Last passage percolation (in a strip)

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October 24, 2024

Discrete Integrable Systems,
ICTS

Schur polynomials

The Schur polynomials

$$s_\lambda(x) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\det\left(x_i^{n-j}\right)_{i,j=1}^{n}},$$

where $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0)$ is an integer partition and $x = \{x_1, \ldots, x_n\}$ a set of variables, are symmetric polynomials.

They form a basis of the algebra of symmetric functions and satisfy the Cauchy summation identity

$$\sum_\lambda s_\lambda(a_1, \ldots, a_n) s_\lambda(b_1, \ldots, b_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - a_i b_j} =: \Pi(a; b).$$

Proof: Binet-Cauchy formula

Branching rule

Schur functions satisfy a branching rule

$$s_\lambda(a_1, \dots, a_n) = \sum_\mu s_{\lambda/\mu}(a_1) s_\mu(a_2, \dots, a_n),$$

where for $a \in \mathbb{R}$,

$$s_{\lambda/\mu}(a) = \mathbb{1}_{\mu \prec \lambda} a^{|\lambda| - |\mu|}$$

with $|\lambda| = \sum_i \lambda_i$ and we write $\mu \prec \lambda$ for $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \dots$

This allows to expand Schur functions in monomials

$$s_\lambda(x) = \sum_{\emptyset \prec \lambda^{[1]} \prec \cdots \prec \lambda^{(n)} = \lambda} \prod_{i=1}^n x_i^{|\lambda^{(i)}| - |\lambda^{(i-1)}|}$$

where $\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} = \lambda$ is a sequence of interlaced partitions.

RSK correspondence

Let us expand in monomials each side of the Cauchy identity

$$\sum_{\lambda} s_{\lambda}(a_1, \ldots, a_n) s_{\lambda}(b_1, \ldots, b_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - a_i b_j},$$

RSK correspondence

Let us expand in monomials each side of the Cauchy identity

$$\sum_{\lambda} s_\lambda(a_1,\ldots,a_n)s_\lambda(b_1,\ldots,b_m) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-a_ib_j},$$

LHS

The LHS can be expanded using the monomial expansion of Schur polynomials. Thus, it is indexed by sequences

$$\begin{gathered}\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} = \lambda \\ \emptyset \prec \mu^{(1)} \prec \cdots \prec \mu^{(m)} = \lambda\end{gathered}$$

and

$$\sum_{\lambda} s_\lambda(a_1,\ldots,a_n)s_\lambda(b_1,\ldots,b_m) =$$

$$\sum_{\substack{\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} = \lambda \\ \emptyset \prec \mu^{(1)} \prec \cdots \prec \mu^{(m)} = \lambda}} \prod_{i=1}^{n} a_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|}\prod_{j=1}^{m} b_j^{|\mu^{(j)}|-|\mu^{(j-1)}|}.$$

RSK correspondence

Let us expand in monomials each side of the Cauchy identity

$$\sum_{\lambda} s_\lambda(a_1, \ldots, a_n) s_\lambda(b_1, \ldots, b_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - a_i b_j},$$

RHS

The RHS is expanded using

$$\frac{1}{1 - a_i b_j} = \sum_{w=0}^{+\infty} (a_i b_j)^w$$

so that the sum is indexed by matrices $W \in \mathbb{N}^{n \times m}$:

$$\prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - a_i b_j} = \prod_{i=1}^{n} \prod_{j=1}^{m} \sum_{w_{i,j}=0}^{+\infty} (a_i b_j)^{w_{i,j}}$$

$$= \sum_{W=(w_{i,j}) \in \mathbb{N}^{n \times m}} \prod_{i=1}^{n} \prod_{j=1}^{m} (a_i b_j)^{w_{i,j}}$$

RSK correspondence

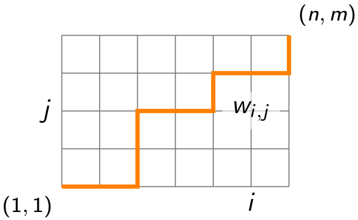
Let us expand in monomials each side of the Cauchy identity

$$\sum_{\lambda} s_\lambda(a_1,\dots,a_n)s_\lambda(b_1,\dots,b_m) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-a_ib_j},$$

