Macdonald processes and KPZ equation in a half-space

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May 14, 2018

An SPDE motivation

- Consider the (multiplicative noise) stochastic heat equation,

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + Z\,\xi, \quad \text{where } Z(t,x),\ x \in \mathbb{X}, t > 0,$$

  where ξ is a Gaussian space time white noise. In one spatial dimension, one can make sense of it in an integrated sense.

- $h := \log(Z)$ is the solution to the KPZ equation

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}\left(\partial_x h\right)^2 + \xi.$$

- **Question:** What is the probability distribution of the solution?
- The answer depends on
  1. The initial condition. In this talk we restrict to delta initial data.
  2. The space $\mathbb{X}$. It may be the whole line $\mathbb{R}$, the torus $\mathbb{R}/\mathbb{Z}$ or an interval with Neumann/Dirichlet/other boundary conditions.

A beautiful answer when $\mathbb{X} = \mathbb{R}$

Consider the solution to the multiplicative SHE on the whole line $\mathbb{R}$,

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + Z\,\xi, \quad \text{where } x \in \mathbb{R}, t > 0,$$

with delta initial condition $Z(0,\cdot) = \delta_0$.

Theorem (Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn 2010)

For $u \in \mathbb{C}$ with $\Re\mathfrak{e}(u) > 0$,

$$\mathbb{E}\left[e^{-uZ(t,0)e^{t/24}}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{1+u\ \exp\left((t/2)^{1/3}\mathfrak{a}_i^{\mathrm{GUE}}\right)}\right].$$

where $\{\mathfrak{a}_i^{\mathrm{GUE}}\}_{i\geqslant 1}$ are the limiting eigenvalues of the GUE scaled at the edge (Airy point process).

- The RHS is explicit and can be computed numerically. One can deduce large time limits, large deviations estimates.

Analogue when $\mathbb{X} = \mathbb{R}_+$?

Consider now the solution $Z(\tau, x)$ to the multiplicative SHE in a half-space,

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + Z\,\xi, \quad \text{where } x \in \mathbb{R}_{\geqslant 0}, t > 0,$$

with delta initial condition $Z(0,\cdot) = \delta_0$ for some boundary condition at $x = 0$ (Neumann, Dirichlet, mixed...).

What is the law of the solution? Can one find a function f and a matrix ensemble such that

$$\mathbb{E}\left[e^{-uZ(t,x)}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} f\left(u\, e^{(t/2)^{1/3}\mathfrak{a}_i}\right)\right]?$$

Based on results on symmetrized last passage percolation [Baik-Rains 2001], we expect a transition depending on boundary parameters between Tracy-Widom and Gaussian type statistics.

KPZ equation through discrete models

Moments

Consider mixed moments of $\mathbb{E}[Z(t,x_1)\dots Z(t,x_k)]$. These are solutions of the delta Bose gas, which can be solved by Bethe ansatz. However, the moments grow too fast to characterize the distribution.

There are two rigorous approaches.

KPZ equation through ASEP

The height function of the asymmetric simple exclusion process (ASEP) converges to the KPZ equation [Bertini-Giacomin 1997] under a certain weak asymmetry scaling.

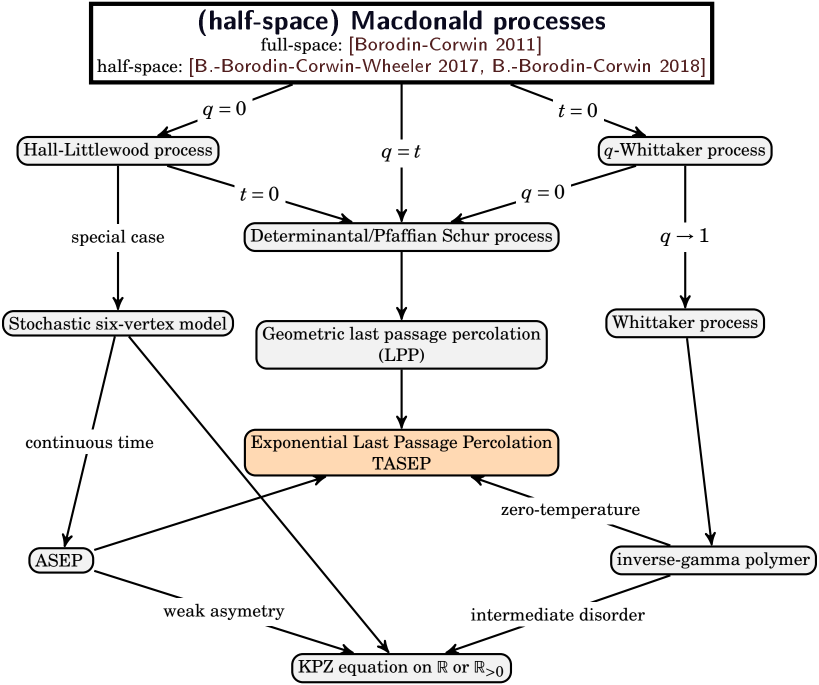
KPZ equation through discrete directed polymers

The free energy of directed polymer in $\mathbb{Z}^2$ at high temperature converges to the KPZ equation [Alberts-Khanin-Quastel 2010].

Both discretizations are marginals or limits of more general measures on partitions called **(half-space) Macdonald processes**.

