Random walks in Beta random environment

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Outline of the talk

This talk is based on a work in collaboration with Ivan Corwin.
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1. **Introduction** to random walks in space-time random environment (RWRE).
2. An exactly solvable model : the **Beta RWRE**.
3. Main result: a **limit theorem** for the second order corrections to the large deviation principle.
4. Some ideas behind the **proof**: Bethe ansatz, a non-commutative binomial formula, and integral formulas from "Macdonald processes".
5. **Origin**: The model is a limit of the q-Hahn interacting particle system.
6. **Consequences** (KPZ universality, extreme value theory, Zero-temperature limit)

Consider the simple random walk X_t on $\mathbb{Z}$, starting from 0. We note

$$\mathbb{P}\left(X_{t+1}=X_t+1\right)=\frac{\alpha}{\alpha+\beta},\quad \mathbb{P}\left(X_{t+1}=X_t-1\right)=\frac{\beta}{\alpha+\beta}.$$

The CLT says that

$$\frac{X_t - t\frac{\alpha-\beta}{\alpha+\beta}}{\sigma\sqrt{t}} \Longrightarrow \mathcal{N}(0,1).$$

where $\sigma = 2\sqrt{\alpha\beta}/(\alpha+\beta)$.

Theorem (Cramér)

For $\frac{\alpha-\beta}{\alpha+\beta} < x < 1$,

$$\frac{\log\left(\mathbb{P}(X_t > xt)\right)}{t} \xrightarrow[t\to\infty]{} -I(x),$$

where $I(x)$ *is the Legendre transform of*

$$\lambda(z) := \log\left(\mathbb{E}\left[e^{zX_1}\right]\right) = \log\left(\frac{\alpha e^{z}+\beta e^{-z}}{\alpha+\beta}\right).$$

In random environment ?

Question

What can we say for a random walk in random environment ?

In this talk, we consider simple random walks on $\mathbb{Z}$ in space-time i.i.d. environment:

$$\mathsf{P}(X_{t+1} = x+1 | X_t = x) = B_{t,x}, \quad \mathsf{P}(X_{t+1} = x-1 | X_t = x) = 1 - B_{t,x},$$

where $(B_{t,x})_{t,x}$ is i.i.d., distributed according to some law with support on $[0,1]$.
We note P, E (resp. $\mathbb{P}, \mathbb{E}$) the measure and expectation with respect to the random walk (resp. the environment)

Answer

All results from the previous slide still hold, even conditionally on the environment, for almost every realization of the environment.

Quenched central limit theorem and invariance principle

Theorem (Rassoul-Agha and Seppäläinen, 2004)

Assume that $\mathbb{P}\big(0 < B_{t,x} < 1\big) > 0$. *We denote*

$$v = \mathbb{E}\big[\mathsf{E}[X_1]\big] = 2\mathbb{E}[B_{t,x}] - 1,$$

the expected drift. Let

$$W_n(t) = \frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{\sigma\sqrt{n}} \quad \textit{(same } \sigma \textit{ as before)}.$$

Then, for $\mathbb{P}$*-almost every environment, when* $n \to \infty$*, we have the convergence in distribution*

$$\big(W_n(t)\big)_t \Longrightarrow \big(W_t\big)_t \ \textit{(Brownian motion)}.$$

The results holds more generally for unbounded steps, in any dimension, and even for any ballistic random walk in random environment under additional conditions.

Quenched large deviation principle

Theorem (Rassoul-Agha, Seppäläinen and Yilmaz, 2013)

Assume that $\log(B_{t,x})$ *have a finite third moment. Then, the limiting moment generating function*

$$\lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log\Big(\mathsf{E}\big[e^{zX_t}\big]\Big),$$

exists a.s., and

$$\frac{\log\Big(\mathsf{P}(X_t > xt)\Big)}{t} \xrightarrow[t\to\infty]{a.s.} -I(x).$$

where $I(x)$ *is the Legendre transform of* λ*.*

The result holds more generally in any dimension, for any random walk in random potential, with not-necessarily i.i.d. weights, under some condition on the mixing properties of the environment.

