ASEP with boundary, stationary measures and Markov duality

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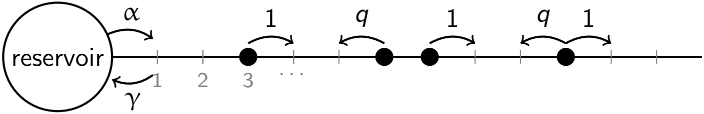
Recent Developments in Stochastic Duality

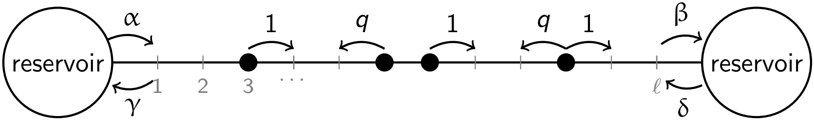
ASEP with boundary

This talk is about asymmetric simple exclusion process with boundaries:

- ASEP on $\mathbb{Z}_{\geqslant 1}$



- Open ASEP on a segment $\{1,\ldots,\ell\}$



These processes were first studied by [Liggett 1975], who imposed conditions on the rates:

$$\gamma = q(1-\alpha), \quad \delta = q(1-\beta)$$

so that the density parameters are related to the jump rates by

$$\varrho_0 = \alpha, \quad 1-\varrho_\ell = \beta.$$

Liggett's condition

It can be reformulated as imposing that at the left boundary,

$$\alpha = \varrho_0, \quad \gamma = q(1 - \varrho_0)$$

while at the right boundary

$$\beta = 1 - \varrho_\ell, \quad \delta = q\varrho_\ell.$$

> *"The creation and destruction rates at one depend on only one parameter $\varrho_0 \in [0, 1]$, and are chosen in such a way that they can be viewed as arising from a fictitious state 0 with the property that $\eta_0(t) = 1$ with probability ϱ_0 and $\eta_0(t)$ is independent of the process $\{\eta_x(t), x \in \mathbb{Z}_{\geqslant 1}\}$ for all $t > 0$."*
>
> *Ergodic Theorems for the asymmetric exclusion process*, Trans. A.M.S., 1975.

- It is not very clear what this sentence means...
- The condition is such that the truncation of an unbounded system behaves very similarly as the bounded system. This is useful to study stationary measures and convergence towards them.

Outline

Two problems get much simpler under Liggett's condition:

1. Markov duality for half-line ASEP and open ASEP on a segment.
   Joint work with Ivan Corwin
   Application: computing moments of the half-line KPZ equation.

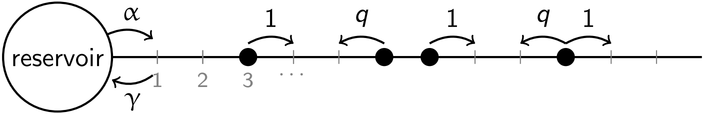
2. Stationary measure for open ASEP (works as well for half-line ASEP)
   Joint work with Pierre Le Doussal.
   Application: Stationary measures of the open KPZ equation.

Part 1: Markov duality for ASEP with boundaries

Coordinate Bethe ansatz

and the KPZ equation

half-line ASEP



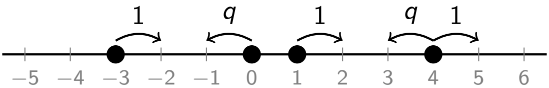
Recall that we impose Liggett's condition

$$\gamma = q(1-\varrho), \quad \alpha = \varrho.$$

The system is described by $\eta = (\eta_1, \eta_2, \dots) \in \{0,1\}^{\mathbb{Z}_{\geqslant 1}}$. The generator is

$$\mathcal{L}^{\mathbb{Z}_{>0}} f(\eta) = \alpha(1-\eta_1)\left(f(\eta^+) - f(\eta)\right) + \gamma\eta_1\left(f(\eta^-) - f(\eta)\right) + \sum_{x=1}^{\infty} \left(\eta_x(1-\eta_{x+1}) + q\eta_{x+1}(1-\eta_x)\right)\left(f(\eta^{x,x+1}) - f(\eta)\right).$$

Full-space ASEP duality



Define

$$N_x(\eta) = \sum_{i=x}^{+\infty} \eta_x, \quad Q_x(\eta) = q^{N_x(\eta)}, \quad \widetilde{Q}_x(\eta) = \eta_x q^{N_{x+1}(\eta)}$$

Define the operator, acting on functions of $x_1 < \cdots < x_n$,

$$\mathcal{D}^{(n)} f(\vec{x}) = \sum_{\substack{1 \leqslant i \leqslant n \\ x_i - x_{i-1} > 1}} \left(f(\vec{x}_i^-) - f(\vec{x}) \right) + \sum_{\substack{1 \leqslant i \leqslant n \\ x_{i+1} - x_i > 1}} q \left(f(\vec{x}_i^+) - f(\vec{x}) \right),$$

which describes a n-particle ASEP with exchanged jump parameters.

