z}(n, n + xn^{2/3}) - cn}{n^{1/3}}$$

is tight as $n \to \infty$.

Let us comment further on the origins of the identity (1.7.2). In the case $m = 0$, it translates in terms of Whittaker functions as follows:

$$r^{\sum_{i=1}^n \alpha_i} \int_{\mathbb{R}_+^n} e^{-r/x_n} \left(\prod_{i=1}^n (x_i/r)^{(-1)^i} \right)^u \Psi_{\boldsymbol{\alpha}}^{\mathfrak{so}_{2n+1}}(\boldsymbol{x}) \prod_{i=1}^n \frac{dx_i}{x_i}$$
$$= \int_{\mathbb{R}_+^n} e^{-rx_1 - 1/x_n} \Psi_{u,\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(\boldsymbol{x}) \Psi_{\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(\boldsymbol{x}) \prod_{i=1}^n \frac{dx_i}{x_i}. \quad (1.7.3)$$

where the $\mathfrak{so}_{2n+1}$-Whittaker functions are defined through an integral formula similar to the definition of $\mathfrak{gl}_n$-Whittaker function in (1.4.4), except that the summation over edges in Figure 8A is replaced by a summation over the edges of the half-triangular array in Figure 8B. The function $\Psi_{u,\boldsymbol{\alpha}}^{\mathfrak{gl}_n}(\boldsymbol{x})$ is a $\mathfrak{gl}_n$-Whittaker function depending on $n+1$ parameters (in general one can define $\Psi_{\beta_1,\dots,\beta_m}^{\mathfrak{gl}_n}$ for any $m \geqslant n$).

1.8. Outlook

To conclude this chapter, we describe potential developments related to half-space Macdonald processes and some ideas for future work.

1.8.1. An identity involving Koornwinder functions. — At zero temperature limit, when one considers geometric last passage percolation instead of the log-gamma polymer, the analogue of identity (1.7.2) was obtained in [BR01a, (7.59), (7.60)]. Its Symmetric function counterpart, i.e. the analogue of (1.7.3) was discussed later in [BZ22, Section 4] and [BZ16]. Integrals in (1.7.3) become sums over integer partitions and Whittaker functions become Schur functions. For even n, and when $m = 1$, the identity (1.7.3) becomes – in the special case $m = 0$ –

$$\frac{1}{H^{\boxtimes}(a;c)} \sum_{\lambda_1 \leqslant s} s_\lambda(a_1,\dots,a_n) s_\lambda(a_1,\dots,a_n,c) = \left(\prod_{i=1}^n a_i\right)^s \sum_{\lambda_1 \leqslant s} \tau_\lambda(c) sp_\lambda(a_1,\dots,a_n) \quad (1.8.1)$$

where sp_λ are symplectic Schur functions

$$sp_\lambda(x) = \frac{\det\left(x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)}\right)_{i,j=1}^n}{\det\left(x_i^{n-j+1} - x_i^{-(n-j+1)}\right)_{i,j=1}^n}$$

and

$$H^{\boxtimes}(a;c) = H(a;a,c) = \prod_{i<j} \frac{1}{(1-a_i a_j)^2} \prod_{i=1}^n \left(\frac{1}{1-a_i^2} \frac{1}{1-a_i c} \right).$$

Another combinatorial proof of this identity at the Schur function level was proposed in [BZ22], using generalizations of bounded Littlewood identities from [Ste90; Mac95; Oka98; Kra98]. Bounded Littlewood identities also appear in [Bet18] in a very similar context. Identities (1.7.3) and (1.8.1) suggest to look for generalizations:

- Even at the Schur functions level, it would be interesting to write and prove generalizations of (1.8.1) depending on a parameter m, i.e. where $\tau_\lambda(c)$ is replaced by $\tau_\lambda(c, b_1, \dots, b_m)$.
- It would be interesting to investigate whether the properties of Schur functions used in [Bet18; BZ22] may be generalized to Macdonald polynomials and Whittaker functions (see [RW21] for partial answers). The Macdonald analogue of the left hand side of (1.8.1) is
$$\frac{1}{H(a; a, c|q, t)} \sum_{\lambda_1 \leqslant s} P_\lambda(a_1, \dots, a_n) Q_\lambda(a_1, \dots, a_n, c).$$
The right hand side would be more interesting, it should be proportional to
$$\sum_{\lambda_1 \leqslant s} \mathcal{E}_\lambda(c) K_\lambda(a_1, \dots, a_n)$$
where K_λ is an appropriately specialized Koornwinder polynomial. As a first step towards understanding such an identity at the Macdonald functions level, it would be reasonable to start with studying the case $q = 0$, i.e. the case of Hall-Littlewood polynomials, where Koornwinder polynomials are called BC type Hall-Littlewood polynomials [Ven15]. Both sides of the potential identity would be related to partition functions of vertex models [Bet18; WZ16].

Remark 1.24. — Because Koornwinder polynomials satisfy less summation properties and cause issues of positivity, they have not been used much in integrable probability yet, but this is a compelling direction of research. A Macdonald analogue of (1.8.1) would be a step further towards that goal.

1.8.2. Around the Pfaffian refined identity. — Recall the Macdonald operator $\mathbf{D}_r$ in (1.3.5) and define the operator $\mathbf{D}(u) = \sum_{r=0}^n (-u)^r \mathbf{D}_r$. Macdonald functions P_λ are eigenfunctions of $\mathbf{D}(u)$ with eigenvalue
$$\sum_{r=0}^n (-u)^r e_r(q^{\lambda_1} t^{n-1}, q^{\lambda_2} t^{n-2}, \dots, q^{\lambda_n}) = \prod_{i=1}^n (1 - u q^{\lambda_i} t^{n-i}).$$
Consider the Cauchy identity for Macdonald polynomials
$$\sum_\lambda P_\lambda(x)\, Q_\lambda(y) = H(x; y|q, t). \tag{1.8.2}$$
Applying the operator $\mathbf{D}(u)$ to both sides yields
$$\frac{1}{H(x; y|q, t)} \sum_\lambda \prod_i \left(1 - u q^{\lambda_i} t^{n-i}\right)\, P_\lambda(x)\, Q_\lambda(y) = \frac{\det\left[\dfrac{1}{1 - x_i y_j} - u \dfrac{1}{1 - t x_i y_j}\right]}{\det\left[\dfrac{1}{1 - x_i y_j}.\right]}, \tag{1.8.3}$$
Indeed, $\mathbf{D}(u)$ is a combination of q-shifts of the form $\prod_{i \in I} \mathbf{T}_i$ with coefficients depending on t but not on q. Hence $\mathbf{D}(u) H(x; y|q, t)/H(x; y|q, t)$ does not depend on q, so that one

can compare the case where q is arbitrary and the case where $q=t$. This implies that

$$\frac{1}{H(x;y|q,t)}\sum_{\lambda}\prod_{i}\left(1-uq^{\lambda_i}t^{n-i}\right)\ P_\lambda(x)\ Q_\lambda(y)=$$
$$\prod_{i,j}(1-x_iy_j)\sum_{\lambda}\prod_{i}\left(1-ut^{n+\lambda_i-i}\right)\ s_\lambda(x)\ s_\lambda(y). \quad (1.8.4)$$

The right hand side can be computed explicitly, using the fact that Schur functions are ratios of determinants and Cauchy-Binet identity, and one arrives at the RHS of (1.8.3).

An analogue of (1.8.3) for the Littlewood identity is less well-understood. We have the Littlewood identity

$$\sum_{\lambda' \text{ even}} b_\lambda^{\mathrm{el}} P_\lambda(x)=\Phi(x|q,t),$$

and the refined version (already stated in (1.5.7))

$$\frac{1}{\Phi(x|q,t)}\sum_{\lambda' \text{ even}}\ \prod_{i \text{ even}}\left(1-uq^{\lambda_i}t^{n-i}\right)\ b_\lambda^{\mathrm{el}}P_\lambda(x_1,\dots,x_n)=\frac{\mathrm{Pf}\left[\dfrac{x_i-x_j}{1-x_ix_j}-u\dfrac{x_i-x_j}{1-tx_ix_j}\right]}{\mathrm{Pf}\left[\dfrac{x_i-x_j}{1-x_ix_j}\right]}.$$

It was first conjectured in [BWZ15] from the special case $q=0$, and proved soon after in [Rai18] using elliptic symmetric functions.

Open Problem 1.25. — *Can one define an operator $\mathbf{O}(u)$ acting on symmetric functions such that one can obtain the refined Littlewood identity by applying the operator on both sides of the Littlewood identity?*

1.8.3. Markov duality. — Many full-space growth models related to Macdonald measures satisfy a form of Markov duality, which may used as an alternative way to prove integral formulas. We start with some background.

Definition 1.26. — Two Markov processes $X(t)$ and $Y(t)$, defined respectively on state-spaces $\mathbb{X}$ and $\mathbb{Y}$, are dual with respect to a function $H:\mathbb{X}\times\mathbb{Y}\to\mathbb{R}$, if for all $x\in\mathbb{X}, y\in\mathbb{Y}$, and all $t\geqslant 0$,

$$\mathbb{E}^x\left[H(X(t),y)\right]=\mathbb{E}_y\left[H(x,Y(t))\right], \quad (1.8.5)$$

where in the L.H.S., $\mathbb{E}^x$ denotes the expectation with respect to the Markov process $X(t)$ started from $X(0)=x$, and in the R.H.S., $\mathbb{E}_y$ denotes the expectation with respect to the Markov process $Y(t)$ started from $Y(0)=y$.

Remark 1.27. — Taking derivatives in (1.8.5) with respect to t, and assuming the existence of generators of these processes, Markov duality implies that

$$\mathscr{L}^X H(x,y)=\mathscr{L}^Y H(x,y), \quad (1.8.6)$$

where $\mathscr{L}^X$ (resp. $\mathscr{L}^Y$) acts on functions from $\mathbb{X}$ to $\mathbb{R}$ (resp. from $\mathbb{Y}$ to $\mathbb{R}$) and denotes the generator of the Markov process $X(t)$ (resp. $Y(t)$). When H is uniformly bounded and belongs to the domain of L^X for any y (resp. to the domain of L^Y for any x), then (1.8.6) also implies (1.8.5) (see [JK14, Prop. 1.2]).

The full-space ASEP is the Markov process on $\{0,1\}^{\mathbb{Z}}$ with generator $\mathscr{L}^{\mathbb{Z}}$, acting on local functions $f:\{0,1\}^{\mathbb{Z}}\to\mathbb{R}$ by

$$\mathscr{L}^{\mathbb{Z}}f(\eta)=\sum_{x\in\mathbb{Z}}\left(p\eta_x(1-\eta_{x+1})+q\eta_{x+1}(1-\eta_x)\right)\left(f(\eta^{x,x+1})-f(\eta)\right). \quad (1.8.7)$$

In order to state ASEP's duality, it is convenient to introduce, for $y \in \mathbb{Z}$, the sets

$$\mathbb{W}^n_{\geqslant y} := \{\vec{x} \in \mathbb{Z}^n : y \leqslant x_1 < \cdots < x_n\}, \quad \mathbb{W}^n := \{\vec{x} \in \mathbb{Z}^n : x_1 < \cdots < x_n\}.$$

We introduce another operator $\mathscr{D}^{(n)}$ acting on functions $f : \mathbb{W}^n \to \mathbb{R}$ by

$$\mathscr{D}^{(n)} f(\vec{x}) = \sum_{\substack{1 \leqslant i \leqslant n \\ x_i - x_{i-1} > 1}} p\left(f(\vec{x}_i^-) - f(\vec{x})\right) + \sum_{\substack{1 \leqslant i \leqslant n \\ x_{i+1} - x_i > 1}} q\left(f(\vec{x}_i^+) - f(\vec{x})\right), \tag{1.8.8}$$

where for $\vec{x} \in \mathbb{W}^n$, $\vec{x}_i^\pm := (x_1, \dots, x_i \pm 1, \dots, x_n)$, and we use the notational convention that $x_0 = -\infty$ and $x_{n+1} = +\infty$. The operator $\mathscr{D}^{(n)}$ is also the generator of a continuous Markov process on the state space $\mathbb{W}^n$: the full-space ASEP with n particles, where the state of the particle system is now described by particle positions x_i rather than occupation variables η_i as previously. Note that in (1.8.7), p and q are the jump rates associated to right and left, while in (1.8.8), we have chosen the opposite orientation.

The full-space ASEP with right and left jump rates p and q (i.e. the Markov process with generator (1.8.7)) and the n-particle ASEP with right and left jump rates q and p (i.e. the Markov process with generator (1.8.8)) satisfy the following duality [BCS14, Theorem 4.2, Proposition 4.7]. Let the duality function be

$$H(\eta, \vec{x}) := \prod_{i=1}^n Q_{x_i}(\eta), \quad Q_x(\eta) = q^{N_x(\eta)}, \quad N_x(\eta) = \sum_{y \geqslant x} \eta_y.$$

For any $\eta \in \{0,1\}^{\mathbb{Z}}$ and $\vec{x} \in \mathbb{W}^n$,

$$\mathscr{L}^{\mathbb{Z}} H(\eta, \vec{x}) = \mathscr{D}^{(n)} H(\eta, \vec{x}). \tag{1.8.9}$$

Remark 1.28. — A different Markov (self-)duality for ASEP was first obtained by [Sch97] with respect to a different function, the function $\widetilde{H}(\eta, \vec{x}) := \prod_{i=1}^n \widetilde{Q}_{x_i}(\eta)$ where $\widetilde{Q}_x(\eta) = \eta_x q^{N_{x+1}(\eta)}$. It was inspired by the analysis of the XXZ spin chain and proved by writing ASEP generator as a tensor product of Pauli matrices and expressing symmetries in terms of the $U_q(\mathfrak{sl}_2)$ algebra. Another proof was provided in [BCS14] by writing out explicitly $\mathscr{L}^{\mathbb{Z}} \widetilde{H}$ and $\mathscr{D}^{(n)} \widetilde{H}$ and comparing the two expressions. We refer to [Kua19, Section 2.6.1] for a discussion of several Markov dualities satisfied by ASEP.

It is shown in [BCS14] that the operator $\mathscr{D}^{(n)}$ can be diagonalized using Bethe ansatz. As a consequence, one obtains closed integral formulas for certain obervables of the current, and using tricks from [IS11], they characterize the probability distribution of the current. These integral formulas obtained via Bethe ansatz can be transformed into sums of integrals of determinants using inspiration from Macdonald processes, and this eventually leads to Fredholm determinants and rigorous asymptotics. Somewhat surprisingly, this method works as well for many other models: the q-TASEP, the q-Hahn TASEP, the log-gamma polymer, etc. In general, models related to Macdonald measures and higher spin stochastic vertex models can be studied using this alternative method, without involving any symmetric function theory.

This side of the story is much less developed for half-space models. An important obstruction is that is it not clear how Pfaffians should appear in the integral formulas – despite a conjecture in [BBC16]. However, half-line ASEP does satisfy a Markov duality, which we obtained in [25]. Its form is extremely similar to (1.8.9), except that the dual system corresponds to n ASEP particles on $\mathbb{Z}_+$ and the system is killed at a certain rate when the first particle is at zero, i.e. when $x_1 = 0$ – we refer to [25, Theorem 2.4] for a precise statement.

Remark 1.29. — Interestingly, the open ASEP on a finite lattice $\{1, 2, \dots, \ell\}$ with arbitrary injection/ejection parameters satisfies as well a similar duality (see [25, Theorem 3.2]). The KPZ equation limit of this duality is equivalent to the following result. Let $Z(t, x)$ be the solution to the multiplicative noise Stochastic Heat Equation on $[0, 1]$ with boundary parameters u (left boundary) and v (right boundary). The model is discussed more precisely in Chapter 2 – see Section 2.3. We have that for $x_1 \leqslant \dots \leqslant x_k$,

$$\mathbb{E}\left[\prod_{i=1}^{k} Z(t, x_i)\right] = \mathbb{E}\left[\prod_{i=1}^{k} Z_0(B_i(t)) e^{-u\ell_t(B_1)(t)} e^{-v\ell_t(B_k)(t)} e^{\sum_{i<j} \ell_t(B_i - B_j)}\right] \tag{1.8.10}$$

where $\ell_t(B)$ denotes the local time of the process B at zero up to time t, and the expectation is taken with respect to k reflecting Brownian motions $0 \leqslant B_1 \leqslant \dots \leqslant B_k \leqslant 1$ on $[0, 1]$ reflected at the boundaries, starting from $x_1 \leqslant \dots \leqslant x_k$.

1.8.4. Half-space universal processes. — Growth processes in the Kardar–Parisi–Zhang class in $1 + 1$ dimensions all converge at large scale to the KPZ fixed point. This means that for a model described by a height function $H(x, t)$, there exists model-dependent constants such that, as a space-time process,

$$\frac{H(L^2x, L^3t) - cL^3}{c_2 L} \xrightarrow[L \to \infty]{} \mathfrak{h}(t, x)$$

where $\mathfrak{h}(t, x)$, called the KPZ fixed point, is a Markov process on the class of upper semi continuous functions. It was introduced in [MQR21] as a limit of TASEP's height function. The large scale-limit of models in the KPZ class may also be described using the directed landscape. Consider a model described by a free energy $F(x; y)$ between two points x and y in $\mathbb{R}^2$ or $\mathbb{Z}^2$ — $F(x; y)$ could also be some last-passage-time, or first passage time, or some other random pseudo distance function. In general, we expect that for any direction $\vec{u}_\circ$, there exists vectors $\vec{u} \parallel \vec{u}_\circ$ and $\vec{v} \perp \vec{u}_{circ}$ and a model dependent constant c so that

$$F(n^3 s\vec{u} + n^2 x\vec{v}; n^3 s\vec{u} + n^2 x\vec{v}) = (t - s)cn^3 - n\left(\mathcal{L}(x, s; y, t) + o_n(x, s; y, t)\right)$$

where $o_n(x, s; y, t)$ is a random function of $(x, s; y, t)$ which goes to zero uniformly on compacts in probability and $\mathcal{L}(s, x; t, y)$ is a four parameter stochastic process called the directed landscape. It was introduced in [DOV22] as a limit of Brownian last-passage percolation (the O'Connell-Yor model [OY02]). The relation between these two universal objects, KPZ fixed point and directed landscape, is given by the following variational formula

$$\mathfrak{h}(x, t) = \sum_{y \in \mathbb{R}} \left\{\mathfrak{h}(y, 0) + \mathcal{L}(y, 0; x, t)\right\}.$$

The definition of the directed landscape in [DOV22] is based on the limit of some last passage percolation time through a semi discrete lattice $\mathbb{R} \times \mathbb{Z}$ where the disorder is given by the Airy line ensemble $\{\mathcal{A}_i(x)\}_{i \geqslant 1, x \in \mathbb{R}}$ – see Figure 14. These curves arise as the universal scaling limit of many objects in the KPZ universality class and random matrices: Dyson Brownian motion, Dyson Brownian bridges, random tiling models, interacting random walks related to Schur or Macdonald processes, etc.

Let us discuss some properties

1. The Airy line ensemble satisfies a kind of spatial Markov property called Brownian Gibbs property. For all positive integers $n \leqslant m$ and reals $a < b$, conditionally on the values of $\mathcal{A}_i(t)$ for $i \notin [\![n, m]\!]$ or $t \notin (a, b)$, the law of $\mathcal{A}_n, \dots, \mathcal{A}_m$ on the interval $[a, b]$ is that of Brownian bridges (whose endpoints are determined by conditioning) conditioned not to intersect between themselves or with $\mathcal{A}_{n-1}$ and $\mathcal{A}_{m+1}$. This property is actually very strong: [AH23] showed that the Airy line ensemble is the only stochastic

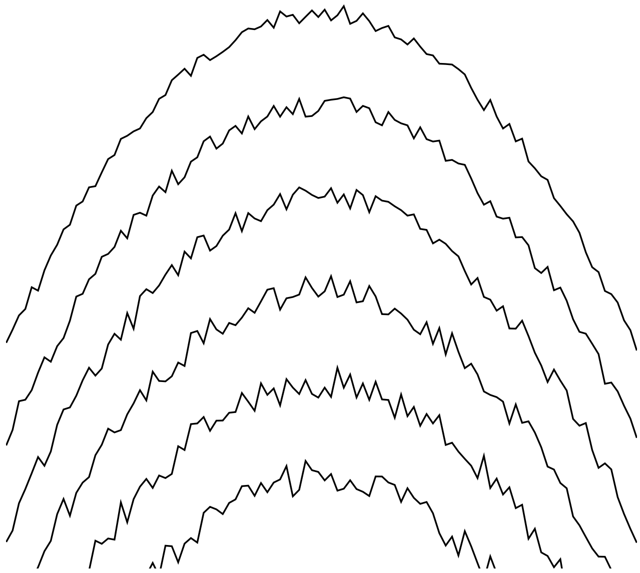


FIGURE 14. Ordered parabolic shaped random processes illustrating the Airy line ensemble

process on $\mathcal{Z}_{\geqslant 1} \times \mathbb{R}$ satisfying the Brownian Gibbs property and so that $\mathcal{A}_1(t) = -t^2 + o(t^2)$.

2. For any x, the probability distribution of $\mathcal{A}_1(x) + x^2$ is the Tracy Widom distribution F_2. The process $x \mapsto \mathcal{A}_1(x)$ is sometimes called the Airy$_2$ process (here the superscript 2 refers to the value of the inverse temperature β in the β ensemble framework, as in the notation F_2). This process describes the spatial correlations of the KPZ fixed point. In particular, starting from the narrow wedge initial condition, the probability distribution of the KPZ fixed point $\mathfrak{h}(t, x)$ is a scaled version of the AIry$_2$ process.
3. For any $t_1 < \cdots < t_k$ the probability distribution of the points $(i, \mathcal{A}_j(t_i))_{1 \leqslant i \leqslant k, j \geqslant 1}$ is a determinantal point process with an explicit kernel $K(i_{,1}, t_1; i_2, t_2)$ called the extended Airy kernel, introduced in [PS02].

In general, Schur processes from Section 1.1.5 as well as Macdonald processes may be viewed as a collection partitions $\lambda^{(1)}, \lambda^{(2)}, \dots$ or a collection of interacting random walks $x \mapsto \lambda_1^{(x)}, x \mapsto \lambda_2^{(x)}$ etc. They also satisfy some discrete version of the Brownian Gibbs property, are related to determinantal processes in the Schur case, and converge to the Airy line ensemble under various specializations. Viewing them as a line ensemble rather than a probability measure on partitions is sometimes useful in order to apply probabilistic tools. This is often used in order to study spatial correlations for instance: see [20] for an example related to the log-gamma polymer, though there are many other references such as [CH14; CH16; CD18; Ham19; JO20; Wu20b; CHH23] (this list if far from complete).

In [30] we initiated the study of half-space Gibbs line ensembles. We focused on the half-space log-gamma line ensemble, which is nothing but a probabilistic way to view the half-space Whittaker process as a collection of interacting random walks with log-gamma distributed increments. The main novelty is that the Gibbs resampling property has to be modified so as to include some boundary weight which is related to the function $\mathcal{E}_{\lambda^\circ}(\varrho_\circ)$ in the definition of the half-space Macdonald process in Section 1.3.4.

More precisely, the random walks $x \mapsto \lambda_i^{(x)}$ are defined for $x \in \{0, 1, 2, \dots\}$ and the interaction in the bulk is the same as for the full-space log-gamma line ensemble, but the probability distribution of the starting points is proportional to

$$e^{-u(\lambda_1^{(0)}-\lambda_2^{(0)}+\lambda_3^{(0)}-\lambda_4^{(0)}+\dots)}.$$

In the scalings so that the log-gamma polymer free energy converges to the KPZ equation, the half space log-gamma converges to a half-space KPZ line ensemble studied recently [DS25]. More importantly, half-space line ensembles associated to any half-space Macdonald processes are other KPZ class models should converge to a universal object, the half-space Airy line ensemble. This object was constructed recently in [DY25] by taking a limit of the Pfaffian Schur process from [BR05; 6].

Anticipating a bit, we expect that the Airy line ensemble will be key to defining a half-space directed landscape. It should be defined as the scaling limit of a last passage percolation model through a symmetrized half-space Airy line ensemble. We note that there exists another approach to defining the half-space analogue of the KPZ fixed point, via the study of half-space TASEP [Zha24].

CHAPTER 2

STATIONARY MEASURES OF GROWTH PROCESSES ON BOUNDED DOMAINS

In this chapter, I present several works on stationary measure on growth processes, especially models defined on a bounded domain with open boundary conditions. My first work in this area is [16], in collaboration with Pierre Le Doussal, published in a Physics journal. From a mathematical point of view, it contains a few mathematical observations and many conjectures. Conjectures related to half-space KPZ equation were proven in [19] with Ivan Corwin, using a symmetry property of half-space Macdonald processes. The method was also used in [18] in another context. Moreover, the form of stationary measures of the KPZ equation [16] and ASEP [21] motivated the introduction of the more general framework of two-layer Gibbs measures [26; 24].

2.1. Stationary measures of the KPZ equation

It is in general a difficult problem to determine the stationary measures of stochastic partial differential equations of the form

$$\partial_t u(t,x) = Hu(t,x) + \sqrt{2}\xi(t,x)$$

when ξ is a space-time white noise. There are at least two notable exceptions. First, if H is a linear operator, stationary measures should be Gaussian with a covariance structure that can be calculated – see for instance the discussion of invariant measures in Chapter 6 of Martin Hairer's lecture notes [Hai09]. Second, there is another special case, when Hu is the functional derivative of some action

$$Hu = -\frac{\delta S}{\delta u}(t,x). \tag{2.1.1}$$

In that case, the stochastic PDE should often be interpreted as the continuous limit of Glauber dynamics and the stationary measure is formally given by the Gibbs measure

$$e^{-S[u]}\mathcal{D}u$$

where $\mathcal{D}u$ is a fictitious Lebesgue measure on functions $x \mapsto u(x)$. Of course, such an object does not exist, but one can sometimes use the Brownian measure as a reference measure instead. This is the case for parabolic stochastic PDEs of the form

$$\partial_t u = \frac{1}{2}\Delta u - \frac{\delta F[u]}{\delta u} + \sqrt{2}\xi \tag{2.1.2}$$

in which case the corresponding action is $S[u] = \frac{1}{2}\int (u'(x))^2 dx + F[u]$ and one identifies

$$e^{\frac{-1}{2}\int (u'(x))^2 dx}\mathcal{D}u$$

with the Brownian probability measure $\mathbb{P}^{\text{Brown}}$ on continuous functions u, so that the stationary measure of (2.1.2) is

$$\frac{1}{\mathcal{Z}} e^{-F[u]} \mathbb{P}^{\text{Brown}}(u). \tag{2.1.3}$$

Remark 2.1. — In higher spatial dimension, the Brownian measure should be replaced by the Gaussian free field. Its lack of regularity typically makes it very difficult to define (2.1.3) and one often adopts the opposite point of view, as proposed by [PW80]. The Gibbs measure (2.1.3) is defined as the unique stationary measure of the stochastic PDE (2.1.2).

In this text, we are rather interested in the KPZ equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x) \tag{KPZ}$$

where we recall that the solution should be interpreted in the Cole-Hopf sense, i.e. $h(t,x) = \log Z(t,x)$ where

$$\partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + Z(t,x)\xi(t,x) \tag{SHE}$$

and the product $Z\xi$ is interpreted using Itō calculus conventions. The stochastic PDE (KPZ) is obviously not linear, and cannot be put in the form (2.1.1). Actually, $h(t,x)$ should be interpreted physically as the height function associated to some interface at time t which typically evolves in an irreversible manner. While the equation well-defines a Markov process on continuous trajectories. It should not have stationary measures. And indeed, one can show, at least for some initial conditions, that $h(t,x) \sim \frac{-t}{24}$ as t goes to infinity.

The KPZ equation admits however stationary measures modulo a global shift. Or equivalently, the process of spatial increments admits stationary measures.

Definition 2.2. — We say that the law of $h^{\text{statr}} \in \mathbb{C}(\mathbb{R},\mathbb{R})$ is stationary for the KPZ equation (KPZ) if the following implication holds:
If $h(0,\cdot) = h^{\text{stat}}$ then for all $t > 0$, $h(t,\cdot) - h(t,0) \overset{(d)}{=} h^{\text{stat}}$.

It turns out that the law of $h^{\text{stat}}(x) = B(x) + \mu x$, where $\mu \in \mathbb{R}$ is arbitrary, is stationary for the KPZ equation. This is also the stationary measure of the linear equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \xi(t,x)$$

and surprisingly, the addition of the nonlinear term $\left(\partial_x h(t,x)\right)^2$ does not change the stationary measure. The first argument supporting this claim was proposed in [Par90]. The first mathematically rigorous proof was proposed in [BG97]. There exists now several other proofs [FQ15; GQ25].

In [BG97], this result is phrased as a corollary of their main result, which is a convergence of ASEP to the KPZ equation. If $H(t,x)$ denotes the hight function associated to the asymmetric simple exclusion process on $\mathbb{Z}$ with jump rates e^{ε} to the right and $e^{-\varepsilon}$ to the left, then $e^{-\varepsilon H(\varepsilon^{-4}t,\varepsilon^{-2}x)}$ converges to $Z(t,x)$, solution of (SHE) as ε goes to zero. Since the configuration of particles where each site is occupied independently with probability ϱ is stationary for ASEP, i.e. when the associated height function is a random walk, possibly with some nonzero drift, and given that random walks converge to Brownian motions in the diffusive scaling, we conclude immediately that the Brownian motion is stationary for the KPZ equation.

Thus, if one is interested in the stationary measure of the KPZ equation on other domains, such as $\mathbb{R}_+$ or a segment $[0,L]$, a natural method should be to analyze first the

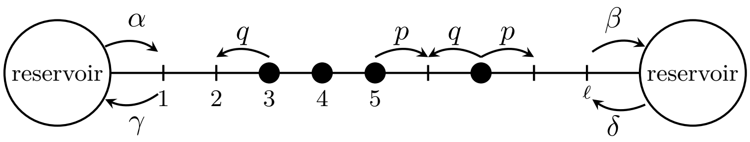


FIGURE 1. Jump rates of Open ASEP on $\ell - 1$ sites.

corresponding discrete exclusion process, and take a scaling limit of its stationary measure. Indeed, it is proved in [CS18] that the height function of open ASEP, i.e. the ASEP on a finite number of sites connected to boundary reservoirs, or the ASEP on the positive integers, converge to the KPZ equation on $[0, 1]$ or $\mathbb{R}_+$.

2.2. The matrix product ansatz

2.2.1. The open ASEP. — Let us start with a more precise definition of open ASEP.

Definition 2.3. — Let $[\![a, b]\!]$ denote the set of integers from a to b included. The open ASEP on a segment is a Markov process on the state-space $\{0, 1\}^{[\![1,\ell]\!]} \times \mathbb{Z}_{\geqslant 0}$ describing the dynamics of particles between two reservoirs, according to the jump rates depicted in Figure 1. Configurations are described by occupation variables $\eta = (\eta_1, \dots, \eta_\ell) \in \{0, 1\}^{[\![1,\ell]\!]}$ together with the number $N = N_{\ell+1} \in \mathbb{Z}$ of particles that have gone through the system. For any $x \in [\![1, \ell - 1]\!]$, a particle jumps from site x to $x + 1$ or from site $x + 1$ to x at the rates p and q. A particle is created or annihilated at the site 1 at exponential rates

$$\alpha\ (1 - \eta_1) \quad \text{and} \quad \gamma\ \eta_1.$$

Moreover, a particle is created or annihilated at the site $\ell - 1$ at exponential rates

$$\delta\ (1 - \eta_1) \quad \text{and} \quad \beta\ \eta_1.$$

All these events are independent. We define the integrated current at site x by

$$N_x(\eta, N_{\ell+1}) = \sum_{i=x}^{\ell} \eta_i + N_\ell,$$

where $N_{\ell+1}(t)$ denotes the number of particles that have been annihilated at site ℓ, minus the number of particles that have been created at site ℓ (this corresponds to the total number of particles that have traveled through the system).