An exactly solvable model

We introduce the *Beta RWRE* as a space-time simple RWRE such that the transition probabilities $(B_{t,x})$ are i.i.d. distributed according to the $Beta(\alpha, \beta)$ distribution.

$$\mathbb{P}\big(B \in [x, x+\mathrm{d}x]\big) = x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mathrm{d}x.$$

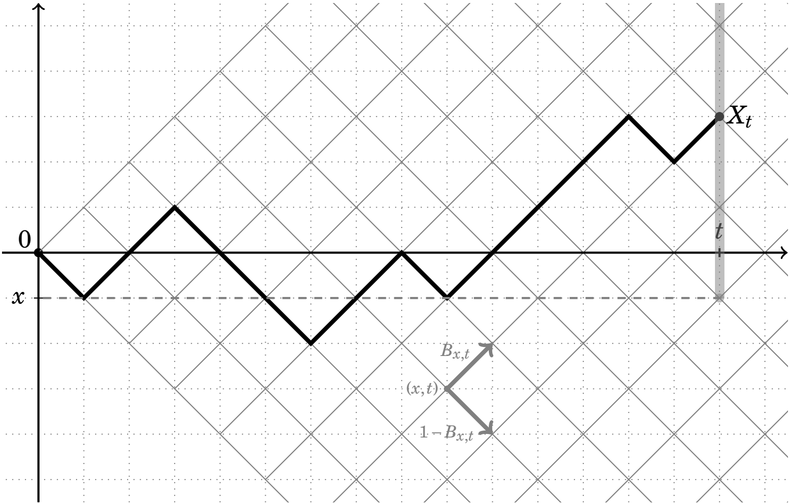
In particular, if $\alpha = \beta = 1$, we recover the uniform distribution.

Notations

- P, E (resp. $\mathbb{P}, \mathbb{E}$) are the measure and expectation with respect to the random walk (resp. the environment)
- We are interested in the random variable $\mathsf{P}(X_t > xt)$.

Exactly solvable means that we can exactly compute the law of $\mathsf{P}(X_t > xt)$.

In principle, one can also characterize the law of $\mathsf{E}[f(X_t)]$ for functions $f : \mathbb{Z} \to \mathbb{R}$.



Remark: The trajectory of (t,X_t) defines also a RWRE in $\mathbb{Z}^2$.

Basic properties

Denote $\mathscr{E}$ and $\mathscr{P}$ the product expectation and measure on {environments} × {paths}.

- If $B \sim Beta(\alpha,\beta)$ then $\mathbb{E}[B] = \frac{\alpha}{\alpha+\beta}$, so that the expected position of the walker is
$$\mathscr{E}[X_t] = \frac{\alpha-\beta}{\alpha+\beta}t.$$
- The second moment is
$$\mathscr{E}\left[X_t^2\right] = \left(\frac{\alpha-\beta}{\alpha+\beta}t\right)^2 + \frac{4\alpha\beta}{(\alpha+\beta)^2}t.$$
- The law of the annealed random walk (under $\mathscr{P}$) is that of the simple random walk from the first slide.
- Let X_t and Y_t two Beta RWRE, drawn independently in the same environment. (Hence X_t and Y_t are not independent!). We have
$$\mathscr{E}\left[X_tY_t\right] = \left(\frac{\alpha-\beta}{\alpha+\beta}t\right)^2 + \frac{4\alpha\beta\sum_{s=0}^{t-1}\mathscr{P}(X_s=Y_s)}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

For simplicity, let's focus for the moment on the case where transition probabilities are uniformly distributed (case $\alpha = \beta = 1$).

Theorem (B.-Corwin)

The LDP rate function is

$$I(x) = 1 - \sqrt{1 - x^2}.$$

We have the convergence in distribution as $t \to \infty$,

$$\frac{\log\Big(\mathsf{P}\big(X_t > xt\big)\Big) + I(x)t}{\sigma(x) \cdot t^{1/3}} \Longrightarrow \mathcal{L}_{GUE},$$

where $\mathcal{L}_{GUE}$ is the GUE Tracy-Widom distribution, and

$$\sigma(x)^3 = \frac{2I(x)^2}{1 - I(x)},$$

under the (technical) hypothesis that $x > 4/5$.