Plan of the talk

1. Last passage percolation in a half-quadrant: the Baik-Rains transition
2. Definition of models and results
   - Limit theorems for Half-line ASEP
   - Law of the KPZ equation
   - Formulas for the inverse gamma polymer partition function in a half-quadrant
3. Half-space Macdonald measures
   - Definition
   - Hall-Littlewood measures and the stochastic six vertex model
   - Markov Dynamics and half-space Macdonald processes
   - Computing observables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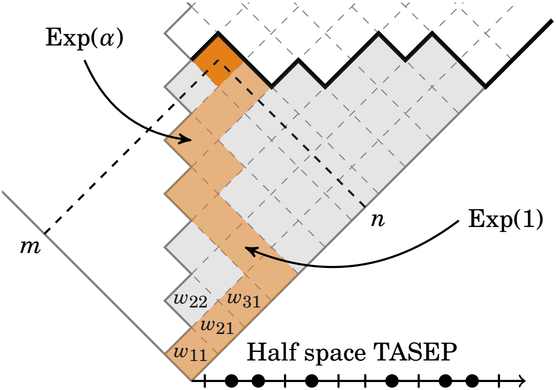
Last Passage Percolation in a half quadrant

Let w_{ij} a family of i.i.d. exponential random variables with rate 1 when $i > j$ and α when $i = j$.



Consider directed paths π from the box $(1,1)$ to (n,m) in the half quadrant. We define the last passage percolation time $H(n,m)$ by

$$H(n,m) = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}.$$

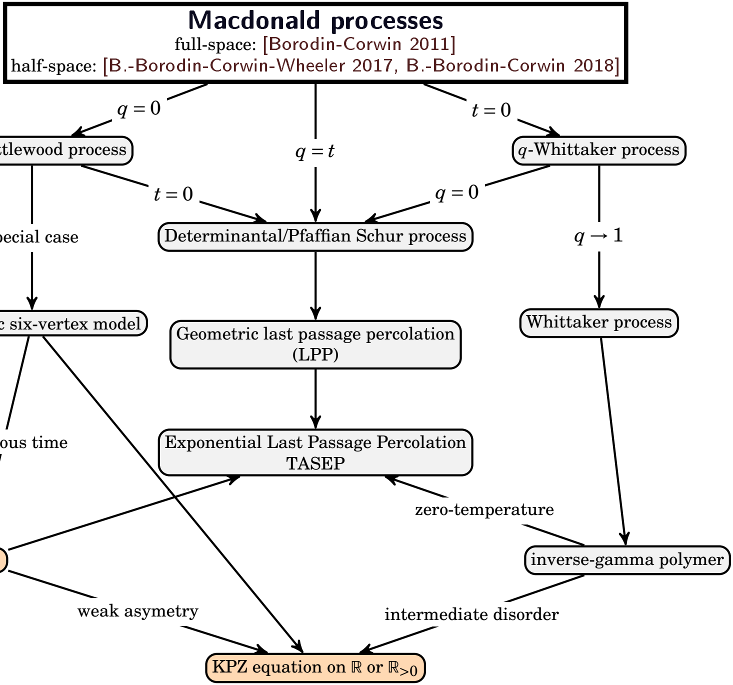
Baik-Rains transition

Last passage percolation can be studied within the framework of determinantal/Pfaffian point processes.

Theorem (Baik-Rains 2001, Baik-B.-Corwin-Suidan 2016)

- *When* $\alpha > 1/2$,
$$\frac{H(n,n)-4n}{2^{4/3}n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GSE}},$$
- *When* $\alpha = 1/2$,
$$\frac{H(n,n)-4n}{2^{4/3}n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GOE}},$$
- *When* $\alpha < 1/2$,
$$\frac{H(n,n)-cn}{c'n^{1/2}} \Longrightarrow \mathcal{N},$$

- Far from the diagonal, we obtain a transition between Tracy-Widom GUE and Gaussian statistics.
- Scaling α close to 1/2 and (n,m) close to the boundary, we obtain crossover ensemble related to RMT.
- Multipoint correlations (along space-like paths) can also be characterized.



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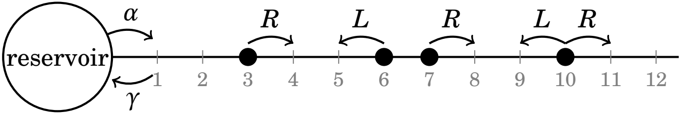
Half-line ASEP

Let $R > L \geqslant 0$, and consider the asymmetric simple exclusion process (**ASEP**) on the positive integers with open boundary condition:



One can characterize the system by the function

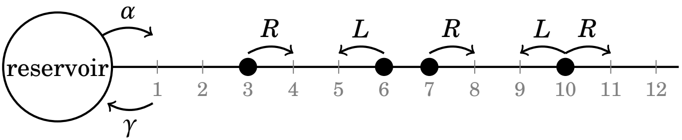
$$N_x(\tau) = \#\{\text{particles on the right of site } x \text{ at time } \tau\}.$$

Under weakly asymmetric scaling, [Corwin-Shen 2016, Parekh 2017] (**see Hao's talk**) showed that $N_x(t)$ converges to the KPZ equation on the positive reals with Neumann boundary condition,

$$\begin{cases} \partial_\tau h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}\left(\partial_x h\right)^2 + \xi \\ \partial_x h(\tau, x)\Big|_{x=0} = a \in \mathbb{R}. \end{cases}$$

This is equivalent to saying that $h = \log Z$ where Z solves the mSHE with mixed Robin boundary condition Z, $\partial_x Z(\tau, x)\Big|_{x=0} = a\, Z(\tau, 0)$.

Previous results on half-line ASEP



- [Liggett 1975] classified the stationary measures when

$$\frac{\alpha}{R} + \frac{\gamma}{L} = 1.$$

  Then $\rho = \alpha/R$ is the average density enforced at the boundary. There is a transition at $\rho = 1/2$ between product-form Bernoulli measure and spatially correlated stationary measures (which can be expressed using Matrix Product Ansatz).

