Extreme Diffusion

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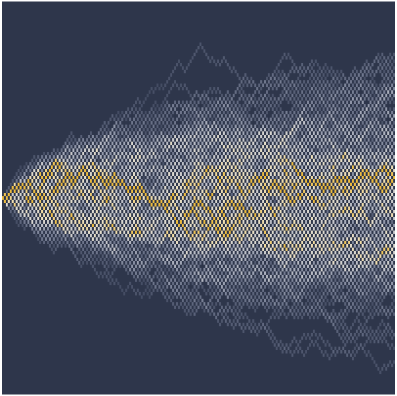
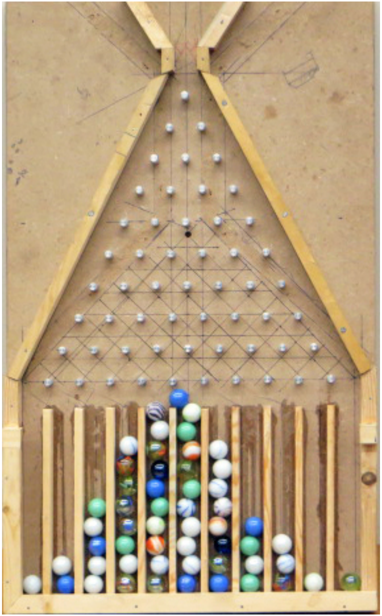


Example 1: Disordered Galton's board

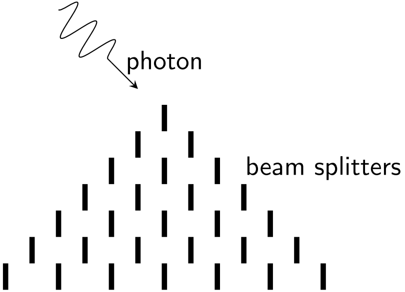
Galton's board [Galton 1889] is often employed to illustrate the convergence of the binomial distribution [De Moivre–Laplace 1733/1812]

What if the board is not perfect and pegs are placed in a disordered fashion?



Example 2: Photonic Galton board

Consider a photon navigating through an quincunx array of beam splitters (a setup considered in quantum information theory)



In a quantum model, the probability distribution is replaced by a wavefunction. Each beam splitter transforms the wave function via an elementary unitary matrix.

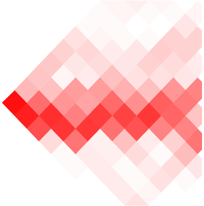
When unitary matrices are chosen at random in the unitary group, the photon effectively performs a random walk in random environment [Saul–Kardar–Read 1992].

Diffusion in random environment

[Einstein 1905]'s theory of diffusion says that up to some physical constant – the diffusion coefficient D – diffusing particle are well-approximated by a Brownian motion, their density solves the heat equation, etc.

How is this theory robust to the presence of a random environment?

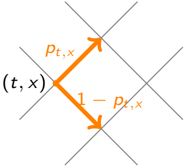
1. the **typical** behaviour is little affected by the environment;
2. the **extreme** behaviour of $N \gg 1$ diffusing particles depends very much on the environment. It is related to the scaling and statistics arising in the **Kardar-Parisi-Zhang universality class**.



Random walk in random environment (RWRE)

Consider a discrete time random walk $X(t)$ on $\mathbb{Z}$ such that

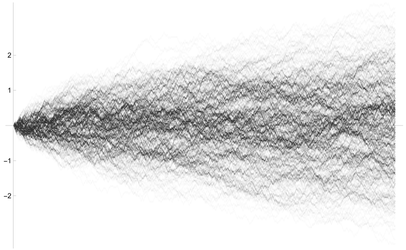
$$P\left(X(t+1) = x+1 | X(t) = x\right) = p_{t,x}$$
$$P\left(X(t+1) = x-1 | X(t) = x\right) = 1 - p_{t,x}$$



Assume that $(p_{t,x})$ are independent over space **and** time.

The average $\mathbb{E}[X(t)]$ over the noise is a simple random walk. But several random walks in the same environment are correlated.

heat equation



heat equation + local fluctuations

The simple random walk

To fix ideas, consider the case $p_{t,x} \equiv 1/2$.

$$\frac{X(t\tau)}{\sqrt{Dt}} \xrightarrow[t\to\infty]{} \text{Brownian motion}(\tau),$$

where the diffusion coefficient is $D = 1$.