Theorem ([Borodin-Corwin-Sasamoto 2012])

Let $H(\eta, \vec{x}) = \prod_{i=1}^n Q_{x_i}(\eta)$. Then we have the Markov duality

$$\mathcal{D}^{(n)} H(\eta, \vec{x}) = \mathcal{L}^{\mathbb{Z}} H(\eta, \vec{x}),$$

where $\mathcal{L}^{\mathbb{Z}}$ is the generator of ASEP on $\mathbb{Z}$.

First proved for $\widetilde{Q}_x$ by [Schütz 1997].

Fictitious site

$$\mathcal{L}^{\mathbb{Z}_{>0}} f(\eta) = \varrho\,(1-\eta_1)\left(f(\eta^+) - f(\eta)\right) + q(1-\varrho)\,\eta_1\left(f(\eta^-) - f(\eta)\right)$$
$$+ \sum_{x=1}^{\infty} \left(\eta_x(1-\eta_{x+1}) + q\eta_{x+1}(1-\eta_x)\right)\left(f(\eta^{x,x+1}) - f(\eta)\right).$$

Introduce a fictitious site η_0, Bernoulli random variable with proba ϱ, so that $E[\eta_0] = \varrho$. Then, for $\eta \in \{0,1\}^{\mathbb{Z}_{\geqslant 1}}$,

$$\mathcal{L}^{\mathbb{Z}_{>0}} f(\eta) = E\left[\mathcal{L}^{\circ} f(\eta)\right],$$

where

$$\mathcal{L}^{\circ} f(\eta) = \sum_{x=0}^{\infty} \left(\eta_x(1-\eta_{x+1}) + q\eta_{x+1}(1-\eta_x)\right)\left(f(\eta^{x,x+1}) - f(\eta)\right),$$

that is the generator of ASEP on $\mathbb{Z}_{\geqslant 0}$ with closed boundary conditions.
This is the correct meaning of Liggett's sentence...

By the full space ASEP duality, for $1 \leqslant x_1 < \cdots < x_n$,

$$\mathcal{L}^{\circ} H(\eta, \vec{x}) = \mathcal{D}^{(n)} H(\eta, \vec{x}),$$

so that

$$\mathcal{L}^{\mathbb{Z}_{>0}} H(\eta, \vec{x}) = E\left[\mathcal{D}^{(n)} H(\eta, \vec{x})\right].$$

Markov duality for half-line ASEP

Define the operator acting on functions of $1 \leqslant x_1 < \cdots < x_n$

$$\mathcal{D}^{(n,\varrho)} f(\vec{x}) = \sum_{\substack{2 \leqslant i \leqslant n \\ x_i - x_{i-1} > 1}} \left(f(\vec{x}_i^-) - f(\vec{x}) \right) + \sum_{\substack{1 \leqslant i \leqslant n \\ x_{i+1} - x_i > 1}} q \left(f(\vec{x}_i^+) - f(\vec{x}) \right)$$
$$+ \mathbb{1}_{x_1 > 1} \left(f(\vec{x}_1^-) - f(\vec{x}) \right) - \mathbb{1}_{x_1 = 1} (1-q) \varrho f(\vec{x}),$$

This describes an ASEP with n particles where the first particle is killed at rate $(1-q)\varrho$ when $x_1 = 1$.

Theorem ([B.-Corwin 2022])

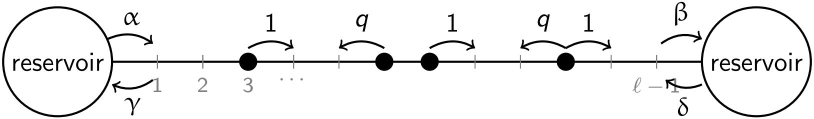
For any $\eta \in \{0,1\}^{\mathbb{Z}_{\geqslant 1}}$ and $1 \leqslant x_1 < \cdots < x_n$,

$$\mathcal{L}^{\mathbb{Z}_{>0}} H(\eta, \vec{x}) = \mathcal{D}^{(n,\varrho)} H(\eta, \vec{x}),$$

where we recall that $\mathcal{L}^{\mathbb{Z}_{>0}}$ is the generator of half-line ASEP and acts on functions of the variable η.