Physically the reservoirs at both ends impose densities ϱ_a, ϱ_b satisfying the mean current conservation conditions $j_a = (p-q)\varrho_a(1-\varrho_a) = \alpha(1-\varrho_a)-\gamma\varrho_a$ and $j_b = (p-q)\varrho_b(1-\varrho_b) = \beta\varrho_b - \delta(1 - \varrho_b)$ [DEHP93]. Hence, it is convenient to introduce the roots

$$\kappa^{\pm}(\alpha, \gamma) = \frac{1}{2\alpha}\left((p - q - \alpha - \gamma) \pm \sqrt{(p - q - \alpha - \gamma)^2 + 4\alpha\gamma}\right)$$

and new parameters

$$\begin{aligned} a &= \kappa^+(\alpha, \gamma) \\ b &= \kappa^+(\beta, \delta) \\ c &= \kappa^-(\alpha, \gamma) \\ d &= \kappa^-(\beta, \delta) \end{aligned}$$

so that

$$\varrho_a = \frac{1}{1 + a}, \quad \varrho_b = \frac{b}{1 + b}.$$

In other terms, we have parametrized the jump rates by a, b, c and d as

$$\alpha = \frac{(p-q)}{(1+a)(1+c)}, \qquad \gamma = -\frac{ac(p-q)}{(1+a)(1+c)},$$
$$\beta = \frac{(p-q)}{(1+b)(1+b)}, \qquad \delta = -\frac{bd(p-q)}{(1+b)(1+d)}.$$

2.2.2. Open ASEP stationary measure. — If $p \neq q$, the integrated current N will diverge to infinity, and the couple $(\eta(t), N(t))$ does not admit any stationary measure. However, the marginal $\eta(t)$ is a Markov process on a finite state space, which admits a unique stationary measure. First investigations into this stationary measure dates back to [MGP68; Lig75], but it was fully characterized in the breakthrough article [DEHP93]. The stationary measure $\mathbb{P}^{\text{stat}}$ can be written as

$$\mathbb{P}^{\text{stat}}(\eta) = \frac{1}{Z_\ell} \langle \mathsf{w}| \prod_{i=1}^{\ell} (\eta_i \mathsf{D} + (1-\eta_i)\mathsf{E})\ |\mathsf{v}\rangle \tag{2.2.1}$$

where the normalization constant is

$$Z_\ell = \langle \mathsf{w}| \,(\mathsf{E}+\mathsf{D})^\ell\, |\mathsf{v}\rangle\,,$$

provided E, D and $\langle \mathsf{w}|\,, |\mathsf{v}\rangle$ satisfy the relations

$$\mathsf{DE} - q\mathsf{ED} = \mathsf{D} + \mathsf{E}$$
$$\langle \mathsf{w}|\,(\alpha\mathsf{E} - \gamma\mathsf{D}) = \langle \mathsf{w}|$$
$$(\beta\mathsf{D} - \delta\mathsf{E})\,|\mathsf{v}\rangle = |\mathsf{v}\rangle$$

Here, D and E are matrices or operators acting on vectors denoted by kets $|\mathsf{v}\rangle$. The bras $\langle \mathsf{w}|$ denote linear forms (which can be identified in practice with a row vector). It is already possible, from this purely abstract statement, to deduce some interesting information. The so-called Matrix Product Ansatz (2.2.1) is even more useful if one can find concrete representations of the commutation relation, and indeed, there are many. In general, E and D are infinite matrices.

The parametrization involving a, b, c, d is convenient to express certain representations. For instance, one may chose matrices $\mathsf{D} = 1 + \mathsf{d}$, $\mathsf{E} = 1 + \mathsf{e}$, where

$$\mathsf{de} - q\mathsf{ed} = 1 - q$$

that is the q-oscillator algebra. Explicitly, one may chose $\mathsf{d} = \sum_{n=1}^{\infty}(1-q^n)\,|n-1\rangle\,\langle n|$ and $\mathsf{e} = \sum_{n=1}^{\infty} |n+1\rangle\,\langle n|$. With this choice, the boundary vectors are simply chosen [Sas99] as $\langle \mathsf{w}| = \sum_{n=1}^{\infty} w_n \langle n|$ and $|\mathsf{v}\rangle = \sum_{n=1}^{\infty} v_n\,|n\rangle$ with

$$w_n = H_n(a,c), \quad v_n = \frac{H_n(b,d)}{(q)_n},$$

where the H_n are essentially q-Hermite polynomials, defined as

$$H_n(X,Y) = \sum_{k=0}^{n} \frac{(q)_n}{(q)_k (q)_{n-k}} X^k Y^{n-k} \tag{2.2.2}$$

We refer to Lazarescu's PhD thesis [Laz13, chapter 2] for more references and details. For now, we just note that the same boundary weights also arise in the half-space Hall-Littlewood process [He24] – this coincidence will be explained in Section 2.6.2.

Remark 2.4. — There exist many other representations of the Matrix Product Ansatz for open ASEP. Furthermore, this method extends to other models, such as exclusion processes with particles of multiple types, processes with pushing mechanisms, non-conservative variants, etc. When one is searching for tridiagonal representations of the

MPA, three term recurrence relations naturally emerge. Such recurrences are typically related to orthogonal polynomials. In particular, [USW04] introduced representations related to Askey-Wilson polynomials (depending on parameters q, a, b, c, d). The MPA also spurred many developments in combinatorics [CW07; CW11; CW10; SW14].

2.3. Stationary measures for the open KPZ equation

The open KPZ equation on the segment $[0, L]$, with boundary parameters $u, v \in \mathbb{R}$ is the stochastic PDE

$$\begin{cases} \partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x), \\ \partial_x h(t,x)\big|_{x=0} = u, \\ \partial_x h(t,x)\big|_{x=L} = -v, \end{cases} \tag{KPZ$_{u,v}$}$$

where ξ is a space-time white noise. As for the KPZ equation on $\mathbb{R}$, solutions can be defined through the Hopf-Cole transform. We say that $h \in C(\mathbb{R}_{\geqslant 0} \times [0, L], \mathbb{R})$ is a solution to (KPZ$_{u,v}$) with initial condition h_0 if we have $h(t,x) = \log Z(t,x)$ where $Z(t,x)$ solves the multiplicative noise stochastic heat equation

$$\begin{cases} \partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + Z(t,x)\xi(t,x), \\ \partial_x Z(t,x)\big|_{x=0} = (u-1/2)Z(t,0), \\ \partial_x Z(t,x)\big|_{x=L} = -(v-1/2)Z(t,L), \end{cases} \tag{SHE$_{u,v}$}$$

where we interpret the product $Z(t,x)\xi(t,x)$ in the Ito sense, with initial condition $Z(0,x) = \log h_0(x)$. We refer to [CS18; Par19b] for details about what this equation means and for proofs of the existence, uniqueness and positivity of solutions.

Definition 2.5. — Let $\mathbb{P}^{u,v}_{\mathrm{KPZ}}$ be a probability distribution on continuous functions B_1, B_2 on $[0, L]$ defined by

$$\mathbb{P}^{u,v}_{\mathrm{KPZ}}(B_1, B_2) = \frac{1}{\mathcal{K}_{u,v}(L)} e^{(u+v)B_1 \otimes B_2(L)}\, \mathbb{P}^{-v}_{\mathrm{Brown}}(B_1)\, \mathbb{P}^{-u}_{\mathrm{Brown}}(B_2) \tag{2.3.1}$$

where the reference measure $\mathbb{P}^{d}_{\mathrm{Brown}}$ is the law of a Brownian motion with drift d starting from 0, and the composition $B_1 \otimes B_2$ is the geometric Pitman transform defined as

$$B_1 \otimes B_2\,(t) = -\log \int_0^t e^{-(B_1(s)+B_2(t)-B_2(s))} ds,$$

The normalization function satisfies

$$\mathcal{K}_{u,v}(L) = \mathbb{E}\left[\left(\int_0^L e^{-(B_1(s)+B_2(L)-B_2(s))} ds\right)^{-u-v}\right],$$

where the expectation is taken over the law of independent standard Brownian motions B_1, B_2.

Theorem 2.6 **([CK24][BKWW23][16]).** — *If $u + v \geqslant 0$, the marginal distribution of B_1 under the probability measure $\mathbb{P}^{u,v}_{\mathrm{KPZ}}$ is the unique stationary measure of the open KPZ equation on $[0, L]$.*

This theorem was obtained through a combination of results from [CK24; BKWW23; 16] based on earlier results of [USW04; DEHP93]. Indeed, in the special case $u + v > 0$, it was shown in [16] that the law of $(B_1(x), B_2(x))$ is the same as the law of the processes

$(\Lambda_1(x)-\Lambda_1(0),\Lambda_2(x)-\Lambda_2(0))$ where Λ_1,Λ_2 are continuous stochastic processes on $[0,L]$ distributed as

$$\mathbb{P}^{u,v}_{\mathrm{KPZ}}(\Lambda_1,\Lambda_2)=\frac{1}{\mathcal{Z}_{u,v}(L)}F_{u,v}(\Lambda_1,\Lambda_2)\delta_0(\Lambda_2(0))\ \mathbb{P}^{\mathrm{free}}_{\mathrm{Brown}}(\Lambda_1)\mathbb{P}^{\mathrm{free}}_{\mathrm{Brown}}(\Lambda_2) \tag{2.3.2}$$

where the reference measure $\mathbb{P}^{\mathrm{free}}_{\mathrm{Brown}}$ is now the Brownian measure with free endpoints (though the presence of the factor $\delta_0(\Lambda_2(0))$ will fix $\Lambda_2(0)=0$), and the functional $F_{u,v}$ is

$$F_{u,v}(\Lambda_1,\Lambda_2)=\exp\left(-\int_0^L e^{-(\Lambda_1(s)-\Lambda_2(s))}ds\right)e^{-u(\Lambda_1(0)-\Lambda_2(0))}e^{-v(\Lambda_1(L)-\Lambda_2(L))}.$$

The normalization constants in (2.3.2) and in (2.3.1) are related by

$$\mathcal{Z}_{u,v}(L)=\Gamma(u+v)e^{\frac{-L}{2}(u^2+v^2)}\mathcal{K}_{u,v}(L). \tag{2.3.3}$$

In order to see that $(\Lambda_1(\cdot)-\Lambda_1(0),\Lambda_2(\cdot)-\Lambda_2(0))$ under (2.3.1) has the same law as (B_1,B_2) under (2.3.2), we first integrate over $\Delta=\Lambda_1(0)-\Lambda_2(0)$ using the integral formula

$$C^{-u-v}=\frac{1}{\Gamma(u+v)}\int_{\mathbb{R}}d\Delta e^{-\Delta(u+v)-e^{-\Delta}C}.$$

This step in referred in the physics literature as summing over the zero mode. Then we may arrange the formula using Girsanov theorem to change the drifts of the Brownian motions. The first averaging yields the factor $\Gamma(u+v)$, and the application of Girsanov's theorem yields the exponential factors in (2.3.3).

We insist that (2.3.2) defines a probability measure only for $u+v>0$. It turns out that this probability measure can be related [BKWW23; 16] to formulas characterizing the open KPZ stationary measures [CK24], so that the law of B_1 (or $\Lambda_1(x)-\Lambda_1(0)$), is the stationary measure of the open KPZ equation. It is known to be unique [KM22; Par22].

After noticing that the marginal distribution of $\Lambda_1(x)-\Lambda_1(0)$ in (2.3.2) can be described by $B_1(x)$ in (2.3.1), [16] conjectured that the law of B_1 is actually the stationary measure for all u,v. This conjecture was motivated by an analytic continuation argument which seems difficult to formalize in the continuous setting. In order to justify this analytic continuation argument, a potential approach is to take a detour through a discrete model. This is the motivation behind Section 2.4. Before we explain this, we will explain how we identified the formulas of [CK24] to the Brownian description above, and we will sketch another way to prove Theorem 2.6 which we proposed in [21].

2.3.1. Liouville quantum mechanics. — The identification between

1. formulas in [CK24] for the multipoint Laplace transform of spatial increments of the open KPZ stationary measasure, and
2. the description (2.3.2),

was made in [BKWW23; 16]. Although the two papers use very different notations and vocabulary, they use the same idea. One recognizes in the formulas from [CK24] certain quantities which often appear when integrating eigenfunctions of the operator

$$H=\frac{1}{4}\Delta-e^{-2x},$$

acting on a suitable L^2 space of functions with enough decay at infinity. From a probabilistic point of view, H is the generator of a Brownian motion $b(x)$ (with diffusion coefficient 1/2) killed at exponential rate $e^{-2b(x)}$. From a physical point of view, $-H$ is the Hamiltonian of a quantum particle in exponential potential.

There exist a distinguished orthonormal basis of eigenfunctions of H indexed by $k \geqslant 0$ with

$$\psi_k(x) = \sqrt{\frac{2}{\pi}} \frac{1}{|\Gamma(ik)|} K_{ik}(2e^{-x})$$

where K is the modified Bessel K function, normalized so that

$$H\psi_k = \frac{k^2}{4}\psi_k, \quad \langle \psi_k | \psi_{k'} \rangle = \delta(k-k').$$

These functions satisfy further properties, such as, for $s > 0$

$$\int_{\mathbb{R}} dx e^{-sx} \psi_k(x) \psi_{k'}(x) = \frac{C}{8\pi} \frac{\boldsymbol{\Gamma\left(\frac{s}{2} \pm \frac{ik}{2} \pm \frac{ik'}{2}\right)}}{\boldsymbol{\Gamma(s)}}, \tag{2.3.4}$$

This identity should be understood as a two-variable degeneration of the Cauchy identity for Whittaker functions (1.4.3). Using the Feynman-Kac formula, the fundamental solution of $\partial_t u = -Hu$ can be written as

$$\int dk \psi_k(x') e^{-t\frac{k^2}{4}} \psi_k(x) = p_t(x,x') \mathbb{E}_B \left[e^{-\int_0^t e^{-2B(s)ds}} \right],$$

where the expectation is w.r.t. a Brownian bridge such that $B(0) = x$ et $B(t) = x'$.

Now, by taking a limit of formulas from [BW17] under the KPZ equation scaling., [CK24] obtained that (in the special case of one point marginal)

$$\mathbb{E}[e^{-sh^L_{u,v}(y)}] = e^{\frac{s^2 y}{4}} \times \iint_0^\infty dt_1\, dt_2 p_0(t_2) p_{0,s}(t_2,t_1) e^{\frac{-1}{4}(t_1 y + t_2(L-y))}, \tag{2.3.5}$$

where $h^L_{u,v}$ denotes the stationary measure of the open KPZ equation (here we have to assume $u, v > 0$ for the formula to be valid), where

$$p_0(t) = \frac{(u+v)(u+v+1)}{8\pi} \frac{\Gamma\left(v \pm i\frac{\sqrt{t}}{2}\right)\Gamma\left(u \pm i\frac{\sqrt{t}}{2}\right)}{\sqrt{t}\Gamma\left(\pm i\sqrt{t}\right)} \tag{2.3.6}$$

$$p_{0,s}(t_2,t_1) = \frac{1}{8\pi} \frac{\Gamma\left(u - \frac{s}{2} \pm i\frac{\sqrt{t_1}}{2}\right)}{\Gamma\left(u \pm i\frac{\sqrt{t_2}}{2}\right)\sqrt{t_1}\Gamma\left(\pm i\sqrt{t_1}\right)} \frac{\boldsymbol{\Gamma\left(\frac{s}{2} \pm i\frac{\sqrt{t_2}}{2} \pm i\frac{\sqrt{t_1}}{2}\right)}}{\boldsymbol{\Gamma(s)}} \tag{2.3.7}$$

Identifying the red terms in (2.3.7) and (2.3.4), one can rewrite the Laplace transform formula (2.3.5) fully in terms of the eigenbasis ψ_k, and using the Feynman-Kac formula, as exponential functionals of a Brownian motion. Eventually, one finds that the stationary measure of the open KPZ equation can be written as

$$h^L_{u,v}(x) = \frac{1}{\sqrt{2}} W(x) + U(x) - U(0)$$

where the process U is absolutely continuous with respect to the law of the Brownian motion with Radon-Nikodym derivative proportional to

$$e^{-2uU(0)-2vU(L)} e^{-\int_0^L e^{-2U(s)}ds} \tag{2.3.8}$$

and W is an independent Brownian motion. If one identifies

$$U(x) = \frac{\Lambda_1(x) - \Lambda_2(x)}{2}, \quad \frac{1}{\sqrt{2}} W(x) = \frac{\Lambda_1(x) + \Lambda_2(x)}{2},$$

one recovers the probability measure (2.3.2). These ideas were used in physics in order to compute exponential functionals of the Brownian motion and the method is referred to as Liouville quantum mechanics [CM96; CMY98; CT98; MC94]. In particular, summing over the zero mode is a standard trick in this context. The identifications between the red terms above was facilitated for Pierre Le Doussal and myself because we used this method only a few month before the paper [CK24] came out, in our paper [14, Appendix F].

2.3.2. A representation of Derrida and Enaud and reweighted random walks. — In this section, we assume that the boundary parameters $\alpha, \gamma, \beta, \delta$ satisfy Liggett's condition [Lig75] on both sides, i.e. we assume that

$$\frac{\alpha}{p} + \frac{\gamma}{q} = 1, \quad \frac{\beta}{p} + \frac{\delta}{q} = 1, \tag{2.3.9}$$

so that $\varrho_a = \alpha/p$ and $\varrho_b = \delta/q$. In terms of the parameters a, b, c and d, it corresponds to assuming $c = d = -q$ so that $\alpha = \varrho_a = \frac{1}{1+a}$ and $\delta = q\varrho_b = \frac{qb}{1+b}$. It is not strictly necessary to make this hypothesis, but it greatly simplify the expressions below.

2.3.2.1. *Enaud-Derrida's representation.* — In [21], we used a specific representation of the MPA due to [ED04] and a method from [DEL04]. We choose D and E as the infinite matrices

$$D = \begin{pmatrix} [1]_q & [1]_q & 0 & 0 & 0 & \cdots \\ 0 & [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & 0 & [3]_q & [3]_q & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}, \qquad E = \begin{pmatrix} [1]_q & 0 & 0 & 0 & \cdots \\ [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & [3]_q & [3]_q & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.3.10}$$

where we use the usual notation for q-numbers

$$[n]_q = \frac{1-q^n}{1-q}.$$

Let us denote by $\{|n\rangle\}_{n\geqslant 1}$ the vectors of the basis in which the matrices are written. One can alternatively write the matrices D and E as

$$D = \sum_{n=1}^{+\infty} [n]_q \,|n\rangle\, (\langle n| + \langle n+1|), \qquad E = \sum_{n=1}^{+\infty} ([n]_q\, |n\rangle + [n+1]_q\, |n+1\rangle)\, \langle n|\,. \tag{2.3.11}$$

In this basis the vectors $|\mathsf{v}\rangle$ and $\langle\mathsf{w}|$ are given by

$$\langle \mathsf{w}| = \sum_{n\geqslant 1} \left(\frac{1-\varrho_a}{\varrho_a}\right)^n \langle n|\,, \qquad |\mathsf{v}\rangle = \sum_{n\geqslant 1} \left(\frac{\varrho_b}{1-\varrho_b}\right)^n [n]_q\, |n\rangle\,. \tag{2.3.12}$$

Because products of tridiagonal matrices in (2.2.1) can always be interpreted as a weighted sum over Motzkin paths, it is possible to reinterpret (2.2.1) in terms of reweighted random walks.

From now on we assume that $q \in [0,1)$ and $\varrho_b < \varrho_a$. Following [DEL04] and [ED04] we introduce discrete walks on the strictly positive integers which, at each step, either increase by one unit, decrease by one unit, or stay constant, with the constraint that they remain strictly positive. They are described by a sequence of $\ell+1$ integers $\vec{n} = (n_i)_{0\leqslant i\leqslant \ell}$ such that $n_i > 0$ and $|n_i - n_{i+1}| \leqslant 1$ for all $0 \leqslant i \leqslant \ell - 1$. It is easy to write

$$Z_\ell(q) = \langle W|\, (D+E)^\ell\, |V\rangle = \sum_{\vec{n}} \langle W|n_0\rangle\langle n_\ell|V\rangle \prod_{i=1}^{\ell} \langle n_{i-1}|D+E|n_i\rangle = \sum_{\vec{n}} \Omega(\vec{n}),$$

where

$$\Omega(\vec{n}) = \left(\frac{1-\varrho_a}{\varrho_a}\right)^{n_0} \left(\frac{\varrho_b}{1-\varrho_b}\right)^{n_\ell} \prod_{i=1}^{\ell} v(n_{i-1}, n_i) \prod_{i=0}^{\ell} [n_i]_q, \tag{2.3.13}$$

where

$$v(n,n') = \begin{cases} 2 & \text{if } n = n', \\ 1 & \text{if } |n-n'| = 1 \\ 0 & \text{else.} \end{cases}$$

Following arguments in [DEL04] which restricted to $q=0$, one may write that under the stationary measure of open ASEP $\mathbb{P}^{\text{stat}}(\eta)$, the height function

$$H(i) = H(0) + \left(\sum_{j=1}^{i}(2\eta_i - 1)\right)_{1\leqslant i\leqslant \ell}$$

decomposes as

$$H(i) - H(0) = (n_i - n_0 + m_i)_{1\leqslant i\leqslant \ell}\,, \tag{2.3.14}$$

where $(n_i, m_i)_{0\leqslant i\leqslant \ell}$ is a two dimensional random walk on $\mathbb{Z}^2$, starting from $(n_0, 0)$, distributed as

$$\mathbb{P}(\vec{n},\vec{m}) = \frac{\mathbb{1}_{n_0>0}}{4^{-\ell}Z_\ell(q)}\left(\frac{1-\varrho_a}{\varrho_a}\right)^{n_0}\left(\frac{\varrho_b}{1-\varrho_b}\right)^{n_\ell}\prod_{i=0}^{\ell}[n_i]_q \mathbb{P}^{SSRW}_{n_0,0}(\vec{n},\vec{m}), \tag{2.3.15}$$

where $\mathbb{P}^{SSRW}_{n_0,0}$ denotes the probability measure of the symmetric simple random walk (SSRW) on $\mathbb{Z}^2$ starting from $(n_0, 0)$, that is the random walk performing steps $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ with equal probability $1/4$. Note that in (2.3.13), the random walk n_i was constrained to the strictly positive integers. However, the weight $\Omega(\vec{n})$ vanishes if $n_i = 0$ for some i, so that one can relax the positivity assumption.

2.3.2.2. *Scaling limit.* — We now use the scalings so that open ASEP's height functions converges to the open KPZ equation. We scale $q = e^{-\varepsilon}$, $\ell = 4L\varepsilon^{-2}$, and we choose boundary rates $\alpha, \beta, \gamma, \delta$ in such a way that

$$\varrho_a = \frac{1}{2} + \frac{u}{4}\varepsilon, \qquad \varrho_b = \frac{1}{2} - \frac{v}{4}\varepsilon.$$

We define functions $U_\varepsilon, V_\varepsilon$ on $[0, L] \cap \frac{\varepsilon^2}{4}\mathbb{Z}$ by

$$U_\varepsilon(x) = \frac{\varepsilon}{2}(n_{4x\varepsilon^{-2}} + \varepsilon^{-1}\log(\varepsilon^2/4)), \qquad V_\varepsilon(x) = \frac{\varepsilon}{2}m_{4x\varepsilon^{-2}}, \tag{2.3.16}$$

that we extend linearly to the whole interval $[0, L]$. The function U_ε lives in the interval $(\log(\varepsilon/2), +\infty)$ which becomes the whole line $\mathbb{R}$ in the $\varepsilon \to 0$ limit. To determine the limit of $U_\varepsilon, V_\varepsilon$ as $\varepsilon \to 0$, we need to examine the behavior of (2.3.15) under these scaling. First, under the reference measure $P^{SSRW}_{n_0,0}$, the couple of random functions $(U_\varepsilon(x), V_\varepsilon(x))$ becomes a two-dimensional Brownian motion.

Next we write

$$\prod_{i=0}^{\ell}[n_i]_q = \exp\left(\sum_{i=0}^{\ell}\log\left(\frac{1-e^{-n_i\varepsilon}}{1-e^{-\varepsilon}}\right)\right) \tag{2.3.17}$$

$$\approx C(\varepsilon)\exp\left(-\int_0^L e^{-2U_\varepsilon(x)}dx\right), \tag{2.3.18}$$

where the constant $C(\varepsilon) = \varepsilon^{-(\ell+1)}$, independent from U_ε, disappears when dividing by the normalization. Under the same scalings, the boundary terms in (2.3.15) yield

$$\left(\frac{1-\varrho_a}{\varrho_a}\right)^{n_0}\left(\frac{\varrho_b}{1-\varrho_b}\right)^{n_\ell} \approx C'(\varepsilon)e^{-2uU_\varepsilon(0)-2vU_\varepsilon(L)},$$

up to another unimportant multiplicative factor $C'(\varepsilon) = (\varepsilon^2/4)^{u+v}$, which also disappears when dividing by the normalization.

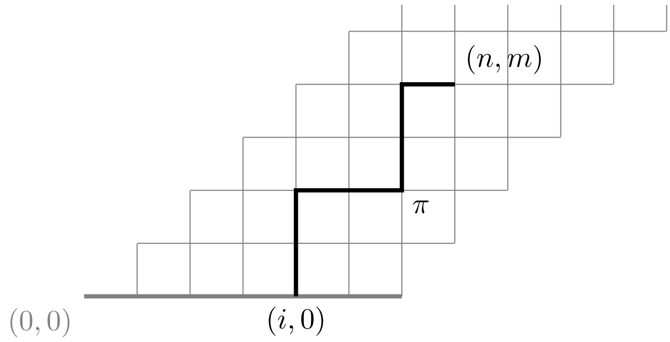


FIGURE 2. The subset $\mathbb{Z}^{\text{strip}}$ of the $\mathbb{Z}^2$ lattice. In black is shown an example of up-right path as used in the definitions of last passage percolation and the log-gamma polymer in the strip.

Thus, the couple of random function $(U_\varepsilon(x), V_\varepsilon(x))$ should converge to a two-dimensional process $(U(x), V(x)) \in \mathbb{R}^2$, absolutely continuous with respect to the law of the two-dimensional Brownian motion with covariance matrix $\frac{1}{2}I_2$, where $V(0) = 0$ and $U(0)$ is distributed according to the Lebesgue measure. The Radon-Nikodym derivative is equal to

$$\frac{1}{\widetilde{\mathcal{Z}}_{u,v}} \exp\left(-2uU(0) - 2vU(L) - \int_0^L dx e^{-2U(x)}\right). \tag{2.3.19}$$

One recovers exactly (2.3.8). The arguments above were presented in the physics article [21] as the simplest way to derive theorem 2.6 – without going through exact integral formulas as in [CK24]. The paper [Him24] subsequently confirmed that these heuristics can be made fully rigorous.

2.4. Two-layer Gibbs measures

In this section, we will see that the structure (2.3.2) is quite general and there are variants for many other solvable models. Similar results on discrete models, after summing over the zero mode, allows to justify that the result of Theorem 2.6 analytically extends to all $u, v \in \mathbb{R}$ as conjectured in [16]. The only caveat is that this approach requires the convergence of the discrete model to the other, which has not been established yet for the log-gamma polymer in a strip.

Hence, we will define discretized versions of the measure (2.3.2), which we will view as variants of the Schur or Whittaker process. These discrete stochastic processes are such that – see [26] for details

1. These measures are stationary for certain Markov dynamics (of push-block type) on sequences of signatures.
2. Under these Markov dynamics, the top layer, i.e. the process $x \mapsto \lambda_1^x$, evolves according to the stochastic recursion satisfied by the LPP time $G(n,m)$ or log-gamma polymer free energies $H(n,m)$.
3. Analytic continuations in boundary parameters can be performed at the discrete level.

2.4.1. Discrete models. — We consider stochastic growth models defined on a strip of the $\mathbb{Z}^2$ lattice denoted $\mathbb{Z}^{\text{strip}} = \{(i,j) \in \mathbb{Z}^2; j \leqslant i \leqslant j+N\}$. The discrete models considered in [26] are the following.

***Definition 2.7* (Last passage percolation).** — Let $a \in (0,1)$ and $c_1, c_2 \geqslant 0$ such that $ac_1, ac_2 < 1$. We associate a family of independent random variables $w_{i,j}$ to the vertices of $\mathbb{Z}^{\text{strip}}$. We assume that in the bulk of the strip, i.e. for $j < i < j+N$, $w_{i,j} \sim \text{Geom}(a^2)$, while on the left boundary $w_{i,i} \sim \text{Geom}(ac_1)$ and on the right boundary, $w_{j+N,j} \sim \text{Geom}(ac_2)$. For a given initial condition $G_0 : [\![0,N]\!] \to \mathbb{R}$ with $G(0) = 0$, we set $G(i,0) = G_0(i)$ and define

$$G(n,m) = \max_{1\leqslant i\leqslant N} \left\{ G_0(i) + \max_{\pi:(i,1)\to(n,m)} \sum_{(i,j)\in\pi} w_{i,j} \right\}, \tag{2.4.1}$$

where the inner maximum runs over up-right paths π from $(i,1)$ to (n,m), as shown in Fig. 2. We may then define the Markov process $G_m = (G_m(i))_{1\leqslant i\leqslant N}$ on $\mathbb{Z}^N$ by setting

$$G_m(i) = G(m+i,m) - G(m,m) \text{ for } 1 \leqslant i \leqslant N.$$

We say that a probability measure $\mathbb{P}^{\text{stat}}$ on $\mathbb{Z}^N$ is a stationary measure for LPP on the strip if $G_0 \sim \mathbb{P}^{\text{stat}}$ implies that $G_m \sim \mathbb{P}^{\text{stat}}$ for all $m \geqslant 0$.

***Definition 2.8* (Log-gamma polymer model).** — Let $\alpha > 0$ and $u, v \in \mathbb{R}$ such that $\alpha + u, \alpha + v > 0$. We consider another family of independent random variables $\omega_{i,j}$ such that for $j < i < j+N$, $\omega_{i,j} \sim \text{Gamma}^{-1}(2\alpha)$, on the left boundary $\omega_{i,i} \sim \text{Gamma}^{-1}(\alpha+u)$ and on the right boundary, $\omega_{j+N,j} \sim \text{Gamma}^{-1}(\alpha+v)$ (here, $\text{Gamma}^{-1}(\theta)$ denotes the inverse Gamma distribution with shape parameter θ). For a given initial condition $H_0 : [\![0,N]\!] \to \mathbb{R}$ with $H(0) = 0$, we set $H(i,0) = H_0(i)$ and define

$$H(n,m) = \log \left(\sum_{1\leqslant i\leqslant N} e^{H_0(i)} \sum_{\pi:(i,1)\to(n,m)} \prod_{(i,j)\in\pi} w_{i,j} \right). \tag{2.4.2}$$

We may then define the Markov process $H_m = (H_m(i))_{1\leqslant i\leqslant N}$ on $\mathbb{R}^N$ as before by setting

$$H_m(i) = H(m+i,m) - H(m,m) \text{ for } 1 \leqslant i \leqslant N$$

and consider its stationary measures.

2.4.2. Probabilistic description of stationary measures. — For both Markov processes G_m and H_m, and for any choice of parameters, an explicit description of their unique ergodic stationary measure is found in [19].