Theorem ([Cramér 1938])

For $0 < x < 1$,

$$\frac{\log\Big(\mathrm{P}(X(t) > xt)\Big)}{t} \xrightarrow[t\to\infty]{} -I(x) = -I_{\mathrm{SRW}}(x),$$

where $I_{\mathrm{SRW}}(x) = \sup_{z\in\mathbb{R}}\big(zx - \lambda(z)\big)$ *is the Legendre transform of*

$$\lambda(z) := \log\Big(\mathbb{E}\big[e^{zX(1)}\big]\Big),$$

leading to $I_{\mathrm{SRW}}(x) = \frac{1}{2}\left((1+x)\log(1+x) + (1-x)\log(1-x)\right)$.

General result I

P: probability measure on paths, with average denoted E.
$\mathbb{P}$: probability measure on the environment, with average denoted $\mathbb{E}$.

Theorem ([Rassoul-Agha–Seppäläinen 2004])

For almost-every environment

$$\frac{X(\tau t) - d\tau t}{\sqrt{Dt}} \xrightarrow{t\to\infty} B(\tau)$$

where the drift $d = \mathbb{E}[2p_{t,x} - 1]$ *and the diffusion coefficient* $D = 1 - d^2$*.*

The limit no longer depends on the environment.

- The results holds more generally (higher dimensions, unbounded steps, mixing environment...)
- Many authors considered similar problems in various contexts, much before 2004...
- One can also proved a local CLT $\rightsquigarrow$ not discussed today.

General result II

P: probability measure on paths, with average denoted E.
$\mathbb{P}$: probability measure on the environment, with average denoted $\mathbb{E}$.

Theorem (Large deviation principle, [Rassoul-Agha–Seppäläinen–Yilmaz 2013])

Assume that $\mathbb{E}(\log p)^3 < \infty$ *and same for* $\log(1-p)$*. For almost-every environment,*

$$\frac{1}{t} \log E\left[e^{zX(t)}\right] \xrightarrow{t\to\infty} \lambda(z)$$

and

$$\frac{\log P(X(t) > xt)}{t} \xrightarrow{t\to\infty} -I(x)$$

where $I(x) = \sup_{z\geqslant 0}\{xz - \lambda(z)\}$*, the Legendre transform of* λ*.*

The result holds more generally.

- Finding an explicit formula for $\lambda(z)$ or $I(x)$ is generally not possible.
- $I_{\mathrm{RWRE}}(x) \geqslant I_{\mathrm{SRW}}(x)$ (by Jensen's inequality).

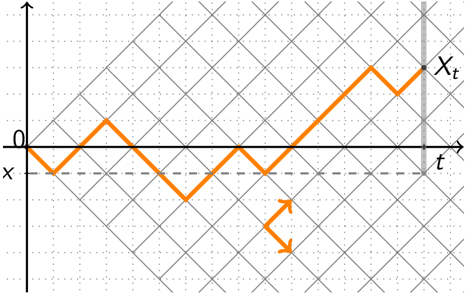
Integrable model: Beta RWRE

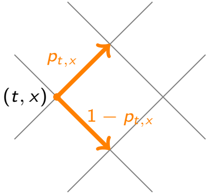
Assume that the family of random variables $(p_{t,x})$ follow the *Beta*(α, β) distribution, i.e. with density

$$p^{\alpha-1}(1-p)^{\beta-1}\mathbb{1}_{p\in[0,1]}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.$$

We will consider the random variable

$$\mathrm{P}(X_t > x).$$





From RWRE to random matrix theory

Assume that $p_{t,x} \sim \mathrm{Uniform}([0,1])$. This corresponds to $\alpha = \beta = 1$.

Theorem ([B.–Corwin 2015])

The large deviation principle rate function is

$$\lim_{t\to\infty} -\frac{\log \mathsf{P}(X_t > xt)}{t} = I(x) = 1 - \sqrt{1-x^2}.$$

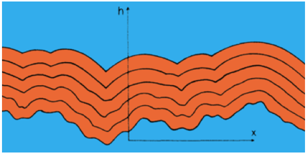
Moreover

$$\mathbb{P}\left(\frac{\log\Big(\mathsf{P}\big(X_t > xt\big)\Big) + I(x)t}{\sigma(x)\cdot t^{1/3}} \leqslant y\right) \xrightarrow[t\to\infty]{} F_{GUE}(y),$$

where F_{GUE} is the GUE Tracy-Widom distribution, describing the fluctuations of the top eigenvalue of large Hermitian random matrices (Gaussian Unitary Ensemble).

Cube-root scale fluctuations and *GUE* limits are a hallmark of the Kardar-Parisi-Zhang (KPZ) universality class.