The two sides can be matched using Robinson-Schensted-Knuth correspondence, a bijection

$$\begin{Bmatrix} \emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} = \lambda \\ \emptyset \prec \mu^{(1)} \prec \cdots \prec \mu^{(m)} = \lambda \end{Bmatrix} \longleftrightarrow W = (w_{i,j}) \in \mathbb{N}^{n\times m}$$

Greene's theorem implies that $\lambda_1 = G(n,m)$ where



$$G(n,m) = \max_{\text{paths } (1,1)\to(m,n)} \left\{ \sum_{(i,j)\in \text{ path}} w_{i,j} \right\}.$$

Schur measure

Assume that when $w_{i,j} \sim \mathrm{Geom}(a_i b_j)$ are independent (we say that $w \sim \mathrm{Geom}(q)$ if $\mathbb{P}(w = k) = (1-q)q^k$). Then,

$$\mathbb{P}\left(G(n,m) \leqslant r\right) = \frac{1}{\Pi(a;b)} \sum_{\lambda:\lambda_1 \leqslant r} s_\lambda(a) s_\lambda(b)$$

In other terms, $G(n,m)$ has the same law as λ_1 when λ is a random partition sampled according to the Schur measure

$$\mathbb{P}(\lambda) = \frac{1}{\Pi(a;b)} s_\lambda(a) s_\lambda(b).$$

Asymptotics

[Johansson, 2001] proved that after appropriate rescaling, $G(n,m)$ fluctuates according to the Tracy-Widom GUE distribution (governing the fluctuations of the largest eigenvalue of large Hermitian matrices).

Aside: Pitman transform

[Pitman (1975)] showed that if X_t is a Brownian motion, and $M_t = \sup_{s\in[0,t]} X_t$, then the process

$$t \mapsto 2M_t - X_t$$

is a Brownian motion conditioned to stay positive for all $t \in \mathbb{R}_+$.

For continuous functions $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}$ define

$$f_1 \otimes f_2(t) = \inf_{0\leqslant s\leqslant t} \{f_1(s) + f_2(t) - f_2(s)\}$$

$$f_1 \odot f_2(t) = \sup_{0\leqslant s\leqslant t} \{f_1(s) + f_2(t) - f_2(s)\}.$$

Then the Pitman transform of $f = (f_1, f_2)$ is

$$\mathcal{P}^{(2)} f = (f_2 \odot f_1, f_1 \otimes f_2).$$

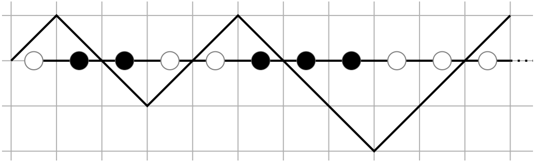
Theorem (Pitman, 1975)

If $B = (B_1, B_2)$ is a Brownian motion in $\mathbb{R}^2$, then $\mathcal{P}^{(2)} B$ is a Brownian motion contrained to stay in the Weyl chamber

$$\mathbb{W}_2 = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > x_2\}$$

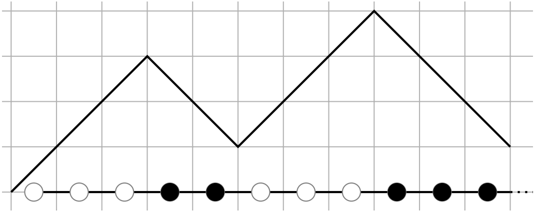
Aside: Pitman transform

Pitman originally proved the result by first considering X_t to be a discrete time random walk with ± 1 steps.



After one step of the Box-Ball-System [Croydon-Kato-Sasada-Tsujimoto 2018], the new configuration is encoded by the Pitman transform of X_t :

$$(\mathcal{P}X)_t = 2 \max_{1 \leqslant s \leqslant t} \{X_s\} - X_t$$



Aside: Pitman transform

Why introducing the operations $f_1 \otimes f_2$ and $f_1 \odot f_2$? It allows to generalize.