The theorem should extend to the general parameter case α, β and when x covers the full range of large deviation events (i.e. $x \in (0,1)$).

Fredholm determinant

Proof ? One makes an asymptotic analysis of a Fredholm determinant formula for the Laplace transform of the r.v. $\mathsf{P}(X_t > xt)$.

Theorem (B.- Corwin)

Let $u \in \mathbb{C} \setminus \mathbb{R}_+$, *and* t, x *with the same parity. Then for any parameters* $\alpha, \beta > 0$ *one has*

$$\mathbb{E}\left[e^{u\mathsf{P}(X_t > x)}\right] = \det(I + K_u^{\mathrm{RW}})_{\mathbb{L}^2(C_0)}$$

where C_0 *is a small positively oriented circle containing* 0 *but not* $-\alpha - \beta$ *nor* -1, *and* $K_u^{\mathrm{RW}} : \mathbb{L}^2(C_0) \to \mathbb{L}^2(C_0)$ *is defined by its integral kernel*

$$K_u^{\mathrm{RW}}(w, w') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)} (-u)^s \frac{g(w)}{g(w+s)} \frac{\mathrm{d}s}{s + w - w'}$$

where

$$g(w) = \left(\frac{\Gamma(w)}{\Gamma(\alpha + w)}\right)^{(t-x)/2} \left(\frac{\Gamma(\alpha + \beta + w)}{\Gamma(\alpha + w)}\right)^{(t+x)/2} \Gamma(w).$$

If C is a contour in the complex plane,

$$\det(I+K)_{\mathbb{L}^2(C)} := 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{1}{2i\pi}\right)^n\int_C\dots\int_C \det\left[K(w_i,w_j)\right]_{i,j=1}^n \mathrm{d}w_1\dots\mathrm{d}w_n.$$

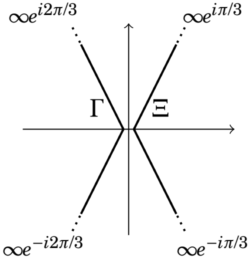
It is convenient to use Fredholm determinants because if X is distributed according to the GUE Tracy-Widom distribution,

$$\mathbb{P}(X\leqslant x) = \det(I-K_{\mathrm{Ai}})_{\mathbb{L}^2(\Gamma)},$$

where

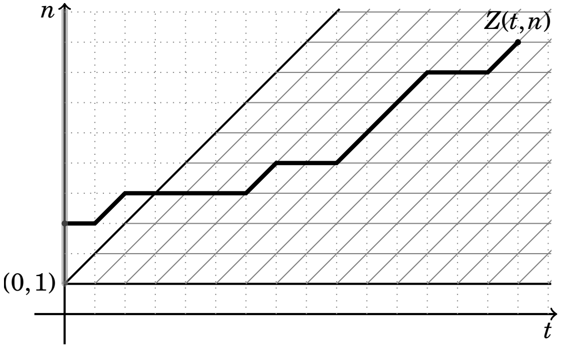
$$K_{\mathrm{Ai}}(w,w') = \frac{1}{2i\pi}\int_{\Xi}\mathrm{d}z\,\frac{e^{z^3/3-zx}}{e^{w^3/3-wx}}\frac{1}{z-w}\frac{1}{z-w'},$$

where Γ and Ξ are infinite contours in the complex plane that do not intersect.



The Beta polymer

The Beta polymer is a measure on right/upright paths from the gray line to (t, n).



The Beta polymer

Measure of a path π:

$$Q_{t,n}(\pi) = \frac{\prod_{e\in\pi} w_e}{Z(t,n)},$$

where

$$w_e = \begin{cases} B_{ij} & \text{if } e \text{ is the horizontal edge } (i-1,j) \to (i,j), \\ 1 - B_{i,j} & \text{if } e \text{ is the diagonal edge } (i-1,j-1) \to (i,j). \end{cases}$$

and $Z(t,n)$ is the partition function .