- [Tracy-Widom 2013] used coordinate Bethe ansatz to find formulas for the transition probabilities, but these do not seem amenable for asymptotic analysis.

- We analyze half-line ASEP through a half space version of the **stochastic six-vertex model**. This approach was first developed in the full-space [Borodin-Corwin-Gorin 2014, Aggarwal-Borodin 2016, Aggarwal 2016, Borodin-Olshanski 2016])

Recall results for TASEP

When $\gamma = L = 0$ (no left jumps), the model is equivalent to last passage percolation. Without loss of generality, we may set $R = 1$ and recall

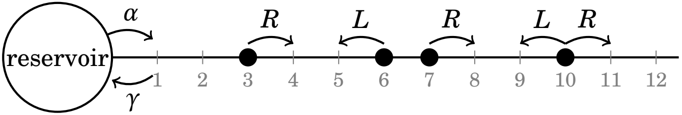
$$N_x(t) = \#\{\text{particles on the right of } x \text{ at time } t\}.$$

Theorem (Baik-Rains 2001, Baik-B.-Corwin-Suidan 2016)

- *When* $\alpha > 1/2$*,*
$$\frac{N_0(t) - \frac{t}{4}}{2^{-4/3}t^{1/3}} \underset{t\to\infty}{\Longrightarrow} -\mathscr{L}_{\mathrm{GSE}}.$$
- *When* $\alpha = 1/2$*,*
$$\frac{N_0(t) - \frac{t}{4}}{2^{-4/3}t^{1/3}} \underset{t\to\infty}{\Longrightarrow} -\mathscr{L}_{\mathrm{GOE}}.$$
- *When* $\alpha < 1/2$*,*
$$\frac{N_0(t) - t\alpha(1-\alpha)}{\sigma t^{1/2}} \underset{t\to\infty}{\Longrightarrow} \mathscr{N}(0,1).$$

$\mathscr{L}_{\mathrm{GSE}}/\mathscr{L}_{GOE}$ is the Tracy Widom GSE / GOE distribution (Gaussian symplectic / orthogonal ensemble).

Laplace transform of ASEP current



We assume

1. Ligget's condition $\frac{\alpha}{R} + \frac{\gamma}{L} = 1$.
2. The boundary enforces a density of particles $\alpha/R = 1/2$ at the origin.

Theorem (B.-Borodin-Corwin-Wheeler 2017)

Let $t = L/R \in (0,1)$. For any time $\tau > 0$ and $u < 0$,

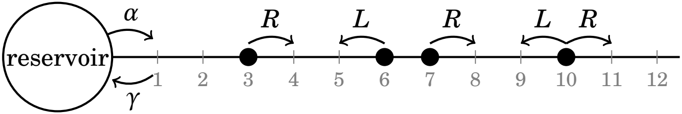
$$\mathbb{E}\left[\frac{1}{(ut^{N_0(\tau)}, t^2)_\infty}\right] = \mathrm{Pf}[J + f_u \cdot K]_{\ell^2(\mathbb{Z}_{\geqslant 0})}$$

$$:= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{x_1,\dots,x_k \in \mathbb{Z}_{\geqslant 0}} f_u(x_1)\dots f_u(x_k) \mathrm{Pf}\left[K(x_i, x_j)\right]_{i,j=1}^{k}$$

where K is a certain kernel (a variant of the Pfaffian Schur process kernel) and

$$f_u(x) = \frac{(ut^{x+1}; t^2)_\infty}{(ut^x; t^2)_\infty} - 1.$$

Large time asymptotics



Asymptotic analysis of the Fredholm Pfaffian yields:

Theorem (B.-Borodin-Corwin-Wheeler 2017)

We assume

1. *Ligget's condition* $\frac{\alpha}{R} + \frac{\gamma}{L} = 1$.
2. *The boundary enforces a density of particles* $\alpha/R = 1/2$ *at the origin.*

Then

$$\frac{N_0\left(\frac{\tau}{R-L}\right) - \frac{\tau}{4}}{2^{-4/3}\tau^{1/3}} \underset{\tau\to\infty}{\Longrightarrow} -\mathscr{L}_{\mathrm{GOE}}.$$

For other values of α, or the current away from the boundary, we expect the same results as for TASEP (but cannot prove it yet).

Law of KPZ equation in half-space

Consider the KPZ equation on $\mathbb{R}_+$ with Neumann boundary condition,

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}\left(\partial_x h\right)^2 + \xi \\ \partial_x h(t,x)\Big|_{x=0} = a \in \mathbb{R}. \end{cases}$$

Using the convergence of half-line ASEP height function to the KPZ equation [Corwin-Shen 2016, Parekh 2017] we obtain:

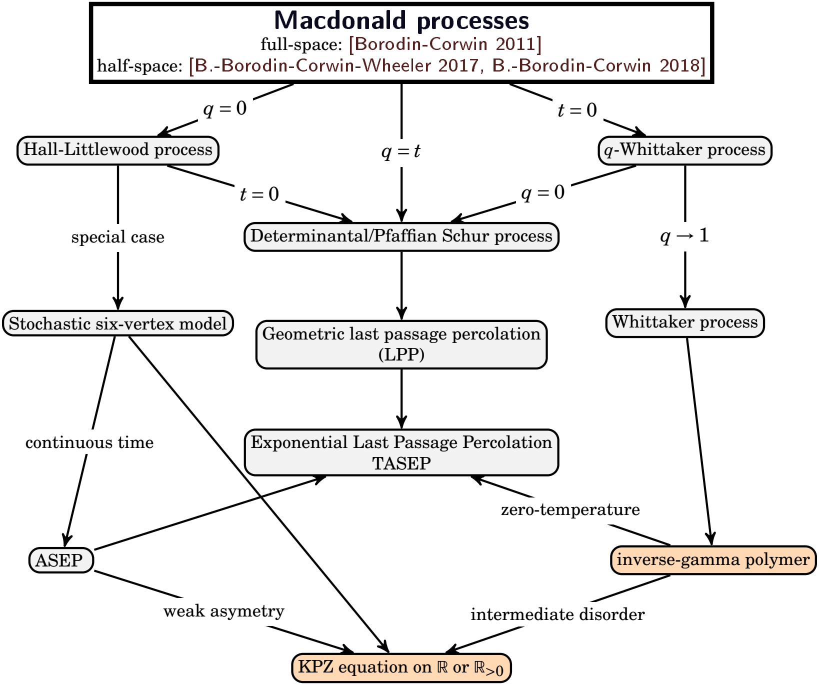
Theorem (B.-Borodin-Corwin-Wheeler 2017)