For $f = (f_1, \dots, f_n)$, let $\mathcal{P}_i^{(n)} f$ be the transformation that replaces (f_i, f_{i+1}) by $(f_{i+1} \odot f_i, f_i \otimes f_{i+1})$.
Then, if

$$(n, n-1, \dots, 2, 1) = s_{i_1} s_{i_n} \dots s_{i_k}$$

is a minimal factorization of the reverse permutation in transpositions $s_i = (i, i+1)$, we define the Pitman transform

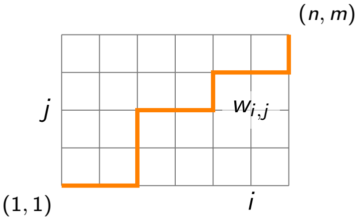
$$\mathcal{P}^{(n)} = \mathcal{P}_{i_1}^{(n)} \dots \mathcal{P}_{i_k}^{(n)}.$$

Theorem ([O'Connell-Yor 2001, Bougerol-Jeulin 2003])

If $B = (B_1, \dots, B_n)$ is a Brownian motion in $\mathbb{R}^n$, then $\mathcal{P}^{(n)} B$ is a Brownian motion conditioned to stay in

$$\mathbb{W}_n = \{(x_1, \dots, x_n) \in \mathbb{R}^2; x_1 > \dots > x_n\}$$

Aside: discrete Pitman transform = RSK



$$G(m, n) = \max_{\text{paths } (1,1)\to(n,m)} \left\{ \sum_{(i,j)\in \text{ path}} w_{i,j} \right\}.$$

Let

$$R_i(m) = \sum_{j=1}^{m} w_{i,j}$$

Then, using the maps

$$R_1 \otimes R_2(m) = \min_{1\leqslant j\leqslant m} \{R_1(j-1) + R_2(m) - R_2(j)\}$$

$$R_1 \odot R_2(m) = \max_{1\leqslant j\leqslant m} \{R_1(j) + R_2(m) - R_2(j-1)\},$$

one can define the Pitman transform $\mathcal{P}^{(n)}R$ of $R = (R_1, \ldots, R_n)$. We then have, for all $m \geqslant 1$,

$$G(m, n) = (\mathcal{P}^{(n)}R)_1(m).$$

Aside: discrete Pitman transform = RSK

More generally, the RSK correspondance

$$\begin{Bmatrix} \emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} = \lambda \\ \emptyset \prec \mu^{(1)} \prec \cdots \prec \mu^{(m)} = \lambda \end{Bmatrix} \quad \longleftrightarrow \quad W = (w_{i,j}) \in \mathbb{N}^{n \times m}$$

can be expressed in terms of discrete Pitman transforms:

We already know that

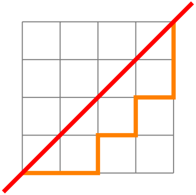
$$\lambda_1 = \lambda_1^{(n)} = \mu_1^{(m)} = G(n,m) = (\mathcal{P}^{(n)} R)_1(m).$$

More generally,

$$(\mu^{(j)})_{1 \leqslant j \leqslant m} = \left(\mathcal{P}^{(n)} R(j)\right)_{1 \leqslant j \leqslant m}$$

and similar relation holds for the $\lambda^{(i)}$ after replacing $w_{i,j}$ by $w_{j,i}$.

Symmetrized last passage percolation



Assume that the weight matrix is symmetric:

$$w_{i,j} = w_{j,i}$$

$$w_{i,j} \sim \mathrm{Geom}(a_i a_j) \text{ for } i > j$$

$$w_{i,i} \sim \mathrm{Geom}(c a_i)$$

$$G^{\boxslash}(n,m) = \max_{\text{paths } (1,1)\to(n,m)} \Big\{ \sum_{(i,j)\in \text{ path}} w_{i,j} \Big\}.$$