Definition 2.9. — Under the same assumptions on parameters a, c_1, c_2 as in Definition 2.7, let us define a probability distribution on couples of integer-valued random walks $\mathbf{L}_1 = (L_1(i))_{0\leqslant i\leqslant N}$ and $\mathbf{L}_2 = (L_2(i))_{0\leqslant i\leqslant N}$ by

$$\mathbb{P}^{a,c_1,c_2}_{\text{Geo}}(\mathbf{L}_1, \mathbf{L}_2) = \frac{1}{\mathcal{K}^{a,c_1,c_2}_{\text{Geo}}} (c_1c_2)^{-\mathbf{L}_1\otimes\mathbf{L}_2(N)} \, \mathbb{P}^{ac_2}_{\text{Geo}}(\mathbf{L}_1) \, \mathbb{P}^{ac_1}_{\text{Geo}}(\mathbf{L}_2), \tag{2.4.3}$$

where the reference measure $\mathbb{P}^q_{\text{Geo}}$ is the law of geometric random walks with parameter q, starting from $L_1(0) = L_2(0) = 0$, i.e.

$$\mathbb{P}^q_{\text{Geo}}(\mathbf{L}) = \prod_{i=1}^N q^{\mathbf{L}(i)-\mathbf{L}(i-1)}(1-q),$$

and the composition $\mathbf{L}_1 \otimes \mathbf{L}_2$ is the discrete Pitman transform

$$\mathbf{L}_1 \otimes \mathbf{L}_2(k) = \min_{1\leqslant j\leqslant k} \{L_1(j-1) + L_2(k) - L_2(j)\}.$$

Theorem 2.10 **([19, Theorem 1.3]).** — *The marginal distribution of* $\mathbf{L}_1$ *under the probability measure* $\mathbb{P}^{a,c_1,c_2}_{\mathrm{Geo}}$ *is the unique stationary measure of the process* G_m *from Definition 2.7.*

Remark 2.11. — Definition 2.9 is a reformulation of [19, Definition 1.2]. We chose this rewriting to make the Pitman transform appear. Following notations from [OY02], we could have defined the composition

$$\mathbf{L}_1 \odot \mathbf{L}_2(k) = \max_{1\leqslant j\leqslant k} \{L_1(j) + L_2(k) - L_2(j-1)\}.$$

The discrete Pitman transform is then generally defined as the map $(\mathbf{L}_1, \mathbf{L}_2) \mapsto (\mathbf{L}_1 \otimes \mathbf{L}_2, \mathbf{L}_2 \odot \mathbf{L}_1)$. Using that $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}_1 \otimes \mathbf{L}_2 + \mathbf{L}_2 \odot \mathbf{L}_1$, we could have defined $\mathbb{P}^{a,c_1,c_2}_{\mathrm{Geo}}$ as

$$\mathbb{P}^{a,c_1,c_2}_{\mathrm{Geo}}(\mathbf{L}_1, \mathbf{L}_2) = \frac{1}{\widetilde{\mathcal{K}}^{a,c_1,c_2}_{\mathrm{Geo}}} \, (c_1c_2)^{\mathbf{L}_2\odot\mathbf{L}_1(N)} \, \mathbb{P}^{a/c_1}_{\mathrm{Geo}}(\mathbf{L}_1)\, \mathbb{P}^{a/c_2}_{\mathrm{Geo}}(\mathbf{L}_2), \tag{2.4.4}$$

where $\widetilde{\mathcal{K}}^{a,c_1,c_2}_{\mathrm{Geo}}$ is another normalization constant. Since we do not assume that a/c_1 and a/c_2 are in $[0, 1)$, we prefer using (2.4.3) than (2.4.4).

We now turn to the log-gamma polymer.

Definition 2.12. — Under the same assumptions on parameters as in Definition 2.8, let us define a probability density for real-valued random walks $\mathbf{L}_1, \mathbf{L}_2$ by

$$\mathbb{P}^{\alpha,u,v}_{\mathrm{LG}}(\mathbf{L}_1, \mathbf{L}_2) = \frac{1}{\mathcal{K}^{\alpha,u,v}_{\mathrm{LG}}} \, e^{(u+v)\mathbf{L}_1\otimes\mathbf{L}_2(N)} \mathbb{P}^{\alpha+v}_{\mathrm{LG}}(\mathbf{L}_1)\mathbb{P}^{\alpha+u}_{\mathrm{LG}}(\mathbf{L}_2),$$

where now the reference measure $\mathbb{P}^{\alpha}_{\mathrm{LG}}$ is the law of log-gamma increment random walk with parameter α, starting from $L_1(0) = L_2(0) = 0$, i.e. with density

$$\mathbb{P}^{\alpha}_{\mathrm{LG}}(\mathbf{L}) = \prod_{i=1}^{N} \frac{1}{\Gamma(\alpha)} \exp\left(-\alpha(\mathbf{L}(i) - \mathbf{L}(i-1)) - e^{-(\mathbf{L}(i)-\mathbf{L}(i-1))}\right), \tag{2.4.5}$$

and, with a slight abuse of notations, the composition $\otimes$ is now the discrete geometric [1] Pitman transform defined as

$$\mathbf{L}_1 \otimes \mathbf{L}_2(k) = -\log\left(\sum_{j=1}^{k} e^{-(L_1(j-1)+L_2(k)-L_2(j))}\right).$$

Theorem 2.13 **([19, Theorem 1.6]).** — *The marginal distribution of* $\mathbf{L}_1$ *under the probability density* $\mathbb{P}^{\alpha,u,v}_{\mathrm{LG}}$ *is the unique stationary measure of the process* H_m *from Definition 2.8.*

2.4.3. Two-layer Schur and Whittaker measures. — Our results are based on description of stationary processes different from Definition 2.9 and Definition 2.12: the two-layer Schur and Whittaker processes. These probability measures are introduced in [19] as Gibbs measures defined on certain graphs. In this text, we will first define them as variants of the Schur and Whittaker processes.

1. Here, geometric refers to the geometric Brownian motion appearing in the study of exponential functionals of the Brownian motion, where positive temperature analogues of the Pitman transform where first studied [MY00], or geometric random walks appearing in geometric liftings of the RSK correspondence [NY04; COSZ14].

2.4.3.1. *Ascending two-layer Schur process.* — Consider an integer $N \geqslant 1$, and parameters a, c_1, c_2 as in Definition 2.7 with the additional assumption that $c_1c_2 < 1$. Let Sign_2 denote the set of $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ with $\lambda_1 \geqslant \lambda_2$. Define a probability measure on sequences $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^N)$ of elements $\lambda^i \in \mathsf{Sign}_2$ such that

$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}^{a,c_1,c_2}_{\mathrm{Geo}}(N)} c_1^{\lambda_1^0-\lambda_2^0} c_2^{\lambda_1^N-\lambda_2^N} \prod_{i=1}^{N} s_{\lambda^i/\lambda^{i-1}}(a) \;\; \mathbb{1}_{\lambda_2^0=0}, \tag{2.4.6}$$

where $s_{\lambda/\mu}$ is a skew Schur function defined in (1.1.3). Then, as proved in [19, Prop. 2.17], we have the equality in law

$$(L_1(x), L_2(x))_{0\leqslant x\leqslant N} \overset{(d)}{=} \left(\lambda_1^x - \lambda_1^0, \lambda_2^x - \lambda_2^0\right)_{0\leqslant x\leqslant N}, \tag{2.4.7}$$

where $\mathbf{L}_1, \mathbf{L}_2$ are distributed as in Definition 2.9 (the equality is shown by explicitly averaging over λ_1^0 in (2.4.6)).

2.4.3.2. *Ascending two-layer Whittaker process.* — Consider an integer $N \geqslant 1$, and parameters α, u, v as in Definition 2.8 with the additional assumption that $u + v > 0$. We consider the probability measure on sequences $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^N) \in (\mathbb{R}^2)^{N+1}$ with density

$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}^{\alpha,u,v}_{\mathrm{LG}}(N)} e^{-u(\lambda_1^0-\lambda_2^0)} e^{-v(\lambda_1^N-\lambda_2^N)} \prod_{i=1}^{N} \Psi_\alpha(\lambda^i/\lambda^{i-1}) \delta_0(\lambda_2^0). \tag{2.4.8}$$

where for $\lambda, \mu \in \mathbb{R}^2$ and $(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$, $\Psi_{\alpha_1,\dots,\alpha_k}(\lambda/\mu)$ is a skew Whittaker function defined below in (2.4.11). In the equation above, δ_0 denotes a Dirac mass at zero, so that throughout the paper, for both models, we always have $\lambda_2^0 = 0$. Then, similarly as for two-layer Schur processes, we have again an equality in law of the form (2.4.7) where now, $\mathbf{L}_1, \mathbf{L}_2$ are distributed as in Definition 2.12 (this is proved in [19, Prop. 3.14]).

2.4.4. General two-layer Schur process. — In order to prove that the probability measure above are stationary, one needs to define a more general object that describes the increments of stationary last passage times of free energies along any down-right paths that joins both sides of the strip.

We start with the Schur case. We define a probability measure on a sequence of signatures in

$$\mathsf{Sign}_2 = \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2; \lambda_1 \geqslant \lambda_2\}$$

depending on some down-right path $\mathcal{P}$ as depicted in Figure 3. It also depends on a sequence $a_1, \dots, a_N$ of inhomogeneity parameters. This probability measure describes the stationary measure of Last Passage Percolation on a strip along any down-right path, and we refer to [19] for more precise statements.

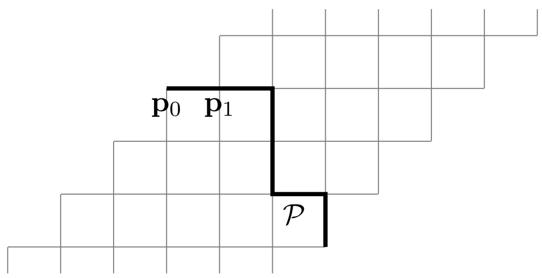


FIGURE 3. A down-right path $\mathcal{P}$ joining left and right boundaries of the strip.

We will consider paths $\mathcal{P}$ joining points $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_N$ where $\mathbf{p}_0 = (x, x)$ for some $x \in \mathbb{Z}$, and $\mathbf{p}_N = (y + N, y)$ for some $y \in \mathbb{Z}$, such as $\mathbf{p}_{i+1} - \mathbf{p}_i$ can be either $(1, 0) =$ " $\rightarrow$ " or

$(0,-1) = \text{“} \downarrow \text{”}$. Up to translations in the diagonal direction, such paths $\mathcal{P}$ are determined by the sequence of increments $\mathbf{p}_{i+1} - \mathbf{p}_i$, which can be seen as a word $w \in \{\rightarrow, \downarrow\}^N$.

Definition 2.14. — Consider an integer $N \geqslant 1$, parameters $a_1, \dots, a_N \in (0,1)$ and boundary parameters c_1, c_2 such that $c_1 c_2 < 1$ and $c_i a_j < 1$ for all i, j. Let $w \in \{\rightarrow, \downarrow\}^N$. The two-layer open Schur process is a probability measure on sequences $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^N) \in (\mathsf{Sign}_2)^{N+1}$ such that

$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}_{\mathrm{Geo}}^{\vec{a},c_1,c_2}(N)} c_1^{\lambda_1^0 - \lambda_2^0} c_2^{\lambda_1^N - \lambda_2^N} \prod_{\substack{1 \leqslant i \leqslant N \\ w_i = \rightarrow}} s_{\lambda^i/\lambda^{i-1}}(a_i) \prod_{\substack{1 \leqslant i \leqslant N \\ w_i = \downarrow}} s_{\lambda^{i-1}/\lambda^i}(a_i)\ \mathbb{1}_{\lambda_2^0 = 0}, \tag{2.4.9}$$

where the normalization constant $\mathcal{Z}_{\mathrm{Geo}}^{\vec{a},c_1,c_2}$ does not depend on the shape of the path w ([19, Proposition 2.15]).

In particular, when $w = (\rightarrow, \dots, \rightarrow)$, that is when $\mathcal{P}$ is a horizontal path, so that

$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}_{\mathrm{Geo}}^{\vec{a},c_1,c_2}(N)} c_1^{\lambda_1^0 - \lambda_2^0} c_2^{\lambda_1^N - \lambda_2^N} \prod_{i=1}^{N} s_{\lambda^i/\lambda^{i-1}}(a_i)\ \mathbb{1}_{\lambda_2^0 = 0}, \tag{2.4.10}$$

and we recover the ascending two layer Schur process of Section 2.4.3.1.

2.4.5. General two-layer Whittaker process. — We define, for $(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$ and $x, y \in \mathbb{R}^n$, *skew* Whittaker functions via the branching rule

$$\Psi^{(n)}_{\alpha_1,\dots,\alpha_k}(x/y) = \int_{\mathbb{R}^{n(k-1)}} dx^1 \dots dx^{k-1} \prod_{i=1}^{k} \Psi^{(n)}_{\alpha_i}(x^i/x^{i-1}), \tag{2.4.11}$$

under the convention that $x^k = x$ and $x^0 = y$. As for the two-layer Schur process, the two-layer Whittaker process in general depends on a down-right path $\mathcal{P}$ as in Figure 3, encoded by a word $w \in \{\rightarrow, \downarrow\}^N$.

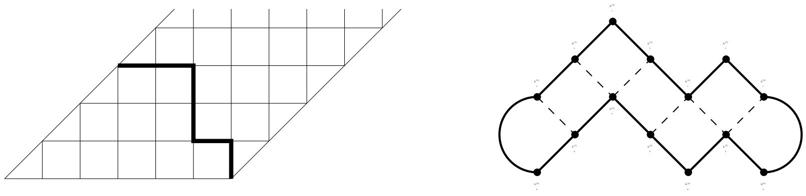
Definition 2.15. — Consider an integer $N \geqslant 1$, parameters $\alpha_1, \dots, \alpha_N \in (0,1)$ and boundary parameters $u, v \in \mathbb{R}$ such that $u + v > 0$ and $u + \alpha_i > 0$ for all i, j. Let $w \in \{\rightarrow, \downarrow\}^N$. The two-layer open Whittaker process is a probability measure on sequences $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^N) \in \mathbb{R}^{2N+2}$ with density

$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}_{\mathrm{LG}}^{\boldsymbol{\alpha},u,v}(N)} e^{-u(\lambda_1^0 - \lambda_2^0)} e^{-v(\lambda_1^N - \lambda_2^N)} \prod_{\substack{1 \leqslant i \leqslant N \\ w_i = \rightarrow}} \Psi^{(2)}_{\alpha_i}(\lambda^i/\lambda^{i-1}) \prod_{\substack{1 \leqslant i \leqslant N \\ w_i = \downarrow}} \Psi^{(2)}_{\alpha_i}(\lambda^{i-1}/\lambda^i)\ \delta_0(\lambda_2^0),$$

where the normalization constant $\mathcal{Z}_{\mathrm{LG}}^{\boldsymbol{\alpha},u,v}(N)$ does not depend on the shape of the path ([19, Proposition 3.12]).

We will focus in particular on the case where $w = (\rightarrow, \dots, \rightarrow)$, where

$$\mathbb{P}^{\boldsymbol{\alpha},u,v}(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}_{\mathrm{LG}}^{\boldsymbol{\alpha},u,v}(N)} e^{-u(\lambda_1^0 - \lambda_2^0)} e^{-v(\lambda_1^N - \lambda_2^N)} \prod_{i=1}^{N} \Psi^{(2)}_{\alpha_i}(\lambda^i/\lambda^{i-1}) \delta_0(\lambda_2^0). \tag{2.4.12}$$



(a) (b)

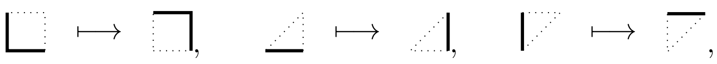
FIGURE 4. (a) A down-right path $\mathcal{P}$. (b) Two-layer graph $\mathcal{GP}$ associated to the path $\mathcal{P}$.

2.4.6. Two-layer Gibbs measure. — Let us consider some down-right path $\mathcal{P}$ as in Figure 3 (a)). To such a down-right path, we associate a two-layer graph $\mathcal{GP}$, depicted in Figure 4 (b), which consists of two copies of the path $\mathcal{P}$, rotated counter-clockwise by $\pi/4$, with certain interaction edges between the two paths.

A two-layer configuration is an assignment of numbers $\lambda_i^{(j)}$, $i = 1, 2$, $1 \leqslant j \leqslant N$ to each vertex of two-layer graph $\mathcal{GP}$ in Figure 4 (b). These take integer values in the case of geometric LPP and real values in the case of log-gamma polymer. We will use the shorthand $\boldsymbol{\lambda}_i := (\lambda_i^{(0)}, \ldots, \lambda_i^{(N)})$ for $i = 1, 2$ and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$.

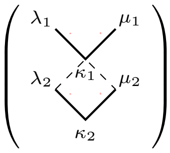
We assign to any two-layer configuration on $\mathcal{GP}$ a weight $\mathrm{wt}^{\mathcal{GP}}(\boldsymbol{\lambda})$ given by the product of (model dependent) Boltzmann weights over all edges in the graph. The specific form of Boltzmann weights (see (2.4.15) and (2.4.18) below) comes from the structure arising in discrete half-space Gibbs line ensemble [6; 30]. We will view the weight of a two-layer configuration as an infinite measure over all configurations on $\mathcal{GP}$ – referred as the two-layer Gibbs measure below – which is translation invariant under adding the same constant to all $\lambda_i^{(j)}$. It may be thought of as a discrete analogue of the measure (2.3.2)

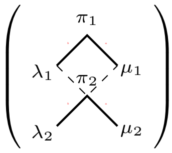
2.4.6.1. *Dynamics.* — It turns out that for both geometric LPP and the log-gamma polymer, the dynamics of the height function can be seen as a marginal of more general Markov dynamics that map the two layer Gibbs measure associated to one path to the two-layer Gibbs measure associated to another path. These Markov dynamics are constructed as the dynamics on half-space Macdonald processes in Chapter 1. First, we notice that any down-right path $\mathcal{P}$ can be updated to any down-right path $\mathcal{Q}$ above it by sequentially performing the following three types of 'local moves':



which we respectively refer to as the bulk/left boundary/right boundary local move. We associate to each local move $\mathcal{P} \mapsto \widetilde{\mathcal{P}}$ of a down-right path the corresponding local move of the two-layer graph $\mathcal{GP} \mapsto \mathcal{G}\widetilde{\mathcal{P}}$. In [26], we define three types of Markov kernels $\mathcal{U}^{\llcorner}$, $\mathcal{U}^{\lrcorner}$ and $\mathcal{U}^{\ulcorner}$ that map two-layer configurations $\boldsymbol{\lambda}$ on $\mathcal{GP}$ to two-layer configurations $\boldsymbol{\lambda}'$ on $\mathcal{G}\widetilde{\mathcal{P}}$. These Markov kernels are local, in the sense that they only update the configuration on the vertex which moves during the local move, and the update only depends on the configuration on neighboring vertices. The bulk kernels $\mathcal{U}^{\llcorner}$,$\mathcal{U}^{\lrcorner}$ and $\mathcal{U}^{\ulcorner}$ that we chose below are very close to push-block dynamics as in Section 1.3.5.

The existence of these kernels follows from certain identities that we show about the local weights in the bulk and boundary for the two-layer Gibbs measures. For instance, in the bulk we have

$\sum_{\kappa_1,\kappa_2} \mathrm{wt}$ 

$= \sum_{\pi_1,\pi_2} \mathrm{wt}$ 

(2.4.13)

where on each side of the equation, the weights are products of Boltzmann weights over all the edges of the subgraphs depicted above. We have only stated here the identity corresponding to a bulk local move, which may be seen as a variation of the (skew) Cauchy identity in symmetric function theory.

Remark 2.16. — It is possible that other type of dynamics, such as the so-called RSK-type dynamics [BP16; MP17; BM18; BP19] could be used as well, though the push-block type kernels seemed to be the most convenient choice for our purposes.

The kernels $\mathcal{U}^{\llcorner}$, $\mathcal{U}^{-}$ and $\mathcal{U}^{|}$ also have the property that the first layer marginal of the two-layer dynamics coincides with the recurrence relations satisfied respectively by geometric LPP passage times and by the log-gamma polymer partition functions. Hence, by construction, those marginal distributions of two-layer Gibbs measures $\mathrm{wt}^{\mathcal{GP}}(\boldsymbol{\lambda})$ are precisely the stationary measures we are after.

In order to arrive at a simple description for the stationary measures, we perform several additional steps, by adapting to the discrete setting the procedure that transforms the measure (2.3.2) to the one in (2.3.1). First of all, the two-layer Gibbs measures are translation invariant infinite measures. To normalize them into probability measures, we notice that under certain restrictions of the range of boundary parameters (analogous to $u+v>0$ in the case of the open KPZ equation), the mass of these infinite measures with fixed value $\lambda_1^{(0)}$ is finite. Therefore one obtains bona fide stationary probability measures under those conditions.

2.4.6.2. *Analytic continuation.* — To get rid of the constraint on boundary parameters, we finally average the aforementioned (stationary) probability measures over the 'zero mode' $\Delta = \lambda_1^{(0)} - \lambda_2^{(0)}$. Remarkably, this yields probability measures on $\mathbb{Z}^N$ or $\mathbb{R}^N$ that are well-defined without any restrictive constraint on boundary parameters. This procedure is analogous to the one performed non-rigorously in [16] for stationary measures of the open KPZ equation in Section 2.3.

The proofs of stationarity of these measures (Theorem 2.10 and Theorem 2.13)) to the full-range of boundary parameters follow analytic continuation arguments. The argument is roughly the following. Let us denote the stationary measure by $\mathrm{P} : \mathbb{K}^N \to [0,1]$, where $\mathbb{K}$ is $\mathbb{Z}$ or $\mathbb{R}$. The dynamics of each model can be encoded by a Markov transition kernel $\mathbf{U}(\mathbf{L}_1'|\mathbf{L}_1)$ for $\mathbf{L}_1, \mathbf{L}_1' \in \mathbb{K}^N$, where $\mathbf{L}_1$ denote the sequence of last passage times or free energies along a path centered by the value on the left-boundary, and $\mathbf{L}_1'$ denote the same sequence along the same path translated by $(1,1)$, i.e., by τ_1. Assume that under some restriction on boundary parameters (say $u+v>0$), for all $\mathbf{L}_1' \in \mathbb{K}^N$, we have an equation of the form

$$\sum_{\mathbf{L}_1' \in \mathbb{K}^N} \mathbf{U}(\mathbf{L}_1'|\mathbf{L}_1)\mathrm{P}(\mathbf{L}_1) = \mathrm{P}(\mathbf{L}_1'), \tag{2.4.14}$$

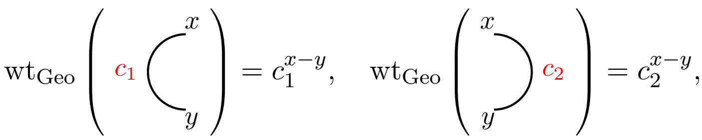
(the summation is an integral when $\mathbb{K} = \mathbb{R}$), where $\mathbf{U}$ and P depend on boundary parameters. The equation may be analytically extended to a larger range of u, v, provided both sides of (2.4.14) are real analytic functions of the variable u. In the case of geometric LPP, we show that the RHS is analytic by direct inspection of an explicit formula, and we prove that the LHS is a power series in the boundary parameter with appropriate radius of convergence [26]. In the log-gamma case the argument is considerably more involved as the sum in (2.4.14) becomes an integral. We prove in [26] a stronger result than real analyticity. Using Morera's theorem and bounds on integrability we show that both sides are holomorphic functions in u in a suitable open set.

Remark 2.17 **(About analyticity of stationary measures)**
For open ASEP a simple argument shows that its stationary measure depends real analytically on bulk and boundary parameters $q, \alpha, \beta, \gamma, \delta$, in the region of the parameter space when the stationary measure is unique (we refer to [DEHP93] for a precise definition of the model). This actually holds for a wide class of Markov chains with finite state space. Suppose $\mathcal{C}_{\boldsymbol{\theta}}$ is a family of continuous-time Markov chains with the same state space $\mathcal{S}$ (where $|\mathcal{S}| < \infty$) and with infinitesimal generator matrices $L_{\boldsymbol{\theta}}$ depending real analytically on a set of parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m) \in \Omega$, where Ω is an open subset of $\mathbb{R}^m$. We also assume that the stationary measure of $\mathcal{C}_{\boldsymbol{\theta}}$, denoted $\mu_{\boldsymbol{\theta}}$, is unique. Then $\mu_{\boldsymbol{\theta}}$ depend real analytically on $\boldsymbol{\theta} \in \Omega$.

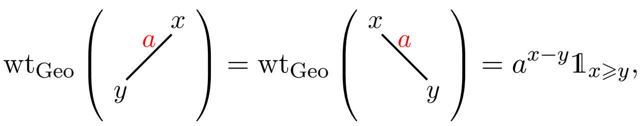
Indeed, by definition, the column vector $\mu_{\boldsymbol{\theta}} \in \mathbb{R}^{|\mathcal{S}|}$ is the unique solution of equations $L^*_{\boldsymbol{\theta}}\mu_{\boldsymbol{\theta}} = 0$ and $[1, \ldots, 1]\mu_{\boldsymbol{\theta}} = 1$. The system can be solved by Gaussian elimination, so that each component of $\mu_{\boldsymbol{\theta}}$ can be expressed as a rational function of the coefficients in $L^*_{\boldsymbol{\theta}}$, hence $\mu_{\boldsymbol{\theta}}$ depends real analytically on $\boldsymbol{\theta} \in \Omega$. This argument does not apply for the geometric LPP and log-gamma polymer models, which have infinite state spaces.

2.4.6.3. *Expression of local weights.* — Now we define the Gibbs measure on the set of two-layer configurations on the two-layer graph $\mathcal{GP}$.

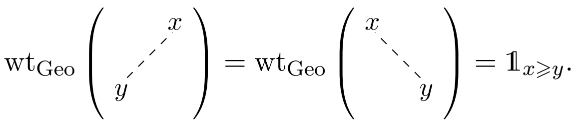
Definition 2.18 **(Two-layer geometric Gibbs measure)**
For $x, y \in \mathbb{Z}$, the weights of solid, dashed and arced edges are



$$\mathrm{wt}_{\mathrm{Geo}}(\cdots) = c_1^{x-y}, \quad \mathrm{wt}_{\mathrm{Geo}}(\cdots) = c_2^{x-y}, \tag{2.4.15a}$$

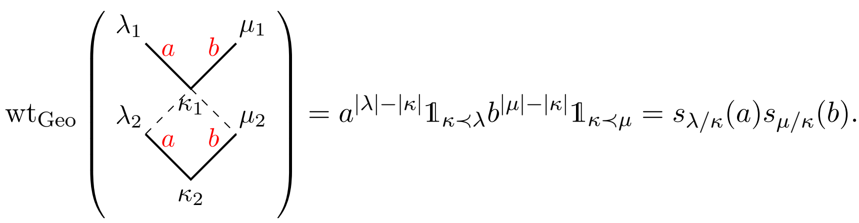


$$\mathrm{wt}_{\mathrm{Geo}}(\cdots) = \mathrm{wt}_{\mathrm{Geo}}(\cdots) = a^{x-y}\mathbb{1}_{x \geqslant y}, \tag{2.4.15b}$$



$$\mathrm{wt}_{\mathrm{Geo}}(\cdots) = \mathrm{wt}_{\mathrm{Geo}}(\cdots) = \mathbb{1}_{x \geqslant y}. \tag{2.4.15c}$$

We define the weight $\mathrm{wt}^{\mathcal{GP}}_{\mathrm{Geo}}(\boldsymbol{\lambda})$ of a two-layer configuration $\boldsymbol{\lambda}$ associated to a two-layer graph $\mathcal{GP}$ to be the product of the above weights over all labeled solid edges, dotted edges and arcs in $\mathcal{GP}$. It is also convenient to write $\mathrm{wt}_{\mathrm{Geo}}$ of a configuration drawn on a sub-graph of $\mathcal{GP}$ to denote the weight of that configuration. For example, we have



$$\mathrm{wt}_{\mathrm{Geo}}(\cdots) = a^{|\lambda|-|\kappa|}\mathbb{1}_{\kappa \prec \lambda} b^{|\mu|-|\kappa|}\mathbb{1}_{\kappa \prec \mu} = s_{\lambda/\kappa}(a)s_{\mu/\kappa}(b). \tag{2.4.16}$$

Notice that $\mathrm{wt}^{\mathcal{GP}}_{\mathrm{Geo}}(\boldsymbol{\lambda})$ is always positive and translation invariant, in the sense that, for all $x \in \mathbb{Z}$,

$$\mathrm{wt}^{\mathcal{GP}}_{\mathrm{Geo}}(\boldsymbol{\lambda}) = \mathrm{wt}^{\mathcal{GP}}_{\mathrm{Geo}}(\boldsymbol{\lambda} + x), \tag{2.4.17}$$

where we write $\boldsymbol{\lambda} + x = (\lambda_1^{(0)} + x, \dots, \lambda_1^{(N)} + x, \lambda_2^{(0)} + x, \dots, \lambda_2^{(N)} + x)$.

We shall view the weight $\mathrm{wt}^{\mathcal{GP}}_{\mathrm{Geo}}(\boldsymbol{\lambda})$ as a measure on $\{\boldsymbol{\lambda} \in \mathbb{Z}^{2N+2}\}$. In fact, the measure is actually supported on Sign_2^{N+1} since each $\lambda^{(j)} \in \mathsf{Sign}_2$ has $\lambda_1^{(j)} \geqslant \lambda_2^{(j)}$. Due to the translation invariance (2.4.17), $\mathrm{wt}^{\mathcal{GP}}_{\mathrm{Geo}}$ must have infinite mass. However, it can be shown [26] that the measure of $\boldsymbol{\lambda}$ with fixed $\lambda_1^{(0)}$ (or any other fixed coordinate $\lambda_i^{(j)}$) is finite under the assumption $c_1c_2 < 1$. Using the definition of the weights above, it is easy to check that this probability measure is exactly (2.4.9).

In the case of the log-gamma polymer and the two-layer Whittaker measure, the story is the same with the following weights:

Definition 2.19 **(Two-layer log-gamma Gibbs measure)**
For $x, y \in \mathbb{R}$, the weights of solid, dashed and arced edges are

$$\mathrm{wt}_{\mathrm{LG}}\left({\color{red}u}\ \overset{x}{\underset{y}{\Big(}} \right) = e^{-u(x-y)}, \quad \mathrm{wt}_{\mathrm{LG}}\left(\overset{x}{\underset{y}{\Big)}}\ {\color{red}v} \right) = e^{-v(x-y)}, \tag{2.4.18a}$$

$$\mathrm{wt}_{\mathrm{LG}}\left({}_{y}\overset{{\color{red}a}\ x}{\diagup} \right) = \mathrm{wt}_{\mathrm{LG}}\left(\overset{x\ {\color{red}a}}{\diagdown}_{y} \right) = e^{-\alpha(x-y) - e^{-(x-y)}}, \tag{2.4.18b}$$

$$\mathrm{wt}_{\mathrm{LG}}\left({}_{y}\overset{x}{\iddots} \right) = \mathrm{wt}_{\mathrm{LG}}\left(\overset{x}{\ddots}_{y} \right) = e^{-e^{-(x-y)}}. \tag{2.4.18c}$$

Notice that the weights in (2.4.18b) are precisely of the form of the density of the log-gamma distribution. This explains why log-gamma random walks will naturally arise in this context.

2.4.7. Markovian description of stationary measures. — The stationary measure for the open KPZ equation can be described in terms of the increments of a Markov process with a specific initial condition [BK21]. This point of view is quite useful when studying various type of limits (large scale limit, limit from the interval to $\mathbb{R}_+$, limit from discrete processes to continuous ones, etc, see for instance [BK21; BW24; BKW24; BW23c]).

In this section, we explain how the probability measures from Section 2.4 can be alternatively viewed as Markov processes. The argument is inspired by earlier results: on the one hand the results of [BKWW23] expressing the open KPZ stationary measure in terms of a Doob transform of a Brownian motion killed at a certain rate, and on the other hand the results of [DS24] which relate certain two-layer Gibbs measures (very similar with our two-layer Whittaker process) and Doob-transformed processes. Moreover, in [24], we show that one can provide fully explicit expressions for the transition probabilities of the Doob-transformed Markov processes involved.

These explicit expressions can in turn be studied as N goes to infinity, or in other asymptotic regimes. This may be used to show that as N goes to infinity, the stationary measures of last passage percolation or the log-gamma polymer in a strip converge to stationary measures of half-space last passage percolation or log-gamma polymer which are described below.