Recurrence formula

$$Z(t,n) = B_{t,n} \cdot Z(t-1,n) + (1 - B_{t,n}) \cdot Z(t-1,n-1).$$

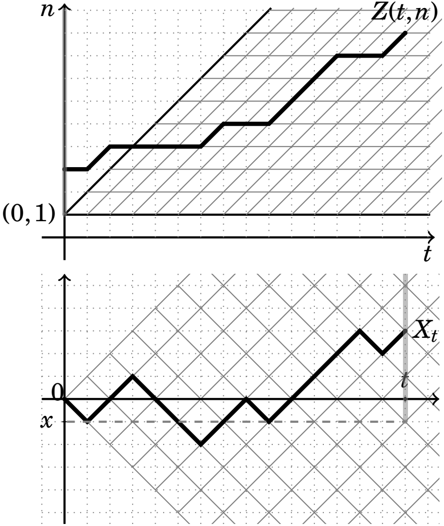
(This Beta polymer is a particular kind of “random average process”.)

Relation with Beta RWRE

- Consider the polymer path from (t, n) to the half line.
- Deform the path to make upright steps.
- One gets a Beta RWRE and

$$Z(t, n) = \mathsf{P}(X_t \geqslant t - 2n + 2).$$

(equality in law for t, n fixed.)



Evolution equation

$$Z(t+1,n) = B_{t+1,n} \cdot Z(t,n) + (1 - B_{t+1,n}) \cdot Z(t,n-1).$$

We want to compute the moments $\mathbb{E}[Z(t,n)^k]$, in order to take the Laplace transform. Let

$$u(t,\vec{n}) := \mathbb{E}[Z(t,n_1)Z(t,n_2)\dots Z(t,n_k)].$$

Evolution equation

For $\vec{n} = (n,\dots,n)$,

$$\begin{aligned} u(t+1,\vec{n}) &= \sum_{j=0}^{k} \binom{k}{j} \mathbb{E}\left[(1-B)^j B^{k-j} Z(t,n_i-1)^j Z(t,n_i)^{k-j}\right] \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{(\beta)_j (\alpha)_{k-j}}{(\alpha+\beta)_k}\, u\big(t,(n,\dots,n,n-1,\dots,n-1)\big). \end{aligned}$$

where

$$(a)_k = a(a+1)\dots(a+k-1).$$

Non-commutative binomial

The Evolution equation says

$$u(t+1,\vec{n}) = \mathscr{L}u(t,\vec{n}),$$

where $\mathscr{L}$ is an operator on functions $\mathbb{W}^k \to \mathbb{C}$, and

$$\mathbb{W}^k = \left\{\vec{n} \in \mathbb{Z}^k : n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k\right\}.$$

In general, $\mathscr{L}$ acts as in the evolution equation for each cluster of equal components in $\vec{n}$.

Lemma (Povolotsky, 2013)

Let X, Y generate an associative algebra such that

$$(1+\alpha+\beta)YX = XX + (\alpha+\beta-1)XY + YY.$$

Then we have the following non-commutative binomial identity:

$$\left(\frac{\beta}{\alpha+\beta}X + \frac{\alpha}{\alpha+\beta}Y\right)^k = \sum_{j=0}^{k} \binom{k}{j} \frac{(\beta)_j(\alpha)_{k-j}}{(\alpha+\beta)_k} X^j Y^{k-j}.$$

Bethe ansatz

One defines a simpler operator L on functions $f : Z^k \to \mathbb{C}$ by

$$\mathsf{L}f(\vec{n}) = \sum_{i=1}^{k} \left(\frac{\alpha}{\alpha+\beta} f(\vec{n}_i^{\,-}) + \frac{\beta}{\alpha+\beta} f(\vec{n}) \right)$$

where $\vec{n}_i^{\,-}$ is obtained from $\vec{n}$ by decreasing the ith coordinate by 1. Then, it is enough to solve

$$\begin{cases} \forall \vec{n} \in \mathbb{Z}^k, & u(t+1,\vec{n}) = \mathsf{L}u(t,\vec{n}) \; (\textit{free evolution}) \\ \forall \vec{n} \in \partial \mathbb{W}^k, & Bu(t,\vec{n}) = 0 \; (\textit{boundary condition}) \end{cases}$$

Moment formula

The solution of (free evolution + boundary equation) can be written as a contour integral, adapting previous works on Macdonald processes.

