Kardar-Parisi-Zhang universality class and integrable probability

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Part I : Kardar-Parisi-Zhang universality

1. Universality and Integrability
2. KPZ universality class and equation
   (initially modelling random growth)
3. Other examples
   (optimization problems, random paths in random environment, tilings, stochastic six-vertex model)
4. Introduction to integrable probability

Universality

- The large scale behavior of random complex systems is often universal, meaning that it does not depend on the details of how these systems are defined microscopically.
- We expect them to converge to universal scale-invariant fixed point (depending on the characteristic scales at which randomness occurs and mild properties of the interactions).
- Models sharing the same large scale properties form **universality classes**.

Integrability

- Probing the properties of these conjectural fixed points can be performed via special models within these classes.
- These exactly solvable or integrable models possess an underlying algebraic structure which allows for precise computations.

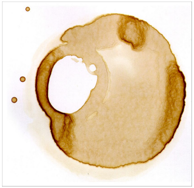
Examples of universality in probability

- **Gaussian universality class.** The sum of independent random variables, properly rescaled, converge to a Gaussian. The Brownian motion is the universal scale invariant process (under diffusive scaling). For space-time processes, the universal fixed point is the stochastic heat equation.
- **Extreme value statistics.** The maximum of independent random variables converge to the Fréchet, Weibull or Gumbel (depending only on the tail distribution of the random variables).
- **Random matrices** For matrices of large size, most spectral properties are independent on the entries distribution.

Deposition of material

Coffee stains

In a coffee stain, before it is dry, particle diffuse and eventually stick to the boundary. This is why the edges of the stain are darker [Yunker-Lohr-Still-Borodin-Durian-Yodh 2013].

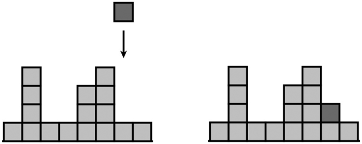


Propagation phenomena

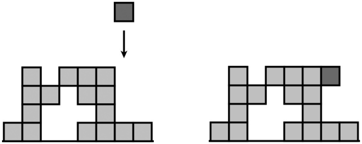
A mathematical model of deposition

Blocks fall on a one dimensional flat substrate :

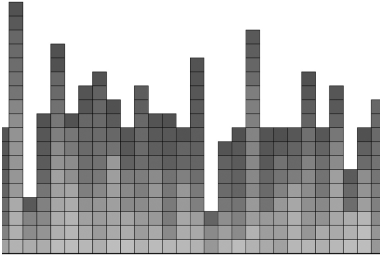
Independently on each column



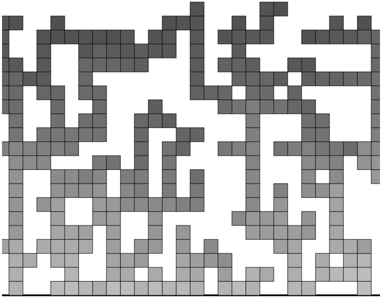
Blocks have sticky edges



After some time



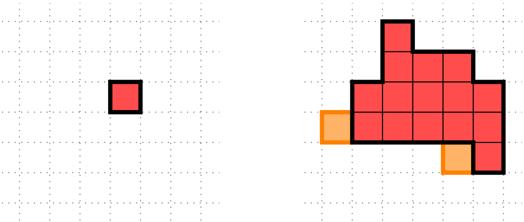
After some time



After a long time

Simulation by [T. Halpin-Healy]. The interface becomes quite smooth, with strong spatial correlations. Mathematical analysis of the model is an open problem.