Assume the boundary parameter $a = -1/2$. For $u \in \mathbb{C}$ with $\mathfrak{Re}(u) > 0$,

$$\mathbb{E}\left[e^{-uZ(t,0)}e^{t/24}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty}\sqrt{\frac{1}{1+4u\exp\left((t/2)^{1/3}\mathfrak{a}_i^{\mathrm{GOE}}\right)}}\right],$$

where $\{\mathfrak{a}_i^{\mathrm{GOE}}\}_{i=1}^{\infty}$ forms the GOE point process (i.e. the sequence of rescaled eigenvalues of a large Gaussian real symmetric matrix).

See also recent results by [Krajenbrink-Le Doussal 2018, Gueudre-Le Doussal 2012] in the case $a = +\infty$.



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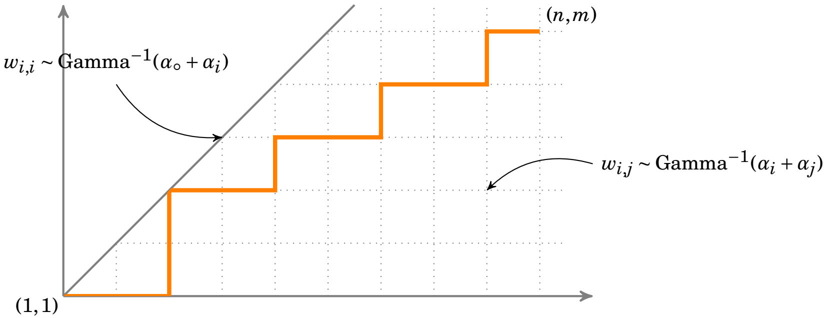
Inverse-gamma directed polymer

Let $\alpha_1, \alpha_2, \dots$ and $\alpha_\circ$ be positive parameters.



The partition function of the half-quadrant inverse-gamma polymer is

$$Z(n,m) = \sum_{\pi:(1,1)\to(n,m)} \prod_{(i,j)\in\pi} w_{i,j}$$

- At zero temperature, the free energy (i.e. $\log(Z)$) converges to passage times of half-space LPP.
- At high temperature, the free energy converges to the KPZ equation [Wu 2018] with Neumann boundary condition (with boundary parameter controlled by $\alpha_\circ$).

Laplace transform formula of the inverse-gamma polymer

- Using geometric RSK, [O'Connell-Seppäläinen-Zygouras, 2012] related the distribution of the partition function to Whittaker functions. A formal application of the Plancherel theorem for Whittaker functions yields integral formulas.
- Using Macdonald processes, these formulas can be proved and generalized.

Theorem ([B-Borodin-Corwin 2018])

For $m \geqslant n$ and any $u > 0$, we have

$$\mathbb{E}\left[e^{-uZ(m,n)}\right] = \frac{1}{n!}\int_{r-i\infty}^{r+i\infty}\frac{\mathrm{d}z_1}{2i\pi}\cdots\int_{r-i\infty}^{r+i\infty}\frac{\mathrm{d}z_n}{2i\pi}\prod_{i\neq j}\frac{1}{\Gamma(z_i-z_j)}\prod_{1\leqslant i<j\leqslant n}\frac{\Gamma(z_i+z_j)}{\Gamma(\alpha_i+\alpha_j)}\prod_{i,j=1}^{n}\Gamma(z_i-\alpha_j)$$
$$\prod_{i=1}^{n}\left(u^{\alpha_i-z_i}\frac{\Gamma(\alpha_\circ+z_i)}{\Gamma(\alpha_\circ+\alpha_i)}\prod_{j=n+1}^{m}\frac{\Gamma(\alpha_j+z_i)}{\Gamma(\alpha_j+\alpha_i)}\right)$$

where r is such that $r+\alpha_\circ > 0$ and $r > \alpha_i$ for all $1 \leqslant i \leqslant n$.