Proposition

For $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$*,*

$$\mathbb{E}\Big[Z(t,n_1)\cdots Z(t,n_k)\Big] =$$

$$\frac{1}{(2i\pi)^k}\int\cdots\int \underbrace{\prod_{1\leqslant A<B\leqslant k}\frac{z_A - z_B}{z_A - z_B - 1}}_{\text{boundary condition}} \underbrace{\prod_{j=1}^{k}\left(\frac{\alpha+\beta+z_j}{z_j}\right)^{n_j}\left(\frac{\alpha+z_j}{\alpha+\beta+z_j}\right)^{t}}_{\text{solution of } u(t+1)=\mathsf{L}u(t)} \underbrace{\frac{\mathrm{d}z_j}{\alpha+\beta+z_j}}_{\text{initial condition}}$$

where the contour for z_k *is a small circle around the origin, and the contour for* z_j *contains the contour for* $z_{j+1}+1$ *for all* $j = 1,\dots,k-1$*, as well as the origin, but all contours exclude* $-\alpha-\beta$*.*

Fredholm determinant

One now has an exact formula for $\mathbb{E}[Z(t,n)^k]$, and equivalently for $\mathbb{E}\left[\mathsf{P}(X_t > xt)^k\right]$. The moments do determine the distribution (!), and one can form the Laplace transform. One obtains Fredholm determinantal formulas from moment formulas via Macdonald processes theory.

Theorem (B.- Corwin)

Let $u \in \mathbb{C} \setminus \mathbb{R}_+$, and t,x with the same parity. Then for any parameters $\alpha, \beta > 0$ one has

$$\mathbb{E}\left[e^{u\mathsf{P}(X_t > x)}\right] = \det(I + K_u^{\mathrm{RW}})_{\mathbb{L}^2(C_0)}$$

where C_0 is a small positively oriented circle containing 0 but not $-\alpha-\beta$ nor -1, and $K_u^{\mathrm{RW}} : \mathbb{L}^2(C_0) \to \mathbb{L}^2(C_0)$ is defined by its integral kernel

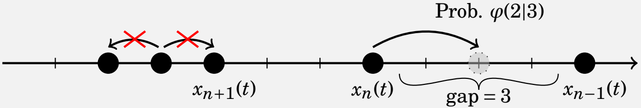
$$K_u^{\mathrm{RW}}(w,w') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)} (-u)^s \frac{g(w)}{g(w+s)} \frac{\mathrm{d}s}{s+w-w'}$$

where

$$g(w) = \left(\frac{\Gamma(w)}{\Gamma(\alpha+w)}\right)^{(t-x)/2} \left(\frac{\Gamma(\alpha+\beta+w)}{\Gamma(\alpha+w)}\right)^{(t+x)/2} \Gamma(w).$$

How we discovered the model

q-Hahn TASEP (Povolotsky 2013, Corwin 2014)



At each integer time t, particles jump by $+j$ when they have a free space m, with probability

$$\varphi(j|m) := q^{j\alpha} \frac{(q^\beta;q)_j (q^\alpha;q)_{m-j}}{(q^{\alpha+\beta};q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q .$$

This probability distribution is a q-analogue of the Beta-Binomial distribution.

Proposition (B.-Corwin)

Starting from initial condition $\forall n \geqslant 1, x_n(0) = -n$, we have as q goes to 1,

$$\left(q^{x_n(t)}\right)_{t,n} \Rightarrow \left(Z(t,n)\right)_{t,n}.$$

Extreme value theory

Fact

The order of the maximum of N i.i.d. random variables is the quantile or order $1-1/N$.

Relation LDP / extreme values

Second order corrections to the LDP have an interpretation in terms of second order fluctuations of the maximum of i.i.d. drawings.

Corollary (B.-Corwin)

Let $X_t^{(1)},\dots,X_t^{(N)}$ be random walks drawn independently in the same environment. Set $N = e^{ct}$. Then, for $\alpha=\beta=1$,

$$\frac{\max_{i=1,\dots,e^{ct}}\left\{X_t^{(i)}\right\} - t\sqrt{1-(1-c)^2}}{d(c)\cdot t^{1/3}} \Longrightarrow \mathscr{L}_{GUE},$$

where $d(c)$ is an explicit function (proved under assumption $c > 2/5$).