A simple model of propagation



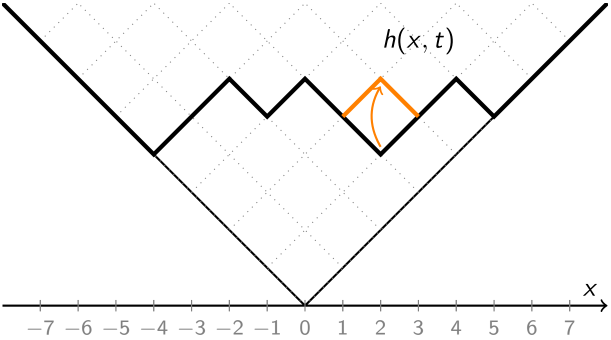
Eden model

- Start with a unit square in the $\mathbb{Z}^2$ lattice
- Add the squares in the border of the cluster randomly at exponential rate 1.
- This is similar to model of a **random metric**. Assign random distances to each edge of the $\mathbb{Z}^2$ lattice and look at the ball of radius t.

This model is still too complicated !

An integrable model : corner growth model

- Consider an interface $h(x, t)$, (where $x \in \mathbb{Z}$) starting from $h(x, 0) = |x|$, and add unit boxes at rate 1 in every valley.



- The interface is mapped to an interacting particle system on $\mathbb{Z}$ called TASEP.

Corner growth model $\Longleftrightarrow$ TASEP

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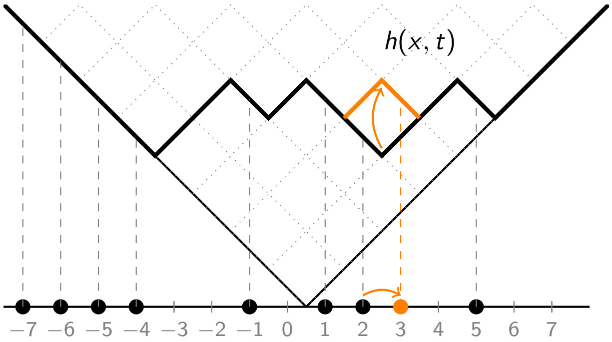


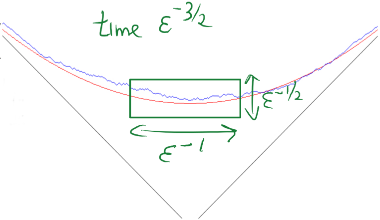
- The interface is mapped to an interacting particle system on $\mathbb{Z}$ called TASEP.

$1:2:3$ scaling

The interface $h(x,t)$ grows linearly with t. Fluctuations are of the order $t^{1/3}$ and the fluctuations at $h(x,t)$ and $h(y,t)$ have non-trivial correlations when $|y-x| = O(t^{2/3})$.

Define the rescaled interface

$$H_\varepsilon(x,t) := \frac{h(\varepsilon^{-1}x, \varepsilon^{-3/2}t) - (t/2)\varepsilon^{-3/2}}{(t/2)^{1/3}\varepsilon^{-1/2}}$$



This is called 1 :2 :3 scaling, because letting $L = \varepsilon^{-1/2}$, we consider the fluctuations of $L^{-1}h(L^2x, L^3t)$.

What is the limiting process $\lim_{\varepsilon\to 0} H_\varepsilon(x,t)$?

Theorem (Baik-Deift-Johansson, Johansson 1999)

For the corner growth model with initial condition $h(x,0) = |x|$,

$$\mathbb{P}(H_\varepsilon(0,1) \geqslant -s) \xrightarrow[\varepsilon\to 0]{} F_{\mathrm{GUE}}(s).$$

Tracy-Widom GUE distribution

Define the Airy function

$$\mathrm{Ai}(x) = \int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} e^{z^3/3 - zx} \mathrm{d}z,$$

and the Airy kernel

$$K_{\mathrm{Ai}}(x, y) = \int_0^{+\infty} \mathrm{Ai}(x+s)\mathrm{Ai}(y+s)\mathrm{d}s,$$

Then

$$\begin{aligned} F_{\mathrm{GUE}}(s) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_s^{+\infty} \mathrm{d}x_1 \dots \int_s^{+\infty} \mathrm{d}x_k \det\left(K_{\mathrm{Ai}}(x_i, x_j)\right)_{i,j}^k \\ &= \det(I - K_{\mathrm{Ai}})_{L^2(s,+\infty)} \end{aligned}$$

The Tracy-Widom distribution F_{GUE} governs fluctuations of the largest eigenvalue of large Hermitian matrices.