[Baik-Rains 2003] proved that

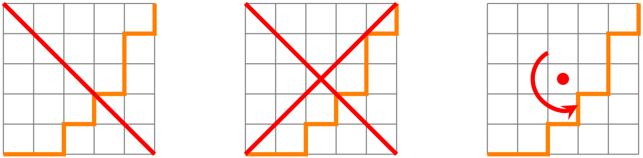
$$\mathbb{P}\left(G^{\boxslash}(n,n) \leqslant r \right) = \frac{1}{\Pi^{\boxslash}(a,c)} \sum_{\lambda:\lambda_1 \leqslant r} c^{\lambda_1-\lambda_2+\lambda_3-\lambda_4+\dots} s_\lambda(a)$$

Asymptotics

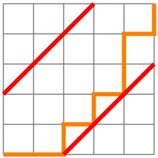
As $n \to \infty$, $G^{\boxslash}(n,n)$ fluctuates according to the Tracy-Widom GSE or GOE distributions (depending if $c = 1$ or $c < 1$).

Other variants

[Baik-Rains 2003] also computed $\mathbb{P}\left(G^{\bullet}(n,n) \leqslant r\right)$ in terms of Schur functions for other symmetry types $\bullet$:



The problem is however open for

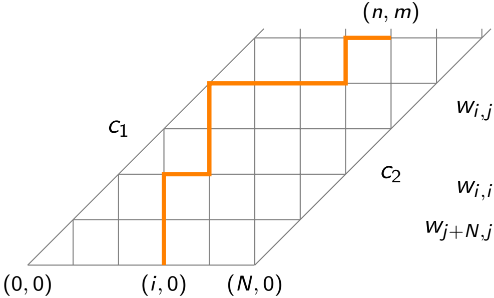


Last passage percolation with walls

Imposing two symmetry axis in the diagonal direction is equivalent to assuming that paths are constrained to remain between two walls.

Last passage percolation in a strip

Let $a_1, \dots, a_N \in (0,1)$, $c_1, c_2 > 0$.



$$w_{i,j} \sim \mathrm{Geom}(a_i a_j) \text{ for } j < i < j+N \text{ (indices modulo } N)$$

$$w_{i,i} \sim \mathrm{Geom}(c_1 a_i)$$

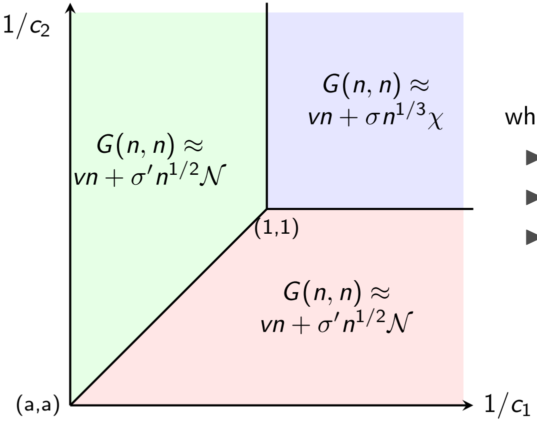
$$w_{j+N,j} \sim \mathrm{Geom}(c_2 a_j)$$

We fix an initial condition $G(i,0) = G_0(i)$ for some function G_0.

$$G(n,m) = \max_{\text{paths } (i,0)\to(n,m)} \left\{ G_0(i) + \sum_{(i,j)\in \text{ path}} w_{i,j} \right\}$$

Conjectural phase diagram

The richest behaviour is when $N = Ln^{2/3}$. For $a_1 = \cdots = a_N = a$, it is expected that



where

- $v = v(a, c_1, c_2)$,
- $\mathcal{N} \sim$ Gaussian
- $\chi \sim$ unknown distribution depending on c_1, c_2, L.

Finding χ is an open problem for any model in the same universality class.

Stationary measure

The process

$$t \mapsto (G(t+i,t))_{0\leqslant i\leqslant N}$$

does not have any stationary measure.

Let

$$G_t(i) = G(t+i,t) - G(t,t).$$

The process $t \mapsto G_t$ is a Markov process on $\mathbb{Z}^N$.