Comparison to correlated Gaussians

By the quenched CLT,

$$\frac{X_t^{(i)}}{\sqrt{t}} \Longrightarrow \mathcal{N}(0,1).$$

The covariance structure is asymptotically known:

$$\mathcal{E}\left[\frac{X_t^{(i)}}{\sqrt{t}}\frac{X_t^{(j)}}{\sqrt{t}}\right] \sim \frac{1}{\alpha\sqrt{\pi t}}.$$

Proposition

Let $(G_1,\ldots,G_N)$ *be a centred Gaussian vector such that* $\mathbb{E}[G_i^2]=1$ *and* $\mathbb{E}[G_iG_j]=\sqrt{\frac{c}{\pi\log(N)}}$.

$$\frac{\max_{i=1,\ldots,N}\left\{G_i\right\}-\sqrt{2\log(N)}+\sqrt{\frac{2c}{\pi}}}{\left(\sqrt{\frac{\pi\log(N)}{c}}\right)^{-1/2}} \Longrightarrow \mathcal{N}(0,1).$$

Zero-temperature model

The parameter $\alpha+\beta$ plays the role of the temperature. For parameters $a,b>0$, we set $\alpha=\epsilon a$ and $\beta=\epsilon b$. As the temperature goes to zero,

$$-\epsilon\log(\mathsf{P}(X_t>xt)) \Longrightarrow \text{passage time.}$$

Bernoulli-Exponential first passage percolation

For parameters $a,b>0$, we define a percolation model on $\mathbb{Z}_+^2$

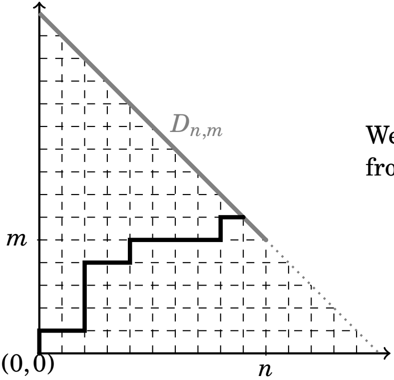
$$E_e \sim \begin{cases} \mathscr{E}(a) \text{ if } e \text{ is vertical,} \\ \mathscr{E}(b) \text{ if } e \text{ is horizontal,} \end{cases}$$

$\xi_{i,j}$ a family of Bernoulli r.v. with parameter $b/(a+b)$. The passage time of an edge is

$$t_e = \begin{cases} \xi_{i,j}E_e \text{ if } e \text{ is the vertical edge } (i,j)\to(i,j+1), \\ (1-\xi_{i,j})E_e \text{ if } e \text{ is the horizontal edge } (i,j)\to(i+1,j). \end{cases}$$

t_e corresponds to the limit of $-\epsilon\log(P)$ as $\epsilon\to 0$, where P is a transition probability, i.e. either B either $1-B$ for $B\sim\text{Beta}(a\epsilon,b\epsilon)$.

Bernoulli-Exponential directed FPP



We define the first passage-time $T(n,m)$ from $(0,0)$ to the half-line $D_{n,m}$ by

$$T(n,m) = \min_{\pi:(0,0)\to D_{n,m}} \sum_{e\in\pi} t_e$$

Limit Theorem

Theorem (B.-Corwin)

For any $\kappa > a/b$ and parameters $a, b > 0$,

$$\frac{T(n,\kappa n) - \tau(\kappa) n}{\rho(\kappa) n^{1/3}} \Longrightarrow \mathcal{L}_{GUE},$$

where $\rho(\kappa)$ and $\tau(\kappa)$ are explicit functions of κ.

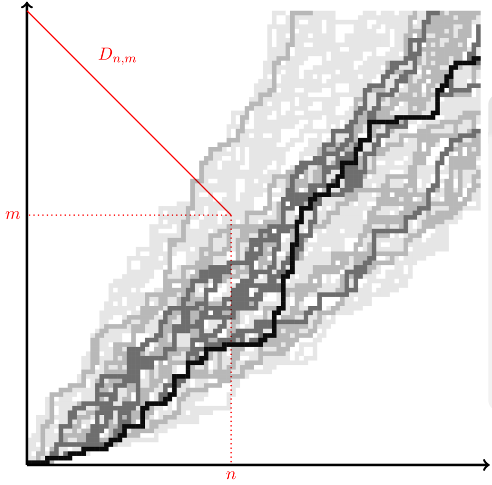
Proof.