2.4.7.1. *Two-layer Schur process: Markovian description.* — The marginal distribution of λ^0 under the probability measure (2.4.6) can be written as $\mathsf{p}_0(\lambda^0)$ where

$$\mathsf{p}_0(\lambda) = c_1^{\lambda_1-\lambda_2} h_{0,N}(\lambda_1 - \lambda_2) \mathbb{1}_{\lambda_2=0},$$

and the functions $h_{x,N} : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_+$ are defined by

$$h_{x,N}(\ell) = \frac{q_{N-x}(\ell)}{\sum_{\ell\geqslant 0} c_1^\ell q_N(\ell)} \tag{2.4.19}$$

with

$$q_M(\ell) = (1-a)^{2M} \sum_{\lambda\in\mathsf{Sign}_2} s_{\lambda/(\ell,0)}\left((a)^M\right) c_2^{\lambda_1-\lambda_2},$$

where $(a)^M = (a, \dots, a) \in [0,1)^M$. We note that the denominator $\sum_{\ell\geqslant 0} c_1^\ell q_N(\ell)$ in (2.4.19) is equal to the partition function $\mathcal{Z}_{\mathrm{Geo}}^{a,c_1,c_2}(N)$ in (2.4.6), but we chose to write it in this way so that it becomes clear that p_0 defines a probability distribution.

For integers $0 = x_0 < x_1 < \cdots < x_k$, the function $h_{x,N}$ is defined precisely so that the marginal distribution of $\lambda^{x_0}, \lambda^{x_1}, \dots, \lambda^{x_k}$ is given by

$$\mathsf{p}_0(\lambda^{x_0}) \prod_{i=1}^k \mathsf{p}_{x_{i-1},x_i}(\lambda^{x_{i-1}}, \lambda^{x_i}),$$

where

$$\mathsf{p}_{x,y}(\lambda,\mu) = (1-a)^{2(y-x)} s_{\mu/\lambda}\left((a)^{y-x}\right) \frac{h_{y,N}(\mu_1-\mu_2)}{h_{x,N}(\lambda_1-\lambda_2)}.$$

Thus, the process $(\lambda^i)_{0\leqslant i\leqslant N}$ defined by (2.4.6) can also be described as a time-inhomogeneous Markov process with initial distribution p_0 and transition kernel $\mathsf{p}_{x,y}(\lambda,\mu)$ (from time x to time y). Furthermore, all quantities appearing in the definition of the kernel admit explicit integral formulas. Skew Schur functions can be written as a contour integral by Lemma, the normalization constant $\mathcal{Z}_{\mathrm{Geo}}^{a,c_1,c_2}(N)$ is given in (2.4.26). Finally, the function q_M admits the following expression (proved in [24]).

For $c_1, c_2 < 1$, and any $\ell \in \mathbb{Z}_{\geqslant 0}$, we have

$$q_M(\ell) = \frac{1}{2}\oint \frac{dz}{2\mathbf{i}\pi}\left(z^{\ell+1} - \frac{1}{z^{\ell+1}}\right)\left(\frac{1}{z^2} - 1\right)\frac{1}{1-c_2 z}\frac{1}{1-c_2/z}\left(\frac{1-a}{1-az}\frac{1-a}{1-a/z}\right)^M, \tag{2.4.20}$$

where the contour is a circle of radius 1 around the origin.

Remark 2.20. — When $N \to \infty$ one can show that for any x and $c_2 < 1$,

$$h_{x,N}(\ell) \xrightarrow[N\to\infty]{} h(\ell) := (\ell+1)(1-c_1)^2. \tag{2.4.21}$$

In particular the limit does not depend on x. Thus, for $c_1, c_2 < 1$, the large N limit of the Markov process with transition probabilities $\mathsf{p}_{x,y}(\lambda,\mu)$ becomes a very simple Doob transformed process, which can be interpreted as a conditioned random walk studied in [OCo03].

Remark 2.21. — The limit (2.4.21) has another application. It allows to define a consistent family of probability measures

$$\mathbb{P}_n\left(\lambda^0,\dots,\lambda^n\right) = c_1^{\lambda_1^0-\lambda_2^0}\mathbb{1}_{\lambda_2^0=0} \times \left(\prod_{i=1}^n (1-a)^2 s_{\lambda^i/\lambda^{i-1}}(a)\right) \times h(\lambda_1^n - \lambda_2^n).$$

They are consistent in the sense that for all $k \geqslant 0$, the marginal distribution of $\lambda^0, \dots, \lambda^k$ in $\mathbb{P}_n$ is the same for all $n \geqslant k$. We believe that these measures are the appropriate object to adapt the method of [19] to half-space models. In general, one should consider a family of possible functions h, parametrized by c_2 with

$$h(\ell) \propto s_{(\ell,0)}(c_2, 1/c_2) = \frac{c_2^{\ell+1} - c_2^{-\ell-1}}{c_2 - 1/c_2}.$$

2.4.7.2. *Two-layer Whittaker process: Markovian description.* — The Whittaker case is similar. Using the branching rule for Whittaker functions, the marginal distribution of λ^0 under the probability measure (2.4.6) has density $\mathsf{P}_0(\lambda^0)$ where

$$\mathsf{P}_0(\lambda) = e^{-u(\lambda_1-\lambda_2)} H_{0,N}(\lambda_1 - \lambda_2)\delta_0(\lambda_2),$$

and the functions $H_{x,N} : \mathbb{R} \to \mathbb{R}_+$ are defined by

$$H_{x,N}(\ell) = \frac{Q_{N-x}(\ell)}{\sum_{\ell \geqslant 0} e^{-u\ell} Q_N(\ell)}$$

with

$$Q_M(\ell) = \frac{1}{\Gamma(\alpha)^{2M}} \int_{\mathbb{R}^2} d\lambda \Psi_{(\alpha)^M}(\lambda/(\ell,0)) e^{-v(\lambda_1-\lambda_2)}.$$

For integers $0 = x_0 < x_1 < \cdots < x_k$, the marginal distribution of $\lambda^{x_0}, \lambda^{x_1}, \ldots, \lambda^{x_k}$ has density

$$\mathsf{P}_0(\lambda^{x_0}) \prod_{i=1}^{k} \mathsf{P}_{x_{i-1},x_i}(\lambda^{x_{i-1}}, \lambda^{x_i}),$$

where

$$\mathsf{P}_{x,y}(\lambda,\mu) = \frac{1}{\Gamma(\alpha)^{2(y-x)}} \Psi_{(a)^{y-x}}(\mu/\lambda) \frac{H_{y,N}(\mu_1-\mu_2)}{H_{x,N}(\lambda_1-\lambda_2)}.$$

Thus, the process $(\lambda^i)_{0 \leqslant i \leqslant N}$ defined by (2.4.8) is a Markov process with initial distribution P_0 and transition kernel $\mathsf{P}_{x,y}$. Again, all quantities appearing in the definition of the kernel admit explicit integral formulas. In particular, the function Q_M admits the following expression. For $u, v > 0$, and any $\ell \in \mathbb{R}$, we have

$$Q_M(\ell) = \int_{\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi} \Psi_z(\ell, 0) \frac{\Gamma(v \pm z)}{2\Gamma(\pm 2z)} \left(\frac{\Gamma(\alpha \pm z)}{\Gamma(\alpha)^2}\right)^M.$$

Remark 2.22. — When $N \to \infty$ an asymptotic analysis of the above integral shows that for any x and $v > 0$,

$$H_{x,N}(\ell) \xrightarrow[N\to\infty]{} H(\ell) := \frac{2K_0(2e^{-\ell/2})}{\Gamma(u)^2},$$

where K_0 is a Bessel K function. Again, the limit does not depend on x, so that for $u, v > 0$, the large N limit of the Markov process with transition probabilities $\mathsf{P}_{x,y}(\lambda,\mu)$ becomes a very simple Doob transformed process, with transition probability (from time x to $x+1$)

$$\overline{\mathsf{P}}(\lambda,\mu) = \frac{H(\mu_1-\mu_2)}{H(\lambda_1-\lambda_2)} \frac{1}{\Gamma(\alpha)^2} \Psi_\alpha(\mu/\lambda). \tag{2.4.22}$$

One remarks that the function H is the same as in the Markovian description of limits of the open KPZ equation stationary measure [BK21, Theorem 2.3] (up to some factors of 2 which result from different conventions used). The same H-transform appeared earlier in several works, for instance in the Matsumoto-Yor theorem [MY00].

2.4.8. Integral formulas. — The definition of stationary measures for last passage percolation and the log gamma polymer in Section2.4.2 and Section 2.4.7 are useful to have a qualitative understanding of the process and take limits. But do they allow to answer quantitative questions? What is the probability that the first step of the stationary increment process for LPP in a strip is equal to 5? This type of questions can be answered using explicit formulas, whose existence comes from the orthogonality and summation properties satisfied by Schur and Whittaker functions. We developed this point of view in [24].

We start with the case of last passage percolation. For integers $0 = x_0 < x_1 < \cdots < x_k = N$ the joint distribution of the increments $\mathbf{L}_1(x_i) - \mathbf{L}_1(x_{i-1})$ for $1 \leqslant i \leqslant N$ is characterized by the generating function

$$\mathcal{L}^{a,c_1,c_2}_{\mathrm{Geo}}(\mathbf{t},\mathbf{x}) := \mathbb{E}^{a,c_1,c_2}_{\mathrm{Geo}}\left[\prod_{i=1}^{k} t_i^{2(\mathbf{L}_1(x_i)-\mathbf{L}_1(x_{i-1}))}\right].$$

Let us introduce the notations

$$\mathfrak{m}_{\mathrm{Geo}}(dz) = \frac{(1-z^2)(1-1/z^2)}{2z}, \quad \phi(z,a) = \phi_{\mathrm{Geo}}(z;a) = \frac{1}{1-za}, \tag{2.4.23}$$

and define the kernel

$$K_{\mathrm{Geo}}(t_1,t_2;z_1,z_2) = \sqrt{1-\left(\frac{t_1}{t_2}\right)^2}\,\phi_{\mathrm{Geo}}(z_2z_1,t_1/t_2)\phi_{\mathrm{Geo}}(z_2/z_1,t_1/t_2). \tag{2.4.24}$$

Theorem 2.23 **([24]).** — *Let a, c_1, c_2 be parameters as in Definition 2.7. Let $t_1,\dots,t_k$ be real numbers such that $c_1 < t_1 < \cdots < t_k < \frac{1}{c_2}$ and $at_k < 1$. For $c_1, c_2 < 1$, we have the formula*

$$\mathcal{L}^{a,c_1,c_2}_{\mathrm{Geo}}(\mathbf{t},\mathbf{x}) = \frac{1}{\mathcal{Z}^{a,c_1,c_2}_{\mathrm{Geo}}(N)}\oint \frac{\mathfrak{m}_{\mathrm{Geo}}(dz_1)}{2\mathbf{i}\pi}\cdots\oint\frac{\mathfrak{m}_{\mathrm{Geo}}(dz_k)}{2\mathbf{i}\pi}\left|\prod_{i=1}^{k}\phi(z_i,at_i)^{x_i-x_{i-1}}\right|^2$$
$$\times\left|\phi(z_1,c_1/t_1)\phi(z_k,c_2t_k)\prod_{i=1}^{k-1}K_{\mathrm{Geo}}(t_i,t_{i+1};z_i,z_{i+1})\right|^2, \tag{2.4.25}$$

where the integration contours are (positively oriented) circles around 0 with radius 1 (for a variable z on such contour, and a rational function f with real coefficients, $|f(z)|^2 = f(z)f(1/z)$, so that the integral in (2.4.25) is a complex contour integral in the usual sense, and it can be evaluated by residues). The normalization constant can be computed explicitly as

$$\mathcal{Z}^{a,c_1,c_2}_{\mathrm{Geo}}(N) = \oint \mathfrak{m}_{\mathrm{Geo}}(dz)\left|\phi(z,c_1)\phi(z,c_2)(\phi(z,a))^N\right|^2. \tag{2.4.26}$$

In the simplest case, that is for $k=1$, Theorem 2.23 implies that for $c_1, c_2 < 1$ with $c_1 < t < \min\{\frac{1}{c_2},\frac{1}{a}\}$,

$$\mathbb{E}^{a,c_1,c_2}_{\mathrm{Geo}}\left[t^{2\mathbf{L}_1(N)}\right] =$$
$$\frac{1}{\mathcal{Z}^{a,c_1,c_2}_{\mathrm{Geo}}(N)}\oint\frac{dz}{2\mathbf{i}\pi z}\frac{(1-z^2)(1-1/z^2)}{2\left((1-taz)(1-ta/z)\right)^N}\frac{1}{1-\frac{zc_1}{t}}\frac{1}{1-\frac{c_1}{zt}}\frac{1}{1-\frac{c_2t}{z}}\frac{1}{1-c_2tz}.$$

We now state an analogous result for the log-gamma polymer stationary measure, more precisely for the joint Laplace transform of its increments

$$\mathcal{L}_{\mathrm{LG}}^{\alpha,u,v}(\mathbf{t},\mathbf{x}) = \mathbb{E}_{\mathrm{LG}}^{\alpha,u,v}\left[\prod_{i=1}^{k} e^{-2t_i(\mathbf{L}_1(x_i)-\mathbf{L}_1(x_{i-1}))}\right].$$

Let us redefine the function ϕ as

$$\phi(z,\alpha) = \phi_{\mathrm{LG}}(z,\alpha) = \Gamma(z+\alpha),$$

and use the notations

$$\mathfrak{m}_{\mathrm{LG}}(dz) = \frac{1}{2\Gamma(2z)\Gamma(-2z)}, \quad K_{\mathrm{LG}}(t_1,t_2;z_1,z_2) = \frac{\phi(z_2+z_1;t_1-t_2)\phi(z_2+z_1;t_1-t_2)}{\sqrt{\Gamma(2t_1-2t_2)}}.$$

***Theorem 2.24* ([24]).** — *Let α, u, v be parameters as in Definition 2.8. Let $t_1,\dots,t_k$ be real numbers such that $u > t_1 > \dots > t_k > -v$ and $\alpha + t_k > 0$. For $u, v > 0$, we have*

$$\mathcal{L}_{\mathrm{LG}}^{\alpha,u,v}(\mathbf{t},\mathbf{x}) = \frac{1}{\mathcal{Z}_{\mathrm{LG}}^{\alpha,u,v}(N)}\int_{\mathbf{i}\mathbb{R}}\frac{\mathfrak{m}_{\mathrm{LG}}(dz_1)}{2\mathbf{i}\pi}\cdots\int_{\mathbf{i}\mathbb{R}}\frac{\mathfrak{m}_{\mathrm{LG}}(dz_k)}{2\mathbf{i}\pi}\left|\prod_{i=1}^{k}\phi(z_i,\alpha+t_i)^{x_i-x_{i-1}}\right|^2$$
$$\times\left|\phi(z_1,u-t_1)\phi(z_k,v+t_k)\prod_{i=1}^{k-1}K_{\mathrm{LG}}(t_i,t_{i+1};z_i,z_{i+1})\right|^2, \quad (2.4.27)$$

where we recall that for $z \in \mathbf{i}\mathbb{R}$, $|f(z)|^2 = f(z)f(-z)$.

The normalization constant is given by

$$\mathcal{Z}_{\mathrm{LG}}^{\alpha,u,v}(N) = \int_{\mathbf{i}\mathbb{R}}\frac{\mathfrak{m}_{\mathrm{LG}}(dz)}{2\mathbf{i}\pi}\left|\phi(z,u)\phi(z,v)(\phi(z,\alpha))^N\right|^2$$
$$= \int_{\mathbf{i}\mathbb{R}}\frac{dz}{2\mathbf{i}\pi}\frac{\Gamma(u\pm z)\Gamma(v\pm z)\left(\Gamma(\alpha\pm z)\right)^N}{2\Gamma(\pm 2z)}. \quad (2.4.28)$$

In the second line, we use the notational convention that each time that $\pm$ appears in the formula, we take the product of the Gamma function with a plus sign and the Gamma function with a minus sign.

In the simplest case, that is when $k = 1$, Theorem 2.24 implies that for $u > t > \max\{-v, -\alpha\}$,

$$\mathbb{E}_{\mathrm{LG}}^{\alpha,u,v}\left[e^{-2t\mathbf{L}_1(N)}\right] = \frac{1}{\mathcal{Z}_{\mathrm{LG}}^{\alpha,u,v}(N)}\int_{\mathbf{i}\mathbb{R}}\frac{dz}{2\mathbf{i}\pi}\frac{\Gamma(u-t\pm z)\Gamma(v+t\pm z)\left(\Gamma(\alpha+t\pm z)\right)^N}{2\Gamma(2z)\Gamma(-2z)}. \quad (2.4.29)$$

Remark 2.25. — Using the change of variables $z_i = e^{\mathbf{i}\pi\theta_i}$ with $\theta_i \in (-1,1]$ in (2.4.25) and interpreting the θ_i as successive values of a Markov process, the Laplace transform in (2.4.25) can be rewritten in terms of the free Askey-Wilson process [BW07; BMW08]. Askey-Wilson processes are stochastic processes taking values in $[-1,1]$, introduced in [BW10] in the context of quadratic harnesses [BMW07]. Their transition density is expressed in terms of the orthogonality measure for Askey-Wilson orthogonal polynomials, hence the name. Askey-Wilson processes also arises in Laplace transform formulas characterizing the stationary measure of open ASEP [BW17]. The limit of these formulas when ASEP scales to the KPZ equation was studied in [CK24], which led to study a limit of Askey-Wilson processes called continuous dual Hahn processes, also considered in [Bry12] (see also [Bry22]). Letting $z_i = \pm\mathbf{i}\sqrt{x_i}/2$ in (2.4.27), we also notice that the formula (2.4.27) can be rewritten as a functional of the continuous dual Hahn process. We believe that the relations between two-layer Gibbs measures and Askey-Wilson processes deserve to be studied more systematically. This point of view has been recently investigated in [BSW25] which re-derived Theorem 2.23, without using Schur functions.

2.5. Stationary measures of models in a half-space

2.5.1. Old results for TASEP. — For the asymmetric simple exclusion process, it was understood as early as in [Lig75] that stationary measures for the system on $\mathbb{Z}_{>0}$ are limits as $\ell \to \infty$ of stationary measures of the system on $[\![1,\ell]\!]$. For simplicity, let us consider the TASEP on $[\![1,\ell]\!]$ with jump rates $p=1, q=0$ and boundary parameters α, β arbitrary and $\gamma=0, \delta=0$. Let us denote my $\mu^{(\ell)}_{\alpha,\beta}$ the stationary measure of the system. Its limit as ℓ goes to infinity satisfies the phase transition:

Maximal current phase : If $\alpha, \beta \geqslant 1/2$, then $\mu^{(\ell)}_{\alpha,\beta}$ converges to a probability measure $\mu^{(\infty)}_{\alpha,1/2}$ depending only on the parameter α.

Low density phase : If $\alpha \leqslant 1/2$ and $\alpha \leqslant \beta$, then $\mu^{(\ell)}_{\alpha,\beta}$ converges to the probability measure $\nu^{(\infty)}_{\alpha}$, that is the law of independent Bernoulli variables with parameter α.

High density phase : If $\beta \leqslant 1/2$ and $\alpha \geqslant \beta$, then $\mu^{(\ell)}_{\alpha,\beta}$ converges to the probability measure $\mu^{(\infty)}_{\alpha,\beta}$ which depends on both parameters α and β. Under this probability measures, the occupation variables $\eta_x, \eta_{x+1}, \dots, \eta_{x+k}$ are distributed as independent Bernoulli variables with parameter β as x goes to infinity.

Moreover, these limiting probability measures are stationary measures for half-line TASEP. Similar results hold for ASEP [Gro04].

2.5.2. A convergence result of Hariya Yor. — In view of the results of Liggett [Lig75] on half-line TASEP, it is likely that the stationary measures of the KPZ equation on $\mathbb{R}_+$ (KPZ$_u$) are given by limits as $L \to \infty$ of the stationary measure of the open KPZ equation given by the law of B_1 in (2.3.1) – see Theorem 2.6. This conjecture was stated in [16], and, quite fortunately, the limit of the probability measure (2.3.1) had been studied previously by Hariya and Yor [HY04], independently of the KPZ context. They were simply interested in understanding Brownian measures reweighted by exponential functional of the Brownian motion, which satisfy many curious exact relations such as the Matsumoto-Yor theorem [MY00]. In order to state the result of [HY04] in a convenient way for our purposes, we introduce the following stochastic processes.

Definition 2.26. — For $u \in \mathbb{R}$ and $v \leqslant \min\{0,u\}$, define the process $\mathcal{H}_{u,v}(x)$ as

$$\mathcal{H}_{u,v}(x) := \log\left(\frac{1}{\varpi}\int_0^x ds e^{B_1(s)+B_2(x)-B_2(s)} + e^{B_2(x)}\right), \tag{2.5.1}$$

where B_1 and B_2 are independent standard Brownian motions with drifts $-v$ and v (resp.), and $\varpi \sim \mathrm{Gamma}^{-1}(u-v)$, independent from B_1 and B_2. When $u=v$, we define $\frac{1}{\varpi}=0$.

Let us mention a few important properties of the process $\mathcal{H}_{u,v}$:

- For general u, v, spatial increments of $\mathcal{H}_{u,v}$ are not independent or translation invariant.
- For $u = v \leqslant 0$, $\mathcal{H}_{u,v} = B_2$ is Brownian motion with drift $u=v$ (follows immediately).
- For $u = -v \geqslant 0$, $\mathcal{H}_{u,v}$ is Brownian motion with drift $u=-v$ (this is not obvious but true [19]).
- The parameter v controls the drift at infinity, in the sense that almost surely,

$$\lim_{x\to\infty} \frac{\mathcal{H}_{u,v}(x)}{x} = -v.$$

- The definition of the process $\mathcal{H}_{u,v}$ in (2.5.1) makes perfect sense for $v > 0$ (provided $u \geqslant v$) in which case the processes $\mathcal{H}_{u,v}$ and $\mathcal{H}_{u,-v}$ have the same distribution (again, this is not obvious, but this is true [19]). Thus, we may restrict the sign of v, and we choose $v \leqslant 0$ in order to be consistent with earlier literature where the process (2.5.1) has appeared [16; BK21].

Let $h^L_{u,v}$ be distributed as B_1 in (2.3.1), that is the unique stationary measure of the open KPZ equation. Then, we have the weak convergence of probability measures in $C(\mathbb{R}_+, \mathbb{R})$:

$$h^L_{u,v}(x) \xrightarrow[L\to\infty]{} \begin{cases} \mathcal{H}_{u,0}(x) & \text{if } u, v \geqslant 0 \\ \mathcal{H}_{u,u}(x) & \text{if } u < 0, v > u \\ \mathcal{H}_{u,v} & \text{if } v < 0, u > v \end{cases}$$

As expected, the large L limits of the stationary measures of open KPZ equation $[0, L]$ are stationary measures for the KPZ equation on $\mathbb{R}_+$.

Theorem 2.27 **([16; 19]).** — *For any $u \in \mathbb{R}$ and any $v \leqslant \min\{0, u\}$, $\mathcal{H}_{u,v}$ is stationary for the half-space KPZ equation* (KPZ_u).

We also conjecture the following classification:

Conjecture 2.28 **([16]).** — *$\{\mathcal{H}_{u,v}\}_{v\leqslant\min\{0,u\}}$ constitutes all extremal stationary measures for* (KPZ) *and for any initial data $\mathcal{H}(0, x)$ with drift $-v$ as $X \to \infty$ we have the following.*

1. *For $u \geqslant 0$: If $v \leqslant 0$, the process $\mathcal{H}(t, \cdot) - \mathcal{H}(t, 0)$ converges in distribution as $t \to \infty$ to the process $\mathcal{H}_{u,v}(\cdot)$; if $v \geqslant 0$, it converges to the process $\mathcal{H}_{u,0}(\cdot)$.*
2. *For $u \leqslant 0$: If $v \leqslant u$ the process $\mathcal{H}(t, \cdot) - \mathcal{H}(t, 0)$ converges in distribution as $t \to \infty$ to the process $\mathcal{H}_{u,v}$; if $v \geqslant u$, it converges to $\mathcal{H}_{u,u}$, that is a Brownian motion with drift u.*

Some additional conditions besides having drift $-v$ may be needed in this conjecture. The phase diagram that is conjectured is illustrated in Figure 5. In the limit where the drift $-v$ goes to $-\infty$, the sequence of initial datum $\mathcal{H}(0, x) = -vx + \log(v)$ approximates of narrow wedge initial data where $e^{\mathcal{H}(0,x)} = \delta(x)$. In this case the conjecture suggests that $\mathcal{H}(t, \cdot)$ weakly converges as $t \to \infty$ to a Brownian motion with drift u when $u \leqslant 0$ and to $\mathcal{H}_{u,0}$ when $u \geqslant 0$. Similarly, for flat initial data $\mathcal{H}(0, \cdot) \equiv 0$ the process $\mathcal{H}(t, \cdot)$ should weakly converge as $t \to \infty$ to a Brownian motion with drift u when $u \leqslant 0$ and to $\mathcal{H}_{u,0}$ when $u \geqslant 0$.

2.5.3. Log-gamma polymer on a half-quadrant. — Theorem 2.27 is proved in [19] by taking the scaling limit of an analogous result for the log-gamma polymer in a half-quadrant – we recall that the convergence of the log-gamma polymer to the KPZ equation is discussed in Section 1.6.3.

The partition function for the log-gamma polymer in a half-quadrant [(2)], defined in Section 1.4, satisfies the following discrete version of the SHE subject to imposing suitable initial data for the distribution of the process $(\mathcal{Z}(n, 0))_{n\geqslant 0}$:

$$\begin{aligned} \mathcal{Z}(n, m) &= \varpi_{n,m}(\mathcal{Z}(n-1, m) + \mathcal{Z}(n, m-1)) \text{ for } n > m \geqslant 1, \\ \mathcal{Z}(n, n) &= \varpi_{n,n}\mathcal{Z}(n, n-1) \text{ for } n \geqslant 1. \end{aligned} \tag{$\mathrm{dSHE}_{u,\alpha}$}$$

2. It was denoted by $\mathcal{Z}^{\boxslash}(n, m)$ in Section 1.4. In this chapter, we drop the superscript.

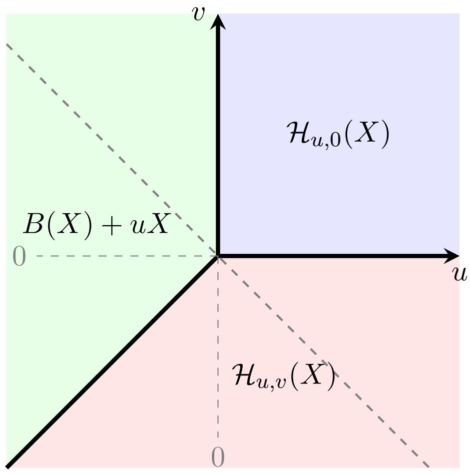


FIGURE 5. Fix $u, v \in \mathbb{R}$. Conjecture 2.28 claims that the solution to (KPZ) converges at large time to one of the spatial process $\mathcal{H}_{u,v}$, $\mathcal{H}_{u,0}$ or a Brownian motion $B(X) + uX$, according to where (u, v) lies in this diagram. The same diagram also describes the large L limits of $h^L_{u,v}$ for different values of (u, v), obtained in [HY04]. Along the antidiagonal line $u+v = 0$, $h^L_{u,v}$ is always Brownian motion with drift $u = -v$.

We assume that the weights $\varpi_{n,m}$ are all independent, and distributed as

$$\begin{cases}\varpi_{n,m} \sim \mathrm{Gamma}^{-1}(2\alpha) & \text{for } n > m \geqslant 1,\\ \varpi_{n,n} \sim \mathrm{Gamma}^{-1}(\alpha+u) & \text{for } n \geqslant 1,\end{cases}$$

where we have a bulk parameter $\alpha > 0$ and a boundary parameter $u > -\alpha$. The free energy $\mathcal{H}(n, m) := \log \mathcal{Z}(n, m)$ plays a similar role here as the KPZ equation. A process $\big(\mathcal{H}(k)\big)_{k\in\mathbb{Z}_{\geqslant 0}}$ is stationary for the half-space log-gamma polymer if the solution to (dSHE$_{u,\alpha}$) with initial data $\mathcal{H}(\cdot, 0) = \mathcal{H}(\cdot)$ has the property that the distribution of $\big(\mathcal{H}(m + k, m) - \mathcal{H}(m, m)\big)_{k\in\mathbb{Z}_{\geqslant 0}}$ in the same for all $m \in \mathbb{Z}_{\geqslant 0}$ (and equal to that of the initial data).

We say that a process $(r(k))_{k\in\mathbb{Z}_{\geqslant 0}}$ is a $\mathrm{Gamma}^{-1}(\theta)$ multiplicative random walk if $\big(r(k + 1)/r(k)\big)_{k\in\mathbb{Z}_{\geqslant 0}}$ are i.i.d. $\mathrm{Gamma}^{-1}(\theta)$.

Definition 2.29. — For $\alpha \in \mathbb{R}_{\geqslant 0}$ and $u, v \in \mathbb{R}$ with $u, v > -\alpha$ and $v \leqslant \min\{0, u\}$, define

$$\mathcal{Z}_{u,v}(k) := r_2(k) + \frac{1}{\varpi}\sum_{\ell=1}^{k} r_1(\ell)\frac{r_2(k)}{r_2(\ell-1)}, \qquad \mathcal{H}_{u,v}(k) := \log \mathcal{Z}_{u,v}(k), \quad \text{for } k \in \mathbb{Z}_{\geqslant 0}, \tag{2.5.2}$$

where r_1 and r_2 are independent $\mathrm{Gamma}^{-1}(\alpha + v)$ and $\mathrm{Gamma}^{-1}(\alpha - v)$ (resp.) multiplicative random walks with $r_1(0) = r_2(0) = 1$, and $\varpi \sim \mathrm{Gamma}^{-1}(u - v)$ is independent (again, when $u = v$, we define $\frac{1}{\varpi} = 0$).

Let us mention a few properties of the process $\mathcal{Z}_{u,v}$:

- For general u, v, $\mathcal{Z}_{u,v}(k + 1)/\mathcal{Z}_{u,v}(k)$ are not independent or translation invariant.
- For $u = v \leqslant 0$, $\mathcal{Z}_{u,u} = r_2$ is a $\mathrm{Gamma}^{-1}(\alpha - u)$ multiplicative random walk.
- For $u = -v \geqslant 0$, $\mathcal{Z}_{u,-u}$ is a $\mathrm{Gamma}^{-1}(\alpha - u)$ multiplicative random walk (this is not obvious, but proved in [19]).

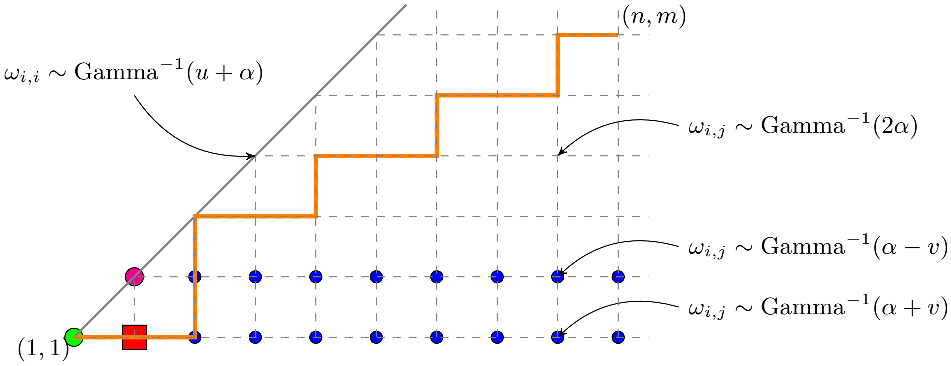


FIGURE 6. The stationary half-space log-gamma polymer with parameters α (in the bulk), u (at the boundary) and v (at infinity).

- The parameter v controls the behaviour at infinity in the sense that $\mathcal{Z}_{u,v}(k+1)/\mathcal{Z}_{u,v}(k) \Rightarrow \mathrm{Gamma}^{-1}(\alpha+v)$ when k goes to infinity (it may be justified using results from [ABO24]).
- The definition of $\mathcal{Z}_{u,v}$ in (2.5.2) makes perfect sense for $v > 0$ (provided $-\alpha < v \leqslant u$) in which case the processes $\mathcal{Z}_{u,v}$ and $\mathcal{Z}_{u,-v}$ have the same distribution [19].

Theorem 2.30. — *For any $\alpha \in \mathbb{R}_{>0}$ and $u, v \in \mathbb{R}$ such that $u, v > -\alpha$ and $v \leqslant \min\{0, u\}$, the process $\mathcal{H}_{u,v}$ is stationary for the half-space log-gamma polymer, i.e., for* $(\mathrm{dSHE}_{u,\alpha})$.