Problem

Find $(G_0(i))_{1\leqslant i\leqslant N}$ such that for all t, $(G_t(i))_{1\leqslant i\leqslant N} \overset{(d)}{=} (G_0(i))_{1\leqslant i\leqslant N}$

- For models such as Asymmetric Simple Exclusion Process, the standard method is the **Matrix Product Ansatz** [Derrida-Evans-Hakim-Pasquier 1993].
- We will illustrate another approach [B.-Corwin-Yang 2023] based on symmetric functions, taking the example of Last Passage Percolation and Schur functions.

Stationary measure (first definition)

Assume for simplicity that $a_1 = \cdots = a_N = a$.

For $R = (R(j))_{0 \leqslant j \leqslant N}$, let

$$\mathbb{P}_q^{\mathrm{RW}}(R) = \prod_{j=1}^{N} (1-q) q^{R(j)-R(j-1)}$$

be the probability that R is a random walk with $\mathrm{Geom}(q)$ increments.

Recall

$$R_1 \otimes R_2(k) = \min_{1 \leqslant j \leqslant k} \{R_1(j-1) + R_2(k) - R_2(j)\}.$$

Consider the probability measure

$$\mathbb{P}_{a,c_1,c_2}(R_1, R_2) = \frac{1}{Z}(c_1 c_2)^{-R_1 \otimes R_2(N)} \times \mathbb{P}_{ac_2}^{\mathrm{RW}}(R_1) \times \mathbb{P}_{ac_1}^{\mathrm{RW}}(R_2).$$

Theorem ([B.-Corwin-Yang 2023])

For any parameters a, c_1, c_2, the marginal law of R_1 under $\mathbb{P}_{a,c_1,c_2}$ is the unique stationary measure of the Markov process G_t.

Scaling limit

In the large N limit, the stationary measure converge to a universal limit, the stationary measure of the *open KPZ fixed point* [B.-Le Doussal 2021]:

It is the marginal B_1 of a couple (B_1, B_2) distributed as

$$\mathbb{P}_{d_1,d_2}(B_1, B_2) = \frac{1}{\mathcal{Z}} e^{(d_1+d_2)B_1 \otimes B_2(L)} \times \mathbb{P}^{\mathrm{Brown}}_{-d_2}(B_1) \times \mathbb{P}^{\mathrm{Brown}}_{-d_1}(B_2)$$

where $\mathbb{P}^{\mathrm{Brown}}_{d}(B)$ is the measure of the Brownian motion on $[0, L]$ with drift d, and

$$B_1 \otimes B_2(t) = \min_{0 \leqslant s \leqslant t} \{B_1(s) + B_2(t) - B_2(s)\}.$$

This probability measure was studied by [Yor, Hariya, Matsumoto, Donati-Martin, etc.] in the early 2000.

A variant of the Schur process

The random walks R_1, R_2 are related to a sequence of interlaced ~~partitions~~ signatures

$$\boldsymbol{\lambda} = \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(N)}$$

where $\lambda^{(j)} = (\lambda_1^{(j)} \geqslant \lambda_2^{(j)}) \in \mathbb{Z}^2$ by

$$R_1(j) = \lambda_1^{(j)} - \lambda_1^{(0)}, \quad R_2(j) = \lambda_2^{(j)} - \lambda_2^{(0)}$$

when $\boldsymbol{\lambda}$ is distributed as

$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{Z_{a,c_1,c_2}(N)} c_1^{\lambda_1^{(0)} - \lambda_2^{(0)}} c_2^{\lambda_1^{(N)} - \lambda_2^{(N)}} \prod_{j=1}^{N} s_{\lambda^{(j)}/\lambda^{(j-1)}}(a_i)$$

and $s_{\lambda/\mu}$ denote skew Schur functions

$$s_{\lambda/\mu}(x) = \mathbb{1}_{\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2} x^{\lambda_1 + \lambda_2 - \mu_1 - \mu_2}.$$

- The construction is similar to the free boundary Schur process [Betea-Bouttier-Nejjar-Vuletic 2017] except that the $\lambda^{(j)}$ are no longer integer partitions.
- The measure $\mathbb{P}$ is infinite but becomes a probability measure if we fix $\lambda_2^{(0)} = 0$.