Other half-line duality results exist [Schütz, previous talk] [Kuan 2021] [Ohkubo 2017]. Similar to [Giardinà-Kurchan-Redig 2007] with asymmetry.

Duality for open ASEP on a segment



Theorem ([B.-Corwin, 2022])

Assume Liggett's condition. For any $\eta \in \{0,1\}^{[\![1,\ell-1]\!]}$ *and* $1 \leqslant x_1 < \cdots < x_n \leqslant \ell - 1$,

$$\mathcal{L}^{[\![1,\ell-1]\!]} H(\eta, \vec{x}) = \mathcal{D}^{(n,\varrho_0,\varrho_\ell)} H(\eta, \vec{x}).$$

$$\mathcal{D}^{(n,\varrho_0,\varrho_\ell)} f(\vec{x}) = \sum_{\substack{2\leqslant i\leqslant n\\ x_i - x_{i-1} > 1}} \left(f(\vec{x}_i^{\,-}) - f(\vec{x})\right) + \sum_{\substack{1\leqslant i\leqslant n-1\\ x_{i+1} - x_i > 1}} q\left(f(\vec{x}_i^{\,+}) - f(\vec{x})\right)$$
$$+ \mathbb{1}_{x_1>1}\left(f(\vec{x}_1^{\,-}) - f(\vec{x})\right) - \mathbb{1}_{x_1=1}(1-q)\varrho_0 f(\vec{x})$$
$$+ \mathbb{1}_{x_n<\ell}\, q\left(f(\vec{x}_1^{\,+}) - f(\vec{x})\right) + \mathbb{1}_{x_n=\ell}(1-q)\varrho_\ell f(\vec{x}).$$

This is the "generator" of a non-stochastic system of n particles on $[\![1,\ell]\!]$ with some killing/duplication at the boundaries.

System of ODEs

Assume Liggett's condition. The function

$$u(t,\vec{x}) = \mathbb{E}\left[H(\eta(t),\vec{x})\right] = \mathbb{E}\left[\prod_{i=1}^{n} q^{N_{x_i}(\eta(t))}\right]$$

is given for $x_1 < \cdots < x_n$ by the solution to:

1. (System of ODEs) For all $\vec{x} \in \mathbb{Z}^n_{\geqslant 1}$,

$$\frac{d}{dt} u(t;\vec{x}) = \Delta^{1,q} u(t;\vec{x}),$$

$$\text{where } \Delta^{1,q} f(\vec{x}) = \sum_{i=1}^{n} f(\vec{x}_i^{-}) + q f(\vec{x}_i^{+}) - (1+q) f(\vec{x});$$

2. (2-body boundary condition) If $x_{i+1} = x_i + 1$,

$$u(t;\vec{x}_{i+1}^{-}) + q u(t;\vec{x}_i^{+}) = (1+q) u(t;\vec{x});$$

3. (Boundary condition at $x_1 = 1$)

$$u(t;0,x_2,\dots) = (\varrho q + 1 - \varrho) u(t;1,x_2,\dots);$$

4. (Initial condition)

$$u(0;\vec{x}) = H(\eta(0),\vec{x}) = \prod_{i=1}^{n} q^{N_{x_i}(\eta(0))}.$$

Convergence ASEP $\to$ KPZ

Let

$$\mathcal{Z}_t(x) = q^{N_x(t)+\frac{x}{2}-t(1-\sqrt{q}^2)}$$

For $q = 1-\varepsilon$, when $\varepsilon \to 0$, [Bertini-Giacomin 1997] showed that

$$\mathcal{Z}_{\epsilon^{-4}t}(\epsilon^{-2}x) \Longrightarrow Z(t,x),$$

the solution of the stochastic PDE

$$\partial_t Z(t,x) = \tfrac{1}{2}\Delta Z(t,x) + Z(t,x)\,\xi(t,x),$$

where ξ is a space-time white noise. Then, $h(t,x) = \log Z(t,x)$ is, by definition, a solution to the [Kardar-Parisi-Zhang 1986] equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}((\partial_x h(t,x))^2 + \xi(t,x)$$

Moments of the half-line KPZ equation

Half-line ASEP converges to the KPZ equation on $\mathbb{R}_{>0}$ with boundary condition [Corwin-Shen 2016]

$$\partial_x h(t,x)\Big|_{x=0} = u,$$

Let $Z(t,x) = e^{h(t,x)}$. Solving explicitly the system of ODEs and letting $q \to 1$, we obtain