KPZ universality class



Initially modeling random growing interfaces [Kardar–Parisi–Zhang 1986], the KPZ class has grown to include:

- Interacting particle systems (their height function is a growth process),
- Directed polymers (their free energy defines a height function),
- Stochastic PDEs such as the celebrated KPZ equation,
- 2D Statistical mechanics models:
  first/last passage percolation, six-vertex model, etc.
- More... even polariton condensates [Bloch-Ravets et al. 2022] !

KPZ characteristic exponents

In dimension 1, fluctuations have size $t^{1/3}$, spatial correlations on the scale $t^{2/3}$, and limiting distributions often related to random matrix theory (GUE, GOE, etc.).

Extreme values in random environment

Second order corrections to the LDP yield estimates for extremes of independent samples:

$$\mathbb{P}(\max\{X^{(i)}(t)\} \leqslant x) = \mathbb{E}\left[P(X(t) \leqslant x)^N\right] \approx \mathbb{E}\left[e^{N\log(1-P(X(t)>x))}\right]$$

Corollary ([B.–Corwin 2015])

Let $X_t^{(1)}, \ldots, X_t^{(N)}$ be random walks sampled independently in the same environment. Set $N = e^{ct}$. Then,

$$\mathbb{P}\left(\frac{\max_{i=1,\ldots,N}\left\{X_t^{(i)}\right\} - t \cdot I^{-1}(c)}{d(c) \cdot t^{1/3}} \leqslant y\right) \xrightarrow[t\to\infty]{} F_{GUE}(y),$$

where $d(c)$ is an explicit function.

For simple random walks, the maximum is $\approx t \cdot \left(I_{\mathrm{SRW}}\right)^{-1}(c)$, and

$$\max_{1\leqslant i\leqslant N} X_t^{(i)} \approx t \cdot I_{\mathrm{SRW}}^{-1}(c) - \frac{1}{\tanh^{-1}(I^{-1}(c))} \cdot \log(t) + dGumbel$$

where *dGumbel* is a discrete variant of the Gumbel distribution.

Recurrence

Let us compute the moments of the random variable

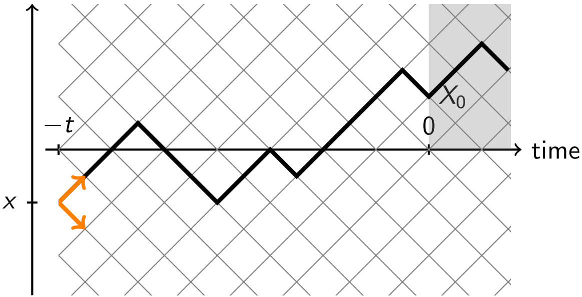
$$P(X(t) > x | X(0) = 0).$$

One should rather study the **time reversal**

$$Q(t, x) = P(X(t) > 0 | X(-t) = x)$$

which satisfies the recurrence

$$\boxed{Q(t, x) = p_{-t,x} Q(t-1, x+1) + (1 - p_{-t,x}) Q(t-1, x-1)}.$$



Moment formula

Define

$$u(t,\vec{x}) = \mathbb{E}\left[Q(t,x_1)\dots Q(t,x_k)\right].$$

The recurrence relation yields an evolution equation for the function u which can be solved.

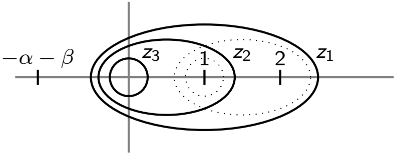
When $x_1 \leqslant \dots \leqslant x_n$,

$$u(\vec{x},t) = \oint \frac{dz_1}{2\mathbf{i}\pi} \cdots \oint \frac{dz_n}{2\mathbf{i}\pi} \prod_{a<b} \frac{z_a - z_b}{z_a - z_b - 1} \prod_{i=1}^{n} f_{t,x_i}(z_i)$$

with

$$f_{t,x}(z) = \left(\frac{(z+\alpha)^2}{z(z+\alpha+\beta)}\right)^{t/2} \left(\frac{z+\alpha+\beta}{z}\right)^{x/2} \frac{1}{z}$$

where the integration contours are nested into each other.



What makes the beta distribution special?

Aside: q-binomial expansion

When $XY = YX$,

$$(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}.$$

When X and Y do not commute but $YX = qXY$, for $q \in \mathbb{C}$,

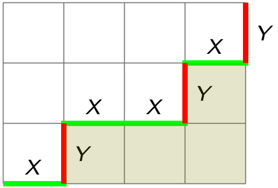
$$(X+Y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q X^k Y^{n-k},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad \text{with } [n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q},$$

It arises in counting

- number of subspaces of dimension k in a space of dimension n over $\mathbb{F}_q$ (when q is a power of a prime number).
- number of lattice paths in a $k \times (n-k)$ grid with fixed area underneath



What makes the beta distribution special?