KPZ universality

- Numerical data shows that the interface of sticky blocks model, Eden model, random metric balls, and many more models has the same fluctuations as in the corner growth model, i.e. Tracy-Widom GUE on the $t^{1/3}$. This depends on the initial condition, GUE arises when starting from a droplet initial shape, and Tracy-Widom GOE fluctuations arise when starting from a flat interface.
- The spatial covariance structure, under the $1:2:3$ scaling is also identical.
- The set of models sharing this scaling behaviour and statistics forms the 1+1 dimensional **Kardar-Parisi-Zhang universality class**.
- For all these models, the rescaled interface

$$H_\varepsilon(x,t) := \frac{h(\varepsilon^{-1}x, \varepsilon^{-3/2}t) - a_{t,x}\varepsilon^{-3/2}}{b_{t,x}\varepsilon^{-1/2}},$$

  where $a_{t,x}, b_{t,x}$ are some model-dependent deterministic constants should converge to a (conjecturally universal) scale invariant space-time random field $\mathcal{H}(x,t)$, called the **KPZ fixed point**.

Turbulent liquid crystals

Liquid Crystals with two phases : light gray in the metastable phase and darker in the stable phase. Initially, crystals are prepared in the metastable phase. After an excitation with a laser in the middle, crystals switch to their stable phase , and the stable phase propagates [Takeuchi-Sano 2010].

Turbulent liquid crystals

Depending on the form of laser beam used for the initial excitation, propagation may start off any initial profile. Here flat.

KPZ universality

It is conjectured that a 1+1 dimensional interface model belongs to the KPZ universality class if it possesses the following (vaguely defined) features :

- **local dynamics**,
- **smoothing mechanism** (deep valleys fill up, peaks erode),
- **radial growth** which is often modeled by a non-linear slope dependent growth rate,
- the system is driven by a noise which decorrelates at large space time scales.

KPZ universality goes far beyond growth processes. One can associate a height field $h(x, t)$ to many statistical physics models.

The KPZ stochastic PDE

- [Kardar-Parisi-Zhang 1986] introduced, as a toy model for interface growth, a stochastic PDE

$$\partial_t h(x,t) = \underbrace{\frac{1}{2}\partial_{xx} h(t,x)}_{\text{smoothing}} + \underbrace{\frac{1}{2}(\partial_x h(x,t))^2}_{\text{non-linear slope dependence}} + \underbrace{\xi(x,t)}_{\text{uncorrelated noise}},$$

  where $\xi(x,t)$ is a Gaussian space-time white noise (see lecture 2). It was predicted to have $t^{1/3}$ fluctuations and $t^{2/3}$ spatial decorrelation scale using (non-rigorous !) renormalization group methods of [Forster-Nelson-Stephen 1977].

- Is it 1 : 2 : 3 scale invariant ? Let $h_\varepsilon(x,t) = \varepsilon^{-1/2} h(\varepsilon^{-1}x, \varepsilon^{-3/2}t)$. Then

$$\partial_t h_\varepsilon(x,t) = \frac{1}{2}\varepsilon^{1/2}\partial_{xx} h_\varepsilon(t,x) + \frac{1}{2}(\partial_x h_\varepsilon(x,t))^2 + \varepsilon^{1/4}\xi(x,t)$$

- The KPZ equation is not invariant under 1 :2 :3 KPZ scaling ! **KPZ equation is not the KPZ fixed point.**