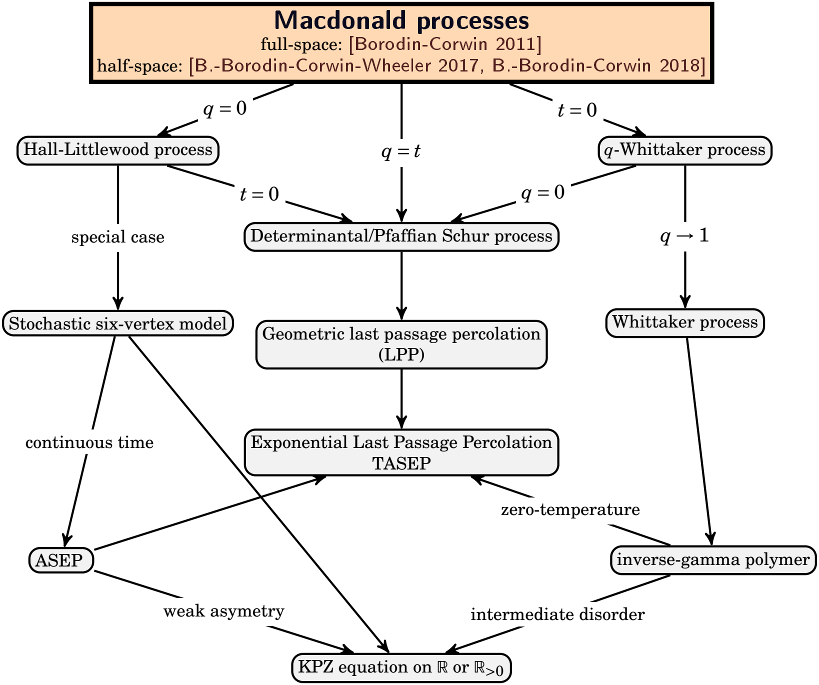
Almost a Fredholm Pfaffian

If the parameters $\alpha_i > 0$ are sufficiently close to each other, for any $m \geqslant n \geqslant 1$, $u > 0$,

$$\mathbb{E}\left[e^{-uZ(m,n,\tau)}\right] = \sum_{k=0}^{n} \frac{1}{k!} \int_{r-i\infty}^{r+i\infty} \frac{\mathrm{d}z_1}{2i\pi} \cdots \int_{r-i\infty}^{r+i\infty} \frac{\mathrm{d}z_k}{2i\pi} \oint \frac{\mathrm{d}v_1}{2i\pi} \cdots \oint \frac{\mathrm{d}v_k}{2i\pi}$$
$$\times \prod_{1\leqslant i<j\leqslant k} \frac{(z_i - z_j)(v_i - v_j)\Gamma(v_i + v_j)\Gamma(-z_i - z_j)}{(z_j + v_i)(z_i + v_j)\Gamma(v_j - z_i)\Gamma(v_i - z_j)}$$
$$\times \prod_{i=1}^{k} \left[\frac{\pi}{\sin(\pi(v_i + z_i))} \frac{G_{n,m}(v_i)}{G_{n,m}(-z_i)} \frac{\Gamma(2v_i)}{\Gamma(v_i - z_i)} \frac{u^{z_i+v_i}}{z_i + v_i} \right],$$

for well chosen r and a certain explicit function $G_{n,m}(z)$.

- Using the approximation $\Gamma(z) \approx 1/z$ close to $z = 0$, the series would become a Fredholm Pfaffian. However, without that approximation, the series is not summable as n goes to infinity.
- A formal saddle point asymptotic analysis leads to the Baik-Rains transition as for LPP.



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Macdonald measures

- An integer partition λ is a sequences of integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$. Symmetric Macdonald polynomials P_λ, Q_λ are multivariate symmetric polynomials whose coefficients are rational functions in two parameters $q, t \in (0,1)$. They degenerate to Schur functions s_λ when $q = t$.
- Macdonald functions satisfy a Cauchy type summation identity. For two sets of parameters $a = \{a_i\}$, $b = \{b_i\}$,

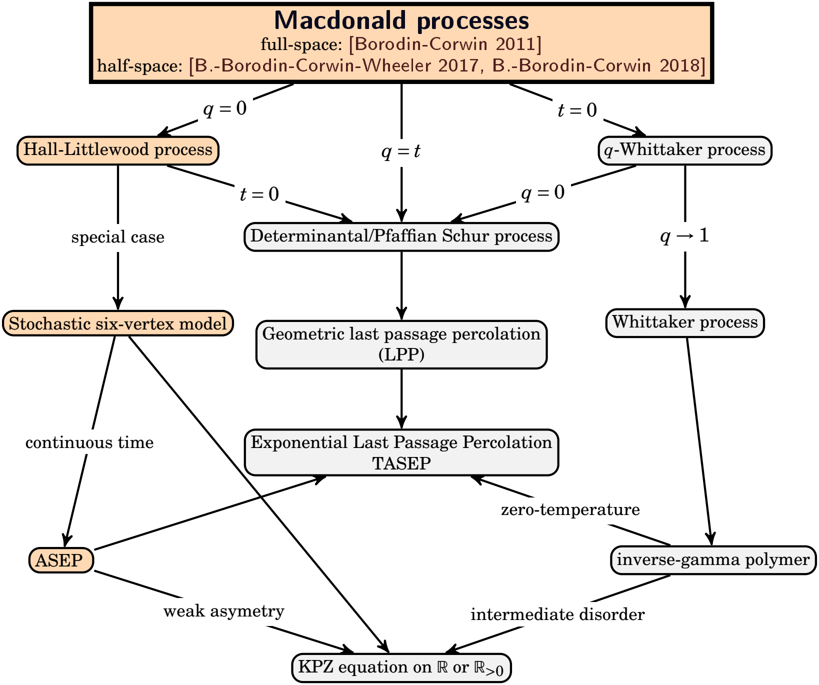
$$\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$$

  where $\Pi(a;b) = \prod_{i,j} \frac{(ta_i b_j;q)_\infty}{(a_i b_j;q)_\infty}$.
- This leads to the definition of **Macdonald measures** [Borodin-Corwin 2011] which generalizes the Schur measure [Okounkov 2001]. These are measures on partitions such that

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Pi(a;b)} P_\lambda(a) Q_\lambda(b).$$