Theorem ([B.-Corwin, 2022])

For any $0 \leqslant x_1 \leqslant \ldots \leqslant x_n$,

$$\mathbb{E}\left[\prod_{i=1}^{n} Z(t,x_i)\right] = 2^n \int_{r_1+\mathbf{i}\mathbb{R}} \frac{dz_1}{2\mathbf{i}\pi} \cdots \int_{r_n+\mathbf{i}\mathbb{R}} \frac{dz_n}{2\mathbf{i}\pi}$$

$$\prod_{i<j} \frac{z_i - z_j}{z_i - z_j + 1} \frac{z_i + z_j}{z_i + z_j - 1} \prod_{i=1}^{n} e^{\frac{tz_i^2}{2} - xz_i} \frac{z_i}{u + 1/2 + z_i}$$

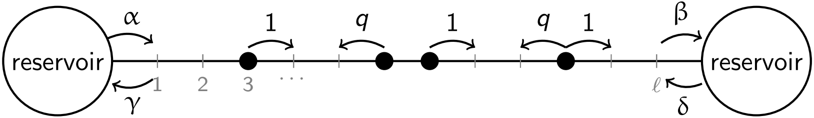
where $0 = r_1 < r_2 - 1 < \cdots < r_n - n + 1$.

The formula was conjectured earlier [Borodin-Bufetov-Corwin 2014].

Part 2: Stationary measures of open ASEP

Matrix product ansatz

Consider ASEP on $\{0,1\}^\ell$ with boundary parameters $\alpha, \beta, \gamma, \delta$.



We describe the state of the system by $\eta \in \{0,1\}^\ell$. The stationary measure $\mathbb{P}$ can be written as [Derrida-Evans-Hakim-Pasquier 1993]

$$\mathbb{P}(\eta) = \frac{1}{Z_\ell} \langle w| \prod_{i=1}^{\ell} (\eta_i D + (1-\eta_i)E) |v\rangle$$

where

$$Z_\ell = \langle w| (E+D)^\ell |v\rangle$$

and E, D are infinite matrices, and $\langle w|, |v\rangle$ are row/column vectors such that

$$DE - qED = D + E$$
$$\langle w| (\alpha E - \gamma D) = \langle w|$$
$$(\beta D - \delta E) |v\rangle = |v\rangle$$

Representations of the MPA

- Finding representations, i.e. matrices E, D and explicit vectors u, v satisfying the relations, is non trivial. Special cases are worked out in [Derrida-Evans-Hakim-Pasquier 1993].
- For TASEP, $q = \gamma = \delta = 0$, we may take

$$D = \begin{pmatrix} 1 & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ 0 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

  and easily find eigenvectors $\langle w| , |v\rangle$.
- [Sandow, 1995] proposed a representation in the most general case. The vectors $\langle w| , |v\rangle$ are complicated.
- Several families of orthogonal polynomials appear. In the most general case, [Uchiyama-Sasamoto-Wadati, 2003] found a representation using Askey-Wilson orthogonal polynomials.
- Another very simple representation was proposed in [Enaud-Derrida, 2003]

Enaud-Derrida's representation

Enaud-Derrida found a very simple representation for any parameters $q, \alpha, \beta, \gamma, \delta$. Under Liggett's condition, it becomes :

$$D = \begin{pmatrix} [1]_q & [1]_q & 0 & 0 & 0 & \cdots \\ 0 & [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & 0 & [3]_q & [3]_q & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}, \quad E = \begin{pmatrix} [1]_q & 0 & 0 & 0 & \cdots \\ [2]_q & [2]_q & 0 & 0 & \cdots \\ 0 & [3]_q & [3]_q & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix}$$

where $[n]_q = \frac{1-q^n}{1-q}$.

Denoting by $\{|n\rangle\}_{n\geqslant 1}$ the vectors of the associated basis, let

$$\langle w| = \sum_{n\geqslant 1} \left(\frac{1-\varrho_0}{\varrho_0}\right)^n \langle n|, \qquad |v\rangle = \sum_{n\geqslant 1} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^n [n]_q \, |n\rangle .$$

Then, $E, D, \langle w|, |v\rangle$ satisfy

$$DE - qED = D + E$$
$$\langle w| (\alpha E - \gamma D) = \langle w|$$
$$(\beta D - \delta E) |v\rangle = |v\rangle$$