Markov chain (second definition)

The kernel

$$q(\mu, \lambda) = (1-a)^2 s_{\lambda/\mu}(a)$$

can be interpreted as a random walk on $\mathbb{Z}^2$ killed when $\mu \not\prec \lambda$. Then, one may define some function $h_{t,N}(\lambda)$ and a time-inhomogeneous kernel

$$p_t(\mu, \lambda) = q(\mu, \lambda) \frac{h_{t,N}(\lambda)}{h_{t-1,N}(\mu)}$$

so that the process $\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(N)}$ has the same distribution as the Markov chain with kernel p_t and initial condition

$$p_0(\lambda) = c_1^{\lambda_1 - \lambda_2} h_{0,N}(\lambda).$$

Limit $N \to \infty$ [B. 2024]

As $N \to \infty$

$$h_{t,N}(\lambda) \to h(\lambda) = (1-c_1)^2 (\lambda_1 - \lambda_2 + 1)$$

so that the process is a Doob transformed random walk (studied in [O'Connell 2003] in more general context).

Formulas (third definition)

Properties of Schur functions yield explicit formulas [B. 2024]:

$$\mathbb{E}\left[t^{2R_1(N)}\right] = \frac{1}{Z_{a,c_1,c_2}(N)} \oint \frac{dz}{2\mathbf{i}\pi z} \left| \frac{1-z^2}{(1-zc_1/t)(1-zc_2 t)} \left(\frac{1-a}{1-atz}\right)^N \right|^2.$$

More generally, for $0 = x_0 < \cdots < x_k = N$, there is a simple formula for

$$\mathbb{E}\left[\prod_{i=1}^{k} t_i^{2(R_1(x_i)-R_1(x_{i-1}))}\right]$$

In particular, one can deduce that starting from the stationary initial condition,

$$\mathbb{E}[G(n,n)] = n \times v(a, c_1, c_2, N)$$

where

$$v(a, c_1, c_2, N) = \frac{Z_{a,c_1,c_2}(N+1) - Z_{a,c_1,c_2}(N)}{Z_{a,c_1,c_2}(N)}$$

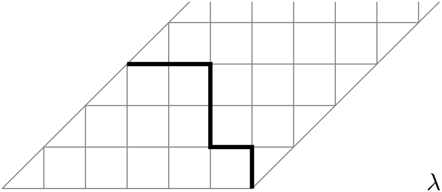
with

$$Z_{a,c_1,c_2}(N) = \oint \frac{dz}{2\mathbf{i}\pi z} \left| \frac{1-z^2}{(1-zc_1)(1-zc_2)} \left(\frac{1-a}{1-az}\right)^N \right|^2.$$

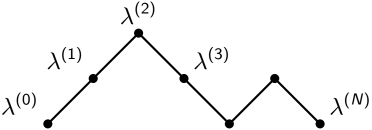
Two ideas in the proof:

1. Decompose the dynamics with elementary moves.
2. View LPP as a marginal of Markovian dynamics in a larger state space.

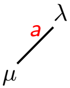
More general two-layer Schur process

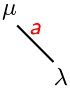


$$\boldsymbol{\lambda} = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(N)}\right)$$



Vertices on the path are decorated by signatures $\lambda = (\lambda_1 \geqslant \lambda_2) \in \mathbb{Z}^2$. We define a probability measure on $\boldsymbol{\lambda}$ by taking the product of Boltzmann weights

wt (

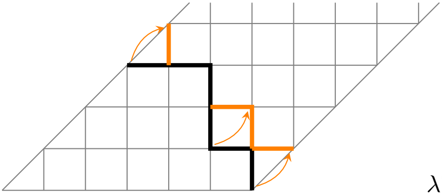
) $= s_{\lambda/\mu}(a)$, wt (

) $= s_{\mu/\lambda}(a)$

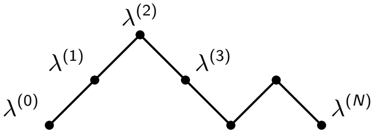
$$\mathbb{P}(\boldsymbol{\lambda}) = \frac{1}{Z_{a,c_1,c_2}(N)} c_1^{\lambda_1^{(0)} - \lambda_2^{(0)}} c_2^{\lambda_1^{(N)} - \lambda_2^{(N)}} \prod_{\text{edges } e} \mathrm{wt}(e)$$

When $c_1 c_2 < 1$, this is a well-defined probability measure.