Half-space Macdonald measures

- Macdonald functions also satisfy a Littlewood type summation identity
$$\sum_{\lambda' \text{even}} b_\lambda^{\text{el}} P_\lambda(a) = \Phi(a)$$
where $\Phi(a) = \prod_{i<j} \frac{(ta_i a_j;q)_\infty}{(a_i a_j;q)_\infty}$ and λ' even means $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4 \dots$
- More generally, we define
$$\mathcal{E}_\lambda(b) = \sum_{\mu' \text{even}} b_\mu^{\text{el}} Q_{\lambda/\mu}(b),$$
so that
$$\sum_\lambda \mathcal{E}_\lambda(b) P_\lambda(a) = \Pi(a;b)\Phi(a).$$
- We define the half-space Macdonald measure by
$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Pi(a,b)\Phi(a)} P_\lambda(a)\mathcal{E}_\lambda(b).$$
Half-space Macdonald measures degenerate when $q = t$ to Pfaffian Schur measures [Borodin-Rains 2005].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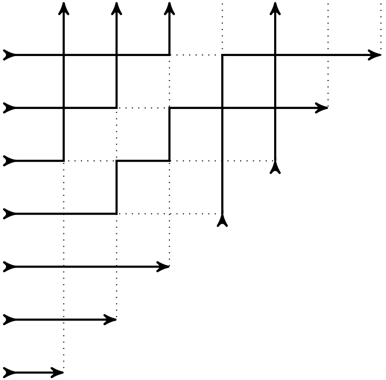
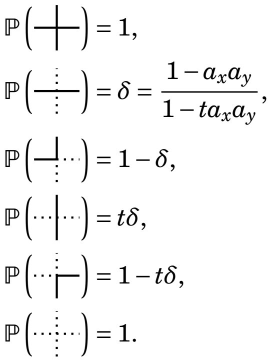
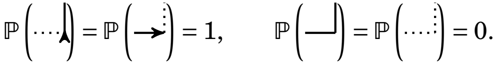
Six vertex model in a half-quadrant

Let $a_1, a_2, \cdots \in (0,1)$ and $t \in (0,1)$ be some parameters.



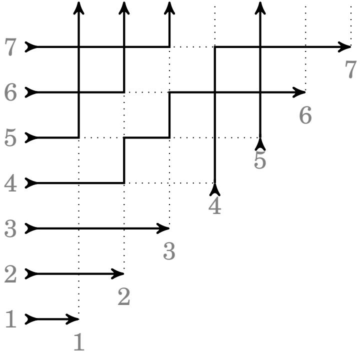


We use the boundary weights [Kuperberg 2000, Wheeler-Zinn-Justin 2016]



For small δ, **paths are discretizations of the trajectories of particles in half-line ASEP.**

Stochastic six vertex model in a half space and Hall-Littlewood functions



- Let $\mathfrak{h}(x,y)$ be the number of outgoing vertical arrows from the vertices on the left of (x,y).
- Let $\ell(\lambda) = \lambda'_1$ be the number of nonzero components in a partition λ following the half-space Hall-Littlewood measure (i.e. Macdonald measure when $q=0$).

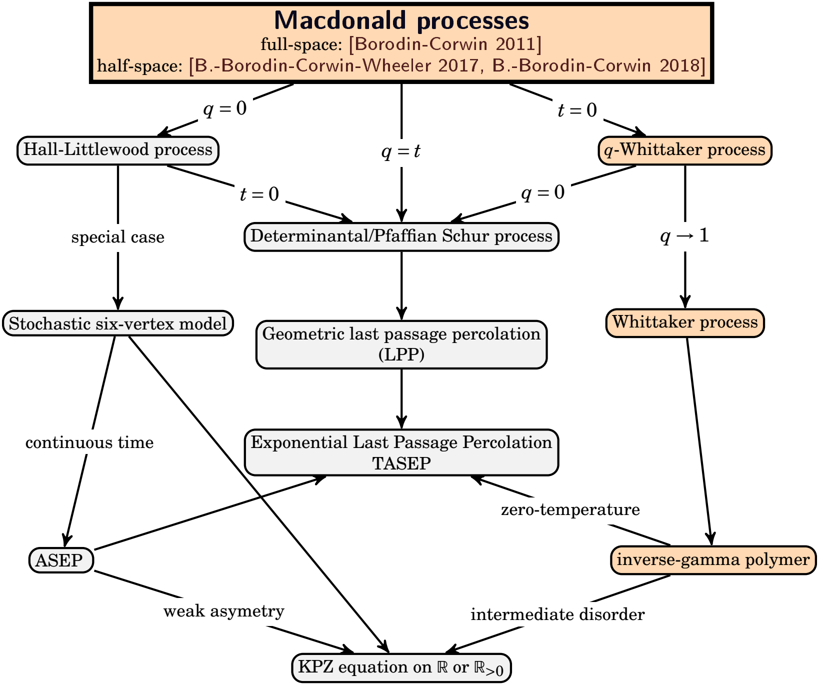
Theorem (B.-Borodin-Corwin-Wheeler 2017)

$$\mathfrak{h}(n,m) \overset{(d)}{=} \lambda'_1,$$

where

$$\mathbb{P}(\lambda) \propto P_\lambda(a_1,\dots,a_n)\mathscr{E}_\lambda(a_{n+1},\dots,a_m).$$

Similar results exist for full-space models [Borodin 2016, Borodin-Bufetov-Wheeler 2016, Bufetov-Matveev 2017].



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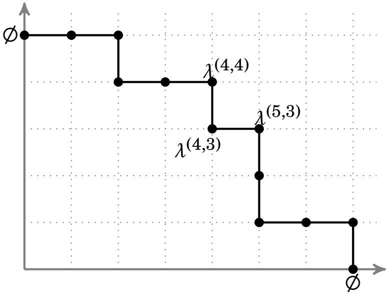
KPZ equation on $\mathbb{R}$ or $\mathbb{R}_{>0}$

(full-space) Macdonald processes



Consider a path γ as on the left

- vertex $v \mapsto \lambda^v$ a random partition,
- edge $e \mapsto \rho_e$ a (set of) variable(s). (More generally a specialization of the symmetric functions).