We also conjecture in [19] an analogue of Conjecture 2.28 for the half-space log-gamma polymer. Theorem 2.30 suggests the following open problem, for which a potential approach is discussed in [24].

Open Problem 2.31. — *Show that the limit of stationary measures of the log-gamma polymer in a strip (Definition 2.12) converge to the process* (2.5.2) *in Definition 2.29, according to the same phase diagram as in [HY04].*

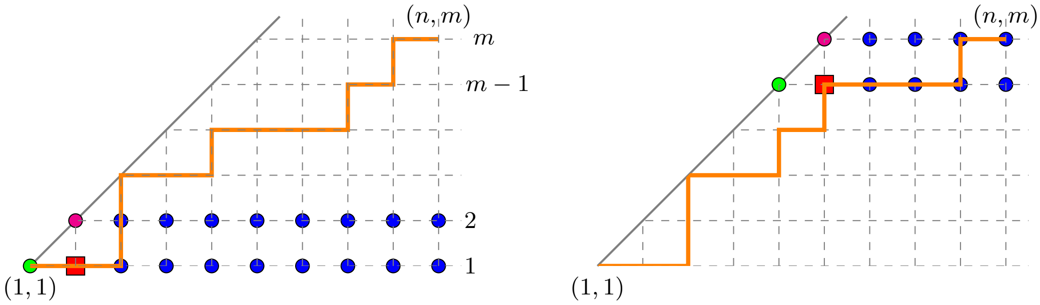
2.5.4. Rerouting paths argument. — The proof of Theorem 2.30 is based on the following argument. Let us consider the log-gamma polymer in a half-quadrant depending on parameters $\alpha_1, \alpha_2, \dots$ as in Section 1.4. Since it is a marginal of a Macdonald processes (as $t = 0$, in the limit $q \to 1$) – see Section 1.3.4 – one can show that the law of

$$(Z(m,m), Z(m+1,m), \dots, Z(m+k,m))$$

is invariant with respect to permutations of parameters $\alpha_1, \dots, \alpha_m$. In order to construct the stationary log-gamma polymer model, we let $\alpha_1 = v, \alpha_2 = -v$ and $\alpha_i = \alpha$ for $i \geqslant 3$, as depicted in Figure 6. We have used the following coloring conventions:

- The weight ■ is distributed as $\mathrm{Gamma}^{-1}(v - v) = +\infty$. In practice, weights should not be infinity, so one has to assume that $\alpha_1 = v + \varepsilon$ and let $\varepsilon \to 0$ at the end of the argument.
- The weights • are distributed as $\mathrm{Gamma}^{-1}(\alpha+v)$ or $\mathrm{Gamma}^{-1}(\alpha-v)$.
- The weight • is distributed as $\mathrm{Gamma}^{-1}(u-v)$. It should be identified with the weight ω in the definition $\mathcal{Z}_{u,v}$ in (2.5.2).

Using the symmetry with respect to permuting parameters, we may exchange rows and columns, so that the following two partition functions have the same distribution:



On the left, we are considering the log-gamma polymer in a half quadrant $\mathcal{Z}(n,m)$ with initial condition $\mathcal{Z}_{u,v}$ defined in (2.5.2). On the right, only the paths going through the huge weight ■ can contribute to the partition function. Hence, when one considers ratios of the form $\frac{\mathcal{Z}(m+k,m)}{\mathcal{Z}(m,m)}$ the contribution of paths from $(1,1)$ to ■ cancels out, and one is left with a ratio of two-row partition functions distributed as $\mathcal{Z}_{u,v}(k)/\mathcal{Z}_{u,v}(k)$. Thus, we conclude that for any m,

$$\left(\frac{\mathcal{Z}(m+k,m)}{\mathcal{Z}(m,m)}\right)_{k\geqslant 0} = \left(\frac{\mathcal{Z}(2+k,2)}{\mathcal{Z}(2,2)}\right)_{k\geqslant 0}$$

which proves Theorem 2.30. It also leads to Theorem 2.27 after a scaling limit.

Remark 2.32. — The argument explained above, based on the symmetry with respect top parameters and rerouting paths, also applies to models of last passage percolation with exponential or geometric weights [19]. The corresponding stationary measures are similar to (2.5.2) except that log-gamma variables are replaced by exponential or geometric ones, and logarithms of sums are replaced by maximums.

Interestingly, the argument based on rerouting paths also applies in other settings. For instance, in [18], we apply it to determine the stationary measure associated with the evolution of partition functions for k non-intersecting polymer paths.

2.6. Outlook

2.6.1. A random matrix model. — Endow the space $\mathcal{P}_d$ of $d\times d$ positive definite symmetric matrices with the product

$$A\star B = \sqrt{B}\,A\,\sqrt{B}.$$

In this section, we consider the following recursion, introduced in [ABO23],

$$Z(n,m) = \Big(Z(n-1,m) + Z(n,m-1)\Big)\star (W_{n,m})^{-1}$$

where $W_{n,m}$ is a Wishart random matrix, that is with density

$$\mathbb{P}(dW) = \frac{1}{\Gamma_d(\alpha)}|W|^{\alpha}e^{-\mathrm{Tr}(W)}\mu(dW),\quad |W| := \det(W),\quad \mu(dW) = \frac{dW}{|W|^{\frac{d+1}{2}}}.$$

When $d=1$, the model reduces to the partition function of the log-gamma polymer. In [33], we consider stationary versions of this model. In particular, we may consider the recursion in a strip with appropriate boundary conditions depending on weights

$$\begin{aligned}
W_{i,j} &\sim \mathrm{Wishart}^{-1}(2\alpha) \text{ for } j<i<j+N\\
W_{i,i} &\sim \mathrm{Wishart}^{-1}(\alpha+u)\\
W_{j+N,j} &\sim \mathrm{Wishart}^{-1}(\alpha+v).
\end{aligned}$$

Assume $u+v>\frac{d-1}{2}$. Then, we may consider the measure

$$\mathbb{P}_{\alpha,u,v}(\Lambda_1,\Lambda_2)=\mu(d\Lambda_1^{(0)})\ \left|\Lambda_2^{(0)}(\Lambda_1^{(0)})^{-1}\right|^u\ \left|\Lambda_2^{(N)}(\Lambda_1^{(N)})^{-1}\right|^v$$
$$\prod_{j=1}^{N} e^{-\mathrm{Tr}\left(\Lambda_2^{(j)}(\Lambda_1^{(j-1)})^{-1}\right)}\ \mathbb{P}_\alpha^{\mathrm{RW}}(\Lambda_1)\mathbb{P}_\alpha^{\mathrm{RW}}(\Lambda_2) \quad (2.6.1)$$

where $\mathbb{P}^{\mathrm{RW}}$ denotes the random walk on $\mathcal{P}_d$ with inverse Wishart increments:

$$\mathbb{P}_\alpha^{\mathrm{RW}}(R)=\prod_{j=1}^{N} P_\alpha\left(R(j-1)R(j)^{-1}\right)\mu(dR(j))$$

with

$$P_\alpha(R)=|R|^\alpha e^{-\mathrm{Tr}R}.$$

The measure (2.6.1) is not a probability measure because $\Lambda_1^{(0)}$ is μ-distributed, and μ is only a Radon measure. However, if we fix $\Lambda_1^{(0)}$, then the measure can be normalized whenever $u+v>\frac{d-1}{2}$.

***Theorem 2.33* ([33]).** — *The law of $x\mapsto\Lambda_1^{(x)}$ under $\mathbb{P}_{\alpha,u,v}$ is a stationary measure of the inverse-Wishart polymer model in a strip.*

Remark 2.34. — The partition function $Z(n,m)$ of the inverse-Wishart polymer is significantly harder to analyze than the log-gamma polymer for several reasons:

- The partition function $Z(n,m)$ cannot be written as a sum over paths. As a consequence, we lack monotonicity results. In particular, it is not clear that we can use sub-additivity arguments to prove law of large numbers.
- When $d>1$, the ratios $(Z(m+k,k)/Z(m,m))_{k\geqslant 0}$ do not form a Markov process as m increases. As a consequence, we may only consider stationary measures for the evolution of partition functions themselves. These are necessarily infinite measures.
- Even in the simplest setting of the model on a quadrant or the model on a strip with $u+v=0$, where inverse-Wishart random walks with μ-distributed starting point are stationary, this is not sufficient to predict the law of large numbers satisfied by the partition function, because there may exist other stationary measures (thanks to Neil O'Connell for useful discussions on that point!).

2.6.2. Hall-Littlewood, q-Whittaker, higher spin, two-layer Gibbs measures. — Since the method explained in Section 2.4 works for two-layer Schur processes and two-layer Whittaker processes, it is likely that it may work as well for other models corresponding to other symmetric functions. For instance, the case of Hall-Littlewood functions should be related to the stochastic six vertex model in a strip, studied in [Yan22] via the Matrix Product Ansatz. The model is a probability measure on up right paths in a strip as depicted in Figure 7, defined using the weights (1.5.1) and (1.5.6) as in Section 1.5.6. The boundary weights may also be reparametered using a,b,c,d as in Section 2.2.1.

The associated two-layer Hall-Littlewood measure takes the form

$$\mathbb{P}_{x,a,b,c,d}(\lambda_1,\lambda_2)=\frac{1}{Z_{x,a,b,c,d}(N)}H_{\lambda_1^{(0)}-\lambda_2^{(0)}}(a,c)H_{\lambda_1^{(N)}-\lambda_2^{(N)}}(b,d)$$
$$\times\prod_{\substack{1\leqslant j\leqslant N\\ \lambda_1^{(j)}-\lambda_2^{(j)}<\lambda_1^{(j-1)}-\lambda_2^{(j-1)}}}\left(1-q^{\lambda_1^{(j-1)}-\lambda_2^{(j-1)}}\right)\ \mathbb{P}_x^{\mathrm{RW}}(\lambda_1)\,\mathbb{P}_x^{\mathrm{RW}}(\lambda_2) \quad (2.6.2)$$

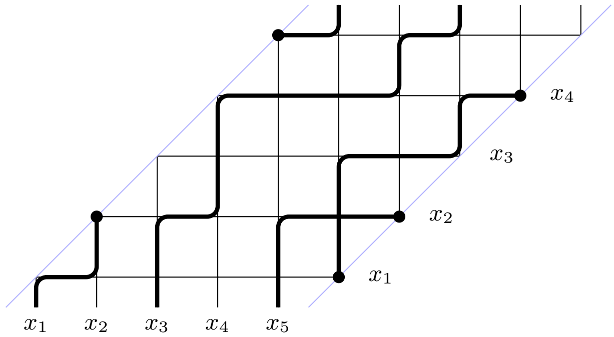


FIGURE 7. Stochastic six-vertex model in a strip, depending on a family of inhomogeneity parameters $x_1, \ldots, x_N$, which should be taken equal in order to consider stationary measures.

where $\mathbb{P}_x^{\mathrm{RW}}$ is the probability measure of a Bernoulli random walk with parameter x, and $H_k(a,c)$ are essentially q-Hermite polynomials, defined in (2.2.2). The probability distribution of $x \mapsto \lambda_1(x)$ is the unique stationary measure for the height function of the stochastic six vertex model in a strip. This description should be equivalent to using the MPA representation in [Yan22] with the representation given in Section 2.2.1. Furthermore, adapting the arguments of [24] to the Hall-Littlewood case should allow to re-derive the results of [BW17]. This is a work in preparation with Ivan Corwin. It would be also interesting to the case of higher spin stochastic six-vertex model [Yan23], as well as the case of (higher spin) q-Whittaker functions.

The essential input in order to make the method work is a skew-Cauchy type identity of the form (2.4.13), as well as a skew Littlewood type identity to deal with the boundary dynamics. In the context of vertex models, such identities are typically consequences of the Yang-Baxter equation, and some of the required identities actually appeared in earlier works [BM18; BP19; He24].

Remark 2.35. — When $q = 0$, the description of the stationary measure (2.6.2) coincides with the random walk description in [DEL04]. In the case of ASEP, this two-layer line ensemble is however different from that of [BZ24; Bry24; HY24] which is actually related to the representation of the MPA found by Derrida-Enaud [ED04].

2.6.3. Deformed MPA, higher cumulants. — In this chapter, we have seen many results about stationary measures of models defined on a bounded domain with open boundary conditions. What can be said beyond stationary measures? For models in the KPZ class, the most general scaling to consider is when the size of the system scales as $\ell = t^{2/3}$. A major open problem is the following:

Open Problem 2.36. — *Let $H(t,x), x \in [\![1,\ell]\!]$ be the height function of open TASEP – or any other model in the KPZ class – with $\ell = t^{2/3}$ and boundary densities scaled as*

$$\varrho_a = \frac{1}{2} + ut^{-1/3}, \ \varrho_b = \frac{1}{2} - vt^{-1/3}.$$

Characterize the fluctuations of the rescaled process

$$\frac{H(t\tau, xt^{2/3}) - C(t,x)}{t^{1/3}},$$

where $C(t,x)$ is some deterministic function to determine, as $t \to \infty$.

If one keeps the size of the system fixed, however, one can say more. Let us consider the open KPZ equation ($\mathrm{KPZ}_{u,v}$) on $[0, L]$. Starting from the stationary initial condition, it is possible to deduce from Theorem 2.24 that

$$\mathbb{E}[h(t,0)] = tc_1(L,u,v)$$

where, if $u, v > 0$, the constant c_1 is expressed as [24]

$$c_1(L,u,v) = \frac{-1}{24} + \frac{1}{2}\partial_L \log \mathcal{Z}_{u,v}(L),$$

where $\mathcal{Z}_{u,v}(L)$ is the normalization constant that appears in (2.3.2), and can be computed explicitly as

$$\mathcal{Z}_{u,v}(L) = \int_{\mathbf{i}\mathbb{R}} \frac{dz}{2\mathbf{i}\pi} \left| \frac{\Gamma(u+z)\Gamma(v+z)}{\Gamma(2z)} \right|^2 \frac{e^{z^2 L}}{2}.$$

Moreover, extrapolating arguments of [GK24] from periodic to open boundary conditions, we expect that the height function satisfies a central limit, so that

$$\frac{h(t,0) - C_1(u,v,L)t}{t^{1/2}} \xrightarrow[t\to\infty]{} \mathcal{N}(0, c_2)$$

for some constant $c_2 = c_2(u,v,L)$. We computed its exact expression in [32].

More generally, let $\kappa_k(X)$ denote the kth cumulant of the random variable X. In [32], we computed the large time limit of cumulants of the open KPZ height function. More precisely, we computed the constants

$$c_k(u,v,L) = \lim_{t\to\infty} \frac{1}{t}\kappa_k(h(t,0)).$$

The coefficients are expressed as power series coefficients of some function defined as the analytic solution to some functional equation involving an explicit integral operator. Our method is based on physics techniques, very far from being mathematically rigorous, but the beautiful structure that emerges calls for a mathematical explanation.

We actually use in [32] two different methods leading to the same formulas. The first method is due to Brunet and Derrida [BD00a; BD00b]. It is based on the analysis of solutions to the Bethe equations associated with the delta Bose gas on $[0, L]$, i.e. the operator acting on functions $[0, L]^k \to \mathbb{R}$

$$H = \Delta + \sum_{i<j} \delta(x_i - x_k)$$

with (u, v)-dependent boundary conditions. This comes from the fact that moments $\mathbb{E}[Z(t,x_1)\dots Z(t,x_k)]$ satisfy the PDE $\partial_t u = Hu$. Then, the computation of the cumulant generating function comes from a highly non-rigorous physics method called the replica trick.

The second method we use is to take a scaling limit of the work of [GLMV12] on the cumulants of the open ASEP height function, based on the deformed matrix product ansatz. Roughly speaking, the MPA allows to find the dominant eigenvector of some Markov matrix. If one deforms the Markov matrix by multiplying some entries by e^z, where z is some parameter, one can count how many times the corresponding transitions happen. The deformed MPA method will no longer yield an eigenvector associated to the eigenvalue 0 (or the eigenvalue 1 in the discrete-time setup). It yields an eigenvector associated to the eigenvalue

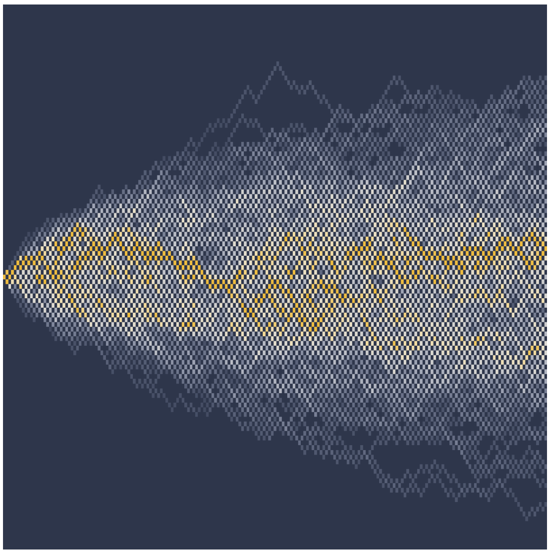
$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[e^{zh(t,0)}\right]$$

and it allows to compute the cumulant generating function. The deformed MPA is studied in details in [Laz13; Laz15; LP14]. In [32], we simply checked that the two methods above lead to compatible formulas. It would be interesting to find a direct connection between certain representations of the MPA and solutions of the Bethe equations.

Moreover, it is shown in [MP18] that the top eigenvalue and associated eigenvector in the deformed MPA can be related to reweighted random walks similar to (2.6.2). It is a compelling direction (on which I am working!) to better understand their connections to Gibbsian line ensembles studied in integrable probability.

CHAPTER 3

DIFFUSION IN RANDOM MEDIA



In this chapter, I start with random walks in Beta distributed random environment (Beta RWRE), a model introduced in [4], part of my PhD thesis. Then, I present subsequent developments:

- a continuous model of diffusion in random uncorrelated environment [13];
- a discussion of moderate deviations and their connection to the KPZ equation [12];
- asymptotics of the zero temperature model [8];
- half-space variants [22];
- and finally, connections to energy exchange models such as the Kipnis-Marchioro-Presutti model [29].

3.1. The beta random walk in random environment

Let $X(t)$ be a discrete-time random walk on $\mathbb{Z}$. Assume that jump probabilities depend on time and space, say that if $X(t) = x$ then $X(t+1) = x+1$ with probability $p_{t,x}$ and $X(t+1) = x-1$ with probability $1-p_{t,x}$. When all $p_{t,x}$ are equal to some $p \in (0,1)$ then $X(t)$ is the simple random walk and one may apply all the standard toolbox of probability theory: law of large numbers, Donsker's central limit theorem, Cramér's theorem to compute the large deviation rate function, etc. However, when jump probabilities are disordered, for instance if they are given by i.i.d. random variables in $(0,1)$, the situation is more delicate.

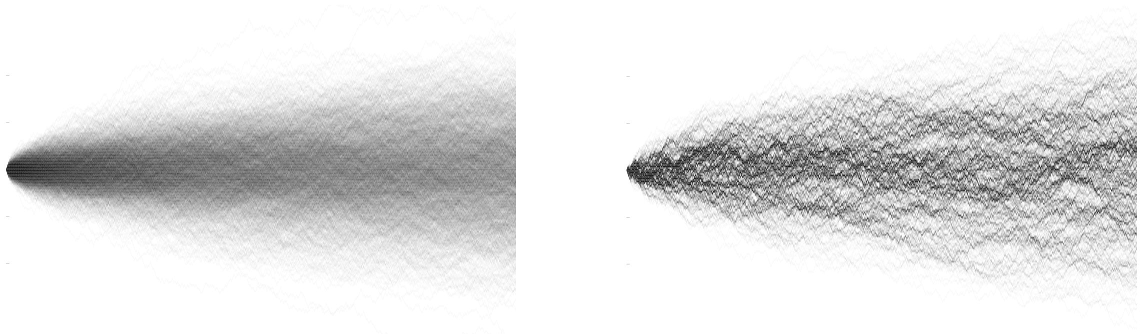


FIGURE 1. We compare trajectories of $N = 10000$ simple random walks (left) and N RWRE in uniform environment (right). The density field is much rougher in presence of a random environment.

We stress that the case where jump probabilities depend only on space is very different from the present setting which is generally referred to as a random walk in dynamic random environment. From a physical point of view, it would be desirable to study random environment with a structure, for instance given by the configuration of some interacting particle system. Below we will restrict to the simplest case of i.i.d. environments.

If we average over the environment, the probability distribution of one random walk in random environment (RWRE) is just the law of the simple random walk with jump probabilities $\mathbb{E}[p]$ and $1 - \mathbb{E}[p]$. We will see that for typical environment, one RWRE behaves at large scale exactly as the average law – called annealed law in the context of directed polymers.

Now consider two RWRE $X_1(t)$ and $X_2(t)$ drawn independently, but in the same environment. Now, if one averages over the environment, the law of the couple (X_1, X_2) is not that of two independent simple random walks. When the two walks meet, they have a slightly higher chance of sticking together. Hence we already see some impact of the environment, and we will see that the change becomes drastic when considering a large number N of RWRE. The simulation shown in Figure 1 shows that the density field of random walks in random environment is much rougher than simple random walks. We will also see the behaviour of the extreme particle, i.e. the asymptotics

$$M_N(t) = \max_{1 \leqslant i \leqslant N} \{X_i(t)\} \tag{3.1.1}$$

depends very much on the presence of a random environment.

3.1.1. General results. — We start with some notations.

Definition 3.1 **(RWRE).** — Let $(p_{t,x})_{t \in \mathbb{Z}, x \in \mathbb{Z}}$ be a family of i.i.d. random variables following some fixed probability distribution on $(0, 1)$. We denote by $\mathbb{P}$ its probability distribution (and $\mathbb{E}$ the associated expectation). We will denote by P (resp. E) the probability distribution (resp. expectation) associated with the random walk with jump probabilities, i.e.

$$\mathsf{P}\left(X(t+1) = x + 1 | X(t) = x\right) = p_{t,x}.$$

We also define a discrete heat kernel

$$\mathsf{P}_{s,t}(x, y) = \mathsf{P}\left(X(t) = y | X(s) = x\right)$$

and we will use the shorthand notations $\mathsf{P}_t(x,y) = \mathsf{P}_{0,t}(x,y)$, and more generally, for $E \subset \mathbb{Z}$,

$$\mathsf{P}_{s,t}(x,E) = \mathsf{P}\left(X(t) \in E | X(s) = x\right).$$

Finally, for $\vec{x}, \vec{y} \in \mathbb{Z}^k$, we will sometimes denote by

$$P^{(k)}(\vec{x},\vec{y}) := \mathbb{E}[\prod_{i=1}^{k} \mathsf{P}(x_i, y_i)]$$

the transition probability of k independent RWRE sampled in the same environment. We call this Markov process the k-point motion.

There exist many results, with various levels of generality, proving that random walks in a random environment obey a law of large numbers and a functional CLT, for almost every environment. This type of question is also relevant in the continuum where one may consider a diffusion in some velocity field, and it has been studied in the context of turbulence. In the discrete setting, early works on this type of problem often assumed an ellipticity condition, i.e. the random variables $p_{t,x}$ are supported on $(\delta, 1-\delta)$ for some $\delta > 0$, and consider various hypotheses on how the environment depends on time.

In our case of space-time i.i.d. environment, it is proved in [RS05] – see also [BMP04; BMP07; RS09; Bér04] – that if $\mathbb{P}(p_{t,x} \neq 0, 1) > 0$, then for almost every environment $\{p_{t,x}\}_{(t,x)\in\mathbb{Z}_+\times\mathbb{Z}}$, we have the weak convergence in $C(\mathbb{R}_+, \mathbb{R})$

$$\frac{X(t\tau) - dt}{\sqrt{1-d}t^{1/2}} \underset{t\to\infty}{\Longrightarrow} B_\tau$$

where B_τ denotes a Brownian motion, and the drift is $d = \mathbb{E}[2p-1]$.

Random walks in random environment also satisfy a large deviation principle [RSY13; RS14] under general conditions. In particular, if $\log p$ and $\log(1-p)$ have a finite third moment, then the limit

$$\lambda(z) = \lim_{t\to\infty} \frac{1}{t} \log E\left[e^{zX_t}\right]$$

exists almost surely and for $x > d$,

$$\lim_{t\to\infty} \log \mathsf{P}(X(t) > xt) = -I(x)$$

where the rate function $I(x)$ is the Legendre transform of the limiting cumulant generating function λ, i.e.

$$I(x) = \sup_{z\in\mathbb{R}_+} \{xz - \lambda(z)\}.$$

In general, the functions λ and I are characterized through some variational principle but computing an exact expression is not easy.

3.1.2. An exactly solvable model – the Beta RWRE. — In order to go beyond the general results stated above, one needs an integrable model, introduced in [4], taking a limit of a model integrable particle system from [Pov13].

Definition 3.2 **(Beta RWRE [4]).** — We keep the same notations as in Definition 3.1 and we now assume that $(p_{t,x})_{t\in\mathbb{Z},x\in\mathbb{Z}}$ follow the Beta(α,β) distribution, i.e. with density

$$p^{\alpha-1}(1-p)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mathbb{1}_{p\in[0,1]}.$$

We remind that the case $\alpha = \beta = 1$ corresponds to the uniform distribution on $[0,1]$. Below when we need to specify the value of parameters α, β, we will refer to the model as the Beta(α,β)−RWRE.

The model is exactly solvable in the following sense

Proposition 3.3 **([4]).** — *For $x_1 \geqslant \ldots \geqslant x_k$ with $t + x_i$ even, we have*

$$\mathbb{E}\left[\mathsf{P}_t(x_1, \mathbb{Z}_+) \ldots \mathsf{P}_t(x_k, \mathbb{Z}_+)\right] = \int_{\gamma_1} \frac{dz_1}{2\pi\mathbf{i}} \cdots \int_{\gamma_k} \frac{dz_k}{2\pi\mathbf{i}} \prod_{1 \leqslant a < b \leqslant k} \frac{z_a - z_b}{z_a - z_b - 1}$$
$$\times \prod_{i=1}^{k} \left(\frac{(\alpha + z_i)^2}{z_i(z_i + \alpha + \beta)} \right)^{t/2} \left(\frac{z_i + \alpha + \beta}{z_i} \right)^{\frac{x_i}{2} - 1} \frac{1}{z_i + \alpha + \beta}, \quad (3.1.2)$$

where the contours $\gamma_1, \ldots, \gamma_k$ all contain 0, exclude $-\alpha - \beta$, and are nested in such a way that γ_i contains $\gamma_j + 1$ for all $1 \leqslant i < j \leqslant k$.

Under the same hypotheses, we also have

$$\mathbb{E}\left[\mathsf{P}_t(x_1, 0) \ldots \mathsf{P}_t(x_k, 0)\right] = (\alpha + \beta)_k \times \int_{\gamma_1} \frac{dz_1}{2\pi\mathbf{i}} \cdots \int_{\gamma_k} \frac{dz_k}{2\pi\mathbf{i}} \prod_{1 \leqslant a < b \leqslant k} \frac{z_a - z_b}{z_a - z_b - 1}$$
$$\times \prod_{i=1}^{k} \left(\frac{(\alpha + z_i)^2}{z_i(z_i + \alpha + \beta)} \right)^{t/2} \left(\frac{z_i + \alpha + \beta}{z_i} \right)^{\frac{x_i}{2} + 1} \frac{1}{(z_i + \alpha + \beta)^2}, \quad (3.1.3)$$

This last formula is a variant given in [TL16], see also [CG17, (3.1)], proved in [22, Appendix B]. Since $\mathsf{P}_t(x_1, \mathbb{Z}_+)$ in a bounded random variable in $[0, 1]$, its moments characterize its distribution. Using ideas from [BC14; BCS14] one establishes Fredholm determinant formulas for the Laplace transform of the random variable $\mathsf{P}_t(x_1, \mathbb{Z}_+)$, and one can deduce from their asymptotic analysis that for the uniform environment the rate function is simply expressed as

$$I(x) = 1 - \sqrt{1 - x^2}. \quad (3.1.4)$$

The general α, β case can be treated as well but formulas are more complicated [4; Kor22; OPR21]. Moreover, we can characterize the law of fluctuations of lower order corrections to the large deviation principle.

Theorem 3.4 **([4]).** — *Consider the* Beta$(1, 1)-$*RWRE (i.e. the case of uniformly distributed disorder). For $x > 0$,*

$$\mathbb{P}\left(\frac{\log \mathsf{P}(X(t) > xt) + I(x)t}{\sigma(x) t^{1/3}} \leqslant s \right) \xrightarrow[t \to \infty]{} F_2(s)$$

where

$$\sigma(x) = \sqrt[3]{\frac{2I(x)^2}{1 - I(x)}}.$$

The study of lower order corrections to some large deviation principle, which concerns events of exponentially small probability, may seem, at first sight, a purely mathematical curiosity. But it is actually very interesting: in general, second order fluctuations in the tail of some probability distribution translate directly in fluctuations for the maximum of i.i.d. samples, which is an extremely common situation in practical modeling.

Corollary 3.5 **([4]).** — *Consider N independent samples of the* Beta$(1, 1)-$*RWRE, i.e. sampled independently in the same environment, and let $\mathrm{P}^{(N)}$ denote their distribution averaged over the environment (i.e. the N-point motion, as in Definition 3.1). Recall that $M_N(t)$ denotes the maximum of the RWREs, as in* (3.1.1)*. If $N = e^{ct}$, $c \in (0, 1)$,*

$$\mathrm{P}^{(N)}\left(\frac{M_N - I^{-1}(c)t}{\frac{\tilde{\sigma}}{I^{-1}(c)} t^{1/3}} \leqslant s \right) \xrightarrow[t \to \infty]{} F_2(s),$$

where the rate function I is given in (3.1.4) *and $\tilde{\sigma} = \sqrt[3]{2c^2(1 - c)^2}$.*

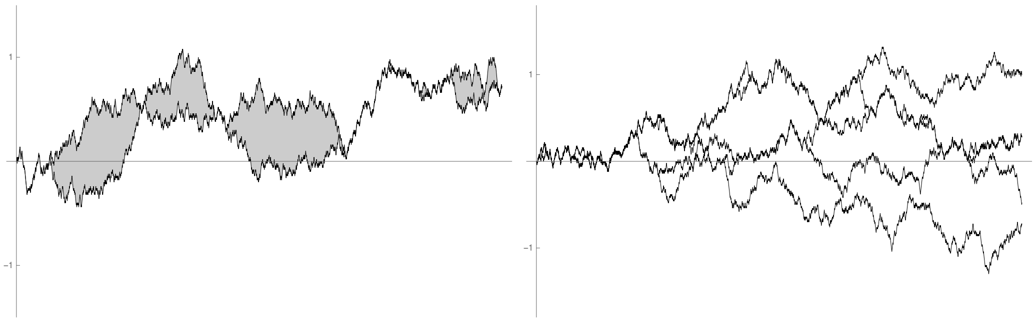


FIGURE 2. Left: Simulation of 2 sticky Brownian motions $x_1(t)$ and $x_2(t)$ starting from $x = 0$ at $t = 0$. Right: Simulation of 5 sticky Brownian motions. For most of the time, at least two trajectories are stuck together.

3.2. A continuous model

In [13], we considered a continuum analogue of the results above. We study an exactly solvable model of diffusion in random environment introduced by Le Jan and Raimond [LR04b; LR04a] and further studied in [LL04; HW09; SSS15].

3.2.1. Diffusion in a white noise velocity field. — The simplest way to define a continuum model of diffusion in random environment would be to replace random walks by Brownian motions, and to impose on them a random drift that is independent in space and time, that is, to consider the diffusion process

$$dx(t) = \xi(t, x(t))\, dt + dB_t \tag{3.2.1}$$

where B_t is a Brownian motion and ξ is a space–time white noise. This diffusion has been considered in some works in physics, in particular [LT17], but from a mathematical point of view it is not well defined, due to the irregularity of the white noise. Nevertheless, if the velocity field ξ were smooth, then the associated transition probabilities $\mathsf{p}_{s,t}(x, y)$ would satisfy the stochastic PDE

$$\partial_t p(t, x) = \tfrac{1}{2}\, \partial_{xx} p(t, x) + \partial_x p(t, x)\, \xi(t, x), \tag{3.2.2}$$

which suggests a stochastic PDE approach to define the model. But again, this stochastic PDE is also not well defined when ξ is white noise, so one needs to regularize the noise by assuming that its correlation is given formally by

$$\mathbb{E}\left[\xi(t, x)\xi(s, y)\right] = \delta(s - t)\frac{1}{r} R\left(\frac{x - y}{r}\right)$$

where r is some length scale which will be eventually sent to zero, and R is some nonnegative smooth function such that $\int R(x)dx = 1$.