Sum over paths

Due to the bidiagonal structure, the normalization constant $Z_\ell = \langle w| (D+E)^\ell |v\rangle$ can be written as a sum over lattice paths $\vec{n} = (n_0, n_1, \dots, n_\ell) \in \mathbb{N}^\ell$ of the form

$$Z_\ell = \sum_{\vec{n}} \Omega(\vec{n})$$

where

$$\Omega(\vec{n}) = \left(\frac{1-\varrho_0}{\varrho_0}\right)^{n_0} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^{n_\ell} \prod_{i=1}^{\ell} v(n_{i-1}, n_i) \prod_{i=0}^{\ell} [n_i]_q,$$

with

$$v(n,n') = \begin{cases} 2 & \text{if } n = n', \\ 1 & \text{if } |n-n'| = 1 \\ 0 & \text{else.} \end{cases}$$

- This introduces a natural probability measure on random walk paths $\vec{n}$. The stationary measure $\mathbb{P}(\eta)$ can be recovered from this measure.

Open ASEP invariant measure

Following arguments similar as [Derrida-Enaud-Lebowitz 2004], one arrives at

Theorem ([B.-Le Doussal 2022])

Under the stationary measure $\mathbb{P}(\tau)$, ASEP height function $H(x) = \sum_{j=1}^{x}(2\eta_i - 1)$ is such that

$$(H(i))_{1\leqslant i\leqslant \ell} \overset{(d)}{=} (n_i - n_0 + m_i)_{1\leqslant i\leqslant \ell}\,,$$

where $(n_i, m_i)_{0\leqslant i\leqslant \ell}$ is a two dimensional random walk on $\mathbb{Z}^2$, starting from $(n_0, 0)$, distributed as

$$P(\vec{n}, \vec{m}) = \frac{\mathbb{1}_{n_0>0}}{4^{-\ell} Z_\ell} \left(\frac{1-\varrho_0}{\varrho_0}\right)^{n_0} \left(\frac{\varrho_\ell}{1-\varrho_\ell}\right)^{n_\ell} \prod_{i=0}^{\ell} [n_i]_q \times P^{SSRW}_{n_0,0}(\vec{n}, \vec{m}),$$

where $P^{SSRW}_{n_0,0}$ denotes the probability measure of the symmetric simple random walk (SSRW) on $\mathbb{Z}^2$ starting from $(n_0, 0)$.

Scaling limit to the KPZ equation

Under the scalings such that ASEP's height function converges to KPZ, in particular

$$q = 1 - \varepsilon, \ \ \ell = \varepsilon^{-2}, \ \ \varrho_0 = \frac{1}{2}(1 + u\varepsilon), \ \ \varrho_\ell = \frac{1}{2}(1 - v\varepsilon)$$

we find, denoting by Y_x the rescaled version of the random walk n_i

$$\prod_{i=0}^{\ell} [n_i]_q \to e^{-\int_0^L e^{-2Y_s} ds}$$

$$\left(\frac{1-\varrho_a}{\varrho_a}\right)^{n_0} \left(\frac{\varrho_b}{1-\varrho_b}\right)^{n_\ell} \to e^{-2uY_0 - 2vY_L}$$

so that

$$(m_i, n_i) \Longrightarrow (W_x, Y_x)$$

where W_x is a Brownian motion and Y_x is absolutely continuous to the Brownian measure with Radon Nikodym derivative

$$\frac{1}{\mathcal{Z}_{u,v}} e^{-2uY_0 - 2vY_L} e^{-\int_0^L e^{-2Y_s} ds}.$$

Liouville field theory in dimension 1

Theorem

The KPZ equation on $[0, L]$ with boundary parameters u and v with $u + v > 0$ has a unique stationary measure

$$h^L_{u,v}(x) = W_x + Y_x - Y_0,$$

where

- *W is a Brownian motion,*
- *Y is independent from W, and its law is absolutely continuous w.r.t. to that of a Brownian motion with free starting point. The Radon-Nikodym derivative is*

$$\frac{1}{\mathcal{Z}_{u,v}} \exp\left(-2uY_0 - 2vY_L - \int_0^L e^{-2Y_s} ds\right)$$

It was originally proved by [Bryc-Kuznetsov-Wang-Wesołowski 2021], [B.-Le Doussal 2021] using results from [Corwin-Knizel 2021]. Uniqueness was later proved by [Knizel-Matetski 2022].

Conclusion

Half-line ASEP is Markov dual to a n-particles ASEP on the half-line with killing at the boundary. This leads to a system of ODEs for q-moments of the current, that is solvable by (a variant of) coordinate Bethe ansatz.

Stationary measures for open ASEP on a segment are given by reweighted random walks. In the KPZ limit, we obtain a one dimensional analogue of Liouville CFT.

Thank you