One may consider more general relations relating XX, XY, YX, YY:

Lemma ([Rosengren 2000, Povolotsky 2013])

Let X, Y generate an associative algebra such that

$$0 = XX + (\alpha + \beta - 1)XY - (1 + \alpha + \beta)YX + YY.$$

Then

$$\left(\frac{\alpha}{\alpha+\beta}X + \frac{\beta}{\alpha+\beta}Y\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha)_k(\beta)_{n-k}}{(\alpha+\beta)_n} X^k Y^{n-k},$$

where $(a)_k = a(a+1)\dots(a+k-1)$.

The coefficients

$$\binom{n}{k} \frac{(\alpha)_j(\beta)_{n-k}}{(\alpha+\beta)_n} = \binom{n}{k} \mathbb{E}[p^k(1-p)^{n-k}]$$

are beta-binomial probabilities. **This allows to transform the recurrence relation satisfied by $u(t, \vec{x})$.**

More general point of view: Bethe ansatz

Bethe ansatz is a method from theoretical physics [Bethe 1931] to diagonalize operators: the Hamiltonian of quantum spin chain, the Markov matrix of a particle system, a recurrence relation of the form

$$u(t+1,\vec{x}) = \mathcal{L}u(t,\vec{x}).$$

Roughly speaking, the Bethe ansatz says that if the rules of interactions between particles in our model satisfy some condition (Yang-Baxter equation) then, eigenfunctions are typically expressed in an explicit way

$$\varphi_{\vec{\xi}}(\vec{x}) = \sum_{\sigma\in\mathcal{S}_n} A_\sigma(\xi)\prod_{i=1}^{n}\xi_i^{x_i}, \quad A_\sigma(\xi) = \prod_{a>b}\frac{S(\xi_{\sigma(a)},\xi_{\sigma(b)})}{S(\xi_a,\xi_b)}$$

In our case, we parametrize $\xi_i = \frac{z_i+\alpha+\beta}{z_i}$ so that eigenfunctions are

$$\varphi_{\vec{z}}(\vec{x}) = \sum_{\sigma\in\mathcal{S}_n}\sigma\left(\prod_{a<b}\frac{z_a-z_b-1}{z_a-z_b}\prod_{i=1}^{n}\left(\frac{z_i+\alpha+\beta}{z_i}\right)^{x_i}\right)$$

Solutions to the recursion may be written as contour integrals.

Moderate deviations and the KPZ equation

The [Kardar–Parisi–Zhang 1986] equation is the stochastic PDE

$$\frac{\partial h(x,t)}{\partial t} = \Delta h(x,t) + (\nabla h(x,t))^2 + \xi(x,t)$$

where ξ is a space time white noise.

- It arises as the limit of several types of KPZ class models under special scalings (particle systems with weak asymmetry, directed polymers at high temperature).
- This equation is famoulsy ill-posed and motivated the introduction of [Hairer 2011]'s regularity structures! In dimension 1, we define it as $h(t,x) = \log Z(t,x)$ where

$$\partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + Z(t,x)\xi(t,x)$$

- It arises in moderate deviations probabilities for RWRE [Le Doussal–Thiery 2017], [B.–Le Doussal 2020].

Extreme RWRE and the KPZ equation

Instead of deviations of order t or $t^{1/2}$, consider deviations of order $t^{3/4}$:

$$P\left(X(\tau t) > \tau t^{3/4} + x t^{1/2}\right) C_{\tau,x}(t) \xrightarrow[t\to+\infty]{} Z(\tau, x) = e^{h(\tau,x)}.$$

Let $X^1(t), \dots, X^N(t)$ be independent random walks sampled in the same environment. Scale $N = e^{\sqrt{\tau t}}$.

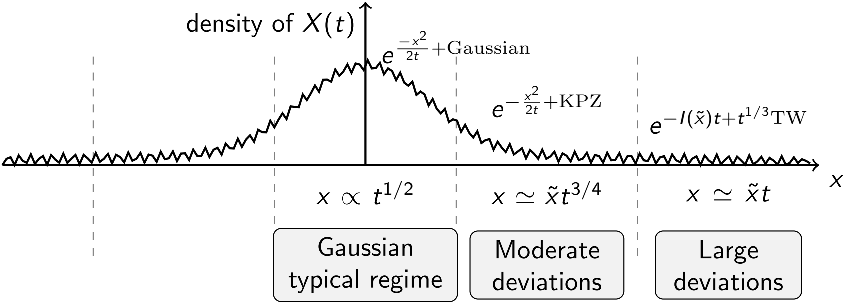
$$\max_{1\leqslant i\leqslant N}\{X^i(t)\} \approx \overbrace{\sqrt{t\log(N)} + \sqrt{\frac{t}{2\log N}}\left(-\log(4\pi\log N) + \mathcal{G}\right)}^{\text{same terms as for independent simple random walks}} + \sqrt{\frac{t}{2\log N}} h(\tau, 0)$$