3.2.2. Sticky Brownian motions. — The limit as $r \to 0$ was investigated in [GH04; War15]. Motivated by the study of Kraichnan's model [Kra68] of passive scalar in turbulence (see also [GK95; GK96; BGK98; GV00]) Gawedzki-Horvai [GH04] showed that two independent diffusions $x_1(t), x_2(t)$ following (3.2.1) with the same drift ξ converge as r goes to 0 to a pair of so called sticky Brownian motions (the sticky interaction was first introduced in [Fel52]). This means that $x_1(t)$ and $x_2(t)$ behave as standard Brownian motions when $x_1(t) \neq x_2(t)$, however, the two trajectories stick together in such a way that the Lebesgue measure of coincidence times between the two trajectories has a positive expectation (see Figure 2, left).

Warren [War15] further proved that n independent diffusions $x_1(t), \dots, x_n(t)$ following (3.2.1) with the same drift ξ converge as r goes to 0 to a n-tuple of sticky Brownian motions. The way these Brownian motions stick together in the $r \to 0$ limit, depends on a family of parameters $\theta(k,l)$ which measure the rate at which $k+l$ paths stuck together will split into groups of k and l paths – see Figure 2 for an example of trajectory of 5 sticky Brownian motions.

In general, families of sticky Brownian motions, can be considered from different points of view:

1. As limits of discrete RWRE. If $\mu^\epsilon(dp)$ denotes the law of the $p_{t,x}$ and
$$\epsilon^{-1}p(1-p)\mu^\epsilon(dp) \Rightarrow \nu(dp) \tag{3.2.3}$$
for some finite measure ν on $(0,1)$ as $\epsilon \to 0^+$, then $x^\epsilon(t) = \epsilon X(\epsilon^{-2}t)$ converges to a diffusion $x(t)$ in a random environment. Likewise, several random walks $X_1(t), X_2(t)\dots$ sampled independently in the same environment converge to diffusions $x_1(t), x_2(t), \dots$ which can be seen as independent diffusions in the same environment. These diffusions will have the same joint distribution as sticky Brownian motions where the splitting rates $\theta(k,l)$ are related to the measure ν via
$$\theta(k,l) = \int_0^1 w^{k-1}(1-w)^{l-1}\nu(dw). \tag{3.2.4}$$
This convergence was first studied in [LL04] in a specific case, and then more generally in [HW09].
2. As independent diffusions in a random environment. The environment was constructed explicitly [SSS14] (see also the review [SSS15]) using results on the so-called Brownian web [Arr79; TW98] and Brownian net [SS08]. The diffusions follow paths of the Brownian web/net, and at special points of the Brownian web/net (classified in [TW98; NRS10; FINR04; SSS09]), trajectories may branch according to a random variable of law ν (potentially renormalized to be a probability measure), we refer to [SSS15] for details.
3. As a diffusion process characterized by a martingale problem, see [SSS15; HW09].

Sticky Brownian motions fit in the theory of stochastic flows of kernels [LR04a; LR02]. Informally, a stochastic flow of kernels is a family of random probability measures $\mathsf{K}_{st}(x,dy)$ which represent the probability for a diffusion starting at x at time s to arrive in $[y, y+dy]$ at time t. For any family of sticky Brownian motions characterized by a given sequence of parameters $\theta(k,l)$ there exist a stochastic flow of kernels such that n sticky Brownian motions corresponds to a n-dimension diffusion having transition probabilities equal to the n-point motion
$$P^{(n)}(\vec{x}, d\vec{y}) = \mathbb{E}\left[\prod_{i=1}^n \mathsf{K}_{st}(x_i, dy_i)\right].$$

3.2.3. An integrable model of stochastic flows of kernels. — We consider now the special case where
$$\theta(k,l) = \frac{\lambda}{2}\frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}, \tag{3.2.5}$$
depending on a single parameter λ. This family of sticky Brownian motions was considered by [LR04b; LL04] in the context of stochastic flows of kernels. Using (3.2.3), we see that the corresponding stochastic flow of kernels $\mathsf{K}_{st}(x,E)$ arises as the limit of the Beta RWRE, when $\alpha = \beta = \varepsilon$. In particular as $\epsilon \to 0$, the transition probability $\mathsf{P}_{s\varepsilon^{-1},t\varepsilon^{-1}}(\varepsilon^{-2}x, \varepsilon^{-2}E)$ for the Beta RWRE converges in law to $\mathsf{K}_{st}(x,E)$. We showed in [13] that this model is exactly solvable.

Proposition 3.6. — *For $x_1 \geqslant \cdots \geqslant x_k$, and $t \geqslant 0$, we have [13, Proposition 1.20]*

$$\mathbb{E}\Big[\mathsf{K}_{-t,0}(x_1, \mathbb{R}_+) \dots \mathsf{K}_{-t,0}(x_k, \mathbb{R}_+)\Big] =$$

$$\int_{r_1+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_1}{2\mathbf{i}\pi} \cdots \int_{r_k+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}z_k}{2\mathbf{i}\pi} \prod_{1 \leqslant A < B \leqslant k} \frac{z_A - z_B}{z_A - z_B - 1} \prod_{j=1}^{k} \exp\left(\frac{t\lambda^2}{2z_j^2} + \frac{\lambda x_j}{z_j}\right) \frac{1}{z_j}, \quad (3.2.6)$$

where for $i < j$, the real parts of contours satisfy $r_i > r_j + 1$.

This allows to prove analogues of Theorem 3.4 and Corollary 3.5 for this continuous model of random motions in random environment.

Remark 3.7. — One may observe [13, Section 6] that the integral formula (3.2.6) is a solution $u(\vec{x}, t)$ of the following heat equation on $\{\vec{x} \in \mathbb{R}^k : x_1 \geqslant \cdots \geqslant x_k\}$ subject to boundary conditions

$$\begin{cases} \partial_t u(\vec{x}, t) = \frac{1}{2}\Delta u(\vec{x}, t), \quad t \geqslant 0, \vec{x} \in \mathbb{R}^k, \\ (\partial_{x_i}\partial_{x_{i+1}} + \lambda(\partial_{x_i} - \partial_{x_{i+1}}))u(\vec{x}, t)|_{x_i = x_{i+1}} = 0. \end{cases} \quad (3.2.7)$$

It is customary to associate to (3.2.7) the following equation on $\mathbb{R}^k$ with point interactions,

$$\partial_t v(t, \vec{x}) = \tfrac{1}{2}\Delta v(t, \vec{x}) + \frac{1}{2\lambda} \sum_{i \neq j} \delta(x_i - x_j) \partial_{x_i}\partial_{x_j} v(t, \vec{x}). \quad (3.2.8)$$

so that $v = u$ on the set $\{\vec{x} \in \mathbb{R}^k : x_1 \geqslant \cdots \geqslant x_k\}$. We note that (3.2.8) is precisely the equation that one would obtain by computing the equation satisfied by moments of the solution of (3.2.2) in the limit where ξ is white noise. This confirms that our model is a correct model for a diffusion in white noise velocity field. That said, as observed by [BW23b], it appears that solutions of (3.2.8) are not unique.

3.3. Moderate deviations

In [12], we refined the results of [4] in another direction, more interesting from a physical point of view. Results also apply to the continuous case, but we present them here only in the discrete setting.

We considered the asymptotics of the transition probabilities $\mathsf{P}_t(0, xt^\alpha)$, where $\alpha \in (0, 1)$, as t tends to infinity. For $\alpha \in (0, 1/2)$, nothing particularly interesting happens. When $\alpha = 1/2$, it was known that the stochastic flow converges to the stochastic heat equation $\partial_t u = \Delta u + \xi$ [Yu16]. When $\alpha = 1$, Theorem 3.4 states that $\log \mathsf{P}_{0,t}(0, xt)$ exhibits fluctuations on the $t^{1/3}$ scale, governed by the Tracy–Widom law. We showed that when $\alpha = 3/4$, there is an intermediate regime in which the fluctuations of $\log \mathsf{P}_{0,t}(0, xt^{3/4})$ have the same exponential moments as the solution of the KPZ equation (KPZ). The exponent $3/4$ was predicted in earlier work of Le Doussal and Thiery [LT17; TL16], using less precise arguments.

3.3.1. Convergence of moments. — Our result in [12] is the following. It can be stated for the general Beta(α, β) RWRE but we will specialize here to $\alpha = \beta$ for simplicity. Consider the scalings

$$X = -2\alpha^2 t L^{3/4} - \alpha x L^{1/2}, \quad (3.3.1)$$

$$T = 2\alpha^2 t L. \quad (3.3.2)$$

Then, taking a limit of Proposition 3.3, we obtain that for $x_1 \leqslant \cdots \leqslant x_k$,

$$\lim_{L\to\infty} \frac{\mathbb{E}\left[\prod_{i=1}^k \mathsf{P}_T(X, \mathbb{Z}_+)\right]}{\prod_{i=1}^k C_{L,t,x_i}} = \int_{r_1+\mathbf{i}\mathbb{R}} \frac{dz_1}{2\mathbf{i}\pi} \cdots \int_{r_k+\mathbf{i}\mathbb{R}} \frac{dz_k}{2\mathbf{i}\pi} \prod_{A<B} \frac{z_A - z_B}{z_A - z_B - 1} \prod_{j=1}^k e^{tz_i^2 + x_i z_i}, \tag{3.3.3}$$

$$= \mathbb{E}\left[\prod_{i=1}^k Z(2t, x_i)\right]. \tag{3.3.4}$$

where $Z(t,x)$ is a solution to (SHE) with delta initial condition, and the renormalization factor $C_{L,t,x}$ is given by

$$C_{L,t,x} = \frac{1}{\alpha L^{1/4}} \exp\left(-\alpha^2 t\left(\sqrt{L} + \tfrac{1}{6}\right) - \alpha L^{1/4} x\right), \tag{3.3.5}$$

and the contours are vertical lines oriented from bottom to top with real parts r_i such that for $i < j$, $r_i > r_j + 1$. The equality between the mixed moments of the multiplicative SHE in (3.3.4) and the integral formula (3.3.3) was shown earlier in [BC14, Proposition 6.2.3]. This convergence of moments strongly suggest the following:

***Conjecture 3.8* ([12]).** — *Fix $\alpha = \beta > 0$. We have the weak convergence of processes*

$$\log \mathsf{P}_{-\tau t,0}\left(-\tau t^{3/4} - x t^{1/2}, \mathbb{Z}_+\right) + \frac{\tau t^{1/2}}{2} + x t^{1/4} + \log\left(t^{1/4}\alpha\right) + \frac{\tau}{12} \xRightarrow[t\to+\infty]{} h\left(\frac{\tau}{\alpha^2}, \frac{x}{\alpha}\right), \tag{3.3.6}$$

where $h(t,x)$ is the solution to the KPZ equation with narrow wedge initial condition. We expect the convergence to hold in the space $C(\mathbb{R}_{>0}, C(\mathbb{R},\mathbb{R}))$.

Remark 3.9. — If one replaces above $\mathsf{P}_{-\tau t,0}\left(-\tau t^{3/4} - x t^{1/2}, \mathbb{Z}_+\right)$ by

$$\sum_{y\in Z} \mathsf{P}_{-\tau t,0}\left(-\tau t^{3/4} - x t^{1/2}, y\right) Z_0\left(\frac{y}{\sqrt{t}}\right) C_{t,y}$$

for some well chosen function $C_{t,y}$, we expect that the convergence holds towards the solution to the KPZ equation with initial condition $\log Z_0$.

Conjecture 3.8 further suggests that if one samples $N = e^{\sqrt{t\tau}}$ random walks in the same environment, then as t tends to infinity,

$$M_N \approx \sqrt{t \log N} + \sqrt{\frac{t}{\log N}}\left(G + h(0,\tau) - \tfrac{1}{2}\log\log N\right), \tag{3.3.7}$$

where G is a Gumbel random variable, independent of the solution h of the KPZ equation. We recover the result of Corollary 3.5 when $\tau \to \infty$ since the KPZ equation at large time converges to the Tracy-Widom distribution [ACQ11; CDR10; SS10; Dot10]. As $\tau \to 0$ one also recovers the results given by classical extreme value theory for independent Gaussian random variables.

These predictions have been discussed extensively in the physics literature as they are relevant for understanding when impurities in a medium start to affect diffusion phenomena [LT17; 12; HCCC23; HCC24; HDCC24; HDCC25; Has25].

In the mathematics literature, partially motivated by conjectures in [13; 12], variants of Conjecture 3.8 have been considered in the context of sticky flows [BW22; DDP24a] and a general class of simple random walks in space time i.i.d random environment [DDP24b].

3.3.2. Local central limit theorem. — Conjecture 3.8 is partially proven. Indeed, [DDP24b] proved that a slightly different observable, $\log \mathsf{P}_{\tau t}(0, \tau t^{3/4} + x t^{1/2})$ – now we fix the starting point and vary the endpoint – converges to the KPZ equation as $t \to \infty$. However, even for fixed time τ, the convergence does not hold in the space $C(\mathbb{R}_+, C(\mathbb{R}, \mathbb{R}))$ but in $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}))$ where $\mathcal{S}'(\mathbb{R})$ is some space of distributions discussed below. Thus, the convergence holds only after integrating against suitable test functions. This is not due to a technical limitation. Actually, one should not expect even point-wise convergence of finite dimensional distributions, because the density field of RWRE is very disordered when viewed as a function of the endpoint – see Figure 1. For large t, the random probabilities $\mathsf{P}_{0,t}(x, y)$ do not depend much on small changes in x, but depend very much on the arrival point y.

This dependence on the arrival point can be characterized more precisely by the following local central limit theorem, which we state as another conjecture, although it is partially proven.

Conjecture 3.10. — *Consider the* Beta(α, β) *RWRE. For any fixed $x \in \mathbb{R}$, we have the convergence in law (in the sense of finite-dimensional marginals)*

$$\sqrt{t}\mathsf{P}_{-t,0}\left(-dt - x\sqrt{t}, y\right) \xRightarrow[t\to\infty]{} g_\sigma(x)\frac{\Gamma_y}{\alpha+\beta}$$

where d is the annealed drift, i.e. $d = \frac{\alpha-\beta}{\alpha+\beta}$, g_σ is the density of the Gaussian distribution with mean 0 *and variance $\sigma^2 = 1 - \mathsf{d}^2$, and $(\Gamma_y)_{y\in\mathbb{Z}}$ is a sequence of i.i.d.* Gamma$(\alpha+\beta)$ *random variables.*

Conjecture 3.10 is compatible with the general results of Section 3.1.1: if one sum y over some large interval, the Gamma variables average out and one is left with the Gaussian density, compatible with the convergence of RWREs to Brownian motions. The existence of a random field Γ_y is ensured by the local central limit theorem proved for general RWRE in [BMP99, Theorem 2]. Moreover, it was predicted in the physics article [TL16] that the local random field is Gamma distributed. This prediction is proved with full rigor in [22, Prop. 1.13]. The local random field of i.i.d. gamma variables should be seen as the invariant measure for the random recurrence that governs the evolution of transition probabilities: For any fixed starting point $x_\circ$,

$$\mathsf{P}_{0,t}(x_\circ, x) = \mathsf{P}_{0,t-1}(x_\circ, x-1)p_{t-1,x-1} + \mathsf{P}_{0,t-1}(x_\circ, x+1)(1 - p_{t-1,x+1}).$$

3.3.3. Universality. — An important question is whether the connection between RWRE and the KPZ is universal.

Open Problem 3.11. — *Consider a simple random walk in space time random environment $\{p_{t,x}\}$ where $p_{t,x} \sim \mu \in \mathcal{P}([0,1])$. What are the assumptions on μ so that up to μ-dependent constants, Theorem 3.4 and Corollary 3.5 are still true?*

It is well-known that this question is extremely difficult: universality has not been proved for any class of models in the KPZ class, despite being the central open problem in the field. A more approachable question is whether (3.3.7) is universal, that is if the asymptotics of

$$\log \mathsf{P}_{\tau t}(0, \tau t^{3/4} + x t^{1/2})$$

as $t \to \infty$ converge to the same limit for a large class of stochastic flows $\mathsf{P}_{s,t}(x, y)$ satisfying mild hypotheses. Interestingly, [HDCC25] suggested that universality seems to hold with some restrictions. If the average of elementary increments, i.e. the random variable

$\mathsf{E}[X(t+1)-X(t)]$, is deterministic, then asymptotics are different and $t^{3/4}$ should be replaced by $t^{7/8}$. This fact is proved in [Par24].

More precisely, [Par24] considers a general family of discrete time stochastic flows $(\mathcal{K}_{s,t}(x,E))_{s,t\in\mathbb{Z},x\in\mathbb{R},E\subset\mathbb{R}}$. The scalings depend on the smallest integer k such that the kth moment of an elementary increment is random. In other terms, if one denotes by $X(t)$ a random walk with kernel $\mathcal{K}$, we should consider the integer

$$k_{\min} = \min\left\{k; \mathsf{E}[(X(1)-X(0))^k] \text{ is random }\right\}.$$

For example, for the Beta RWRE, $\mathsf{E}[X(1)-X(0)] = 2p_{0,0}-1$ is random, so that $k_{\min}=1$. But if we consider a random walk which does not move with probability $1-p_{t,x}$ or jumps by ± 1 with probability $p_{t,x}/2$, in that case $\mathsf{E}[X(1)-X(0)] = 0$ is deterministic, so that $k_{\min}=2$. Back to the result of [Par24], we shall assume the following mild hypotheses (stated here in an informal manner):

1. Transition kernels $\mathcal{K}_{t,t+1}$ are i.i.d. and translation invariant, i.e. $\mathcal{K}(x,A) \overset{(d)}{=} \mathcal{K}(x+a, a+A)$.
2. The averaged kernel $\mathbb{E}[\mathcal{K}(0,\cdot)]$ is a probability measure with finite exponential moments and variance 1.
3. Increments decorrelate for far enough starting points (this is expressed in terms of the first moments of the k point motions).
4. The 2-point motion is irreducible.

Then, the function

$$(t,x) \mapsto F_n(t,x) := \mathcal{K}_{0,nt}(y, n^{\frac{4p-1}{4p}} + n^{\frac{1}{2}}x),$$

properly rescaled, converges to the (SHE) in the space $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}))$. where $\mathcal{S}'$ is a space of distribution, dual to the Schwartz space

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C_c^\infty(\mathbb{R}) \,:\, \forall \alpha,\beta \in \mathbb{Z}_{\geqslant 0},\ \sup_{x\in\mathbb{R}} |x^\alpha (D^\beta f)(x)| < \infty \right\}.$$

Quite surprisingly, the convergence of the moments of $\langle F_n, \phi\rangle$ turn out to be sufficient to prove the convergence in distribution. this is similar in flavour to the earlier work [BW22; DDP24a; DDP24b] on special stochastic flows, and it is also reminiscent to the work [Tsa24] on the stochastic heat flow. it is proved in [Par24] using stochastic analysis arguments that up to some renormalization constants,

$$\mathbb{E}\left[\langle F_n(t,\cdot), \phi\rangle^k\right] \approx \mathbb{E}\left[E^{\otimes k}\left[\prod_{i=1}^k e^{n^{-1/4}X^{(i)}(nt)} \phi\left(\frac{X^{(i)}(nt)}{\sqrt{n}}\right)\right]\right]$$

converges to

$$\mathbb{E}\left[\prod_{i=1}^k \phi(B^{(i)}) \prod_{i<j} e^{\ell_t(B^{(i)}-B^{(j)})}\right] \tag{3.3.8}$$

where $B^{(1)},\dots,B^{(k)}$ are independent Brownian motions and $\ell_t(B^{(1)}-B^{(2)})$ denotes the local time of intersection. The expression (3.3.8) is known to match with the moments of

$$\mathbb{E}\left[\left(\int Z(t,x)\phi(x)dx\right)^k\right]$$

where $Z(t,x)$ is a solution to (SHE).

3.4. Beta random walk on $\mathbb{N}$

We considered a half-space analogue of the Beta RWRE in [22].

Definition 3.12. — The half-space Beta RWRE is a RWRE as in Definition 3.1 except that we assume that the family of transition probabilities $p_{t,x}$ are independent and distributed as

$$\begin{cases} p_{t,x} \sim \mathrm{Beta}(\alpha,\alpha) & \text{if } x \geqslant 2 \\ p_{t,1} \sim \mathrm{Beta}(\alpha,\eta) \\ p_{t,0} = 1. \end{cases}$$

Hence, if at some time t, $X(t)=0$, then almost-surely $X(t+1)=1$, so that all increments are given by ± 1 and X_t remains in $\mathbb{N}=\mathbb{Z}_{\geqslant 0}$.

This model is integrable in the sense that we have the following moment formula.

Theorem 3.13 **([22]).** — *Let $\alpha,\eta>0$. Let $\mathsf{P}_{s,t}(x,y)$ denote transition probabilities for the half-space Beta RWRE. For $1\leqslant x_1\leqslant \ldots \leqslant x_k$ and $t>0$, with $t+x_i$ odd, we have*

$$\mathbb{E}[\mathsf{P}_{0,t}(x_1,1),\ldots,\mathsf{P}_{0,t}(x_k,1)] = (-2)^k(\alpha+\eta)_k \int_{\mathbf{i}\mathbb{R}} \frac{dz_1}{2\pi\mathbf{i}} \cdots \int_{\mathbf{i}\mathbb{R}} \frac{dz_k}{2\pi\mathbf{i}}$$
$$\prod_{1\leqslant a<b\leqslant k} \frac{z_a-z_b}{z_a-z_b-1}\frac{z_a+z_b}{z_a+z_b+1} \prod_{i=1}^{k} \left(\frac{z_i^2}{z_i^2-\alpha^2}\right)^{t/2+1} \left(\frac{z_i-\alpha}{z_i+\alpha}\right)^{(x_i-1)/2} \frac{1}{z_i(z_i+\eta)}. \quad (3.4.1)$$

The starting point of the proof is that random variables $\mathsf{P}_{0,t}(x,1)$ have the same distribution as $\mathsf{P}_{-t,0}(x,1)$, which satisfy the recurrence relation

$$\begin{cases} \mathsf{P}_{-t,0}(x,1) = W_{-t,x}\mathsf{P}_{-(t-1),0}(x+1,1) + (1-W_{-t,x})\mathsf{P}_{-(t-1),0}(x-1,1) \text{ if } x\geqslant 1 \\ \mathsf{P}_{-t,0}(0,1) = \mathsf{P}_{-(t+1),0}(1,1). \end{cases} \quad (3.4.2)$$

The random recurrence relation (3.4.2) implies a recurrence relation for mixed moments in (3.4.1), which we solve explicitly via Bethe ansatz, leading to Theorem 3.13. The approach is similar to the one employed for the full-space beta RWRE [4], except that an extra boundary condition at $x=1$ must be satisfied. Verifying that (3.4.1) satisfies this boundary condition turned out to be particularly challenging, and there should exist a more conceptual method using integrability.

In contrast to the full-space case, the half-space Beta RWRE is much more difficult to analyze asymptotically, as often with half-space models. In particular, we did not obtain Tracy-Widom type asymptotics, but the following local central limit theorem:

Corollary 3.14 **([22]).** — *Assume $\alpha+\eta>\frac{1}{2}$ (this is a technical hypothesis). We have the weak convergence*

$$\sqrt{t}\mathsf{P}_{0,t}(0,1) \underset{t\to\infty}{\Longrightarrow} \frac{2}{\sqrt{2\pi}}\frac{\Gamma_1}{2\alpha}, \quad (3.4.3)$$

where $\Gamma_1 \sim \mathrm{Gamma}(\eta+\alpha)$ *and*

$$\sqrt{t}\mathsf{P}_{0,t}(0,0) \underset{t\to\infty}{\Longrightarrow} \frac{2}{\sqrt{2\pi}}\frac{\Gamma_0}{2\alpha}. \quad (3.4.4)$$

with $\Gamma_0\sim\mathrm{Gamma}(\eta)$.

In (3.4.3) and (3.4.4), we have intentionally not simplified the factors of 2, because the prefactor $\frac{2}{\sqrt{2\pi}}$ should be interpreted as the density at 0 of a reflected Brownian motion. We further conjecture that the half-space Beta RWRE satisfies a local central limit theorem as in Conjecture 3.10, where the stationary random field Γ_x is made of independent Gamma distributed random variables with

$$\Gamma_x \sim \begin{cases} \text{Gamma}(2\alpha) & \text{if } x \geqslant 2, \\ \frac{1}{2\mu}\text{Gamma}(\alpha+\eta) & \text{if } x = 1 \\ \frac{1}{2\mu}\text{Gamma}(\eta) & \text{if } x = 0. \end{cases}$$

3.5. Energy exchange models

3.5.1. The model of Kipnis, Marchioro and Presutti. — Motivated by the dynamics of energy in chains of harmonic oscillators, [KMP82] introduced a simple Markovian model now called the KMP model. Let $G = (V, E)$ be a graph. We assign a random energy $\eta(x)$ to each site $x \in V$. Energies evolve according to a continuous time Markov process as follows. A Poisson process is attached to each bond $(x, y) \in E$ such that at rate 1, energies are redistributed as

$$(\eta(x), \eta(y)) \to (U(\eta(x) + \eta(y)), (1 - U)(\eta(x) + \eta(y)))$$

where $U \sim \text{Uniform}([0, 1])$. More generally, one may assume that $U \sim \text{Beta}(\alpha, \alpha)$. All random variables U involved are independent. In the following, we work in the simplest setup, when the graph G is a one dimensional lattice $G = \mathbb{Z}$.

This model has been studied by several groups of researchers. It is frequently mentioned in the literature on Markov dualities – see e.g. [GRT24] and references therein – because the article [KMP82] was one of the first applications of Markov duality. The model on a finite lattice connected to heat baths (playing the same role as boundary reservoirs), is one of the simplest examples of a diffusive system driven out of equilibrium. It shares many similarities with the Symmetric Simple Exclusion Process (SSEP). Like the SSEP, the large deviation behaviour of KMP's model is studied in the physics litterature via Macroscopic Fluctuation Theory [GPRIB22; RGIB23; BSM22a; BSM22b; BM24]

While the SSEP is known to converge to the (linear) stochastic heat equation, we showed in [29] that the large scale behaviour of KMP's model is much richer, and in particular it converges to the Kardar–Parisi–Zhang equation. We restrict to the initial condition $\eta(0, x) = \mathbb{1}_{x=0}$. As time increases, this initially localized distribution of energy/mass will progressively spread. Most references focus on a region of size $O(\sqrt{t})$ around the origin, but we argue in [29] that it is interesting to consider the behaviour of $\eta(t, x)$ in larger domains – see Figure 3.

3.5.2. Energy exchange models are stochastic flows of kernels. — Define a random field

$$F_n(t, x) := D_{n,t,x}\eta(nt, n^{3/4}t + n^{1/2}x), \text{ where } D_{n,t,x} = e^{tn^{1/2}/2+xn^{1/4}+t/8},$$

which we shall view as a random distribution. A test function $\phi \in \mathcal{C}_c(\mathbb{R})$ can be integrated against the random field F_n as

$$\langle F_n, \phi \rangle := \sum_{x \in \mathbb{Z}} F_n\left(t, \frac{x}{\sqrt{n}}\right)\phi\left(\frac{x}{\sqrt{n}}\right).$$

***Theorem 3.15* ([29] – informal version).** — *For the KMP model on $\mathbb{Z}$ with parameter α, $F_n(t, \cdot)$ converges in distribution as $n \to \infty$ to $Z(t, x)$, solution to*

$$\partial_t Z(t, x) = \frac{1}{2}\partial_{xx}Z(t, x) + \sqrt{D}\xi(t, x)Z(t, x),$$

with noise strength $D = \frac{1}{4\alpha}$ and delta initial condition, in the space $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}))$.

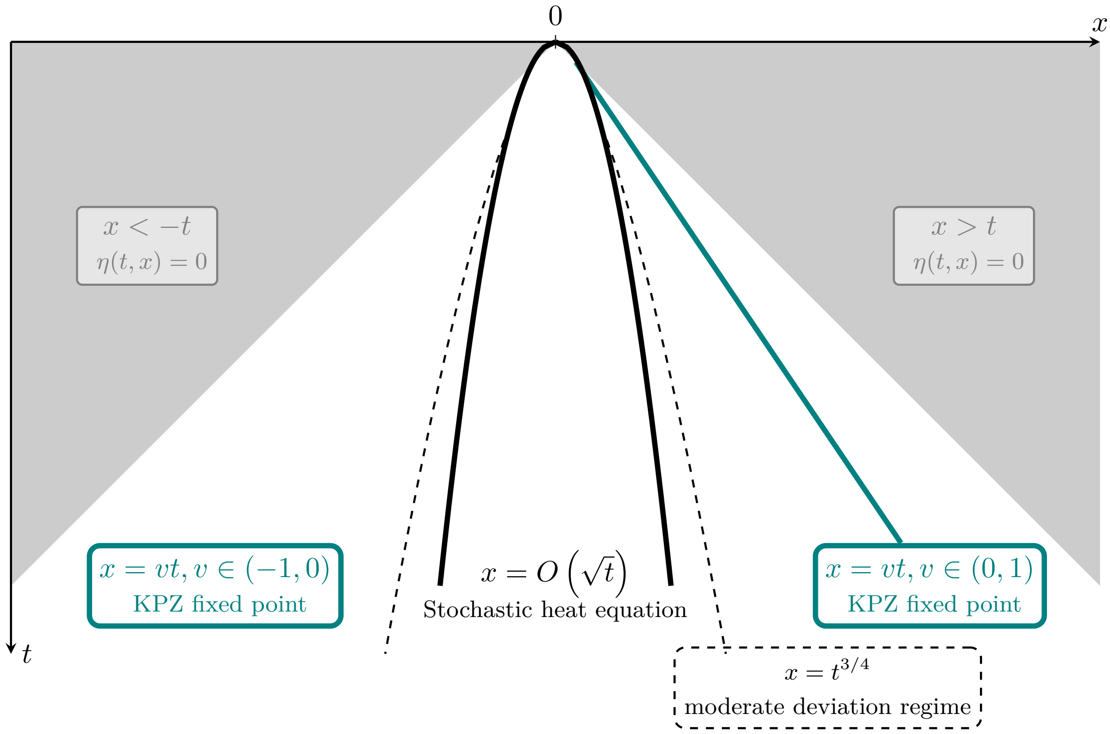


FIGURE 3. Diagram in space time coordinates showing the expected scaling limits of the KMP model with initial condition $\eta(0,x) = \mathbf{1}_{x=0}$. In the gray area, the energy field is frozen (equal to zero with probability exponentially close to 1). In the diffusive scaling window in the middle, the KMP model is expected to converge to the stochastic heat equation (based on results of [BRS06; Yu16] for similar models). Conjecture 3.16 below is about the scaling $x = vt, v \in (0,1)$, in green. Theorem 3.15 shows that between these last two regions, there exist a critical scaling $x \sim t^{3/4}$ where fluctuations are described by the KPZ equation.

Again, the convergence is not expected to hold point-wise because $\eta(t,x)$ locally converges to the stationary measure of the KMP model, which is given by i.i.d. Gamma distributed variables [KMP82]. We show in [29] that Theorem 3.15 is a direct consequence of the universal convergence results of stochastic flows from [Par24], which we explained in Section 3.3.3. Indeed, one can view the evolution of energies in the KMP model as a certain flow of kernels. To see that, we introduce a continuous-time random walk in random environment $X(t)$, which we couple to a KMP process in such a way that when the Poisson clocks rings at time t on the bond $(x, x+1)$, a walker at either x or $x+1$ goes to x with probability $U \sim \text{Beta}(\alpha,\alpha)$ and goes to $x+1$ with probability $1-U$, where the uniform random variable U is the same as in the KMP model.