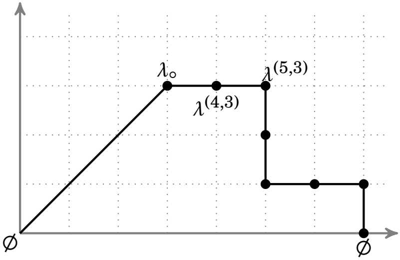
The **Macdonald process** (generalizing the Schur process [Okounkov-Reshetikhin 2003]) is a probability measure on the sequence of partitions $\Lambda := \left(\lambda^v\right)_{v\in\gamma}$ such that

$$\mathbb{P}(\Lambda) \propto \prod_{e\in\gamma} \mathrm{weight}(e),$$

where

$$\mathrm{weight}\left(\overset{\mu \quad \lambda}{\underset{\rho}{\bullet\!\!-\!\!-\!\!\bullet}} \right) = Q_{\lambda/\mu}(\rho) \quad \text{and} \quad \mathrm{weight}\left(\rho \begin{array}{l} \bullet\lambda \\ | \\ \bullet\mu \end{array} \right) = P_{\lambda/\mu}(\rho).$$

Half-space Macdonald process



Consider a path γ as on the left

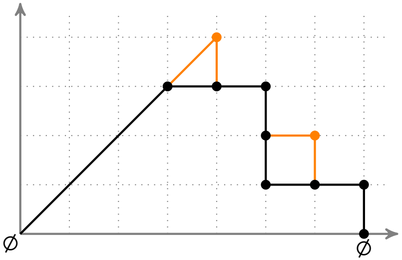
- vertex $v \mapsto \lambda^v$ a random partition,
- edge $e \mapsto \rho_e$ a (set of) variable(s).
- Denote $\rho_\circ$ and $\lambda_\circ$ the specialization and the partition on the diagonal.

The **half-space Macdonald process** is a probability measure on the sequence of partitions $\Lambda := \left(\lambda^v\right)_{v\in\gamma}$ such that

$$\mathbb{P}(\Lambda) \propto \mathscr{E}_{\lambda_\circ}(\rho_\circ) \prod_{e\in\gamma} \text{weight}(e),$$

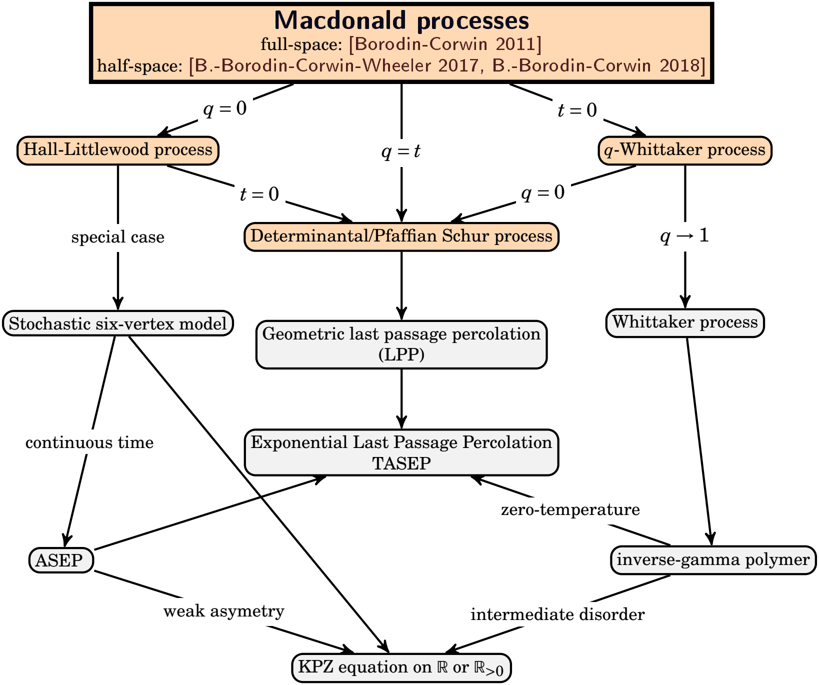
where the weight of off-diagonal edges are chosen as in the Macdonald process.
For any $v \in \gamma$, the law of λ^v is a half-space Macdonald measure.

Dynamics



- Make the path grow by unit boxes in the bulk and half-boxes along the diagonal.
- We define Markov dynamics on sequences of partitions Λ mapping a half-space Macdonald process to another half-space Macdonald process along a new path.
- For well chose-dynamics, $\lambda_1^{(n,m)} \approx \log Z(n,m)$, where $Z(n,m)$ is the inverse-gamma partition function and λ is distributed according to the q-Whittaker process with $q \to 1$.

$$\mathbb{P}(\lambda) \propto P_\lambda(q^{\alpha_1}, \dots, q^{\alpha_n})\, \mathscr{E}_\lambda(q^{\alpha_\circ}, q^{\alpha_{n+1}}, \dots, q^{\alpha_m}).$$



Macdonald processes
full-space: [Borodin-Corwin 2011]
half-space: [B.-Borodin-Corwin-Wheeler 2017, B.-Borodin-Corwin 2018]
$q = 0$
$q = t$
$t = 0$
Hall-Littlewood process
q-Whittaker process
$t = 0$
$q = 0$
special case
Determinantal/Pfaffian Schur process
$q \to 1$
Stochastic six-vertex model
Geometric last passage percolation (LPP)
Whittaker process
continuous time
Exponential Last Passage Percolation TASEP
zero-temperature
ASEP
inverse-gamma polymer
weak asymetry
intermediate disorder
KPZ equation on $\mathbb{R}$ or $\mathbb{R}_{>0}$

Computing observables of Macdonald measures?