where $\mathcal{G}$ denotes a Gumbel Random variable, i.e. $\mathbb{P}(\mathcal{G} < g) = e^{-e^{-g}}$, and $h(\tau, 0)$ is the KPZ equation at the origin and time τ.

General picture

- The KPZ equation limit holds for general distributions of transition probabilities $p_{t,x}$ [Das–Drillick–Parekh 2023] and more general stochastic flows [Parekh 2024].
- **Universality:** The Tracy-Widom limit is also expected to hold more generally, under some moment condition on $\log(p_{x,t})$ and $\log(1-p_{x,t})$.

For random walks or diffusions $X(t)$ in random uncorrelated environment,



A diffusion in random environment

The simplest continuous model would be a diffusion $x(t)$ in a white noise velocity field, i.e.

$$dx(t) = v(x(t), t)dt + dB_t$$

where B_t is a Brownian motion and v is a white noise.
[Le Doussal–Thierry 2017]

Via Kolmogorov's equation, $q(x, t) = P\left(x(0) \geqslant 0 | x(-t) = x\right)$ satisfies

$$\partial_t q = \tfrac{1}{2}\partial_{xx} q + v(t, x)\partial_x q.$$

Similar models are considered in the physics literature on turbulence [Kraichnan 1968, Bernard–Gawedski–Kupiainen 1998]. However, the model with white noise is ill-posed.

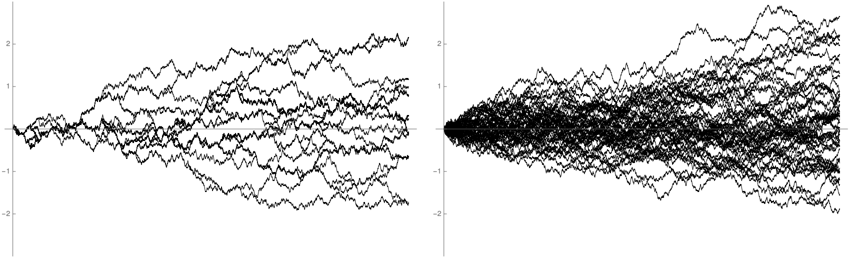
A correct way to define it, coming from [Le Jan–Lemaire 2004] is to scale the Beta RWRE with $\alpha = \beta = \varepsilon$ (the noise becomes strong) and consider the limit

$$\varepsilon X(\varepsilon^{-2} t) \xRightarrow{\varepsilon \to 0} x(t).$$

Sticky Brownian motions

N independent diffusions $x^1, \ldots, x^N$ in the same environment become sticky Brownian motions [Howitt–Warren 2009, Le Jan–Raimond 2004, Gawedski–Horvai 2003]

For $N = 50$, we see that sticky Brownian motions are very different from independent Brownian motions.

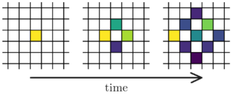


Theorem ([B.–Rychnovsky 2019])

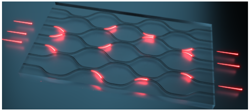
The maximum of $N = e^{ct}$ sticky Brownian motions converges to the Tracy-Widom distribution converges in the $t^{1/3}$ scale.

Open problems

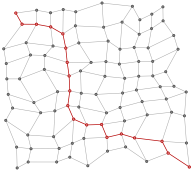
1. What happens in higher dimensions? We expect that the connection between RWRE and KPZ survives, but KPZ in higher dimensions is poorly understood.



2. A single photon going through an array of beam splitters is exactly related to the Beta RWRE, based on [Saul–Kardar–Read 1992]. What about several particles?



3. How does an extreme RWRE path look like? It should converge to the law of a geodesic in the directed landscape.



4. Is this mathematical story related to anything real?
⇝ Experiments in progress by [Eric Corwin–Ivan Corwin–...]

Conclusion

Random walks or diffusions in random environment behave differently from the simple random walk/Brownian motion.

1. The position of the extreme particles fluctuates at a higher scale, related to **Kardar-Parisi-Zhang universality**.
2. This behaviour was probed using an exactly solvable model with Beta distributed transition probabilities, which can be studied using Bethe ansatz.

Thank you