There is a stochastic flow of kernels

$$\mathcal{K}_{s,t}(y,x) = \mathsf{P}(X(t) = x | X(s) = y)$$

naturally associated to the continuous-time RWRE $X(t)$. Then, under the coupling described above, it is not difficult to check that

$$\eta(t,x) = \sum_{y\in\mathbb{Z}} \eta_0(y)\mathcal{K}_{0,t}(y,x)$$

since both side satisfy the same time evolution, conditionally on the times when Poisson clocks rings and the associated uniform random variables. In other terms, the energy $\eta(t,x)$ should be interpreted as the probability to find a particle at x at time t starting from

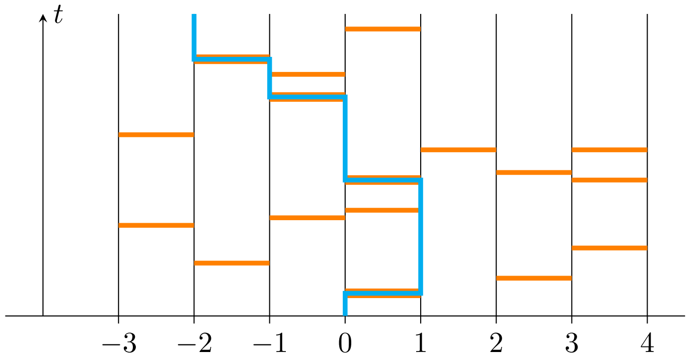


FIGURE 4. Illustration of the continuous time RWRE $X(t)$ associated to the KMP model. Times when a Poisson clock rings are depicted using Blue segments.

distribution $\eta_0(\cdot)$. The conservation of total energy in the KMP model becomes the conservation of probability. Conversely, stochastic flows corresponding to Markov processes in a random environment may be viewed as the evolution of a field of energies/masses evolving according to some local scrambling mechanism that conserves the energy/mass.

At distances of order $x = vt$ from the origin with $|v| < 1$, we expect that energies $\eta(t, x)$ are exponentially small but their fluctuations should be similar as large deviation probabilities of RWRE.

Conjecture 3.16. — *For any $v \in (-1, 1)$ with $v \neq 0$, there exist functions f, g such that*

$$\mathbb{P}\left(\frac{\log \eta(t, vt) + f(v)t}{g(v)t^{1/3}} \leqslant s\right) \xrightarrow[t\to\infty]{} F_{\mathrm{TW}}(s)$$

where F_{TW} is the Tracy-Widom GUE distribution. More generally, we expect that $H_t(\tau, x) = \log \eta(t\tau, vt + t^{2/3}x)$ converges to the KPZ fixed point.

There are other scaling limits that it would be interesting to consider. In particular, in Section 3.2.3 we considered a continuum limit of the Beta RWRE model. It would be interesting to consider the same question for the KMP model.

Open Problem 3.17. — *Can one describe the scaling limit of $\eta(\varepsilon^{-2}t, \varepsilon^{-1}x)$ when $\alpha = \varepsilon$ as ε goes to zero? Is it the same as the Le Jan-Raimond flow discussed in Section 3.2.3? Is it another stochastic flow of Howitt-Warren type [HW09] with different values of $\theta(k, \ell)$?*

3.6. Zero temperature limit

Suppose that a pedestrian wants to go from a point A to a point B in a city where the streets form a rectangular grid – see Figure 5. At each intersection, the configuration of pedestrian crossings is either or with probability $1/2$, where the green edge means that one can cross towards the north without waiting, and the red edge means that one can cross towards the east only after an exponential waiting time Exp(1). All waiting times are independent. Let $T(A \to B)$ be the minimal cumulated waiting time to go from A to B, i.e. the sum of waiting times minimized over all possible paths of minimal distance (the pedestrian does not want to make the walk longer). This model arises as a limit of the Beta RWRE model when $\alpha = \beta = \varepsilon$ and one defines

$$T_\varepsilon(A \to B) = -\varepsilon \log \mathsf{P}(A \to B).$$

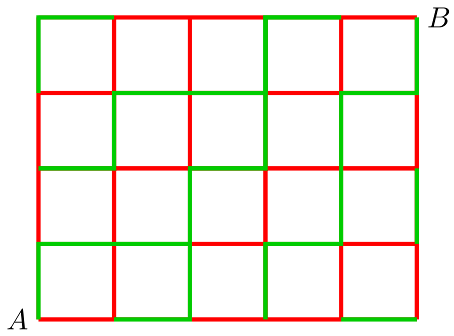


FIGURE 5. The Bernoulli-Exponential first passage percolation model, viewed as a model of optimal pedestrian travel times.

It is shown in [4] that $T_\varepsilon(A \to B)$ weakly converges to $T(A \to B)$. As a consequence of exact solvability, we have the following:

Theorem 3.18 **([4]).** — *The minimal passage time between two points A_n and B_n far away from each other so that $A_n - B_n = (n, \kappa n)$, with $\kappa \in (0, +\infty), \kappa \neq 1$, is such that there exist (explicit) constants $\tau_\kappa, \sigma_\kappa$ such that*

$$\mathbb{P}\left(\frac{T(A_n \to B_n) - \tau_\kappa n}{\sigma_\kappa n^{1/3}} \leqslant s\right) \xRightarrow[n\to\infty]{} F_2(s).$$

Remark 3.19. — The model may be generalized depending on parameters a, b such that with probability $\frac{a}{a+b}$ the upward edge has zero passage time and the rightward edge is $\mathrm{Exp}(b)$ and with probability $\frac{b}{a+b}$ the rightward edge has zero passage time and the upward edge is $\mathrm{Exp}(b)$. Theorem 3.18 still holds in this setting.

Remark 3.20. — When the waiting times are not random but all equal to 1, the model is still in the KPZ universality class. In that case, it is even possible to prove that $T(A \to B)$ converges to the directed landscape [VV23].

3.6.1. Evolution of a percolation cluster. — One may view the model as an evolution of a percolation cluster. If we fix the point A to be the origin $A = \mathbf{O}$, then the set of points M such that $T(\mathbf{O} \to M) = 0$ is given by the trajectory of some random walk – see Figure 6. The set of points M accessible in time $T(\mathbf{O} \to M) < t$ grows as t increases. It grows according to a Markov process where initially, the cluster is formed by a random walk trajectory, and from each point in the cluster, one may branch after an exponentially distributed waiting time, and immediately add to the cluster the trajectory of the branching random walk, which eventually coalesce with the cluster – see Figure 6.

In [8] we considered the asymptotics of the shape of the cluster at finite time t. Then, at distance n to the origin, the cluster has width $n^{2/3}$ around the diagonal, with Tracy-Widom distributed fluctuations.

Moreover, it is showed in [Vet23] that it is interesting to consider the cluster at times of order $t = sn^{-1/2}$ in such case, the width of the cluster is $n^{1/2}$ and fluctuations are different. This suggests the following.

Open Problem 3.21. — *Define a continuum limit of the model in such a way that the portions of random walks added to the cluster become portions of Brownian trajectories.*

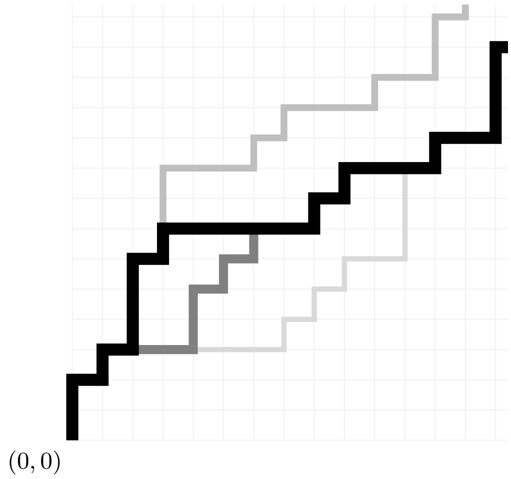


FIGURE 6. A sample of the river delta (Bernoulli-exponential FPP percolation cluster) near the origin. The thick black random walk path corresponds to the river (percolation cluster) at time 0. The other thinner and lighter paths correspond to tributaries added to the river delta (percolation cluster) at later times.

It is likely that the model should be constructed using a Brownian net, or at least a Brownian web, in a similar way as in [VV23] although the distances need to be rescaled differently.

3.7. Outlook

3.7.1. Conjectures. — Cube root fluctuations with Tracy-Widom limit, as in Theorem 3.4, are only a small part of the universal features in the KPZ class. First of all, the exponent 1/3 is complemented by an other exponent 2/3, which describes the decay of correlations ($h(t,x)$ and $h(t,y)$ have nontrivial correlations when $|x-y|O(t^{2/3})$) and the wandering of directed polymer paths (A directed polymer of length n drawn according to the polymer measure typically fluctuates at distance $n^{2/3}$ from the straight line). Furthermore, one can describe much more precisely the correlations of fluctuations.

Regarding the Beta RWRE, the wandering exponent was proven to be 2/3, in the sense that if one conditions a random walk to have a velocity v different from the typical velocity $d = \frac{\alpha-\beta}{\alpha+\beta}$, then RWRE paths wander at distance $t^{2/3}$ from a straight line [BRS19]. Their proof is based on the study of a stationary structure for the (backwards in time) evolution of Beta RWRE transition probabilities. The analysis is parallel to earlier work on the log-gamma polymer [Sep12; GRSY15]. The 2/3 exponent should also arise in the correlations of Beta RWRE transition probabilities.

Open Problem 3.22. — *Prove that for the Beta RWRE*

$$n^{-1/3}\left(\log \mathsf{P}_{sn,tn}(xn^{2/3}, cn+yn^{2/3}) + (t-s)nI(c)\right)$$

converges (up to some constants) to the directed landscape $\mathcal{L}(s,x;t,y)$ introduced in [DOV22].

In order to prove it, the currently most plausible route would be to

1. study a discrete line ensemble associated to the Beta RWRE using its connection to vertex models, as in [AB24];

2. then follow the strategy proposed in [ACH24], exploiting the strong characterization of the Airy line ensemble from [AH23].

3.7.2. A quantum RWRE. — In this section, we consider a discrete version of the Schroedinger equation

$$\mathbf{i}\partial_t \psi(t,x) = (-\Delta + \xi(t,x))\psi(t,x). \tag{3.7.1}$$

The stochastic PDE (3.7.1) is known to be ill-posed – see the references [BKR10; GK21] for a discussion of a possible renormalization procedure. At least at a formal level (in the sense that we apply the rules of differentiation without care for issues of regularity) we see that the evolution is unitary: the L^2 norm

$$\int_{\mathbb{R}} |\psi(t,x)|^2 dx$$

remains constant. Since the ill-posedness was provoking some conflicting results in the physics literature, it was argued in [SKR92] that one should work with a discrete analogue. Authors further argued that the directed polymer interpretation of the usual stochastic heat equation (SHE) is key, and that unitary analogues should likewise be defined through a path integral, i.e. a sum over lattice paths. These considerations led them to the following discrete model. We consider a wave function $\Psi_t(x) \in \mathbb{C}^2$ evolving at discrete times as follows. It is convenient to write

$$\Psi_t(x) = \begin{pmatrix} \Psi_t^+(x) \\ \Psi_t^-(x) \end{pmatrix}$$

and imagine that the components $\Psi_t^\pm(x)$ are attached to the edges adjacent to (t,x) in the lattice, $\Psi_t^+(x)$ is the component of wave function coming from above, i.e. coming to $(t-1, x+1)$, so it should be associated with the edge from $(t-1, x+1)$ to (t,x), and similarly for $\Psi_t^-(x)$. The time evolution is determined by the following. It is convenient to write

$$\Psi_t^{\mathrm{in}}(x) = \begin{pmatrix} \Psi_t^{\mathrm{in},+}(x) \\ \Psi_{\mathrm{in}}^{\mathrm{in},-}(x) \end{pmatrix} := \begin{pmatrix} \Psi_t^+(x) \\ \Psi_t^-(x) \end{pmatrix}, \quad \Psi_t^{\mathrm{out}}(x) = \begin{pmatrix} \Psi_t^{\mathrm{out},+}(x) \\ \Psi_t^{\mathrm{out},-}(x) \end{pmatrix} := \begin{pmatrix} \Psi_{t+1}^+(x-1) \\ \Psi_{t+1}^-(x+1) \end{pmatrix}.$$

Then,

$$\Psi_t^{\mathrm{out}}(x) = U_{t,x}\, \Psi_t^{\mathrm{in}}(x)\,.$$

where $U_{t,x}$ is a Haar distributed 2×2 unitary matrix (and the family $U_{t,x}$ is i.i.d.). Pictorially,

$$\begin{array}{ccccc} \Psi_t^+(x) & & & & \Psi_{t+1}^-(x+1) \\ & \diagdown & & \diagup & \\ & & U_{t,x} & & \\ & \diagup & & \diagdown & \\ \Psi_t^-(x) & & & & \Psi_{t+1}^+(x-1) \end{array}$$

Since $U_{t,x}$ is unitary, we have

$$\|\Psi^{\mathrm{out}}\|^2 = \|\Psi^{\mathrm{in}}\|^2.$$

Moreover, as argued in [SKR92], the law of

$$p_{t,x} := \frac{|\Psi_t^{\mathrm{out},-}(x)|^2}{\|\Psi_t^{\mathrm{out}}(x)\|^2} \tag{3.7.2}$$

does not depend on Ψ^{in} (because the Haar measure is translation invariant) and is uniformly distributed (because columns of Haar unitary matrices are uniformly distributed on the complex sphere).

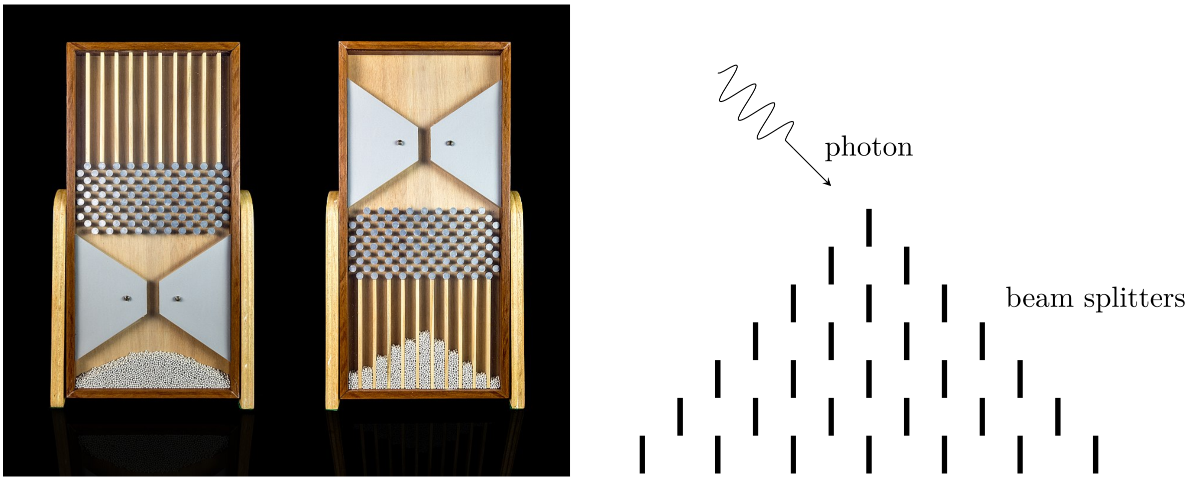


FIGURE 7. Left: Before and After. This Galton board was made by Mathemateca (Universidade de São Paulo) CC-BY-SA 4.0 Matemateca (IME/USP)/Rodrigo Tetsuo Argenton.
Right: a very schematic description of a photonic Galton board. Similar devices are considered in physics articles, see [ADZ93; Kar+09; Sch+10; Per+10; Grä+16; ZW24].

Thus, the evolution of the norm of the wave function is exactly the same as transition probabilities for the Beta RWRE when $\alpha = \beta = 1$.

Proposition 3.23 **([SKR92; 29]).** — *Assume that* $\|\Psi_0(y)\|^2 = \mathbb{1}_{y=x}$. *Then, for all* $t > 0$,

$$\|\Psi_n(y)\|^2 = \mathsf{P}_{0,t}(x, y).$$

where $\mathsf{P}_{s,t}(x, y)$ *denote the* $\mathrm{Beta}(1, 1)$-*RWRE transition probabilities.*

As a consequence, all results on the Beta RWRE also apply to this model, which physically corresponds to the wavefunction of a single photon in a quantum Galton board (Figure 7), that is an array of beam splitters whose behavior is determined by independent unitary matrices.

The model may also be viewed as a random quantum circuits, since the updating rule can be rephrased in a unitary dynamic evolving quantum states – see for instance [FKNV23]. Interestingly, entanglement in random quantum circuits is also expected to be related to the Kardar–Parisi–Zhang class [NRVH17]. More generally, it would be interesting to study how extreme statistics in stochastic quantum systems may be related to the Kardar–Parisi–Zhang class.

BIBLIOGRAPHY

[AB24] A. Aggarwal and A. Borodin. « Colored line ensembles for stochastic vertex models ». *Selecta Math. (N.S.)* 30.5 (2024).

[ACH24] A. Aggarwal, I. Corwin, and M. Hegde. « Scaling limit of the colored ASEP and stochastic six-vertex models » (Mar. 2024). arXiv: `2403.01341`.

[AH23] A. Aggarwal and J. Huang. *Strong Characterization for the Airy Line Ensemble*. Preprint, arXiv:2308.11908 [math.PR] (2023). 2023. eprint: `2308.11908`.

[ADZ93] Y. Aharonov, L. Davidovich, and N. Zagury. « Quantum random walks ». *Physical Review A* 48.2 (1993).

[AKQ14] T. Alberts, K. Khanin, and J. Quastel. « The intermediate disorder regime for directed polymers in dimension $1+1$ ». *Ann. Probab.* 42.3 (2014).

[ACQ11] G. Amir, I. Corwin, and J. Quastel. « Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions ». *Comm. Pure Appl. Math.* 64.4 (2011). eprint: `1003.0443`. URL: `http://dx.doi.org/10.1002/cpa.20347`.

[ABO23] J. Arista, E. Bisi, and N. O'Connell. « Matrix Whittaker processes ». *Probab. Theory Rel. Fields* 187.1-2 (2023).

[ABO24] J. Arista, E. Bisi, and N. O'Connell. « Matsumoto-Yor and Dufresne type theorems for a random walk on positive definite matrices ». *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques* 60.2 (2024).

[Arr79] R. A. Arratia. « Coalescing Brownian motions on the line ». PhD thesis. University of Wisconsin-Madison, 1979.

[BBP05] J. Baik, G. Ben Arous, and S. Péché. « Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices ». *Ann. Probab.* 33.5 (2005).

[BR00] J. Baik and E. M. Rains. « Limiting Distributions for a Polynuclear Growth Model with External Sources ». *J. Stat. Phys.* 100 (2000). eprint: `math/0003130`.

[BDJ99] J. Baik, P. Deift, and K. Johansson. « On the distribution of the length of the longest increasing subsequence of random permutations ». *J. Amer. Math. Soc.* 12.4 (Oct. 1999). arXiv: `math/9810105`.

[BR01a] J. Baik and E. M. Rains. « Algebraic aspects of increasing subsequences ». *Duke Math. J.* 109.1 (2001).

[BR01b] J. Baik and E. M. Rains. « The asymptotics of monotone subsequences of involutions ». *Duke Math. J.* 109.2 (2001).

[BKR10] G. Bal, T. Komorowski, and L. Ryzhik. « Kinetic limits for waves in a random medium ». *Kinet. Relat. Models* 3.4 (2010).

[BRS06] M. Balázs, F. Rassoul-Agha, and T. Seppäläinen. « The random average process and random walk in a space-time random environment in one dimension ». *Comm. Math. Phys.* 266.2 (2006).

[BRS19] M. Balázs, F. Rassoul-Agha, and T. Seppäläinen. « Large deviations and wandering exponent for random walk in a dynamic beta environment ». *Ann. Probab.* 47.4 (2019).

[Bax82] R. J. Baxter. *Exactly solvable models in statistical mechanics*. Academic Press London, 1982.

[Bér04] J. Bérard. « The almost sure central limit theorem for one-dimensional nearest-neighbour random walks in a space-time random environment ». *J. Appl. Probab.* 41.1 (Mar. 2004). URL: `http://dx.doi.org/10.1239/jap/1077134669`.

[BGK98] D. Bernard, K. Gawedzki, and A. Kupiainen. « Slow modes in passive advection ». *J. Stat. Phys.* 90.3-4 (1998).

[BG97] L. Bertini and G. Giacomin. « Stochastic Burgers and KPZ equations from particle systems ». *Comm. Math. Phys.* 183.3 (1997).

[BFO20] D. Betea, P. L. Ferrari, and A. Occelli. « Stationary half-space last passage percolation ». *Comm. Math. Phys.* 377.1 (2020).

[BWZ15] D. Betea, M. Wheeler, and P. Zinn-Justin. « Refined Cauchy/Littlewood identities and six-vertex model partition functions. II: Proofs and new conjectures ». *J. Algebr. Comb.* 42.2 (2015).

[Bet18] D. Betea. « On the combinatorics of last passage percolation in a quarter square and GOE^2 fluctuations ». *preprint arXiv:1809.06792* (2018). eprint: `1809.06792`.

[BBNV18] D. Betea, J. Bouttier, P. Nejjar, and M. Vuletić. « The free boundary Schur process and applications. I ». *Annales Henri Poincaré* 19.12 (2018).

[BM24] E. Bettelheim and B. Meerson. « Full statistics of regularized local energy density in a freely expanding Kipnis–Marchioro–Presutti gas ». *Journal of Statistical Mechanics: Theory and Experiment* 2024.11 (2024).

[BSM22a] E. Bettelheim, N. R. Smith, and B. Meerson. « Full statistics of nonstationary heat transfer in the Kipnis–Marchioro–Presutti model ». *Journal of Statistical Mechanics: Theory and Experiment* 2022.9 (2022).

[BSM22b] E. Bettelheim, N. R. Smith, and B. Meerson. « Inverse scattering method solves the problem of full statistics of nonstationary heat transfer in the Kipnis-Marchioro-Presutti model ». *Physical Review Letters* 128.13 (2022).

[BBO05] P. Biane, P. Bougerol, and N. O'Connell. « Littelmann paths and Brownian paths ». *Duke Math. J.* 130.1 (2005).

[BZ16] E. Bisi and N. Zygouras. « GOE and $\mathrm{Airy}_{2\to1}$ Marginal Distribution via Symplectic Schur Functions ». *International Conference in Honor of the 75th Birthday of SRS Varadhan*. Springer. 2016.

[BZ19] E. Bisi and N. Zygouras. « Point-to-line polymers and orthogonal Whittaker functions ». *Trans. Amer. Math. Soc.* 371.12 (2019). eprint: 1703.07337.

[BZ22] E. Bisi and N. Zygouras. « Transition between characters of classical groups, decomposition of Gelfand-Tsetlin patterns and last passage percolation ». *Adv. Math.* 404 (2022). eprint: 1905.09756.

[BMP04] C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. « Random walks in quenched iid space-time random environment are always as diffusive ». *Probab. Theory Rel. Fields* 129.1 (2004).

[BMP99] C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. « Central limit theorem for a random walk in dynamical environment: integral and local ». *Theor. Stoch. Processes* 5.21 (1999).

[BMP07] C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. « Random walks in a random (fluctuating) environment ». *Russian Mathematical Surveys* 62.4 (2007).

[Bor11] A. Borodin. « Schur dynamics of the Schur processes ». *Adv. Math.* 4.228 (2011). eprint: 1001.3442.

[BBC16] A. Borodin, A. Bufetov, and I. Corwin. « Directed random polymers via nested contour integrals ». *Ann. Phys.* 368 (2016). arXiv: 1511.07324.

[BBW16] A. Borodin, A. Bufetov, and M. Wheeler. « Between the stochastic six vertex model and Hall-Littlewood processes ». *J. Comb. Th. A, arXiv:1611.09486* (2016).

[BCFV15] A. Borodin, I. Corwin, P. Ferrari, and B. Vető. « Height fluctuations for the stationary KPZ equation ». *Math. Phys. Anal. Geom.* 18 (2015). eprint: 1407.6977.

[BCGS16] A. Borodin, I. Corwin, V. Gorin, and S. Shakirov. « Observables of Macdonald processes ». *Trans. Amer. Math. Soc.* 368.3 (2016). arXiv: 1306.0659.

[BCR13] A. Borodin, I. Corwin, and D. Remenik. « Log-gamma polymer free energy fluctuations via a Fredholm determinant identity ». *Comm. Math. Phys.* 324.1 (2013). arXiv: 1206.4573.

[BF14] A. Borodin and P. L. Ferrari. « Anisotropic growth of random surfaces in $2+1$ dimensions ». *Comm. Math. Phys.* 325.2 (2014). eprint: 0804.3035.

[BG16] A. Borodin and V. Gorin. « Moments Match between the KPZ Equation and the Airy Point Process ». *SIGMA* 12 (2016). eprint: 1608.01557.

[BO17] A. Borodin and G. Olshanski. « The ASEP and determinantal point processes ». *Comm. Math. Phys.* 353.2 (2017). eprint: 1608.01564.

[BP16] A. Borodin and L. Petrov. « Nearest neighbor Markov dynamics on Macdonald processes ». *Adv. Math.* 300 (2016).

[Bor18] A. Borodin. « Stochastic higher spin six vertex model and Macdonald measures ». *J. Math. Phys.* 59.2 (2018).

[BC14] A. Borodin and I. Corwin. « Macdonald processes ». *Probab. Theory Rel. Fields* 158.1-2 (2014).

[BCG16] A. Borodin, I. Corwin, and V. Gorin. « Stochastic six-vertex model ». *Duke Math. J.* 165.3 (2016).

[BCS14] A. Borodin, I. Corwin, and T. Sasamoto. « From duality to determinants for q-TASEP and ASEP ». *Ann. Probab.* 42.6 (2014).

[BK08] A. Borodin and J. Kuan. « Asymptotics of Plancherel measures for the infinite-dimensional unitary group ». *Adv. Math.* 219.3 (2008).

[BOO00] A. Borodin, A. Okounkov, and G. Olshanski. « Asymptotics of Plancherel measures for symmetric groups ». *J. Amer. Math. Soc.* 13.3 (2000).

[BP18] A. Borodin and L. Petrov. « Higher spin six vertex model and symmetric rational functions ». *Selecta Math. (N.S.)* 24.2 (2018).

[BR05] A. Borodin and E. M. Rains. « Eynard-Mehta theorem, Schur process, and their Pfaffian analogs ». *J. Stat. Phys.* 121.3-4 (2005).

[BJ02] P. Bougerol and T. Jeulin. « Paths in Weyl chambers and random matrices ». *Probab. Theory Rel. Fields* 124.4 (2002).

[BW22] D. Brockington and J. Warren. « At the edge of a cloud of Brownian particles ». *arXiv preprint arXiv:2208.11952* (2022).

[BW23b] D. Brockington and J. Warren. « The Bethe ansatz for sticky Brownian motions ». *Stochastic Processes and their Applications* 162 (2023).

[BD00a] E. Brunet and B. Derrida. « Probability distribution of the free energy of a directed polymer in a random medium ». *Phys. Rev. E* 61.6 (2000). eprint: cond-mat/0005352.

[BD00b] É. Brunet and B. Derrida. « Ground state energy of a non-integer number of particles with δ attractive interactions ». *Physica A* 279.1-4 (2000).

[Bry24] W. Bryc. « Stationary Distribution of open Asymmetric Simple Exclusion Processes on an Interval as a marginal of a two-layer ensemble ». *ALEA, Lat. Am. J. Probab. Math. Stat. 22, 131-148 (2025)* 22.1 (Aug. 2024). arXiv: 2408.06535.

[Bry12] W. Bryc. « Quadratic harnesses from generalized beta integrals ». *Noncommutative harmonic analysis with applications to probability. III: Papers presented at the 13th workshop, Bedlewo, Poland, July 11–17, 2010.* Warszawa: Polish Academy of Sciences, Institute of Mathematics, 2012.

[Bry22] W. Bryc. « On the continuous dual Hahn process ». *Stoch. Proc. Appl.* 143 (2022).

[BK21] W. Bryc and A. Kuznetsov. « Markov limits of steady states of the KPZ equation on an interval ». *ALEA, Lat. Am. J. Probab. Math. Stat. 19, 1329-1351 (2022)* 19.2 (Sept. 2021). arXiv: 2109.04462.

[BKWW23] W. Bryc, A. Kuznetsov, Y. Wang, and J. Wesołowski. « Markov processes related to the stationary measure for the open KPZ equation ». *Probab. Theory Rel. Fields* 185.1-2 (2023).

[BSW25] W. Bryc, K. Szpojankowski, and J. Wesolowski. « Free Askey–Wilson functionals and geometric last passage percolation on a strip » (June 2025). arXiv: 2506.01879.

[BW10] W. Bryc and J. Wesołowski. « Askey-Wilson polynomials, quadratic harnesses and martingales ». *Ann. Probab.* 38.3 (2010).

[BW17] W. Bryc and J. Wesołowski. « Asymmetric simple exclusion process with open boundaries and quadratic harnesses ». *J. Stat. Phys.* 167.2 (2017).

[BZ24] W. Bryc and P. Zatitskii. « A two-line representation of stationary measure for open TASEP ». *Electronic Journal of Probability 2024, Vol. 29, paper no. 199, 1-24* 29.none (Mar. 5, 2024). arXiv: 2403.03275.

[BKW24] W. Bryc, A. Kuznetsov, and J. Wesolowski. « Limits of Random Motzkin paths with KPZ related asymptotics ». *arXiv:2402.00265* (2024).

[BMW07] W. Bryc, W. Matysiak, and J. Wesołowski. « Quadratic harnesses, q-commutations, and orthogonal martingale polynomials ». *Trans. Amer. Math. Soc.* 359.11 (2007).

[BMW08] W. Bryc, W. Matysiak, and J. Wesołowski. « The bi-Poisson process: A quadratic harness ». *Ann. Probab.* 36.2 (2008).

[BW24] W. Bryc and Y. Wang. « Limit theorems for random Motzkin paths near boundary ». *Bernoulli* 30.3 (2024). URL: `projecteuclid.org/journals/bernoulli/volume-30/issue-3/Limit-theorems-for-random-Motzkin-paths-near-boundary/10.3150/23-BEJ1669.full`.

[BW23c] W. Bryc and J. Wesolowski. « Pitman's discrete $2M - X$ theorem for arbitrary initial laws and continuous time limits ». *arXiv:2304.07144* (2023).

[BW07] W. Bryc and J. Wesołowski. « Bi-Poisson process ». *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 10.2 (2007).

[BP19] A. Bufetov and L. Petrov. « Yang–Baxter field for spin Hall–Littlewood symmetric functions ». *Forum Math. Sigma* 7 (2019).

[BM18] A. Bufetov and K. Matveev. « Hall-Littlewood RSK field ». *Selecta Math. (N.S.)* 24.5 (2018).

[CDR10] P. Calabrese, P. L. Doussal, and A. Rosso. « Free-energy distribution of the directed polymer at high temperature ». *EPL (Europhysics Letters)* 90.2 (2010). eprint: `1002.4560`.

[CHH23] J. Calvert, A. Hammond, and M. Hegde. « Brownian structure in the KPZ fixed point ». *Astérisque* 441 (2023). eprint: `1912.00992`.

[Cau41] A. L. Cauchy. *Exercices d'analyse et de physique mathematique: 2*. Vol. 2. Bachelier, imprimeur-libraire, 1841.

[Chh13] R. Chhaibi. « Modèle de Littelmann pour cristaux géométriques, fonctions de Whittaker sur des groupes de Lie et mouvement brownien. » PhD thesis. Université Pierre et Marie Curie-Paris VI, 2013.

[CM96] A. Comtet and C. Monthus. « Diffusion in a one-dimensional random medium and hyperbolic Brownian motion ». *J. Phys. A: Math. Gen.* 29.7 (1996). eprint: `cond-mat/9506024`.

[CMY98] A. Comtet, C. Monthus, and M. Yor. « Exponential functionals of Brownian motion and disordered systems ». *J. Appl. Probab.* 35.2 (1998). arXiv: `cond-mat/9601014`.