The Fredholm determinant formula for the Beta RWRE degenerates in the scaling limit, and one gets a formula

$$\mathbb{P}\big(T(n,m) \leqslant r\big) = \det\big(I - K^{FPP}\big).$$

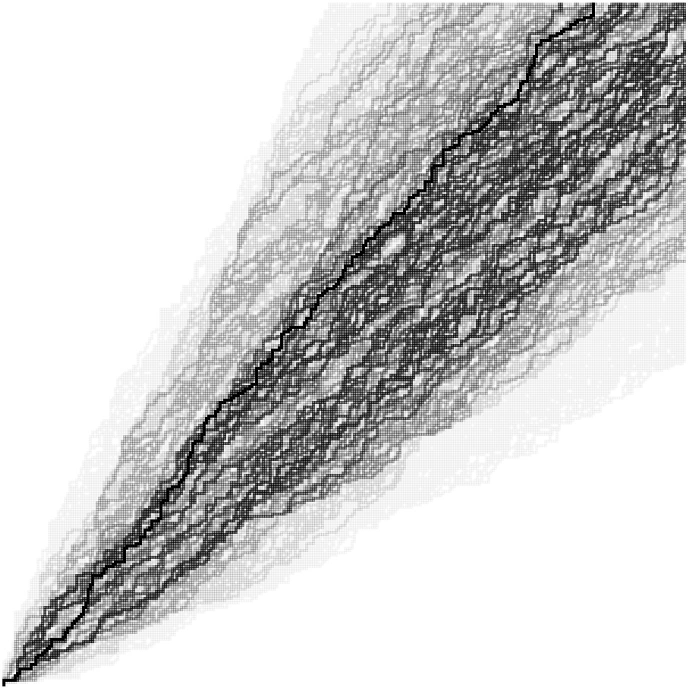
A saddle point analysis yields the limit theorem. □

Dynamical construction



Alternative description

- At time 0, only one random walk trajectory (in black).
- One adds to the percolation cluster portions of branching-coalescing random walks at exponential rate, at each branching point.



Relation to RMT ?

When $b \to \infty$, one recovers a previously known FPP model, which is itself a limit of a polymer model studied by O'Connell and Ortmann.

Proposition (Draief-Mairesse-O'Connell + O'Connell-Ortmann)

When $b \to \infty$, $T(n,m)$ is distributed as the smallest eigenvalue of the Laguerre ensemble, i.e. with density proportional to

$$\prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{n} \lambda_i^{m-1} e^{-\lambda_i} \mathrm{d}\lambda_i.$$

Question

Is there such a random matrix interpretation in the general two-parameter case ?

KPZ universality

One expects that for any $\alpha, \beta > 0$,

$$\frac{\log\left(\mathsf{P}\left(X_t > xt\right)\right) + I(x)t}{\sigma(x) \cdot t^{1/3}} \Longrightarrow \mathscr{L}_{GUE}.$$

Critical point analysis of the Fredholm determinant formula yields expressions for $I(x)$ and $\sigma(x)$, but the proof is technical (not complete).

Universality ?

$\log\left(\mathsf{P}\left(X_t > xt\right)\right)$ is the analogue of the free energy for directed polymers.
The limit above give free energy fluctuations.
Universal behaviour for RWRE ? Under which hypotheses ?

Outlook

We have seen

- A first exactly solvable model of space-time RWRE.
- Second order corrections to the LDP converge to $\mathscr{L}_{\text{GUE}}$.
- Limit theorem for the max of $N = e^{ct}$ trajectories.
- Results propagate to the zero temperature model.

Questions

- KPZ universality for RWRE and random average process, to which extent ?
- Integrability : determinantal structure ? Analogue of Schur/Macdonald processes ? Link with RMT ?
- Tracy-Widom distribution and extreme value theory...

Merci !