Assume that we have an operator $\mathbf{D}_n$ acting on functions of n variables $a_1,\dots,a_n$, diagonalized by Macdonald polynomials $P_\lambda(a)$, with eigenvalue d_λ. By acting on both sides of the Cauchy-Littlewood identity

$$\sum_\lambda P_\lambda(a)\,\mathscr{E}_\lambda(b) = \Pi(a;b)\Phi(a),$$

and dividing by $\Pi(a;b)\Phi(a)$, we obtain

$$\mathbb{E}^{q,t}[d_\lambda] = \frac{\mathbf{D}_n \Pi(a;b)\Phi(a)}{\Pi(a;b)\Phi(a)}.$$

Families of such operators exist (Macdonald difference operators, Noumi's q-integral operator) and the resulting observables characterize the law of λ [Borodin-Corwin 2011, Borodin-Corwin-Gorin-Shakirov 2012, B.-Borodin-Corwin 2018].

A variant of Noumi's q-integral operator

We define an operator $\mathbf{M}_n^z$ on symmetric functions of n variables by

$$\mathbf{M}_n^z f(x_1,\dots,x_n) = \int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{\mathrm{d}s_1}{2i\pi}\cdots\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\frac{\mathrm{d}s_n}{2i\pi}(-z)^{s_1+\dots+s_n}\prod_{i<j}\frac{q^{s_j}x_i-q^{s_i}x_j}{x_i-x_j}$$
$$\times\prod_{i,j}\frac{(tx_i/x_j)_\infty}{(qx_i/x_j)_\infty}\frac{(q^{s_j+1}x_i/x_j)_\infty}{(tq^{s_j}x_i/x_j)_\infty}\prod_{i=1}^{n}\Gamma(-s_i)\Gamma(1+s_i)\,f(q^{-s_1}x_1,\dots,q^{-s_n}x_n),$$

for $\epsilon>0$ is small enough. We have the eigenrelation

$$\mathbf{M}_n^z P_\lambda(x_1,\dots,x_n) = \prod_{i=1}^{n}\frac{(q^{-\lambda_i}t^i z)_\infty}{(q^{-\lambda_i}t^{i-1}z)_\infty}P_\lambda(x).$$

This comes from

- The Pieri rule for Macdonald polynomials $P_\lambda Q_{(r)}$ which yields an operator written as a linear combination of q-shifts.
- Rewriting sums as Mellin-Barnes type integrals (to avoid dealing with issues of divergence of moments).

Applying $\mathbf{M}_n^z$ to $\Pi(x;b)\Phi(x)$ yields the Laplace transform formulas for the inverse gamma polymer partition function.

Refined Cauchy identity

Recall that for usual Macdonald measures,

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Pi(a;b)} P_\lambda(a_1,\dots,a_n)\, Q_\lambda(b_1,\dots,b_m),$$

Proposition ([Warnaar 2008])

For $u \in \mathbb{C}$,

$$\frac{1}{\Pi(a,b)} \sum_\lambda \prod_i \left(1 - u q^{\lambda_i} t^{n-i}\right) P_\lambda(a)\, Q_\lambda(b) = \frac{\det\left[\frac{1}{1-a_i b_j} - u \frac{1}{1-t a_i b_j}\right]}{\det\left[\frac{1}{1-a_i b_j}\right]}.$$

It implies that

$$\mathbb{E}^{q,t}\left[\prod_{i=1}^{n} \left(1 - u q^{\lambda_i} t^{n-i}\right)\right]$$

does not depend on q! Comparing the $q=0$ and $q=t$ cases yields identities relating functionals of Schur ($q=t$) and Hall-Littlewood ($q=0$) random partitions.

Refined Littlewood identity

For half-space Macdonald measures,

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Phi(a)} P_\lambda(a_1,\dots,a_n)\, b^{\mathrm{el}}_\lambda \mathbb{1}_{\lambda' \mathrm{even}},$$

Proposition ([Rains 2015], [Betea-Wheeler-Zinn-Justin 2015])

For $u \in \mathbb{C}$,

$$\frac{1}{\Phi(a)} \sum_{\lambda' \text{ even}} \prod_{i \text{ even}} \left(1 - u q^{\lambda_i} t^{n-i}\right) b^{\mathrm{el}}_\lambda P_\lambda(a_1,\dots,a_n) = \frac{\mathrm{Pf}\left[\dfrac{a_i - a_j}{1 - a_i a_j} - u\dfrac{a_i - a_j}{1 - t a_i a_j}\right]}{\mathrm{Pf}\left[\dfrac{a_i - a_j}{1 - a_i a_j}\right]}.$$

It implies that

$$\mathbb{E}^{q,t}\left[\prod_{i \text{ even}} \left(1 - u q^{\lambda_i} t^{n-i}\right)\right]$$

does not depend on q! Comparing the $q = 0$ and $q = t$ cases yields identities relating functionals of (half-space) Schur and Hall-Littlewood random partitions.

Conclusion

- Large-scale statistics of the KPZ equation in a half-space and several models in its universality class (ASEP, directed polymers, stochastic six vertex models) can be studied via **half-space Macdonald processes** almost as well as their full-space counterparts.
- Through **half-line ASEP**, one obtains a beautiful formula characterizing the law of the KPZ equation on $\mathbb{R}_+$, but the result is restricted to the height at 0 and a specific boundary corresponding to the critical case in the Baik-Rains transition.
- Through the **inverse gamma polymer** in a half-quadrant, one obtains exact formulas, without restrictions on parameter ranges. However, analyzing these asymptotically remains a challenge.

Thank you