[CT98] A. Comtet and C. Texier. « One-dimensional disordered supersymmetric quantum mechanics: a brief survey ». *Supersymmetry and Integrable Models* (1998). arXiv: `cond-mat/9707313`.

[CW07] S. Corteel and L. K. Williams. « Tableaux combinatorics for the asymmetric exclusion process ». *Adv. appl. math.* 39.3 (2007).

[CW10] S. Corteel and L. K. Williams. « Staircase tableaux, the asymmetric exclusion process, and Askey–Wilson polynomials ». *Proc. Natl. Acad. Sci.* 107.15 (2010).

[CW11] S. Corteel and L. K. Williams. « Tableaux combinatorics for the asymmetric exclusion process and Askey–Wilson polynomials ». *Duke Math. J.* 159.3 (2011).

[CD18] I. Corwin and E. Dimitrov. « Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood Gibbsian line ensembles ». *Comm. Math. Phys.* 363.2 (2018). eprint: 1703.07180.

[CG17] I. Corwin and Y. Gu. « Kardar–Parisi–Zhang equation and large deviations for random walks in weak random environments ». *J. Stat. Phys.* 166.1 (2017).

[CH16] I. Corwin and A. Hammond. « KPZ line ensemble ». *Probab. Theory Rel. Fields* 166.1-2 (2016).

[CS18] I. Corwin and H. Shen. « Open ASEP in the Weakly Asymmetric Regime ». *Comm. Pure Appl. Math.* 71.10 (2018). eprint: 1610.04931.

[CH14] I. Corwin and A. Hammond. « Brownian Gibbs property for Airy line ensembles ». *Invent. Math.* 195.2 (2014).

[CK24] I. Corwin and A. Knizel. « Stationary measure for the open KPZ equation ». *Commun. Pure Appl. Math.* 77.4 (2024).

[COSZ14] I. Corwin, N. O'Connell, T. Seppäläinen, and N. Zygouras. « Tropical combinatorics and Whittaker functions ». *Duke Math. J.* 163.3 (2014).

[CKST18] D. A. Croydon, T. Kato, M. Sasada, and S. Tsujimoto. « Dynamics of the box-ball system with random initial conditions via Pitman's transformation » (June 2018). arXiv: 1806.02147.

[DDP24a] S. Das, H. Drillick, and S. Parekh. « KPZ equation limit of sticky Brownian motion ». *J. Funct. Anal.* 287.10 (2024). eprint: 2304.14279.

[DDP24b] S. Das, H. Drillick, and S. Parekh. « Multiplicative SHE limit of random walks in space–time random environments ». *Probability Theory and Related Fields* (2024).

[DS24] S. Das and C. Serio. « Convergence to stationary measures for the half-space log-gamma polymer ». *arXiv:2203.11037* (2024).

[DS25] S. Das and C. Serio. « The half-space KPZ line ensemble and its scaling limit » (June 2025). arXiv: 2506.07939.

[DOV22] D. Dauvergne, J. Ortmann, and B. Virág. « The directed landscape ». *Acta Mathematica* 229.2 (2022). eprint: 1812.00309.

[DKLT20] J. De Nardis, A. Krajenbrink, P. Le Doussal, and T. Thiery. « Delta-Bose gas on a half-line and the Kardar–Parisi–Zhang equation: boundary bound states and unbinding transitions ». *J. Stat. Mech.: Theor. Exp.* 2020.4 (2020).

[DEL04] B. Derrida, C. Enaud, and J. L. Lebowitz. « The asymmetric exclusion process and Brownian excursions ». *Journal of Statistical Physics* 115.1-2 (2004).

[DEHP93] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier. « Exact solution of a 1D asymmetric exclusion model using a matrix formulation ». *J. Phys. A: Mathematical and General* 26.7 (1993).

[DF90] P. Diaconis and J. A. Fill. « Strong stationary times via a new form of duality ». *Ann. Probab.* 18.4 (1990).

[DY25] E. Dimitrov and Z. Yang. « Half-space Airy line ensembles » (May 2025). arXiv: `2505.01798`.

[Dot10] V. Dotsenko. « Replica Bethe ansatz derivation of the Tracy–Widom distribution of the free energy fluctuations in one-dimensional directed polymers ». *J. Stat. Mech.* 2010.07 (2010).

[ED04] C. Enaud and B. Derrida. « Large deviation functional of the weakly asymmetric exclusion process ». *Journal of statistical physics* 114.3 (2004). eprint: `cond-mat/0307023`.

[Fel52] W. Feller. « The parabolic differential equations and the associated semi-groups of transformations ». *Ann. Math.* (1952).

[FKNV23] M. P. Fisher, V. Khemani, A. Nahum, and S. Vijay. « Random quantum circuits ». *Annual Review of Condensed Matter Physics* 14.1 (2023).

[FINR04] L. R. G. Fontes, M. Isopi, C. M. Newman, and K. Ravishankar. « The Brownian web: characterization and convergence ». *Ann. Probab.* 32.4 (2004). eprint: `math/0311254`.

[FQ15] T. Funaki and J. Quastel. « KPZ equation, its renormalization and invariant measures ». *Stoch. Partial Diff. Eq.: Anal. Computat.* 3.2 (2015). eprint: `1407.7310`.

[GH04] K. Gawedzki and P. Horvai. « Sticky behavior of fluid particles in the compressible Kraichnan model ». *J. Stat. Phys.* 116.5-6 (2004).

[GK95] K. Gawedzki and A. Kupiainen. « Anomalous scaling of the passive scalar ». *Phys. Rev. Lett.* 75.21 (1995).

[GK96] K. Gawedzki and A. Kupiainen. « University in turbulence: An exactly solvable model ». *Low-dimensional models in statistical physics and quantum field theory*. Springer, 1996.

[GV00] K. Gawedzki and M. Vergassola. « Phase transition in the passive scalar advection ». *Phys. D: Nonlinear Phenomena* 138.1-2 (2000).

[GRSY15] N. Georgiou, F. Rassoul-Agha, T. Seppäläinen, and A. Yilmaz. « Ratios of partition functions for the log-gamma polymer ». *Ann. Probab.* 43.5 (2015).

[GLO07] A. Gerasimov, D. Lebedev, and S. Oblezin. « New integral representations of Whittaker functions for classical Lie groups ». *Russian Math. Surv.* 67.1 (2007). eprint: `0705.2886`.

[GLO08] A. Gerasimov, D. Lebedev, and S. Oblezin. « Baxter operator and archimedean Hecke algebra ». *Commun. Math. Phys.* 284.3 (2008).

[GLO12] A. Gerasimov, D. Lebedev, and S. Oblezin. « On a classical limit of q-deformed Whittaker functions ». *Lett. Math. Phys.* 100.3 (2012).

[GH19] M. Gerencsér and M. Hairer. « Singular SPDEs in domains with boundaries ». *Probab. Theor. Rel. Fields* 173.3-4 (2019).

[GRT24] C. Giardinà, F. Redig, and B. van Tol. « Intertwining and propagation of mixtures for generalized KMP models and harmonic models ». *arXiv:2406.01160* (2024).

[Giv97] A. Givental. « Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture ». *Amer. Math. Soc. Transl. Ser. 2* 180 (1997). eprint: `alg-geom/9612001`.

[GLMV12] M. Gorissen, A. Lazarescu, K. Mallick, and C. Vanderzande. *Exact Current Statistics of the ASEP with Open Boundaries.* Preprint, arXiv:1207.6879 [cond-mat.stat-mech] (2012). 2012. eprint: `1207 . 6879`. URL: `https://arxiv.org/abs/1207.6879`.

[GPRIB22] A. Grabsch, A. Poncet, P. Rizkallah, P. Illien, and O. Bénichou. « Exact closure and solution for spatial correlations in single-file diffusion ». *Science Advances* 8.12 (2022).

[Grä+16] M. Gräfe et al. « Integrated photonic quantum walks ». *Journal of Optics* 18.10 (2016).

[Gro04] S. Grosskinsky. « Phase transitions in nonequilibrium stochastic particle systems with local conservation laws ». PhD thesis. PhD thesis, TU Munich, 2004.

[GK21] Y. Gu and T. Komorowski. « Gaussian fluctuations from random Schrödinger equation ». *Communications in Partial Differential Equations* 46.2 (2021). arXiv: `2005.14591`.

[GK24] Y. Gu and T. Komorowski. « KPZ on torus: Gaussian fluctuations ». *Ann. Inst. Henri Poincaré Probab. Stat.* 60.3 (2024).

[GQ25] Y. Gu and J. Quastel. « Integration by parts and invariant measure for KPZ ». *Transactions of the American Mathematical Society* (2025). eprint: `2409.08465`.

[GS92] L. Gwa and H. Spohn. « Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian ». *Phys. Rev. Lett.* 68.6 (1992).

[Hai13] M. Hairer. « Solving the KPZ equation ». *Ann. of Math. (2)* 178.2 (2013). eprint: `1109.6811`.

[Hai09] M. Hairer. « An introduction to stochastic PDEs ». *arXiv preprint arXiv:0907.4178* (2009). eprint: `0907.4178`.

[Ham19] A. Hammond. « A patchwork quilt sewn from Brownian fabric: regularity of polymer weight profiles in Brownian last passage percolation ». *Forum of Mathematics, Pi* 7 (2019). eprint: `1709.04113`.

[HY04] Y. Hariya and M. Yor. « Limiting distributions associated with moments of exponential Brownian functionals ». *Stud. Sci. Math. Hung.* 41.2 (2004).

[Has82] M. Hashizume. « Whittaker functions on semisimple Lie groups ». *Hiroshima Math. J.* 12.2 (1982).

[Has25] J. Hass. « Super-Universal Behavior of Outliers Diffusing in a Space-Time Random Environment ». *arXiv preprint arXiv:2505.01533* (2025).

[HDCC25] J. Hass, H. Drillick, I. Corwin, and E. Corwin. « Universal KPZ Fluctuations for Moderate Deviations of Random Walks in Random Environments ». *arXiv preprint arXiv:2504.00266* (2025).

[HCCC23] J. B. Hass, A. N. Carroll-Godfrey, I. Corwin, and E. I. Corwin. « Anomalous fluctuations of extremes in many-particle diffusion ». *Physical Review E* 107.2 (2023).

[HCC24] J. B. Hass, I. Corwin, and E. I. Corwin. « First-passage time for many-particle diffusion in space-time random environments ». *Phys. Rev. E* 109 (5 May 2024). URL: `https://link.aps.org/doi/10.1103/PhysRevE.109.054101`.

[HDCC24] J. B. Hass, H. Drillick, I. Corwin, and E. I. Corwin. « Extreme Diffusion Measures Statistical Fluctuations of the Environment ». *Phys. Rev. Lett.* 133 (26 Dec. 2024). URL: `https://link.aps.org/doi/10.1103/PhysRevLett.133.267102`.

[He22] J. He. *Shift invariance of half space integrable models*. Preprint, arXiv:2205.13029 [math.PR] (2022). 2022. eprint: `2205.13029`. URL: `https://arxiv.org/abs/2205.13029`.

[He24] J. He. « Boundary current fluctuations for the half-space ASEP and six-vertex model ». *Proc. Lond. Math. Soc. (3)* 128.2 (2024). eprint: `2303.16335`.

[HY24] M. Hegde and Z. Yang. « Large deviation principle for the stationary measures of open asymmetric simple exclusion processes » (Dec. 2024). arXiv: `2412.12026`.

[Him24] Z. Himwich. « Stationary Measure of the Open KPZ Equation through the Enaud-Derrida Representation » (Apr. 2024). arXiv: `2404.13444`.

[HW09] C. Howitt and J. Warren. « Consistent families of Brownian motions and stochastic flows of kernels ». *Ann. Probab.* 37.4 (2009).

[IS11] T. Imamura and T. Sasamoto. « Current moments of 1D ASEP by duality ». *J. Stat. Phys.* 142.5 (2011). eprint: `1011.4588`.

[IMS21a] T. Imamura, M. Mucciconi, and T. Sasamoto. « Identity between Restricted Cauchy Sums for the q-Whittaker and Skew Schur Polynomials ». *SIGMA 20 (2024), 064, 28 pages* (June 22, 2021). arXiv: `2106.11913`.

[IMS21b] T. Imamura, M. Mucciconi, and T. Sasamoto. « Skew RSK dynamics: Greene invariants, affine crystals and applications to q-Whittaker polynomials » (June 2021). arXiv: `2106.11922`.

[IMS22] T. Imamura, M. Mucciconi, and T. Sasamoto. « Solvable models in the KPZ class: approach through periodic and free boundary Schur measures ». *Annals of Probability* (Apr. 2022). arXiv: `2204.08420`.

[IS07] T. Ishii and E. Stade. « New formulas for Whittaker functions on $\mathrm{GL}(n, \mathbb{R})$ ». *J. Funct. Anal.* 244.1 (2007).

[Jac67] H. Jacquet. « Fonctions de Whittaker associées aux groupes de Chevalley ». *Bulletin de la société mathématique de France* 95 (1967).

[JK14] S. Jansen and N. Kurt. « On the notion(s) of duality for Markov processes ». *Probab. Surveys* 11 (2014). URL: `http://dx.doi.org/10.1214/12-PS206`.

[Joh00] K. Johansson. « Shape fluctuations and random matrices ». *Commun. Math. Phys.* 209.2 (2000). eprint: `math/9903134`.

[JO20] S. Johnston and N. O'Connell. « Scaling Limits for Non-intersecting Polymers and Whittaker Measures. » *J. Statis. Phys.* 179 (2020). eprint: `1909.03219`.

[Kar+09] M. Karski et al. « Quantum walk in position space with single optically trapped atoms ». *Science* 325.5937 (2009).

[KL01] S. Kharchev and D. Lebedev. « Integral representations for the eigenfunctions of quantum open and periodic Toda chains from the QISM formalism ». *J. Phys. A* 34.11 (2001). eprint: `hep-th/0007040`.

[KMP82] C. Kipnis, C. Marchioro, and E. Presutti. « Heat flow in an exactly solvable model ». *Journal of Statistical Physics* 27 (1982).

[KM22] A. Knizel and K. Matetski. « The strong Feller property of the open KPZ equation ». *arXiv:2211.04466* (2022). eprint: `2211.04466`.

[Kor22] S. Korotkikh. « Hidden diagonal integrability of q-Hahn vertex model and Beta polymer model ». *Probab. Theory Related Fields* 184.1-2 (2022).

[Kos78] B. Kostant. « On Whittaker vectors and representation theory ». *Inventiones mathematicae* 48.2 (1978).

[Kra68] R. H. Kraichnan. « Small-scale structure of a scalar field convected by turbulence ». *The Physics of Fluids* 11.5 (1968).

[KL20] A. Krajenbrink and P. Le Doussal. « Replica Bethe Ansatz solution to the Kardar-Parisi-Zhang equation on the half-line ». *SciPost Phys.* 8 (2020). arXiv: `1905.05718`.

[Kra98] C. Krattenthaler. « Identities for classical group characters of nearly rectangular shape ». *J. Algebra* 209.1 (1998). eprint: `math/9808118`.

[Kua19] J. Kuan. « Stochastic fusion of interacting particle systems and duality functions ». *arXiv preprint arXiv:1908.02359* (2019). eprint: `1908.02359`.

[Lam13] T. Lam. « Whittaker functions, geometric crystals, and quantum Schubert calculus ». *arXiv preprint arXiv:1308.5451* (2013). eprint: `1308.5451`.

[Laz13] A. Lazarescu. « Exact large deviations of the current in the asymmetric simple exclusion process with open boundaries ». PhD thesis. Université Pierre et Marie Curie Paris VI and Institut de Physique Théorique CEA Saclay, 2013. eprint: `1311.7370`.

[Laz15] A. Lazarescu. « The physicist's companion to current fluctuations: one-dimensional bulk-driven lattice gases ». *J. Phys. A:Math. Theor.* 48.50 (2015).

[LP14] A. Lazarescu and V. Pasquier. « Bethe ansatz and Q-operator for the open ASEP ». *J. Phys. A Math. Theor.* 47.29 (2014).

[LT17] P. Le Doussal and T. Thiery. « Diffusion in time-dependent random media and the Kardar-Parisi-Zhang equation ». *Physical Review E* 96.1 (2017).

[LL04] Y. Le Jan and S. Lemaire. « Products of Beta matrices and sticky flows ». *Probab. Theory Rel. Fields* 130.1 (2004).

[LR02] Y. Le Jan and O. Raimond. « Integration of Brownian vector fields ». *Ann. Probab.* 30.2 (2002).

[LR04a] Y. Le Jan and O. Raimond. « Flows, coalescence and noise ». *Ann. Probab.* 32.2 (2004).

[LR04b] Y. Le Jan and O. Raimond. « Sticky flows on the circle and their noises ». *Probab. Theory Related Fields* 129.1 (2004).

[Lig75] T. M. Liggett. « Ergodic theorems for the asymmetric simple exclusion process ». *Trans. Amer. Math. Soc.* 213 (1975).

[MGP68] C. T. MacDonald, J. H. Gibbs, and A. C. Pipkin. « Kinetics of biopolymerization on nucleic acid templates ». *Biopolymers* 6.1 (1968).

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials, 2nd edition.* Vol. 354. Oxford, 1995.

[Mac88] I. G. Macdonald. « A new class of symmetric functions. » *Séminaire Lotharingien de Combinatoire [electronic only]* 20 (1988).

[MP18] K. Mallick and S. Prolhac. « Brownian bridges for late time asymptotics of KPZ fluctuations in finite volume ». *J. Stat. Phys. 173 (2018) 322-361* 173.2 (May 8, 2018). arXiv: `1805.03187`.

[MQR21] K. Matetski, J. Quastel, and D. Remenik. « The KPZ fixed point ». *Acta Mathematica* 227.1 (2021). eprint: `1701.00018`.

[MY00] H. Matsumoto and M. Yor. « An analogue of Pitman's $2M - X$ theorem for exponential Wiener functionals. I: A time-inversion approach ». *Nagoya Math. J.* 159 (2000).

[Mat19] K. Matveev. « Macdonald-positive specializations of the algebra of symmetric functions: Proof of the Kerov conjecture ». *Ann. Math.* 189.1 (2019). eprint: `1711.06939`.

[MP17] K. Matveev and L. Petrov. « q-randomized Robinson-Schensted-Knuth correspondences and random polymers ». *Ann. Inst. H. Poincaré D* 4.1 (2017). eprint: `1504.00666`.

[MC94] C. Monthus and A. Comtet. « On the flux distribution in a one dimensional disordered system ». *Journal de Physique I* 4.5 (1994).

[NRVH17] A. Nahum, J. Ruhman, S. Vijay, and J. Haah. « Quantum entanglement growth under random unitary dynamics ». *Physical Review X* 7.3 (2017).

[NRS10] C. M. Newman, K. Ravishankar, and E. Schertzer. « Marking (1, 2) points of the Brownian web and applications ». *Ann. IHP: Probab. stat.* Vol. 46. 2010.

[NS20] M. Noumi and A. Sano. « An infinite family of higher-order difference operators that commute with Ruijsenaars operators of type A ». *Letters in Mathematical Physics* 111.4 (Dec. 5, 2020). arXiv: `2012.03135`.

[NY04] M. Noumi and Y. Yamada. « Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group action ». *Representation theory of algebraic groups and quantum groups. Papers from the conference held as the 10th International Research Institute of the Mathematical Society of Japan (MSJ-IRI) at Sophia University, Tokyo, Japan, August 1–10, 2001.* Tokyo: Mathematical Society of Japan, 2004. eprint: `math-ph/0203030`.

[OCo14] N. O'Connell. « Whittaker functions and related stochastic processes ». *MSRI publications, arXiv:1201.4849* 65 (2014).

[OSZ14] N. O'Connell, T. Seppäläinen, and N. Zygouras. « Geometric RSK correspondence, Whittaker functions and symmetrized random polymers ». *Invent. Math.* 197.2 (2014).

[OCo03] N. O'Connell. « Conditioned random walks and the RSK correspondence ». *J. Phys. A: Math. Gen.* 36.12 (2003).

[OCo09] N. O'Connell. « Directed polymers and the quantum Toda lattice ». *Annals of Probability 2012, Vol. 40, No. 2, 437-458* 40.2 (Oct. 1, 2009). arXiv: `0910.0069`.

[OCo22] N. O'Connell. « Discrete Whittaker processes ». *Prob. Math. Phys. 4 (2023) 965-1002* 4.4 (Nov. 10, 2022). arXiv: `2211.05718`.

[OY02] N. O'Connell and M. Yor. « A representation for non-colliding random walks ». *Electron. Commun. Prob.* 7 (2002).

[Oka98] S. Okada. « Applications of minor summation formulas to rectangular-shaped representations of classical groups ». *J. Algebra* 205.2 (1998).

[Oko01] A. Okounkov. « Infinite wedge and random partitions ». *Selecta Math. (N.S.)* 7.1 (2001).

[OR03] A. Okounkov and N. Reshetikhin. « Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram ». *J. Amer. Math. Soc.* 16.3 (2003).

[OPR21] G. Oviedo, G. Panizo, and A. F. Ramírez. « Second order fluctuations of large deviations for perturbed random walks ». *arXiv preprint arXiv:2108.02877* (2021).

[Par19a] S. Parekh. « Positive random walks and an identity for half-space SPDEs ». *arXiv:1901.09449* (2019). arXiv: `1901.09449`.

[Par19b] S. Parekh. « The KPZ Limit of ASEP with Boundary ». *Comm. Math. Phys.* 365.2 (2019). eprint: `1711.05297`.

[Par22] S. Parekh. « Ergodicity results for the open KPZ equation ». *arXiv:2212.08248* (2022).

[Par24] S. Parekh. « Hierarchy of KPZ limits arising from directed random walk models in random media ». *arXiv:2401.06073* (2024).

[PW80] G. Parisi and Y. Wu. « Perturbation theory without gauge fixing ». *Sci. China, A* 24.ASITP-80-004 (1980).

[Par90] G. Parisi. « On the replica approach to random directed polymers in two dimensions ». *Journal de Physique* 51.15 (1990).

[Per+10] A. Peruzzo et al. « Quantum walks of correlated photons ». *Science* 329.5998 (2010).

[Pit75] J. W. Pitman. « One-dimensional Brownian motion and the three-dimensional Bessel process ». *Advances in Appl. Probability* 7.3 (1975).

[Pov13] A. M. Povolotsky. « On the integrability of zero-range chipping models with factorized steady states ». *J. Phys. A* 46.46 (2013).

[PS02] M. Prähofer and H. Spohn. « Scale invariance of the PNG droplet and the Airy process ». *J. Stat. Phys.* 108.5-6 (2002).

[Rai18] E. M. Rains. « Multivariate quadratic transformations and the interpolation kernel ». *SIGMA* 14.0 (2018). eprint: `1408.0305`.

[RW21] E. M. Rains and S. O. Warnaar. « Bounded Littlewood identities ». *Memoirs of the American Mathematical Society.* Mem. Am. Math. Soc. 1317 (2021).

[RS05] F. Rassoul-Agha and T. Seppäläinen. « An almost sure invariance principle for random walks in a space-time random environment ». *Probab. Theory Rel. Fields* 133.3 (2005).

[RS09] F. Rassoul-Agha and T. Seppäläinen. « Almost sure functional central limit theorem for ballistic random walk in random environment ». *Ann. Inst. Henri Poincaré Probab. Stat* 45.2 (2009).

[RS14] F. Rassoul-Agha and T. Seppäläinen. « Quenched point-to-point free energy for random walks in random potentials ». *Probab. Theory Rel. Fields* 158.3-4 (2014).

[RSY13] F. Rassoul-Agha, T. Seppäläinen, and A. Yilmaz. « Quenched free energy and large deviations for random walks in random potentials ». *Comm. Pure Appl. Math.* 66.2 (2013). arXiv: `1104.3110`.

[RGIB23] P. Rizkallah, A. Grabsch, P. Illien, and O. Bénichou. « Duality relations in single-file diffusion ». *Journal of Statistical Mechanics: Theory and Experiment* 2023.1 (2023).

[Rob38] G. Robinson. « On the Representations of the Symmetric Group ». *American Journal of Mathematics* 60.3 (1938). URL: `http://www.jstor.org/stable/2371609` (visited on 08/21/2025).

[SI04] T. Sasamoto and T. Imamura. « Fluctuations of the one-dimensional polynuclear growth model in half-space ». *J. Stat. Phys.* 115.3-4 (2004).

[SS10] T. Sasamoto and H. Spohn. « Exact height distributions for the KPZ equation with narrow wedge initial condition ». *Nuclear Phys. B* 834.3 (2010). eprint: `1002.1879`.

[SW14] T. Sasamoto and L. Williams. « Combinatorics of the asymmetric exclusion process on a semi-infinite lattice ». *Journal of Combinatorics* 5.4 (2014).

[Sas99] T. Sasamoto. « One-dimensional partially asymmetric simple exclusion process with open boundaries: Orthogonal polynomials approach ». *J. Phys. A: Math. Gen.* 32.41 (1999).

[SKR92] L. Saul, M. Kardar, and N. Read. « Directed waves in random media ». *Physical Review A* 45.12 (1992).

[SSS14] E. Schertzer, R. Sun, and J. M. Swart. « Stochastic flows in the Brownian web and net ». *Memoirs Amer. Math. Soc.* 227.1065 (2014).

[SSS09] E. Schertzer, R. Sun, and J. M. Swart. « Special points of the Brownian net ». *Electr. J. Probab.* 14 (2009).

[SSS15] E. Schertzer, R. Sun, and J. M. Swart. « The Brownian web, the Brownian net, and their universality ». *Advances in Disordered Systems, Random Processes and Some Applications* (2015).

[Sch+10] A. Schreiber et al. « Photons walking the line: a quantum walk with adjustable coin operations ». *Physical review letters* 104.5 (2010).

[Sch97] G. M. Schütz. « Duality relations for asymmetric exclusion processes ». *J. Stat. Phys.* 86.5-6 (1997).

[Sem94] M. Semenov-Tian-Shansky. « Quantization of open Toda lattices ». *In V.I. Arnol'd and S.P. Novikov, editors, Dynamical Systems VII: Integrable Systems Nonholonomic Dynamical Systems, volume 16 of Encyclopaedia of Mathematical Sciences* (1994).

[Sep12] T. Seppäläinen. « Scaling for a one-dimensional directed polymer with boundary conditions ». *Ann. Probab.* 40.1 (2012).

[Ste90] J. R. Stembridge. « Nonintersecting paths, Pfaffians, and plane partitions ». *Adv. Math.* 83.1 (1990).

[SS08] R. Sun and J. M. Swart. « The Brownian net ». *Ann. Probab.* 36.3 (2008).

[TL16] T. Thiery and P. Le Doussal. « Exact solution for a random walk in a time-dependent 1D random environment: the point-to-point Beta polymer ». *J. Phys. A: Math. Theor.* 50.4 (2016).

[TW98] B. Tóth and W. Werner. « The true self-repelling motion ». *Probab. Theor. Rel. Fields* 111.3 (1998).

[TW08a] C. A. Tracy and H. Widom. « A Fredholm determinant representation in ASEP ». *J. Stat. Phys.* 132.2 (2008).

[TW08b] C. A. Tracy and H. Widom. « Integral formulas for the asymmetric simple exclusion process ». *Comm. Math. Phys.* 279.3 (2008).

[TW09] C. A. Tracy and H. Widom. « Asymptotics in ASEP with step initial condition ». *Comm. Math. Phys.* 290.1 (2009).

[Tsa24] L.-C. Tsai. « Stochastic heat flow by moments ». *arXiv preprint arXiv:2410.14657* (2024). arXiv: `2410.14657`.

[USW04] M. Uchiyama, T. Sasamoto, and M. Wadati. « Asymmetric simple exclusion process with open boundaries and Askey–Wilson polynomials ». *J. Phys. A* 37.18 (2004). eprint: `cond-mat/0312457`.

[Ven15] V. Venkateswaran. « Symmetric and nonsymmetric Hall-Littlewood polynomials of type BC ». *J. Algebr. Comb.* 42 (2015). eprint: `1209.2933`.

[Vet23] B. Vető. « Fluctuations around the diagonal in Bernoulli-Exponential first passage percolation ». *Electron. Commun. Probab., 29 (2024), no. 31, 1-14* 29.none (June 15, 2023). arXiv: `2306.09073`.

[VV23] B. Vető and B. Virág. « The geometry of coalescing random walks, the Brownian web distance and KPZ universality » (May 2023). arXiv: `2305.15246`.

[War15] *Sticky particles and stochastic flows.* In Memoriam Marc Yor-Séminaire de ProbabilitésXLVII. Springer, 2015.

[WW09] J. Warren and P. Windridge. « Some examples of dynamics for Gelfand-Tsetlin patterns ». *Electronic Journal of Probability* 14 (2009). eprint: `0812.0022`.

[WZ16] M. Wheeler and P. Zinn-Justin. « Refined Cauchy/Littlewood identities and six-vertex model partition functions. III. Deformed bosons ». *Adv. Math.* 299 (2016).

[Wu20a] X. Wu. « Intermediate Disorder regime for half-space directed polymers ». *J. Stat. Phys.* 181 (2020). eprint: `1804.09815`.

[Wu20b] X. Wu. « Discrete Gibbsian line ensembles and weak noise scaling for directed polymers ». PhD thesis. 2020.

[Yan22] Z. Yang. « Stationary measure for six-vertex model on a strip ». *Electron. J. Probab. 29 1 - 28, 2024* 29.none (Dec. 18, 2022). arXiv: `2212.09111`.

[Yan23] Z. Yang. « Stationary measures for higher spin vertex models on a strip » (Sept. 2023). arXiv: `2309.04897`.

[Yu16] J. Yu. « Edwards–Wilkinson fluctuations in the Howitt–Warren flows ». *Stoch. Proc. Appl.* 126.3 (2016).

[Zha24] X. Zhang. « TASEP in half-space ». *arXiv preprint arXiv:2409.09974* (2024). arXiv: `2409.09974`.

[ZW24] X. Zheng and E. Waks. « Strongly interacting photonic quantum walk using single atom beamsplitters ». *Physical Review Research* 6.1 (2024).

RÉSUMÉ

Ce document présente une synthèse de mes travaux de recherche depuis la soutenance de mon doctorat. Ils se situent dans le domaine des probabilités dites intégrables, qui regroupe l'étude de modèles aléatoires possédant une structure bien particulière permettant de les analyser au moyen de calculs exacts. Les applications portent principalement sur des modèles de la classe de Kardar-Parisi-Zhang.
Le premier chapitre introduit les processus de Macdonald en demi-espace. Les processus de Macdonald sont des mesures de probabilité sur les partitions d'entiers basées sur des identités remarquables satisfaites par les fonctions de Macdonald. La variante en demi-espace que nous introduisons permet d'étudier des modèles de la classe KPZ définis sur une moitié de quadrant, ou bien des modèles où le désordre est symétrique par rapport à un axe, ou encore des systèmes de particules reliés à un réservoir.
Le deuxième chapitre porte sur le calcul de mesures stationnaires de modèles de la classe KPZ sur un domaine borné, avec des conditions au bord dites ouvertes, en particulier l'équation KPZ sur un segment.
Le troisième chapitre discute de marches aléatoires et diffusions en environnement aléatoire, dont le comportement extrême n'est pas décrit par la théorie classique de la diffusion mais obéit plutôt aux lois de la classe KPZ.

ABSTRACT

This document provides an overview of my research work since the defense of my PhD. It belongs to the area of integrable probability, which encompasses the study of random systems exhibiting a particular structure that allows for exact analysis. The main applications concern models within the Kardar-Parisi-Zhang (KPZ) universality class.
The first chapter addresses Macdonald processes in a half-space. Macdonald processes are probability measures on integer partitions based on remarkable identities satisfied by Macdonald functions. The half-space variant we present enables the study of KPZ-class models defined on a half-quadrant, models where the disorder is symmetric about an axis, or particle systems connected to a reservoir.

The second chapter focuses on the computation of stationary measures for KPZ-class models on a bounded domain with open boundary conditions, particularly the KPZ equation on a segment.

The third chapter examines random walks and random diffusions in random environment, whose extreme behaviour is not described by classical extreme value theory but instead follows the laws of the KPZ class.