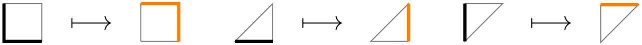
Dynamics on the two-layer Schur process



$$\boldsymbol{\lambda} = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(N)}\right)$$



We construct dynamics on $\boldsymbol{\lambda}$ such that when the path evolves by the elementary moves



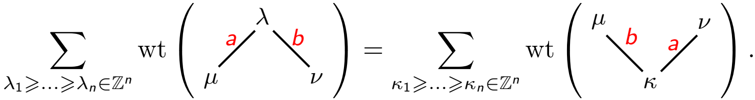
1. The two-layer Schur process is mapped to a two layer Schur process;
2. the λ_1 marginal of the dynamics corresponds to the recurrence of geometric LPP.

Step 1: dynamics preserving two-layer Schur processes

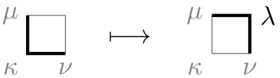
Schur functions satisfy (this is not the usual Cauchy identity)

$$\sum_{\lambda_1 \geqslant \ldots \geqslant \lambda_n \in \mathbb{Z}^n} s_{\lambda/\mu}(a) s_{\lambda/\nu}(b) = \sum_{\kappa_1 \geqslant \ldots \geqslant \kappa_n \in \mathbb{Z}^n} s_{\mu/\kappa}(b) s_{\nu/\kappa}(a),$$

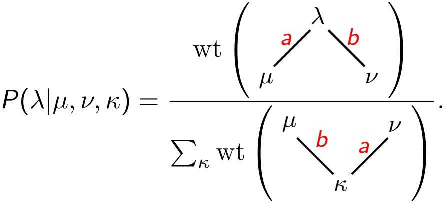
which can be interpreted as



For the dynamics



choose the transition kernel [Borodin-Ferrari 2008, Diaconis-Fill 1980],



Step 2: Marginal distribution

Under the transition kernel $P(\lambda|\mu, \nu, \kappa)$, it is easy to see that if we average over λ_2,

$$\lambda_1 = \max\{\mu_1, \nu_1\} + \mathrm{Geom}(ab),$$

exactly as the recurrence

$$G(n, m) = \max\{G(n-1, m), G(n, m-1)\} + \mathrm{Geom}(ab).$$

satisfied by last passage percolation times.

$\Longrightarrow$ the λ_1 marginal of the two-layer Schur process is stationary for geometric LPP.

Generalizations

The method works as long as we have families of functions satisfying

$$\sum_{\lambda \in \mathbb{X}} f_{\lambda/\mu}(a) g_{\lambda/\nu}(b) = \sum_{\kappa \in \mathbb{X}} g_{\mu/\kappa}(b) f_{\nu/\kappa}(a)$$

and

$$\sum_{\lambda \in \mathbb{X}} f_{\mu/\lambda}(a) c_\lambda = \sum_{\lambda \in \mathbb{X}} g_{\lambda/\mu}(a) c_\lambda.$$

There are many examples of $f_{\lambda/\mu}, g_{\lambda/\nu}$ satisfying the conditions:

- class-one $\mathfrak{gl}_n(\mathbb{R})$-Whittaker functions [B.-Corwin-Yang 2023]
  $\rightsquigarrow$ log-gamma polymer, KPZ equation
- Hall-Littlewood polynomials [Bufetov-Matveev 2018]
  $\rightsquigarrow$ stochastic six-vertex model, ASEP [In progress...]
- Partition functions of vertex models satisfying a **Yang-Baxter equation** (+ a boundary YBE)
  $\rightsquigarrow$ many solvable models in integrable probability.

Conclusion

Summary

The two-layer Schur processes allows to describe the stationary measure of LPP in a strip in terms of reweighted random walks/Markov chains/formulas.

Key idea: view LPP as a marginal of a more general Markov process whose stationary measure is a Gibbs measure.

Outlook

- The method applies to other families of symmetric functions
- Beyond the stationary measure?

Thank you for your attention!