Weak universality

Consider the more general equation

$$\partial_t h(x,t) = \nu\ \partial_{xx} h(t,x) + \lambda\ (\partial_x h(x,t))^2 + \sqrt{D}\xi(x,t)$$

with tunable non-linearity λ and noise variance D. Now consider

$$h_\varepsilon(x,t) = \varepsilon^b h(\varepsilon^{-1}x, \varepsilon^{-z}t).$$

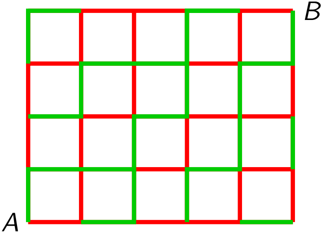
The equation becomes

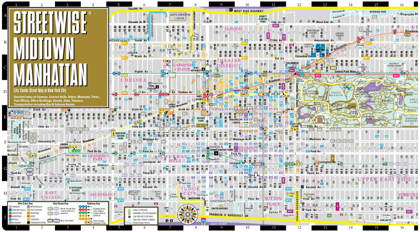
$$\partial_t h_\varepsilon(x,t) = \nu\ \varepsilon^{2-z}\partial_{xx} h_\varepsilon(t,x) + \lambda\ \varepsilon^{2-z-b}(\partial_x h_\varepsilon(x,t))^2 + \sqrt{D}\ \varepsilon^{b-z/2+1/2}\xi(x,t)$$

- **weak non-linearity scaling :** Choose $z=2, b=1/2$ and scale $\lambda = \varepsilon^{1/2}$. We recover the KPZ equation.
- **weak noise scaling :** Choose $z=2, b=0$ and scale $D=\varepsilon$. We recover the KPZ equation.

Models in the KPZ universality class should converge to the KPZ equation under weak non-linearity or weak noise scaling. This is the **weak universality conjecture** (proved for some type of models).

Optimal paths in random environment





- [B.-Corwin 2015] : At each intersection, the configuration is either  or  with probability $1/2$, and the red edge has an exponential waiting time $\mathcal{E}(1)$. All waiting times are independent.
- How long it takes to go from A to B ?
- When the distance d from A to B is large, the minimal passage time behaves as [B.-Corwin 2015]

$$T(A \to B) \approx cd + c'd^{1/3}\chi,$$

where χ is F_{GUE} distributed (under the assumption that the ratio of horizontal/vertical steps is not 1, otherwise the passage time is much smaller).

Random walks in random environment

Let X_t be a random walk on $\mathbb{Z}$, starting from 0, such that when $X_t = x$,

$$X_{t+1} = \begin{cases} x+1 \text{ with probability } p_{x,t}, \\ x-1 \text{ with probability } 1-p_{x,t}. \end{cases}$$

If $p_{x,t} \equiv 1/2$, the model is well-understood. If $p_{x,t}$ are disordered, say independent and uniform in $(0,1)$, then

Theorem (B.-Corwin 2015)

Consider n independent walks in the same environment $X_t^{(1)}, \dots, X_t^{(n)}$, then for $n = e^{ct}$,

$$\max_{i \in \{1,\dots,n\}} \left\{ X_t^{(i)} \right\} \approx c' t + c'' t^{1/3} \chi,$$

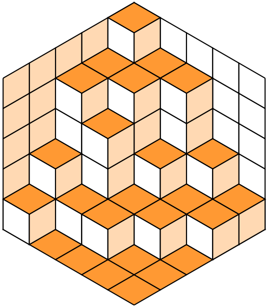
where χ is F_{GUE} distributed.

The statement can be rephrased in terms of large deviations as

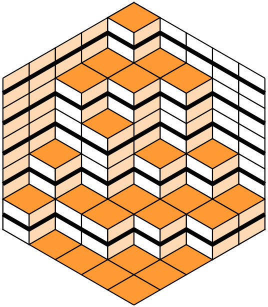
$$-\log \left(\mathbb{P}\left(X_t > xt \middle| \{p_{y,s}\} \right) \right) \approx I(x) \cdot t + c'' \cdot t^{1/3} \cdot \chi.$$

Random tilings

Tile an hexagon using three types of lozenges ,  and  and consider a configuration picked uniformly at random.



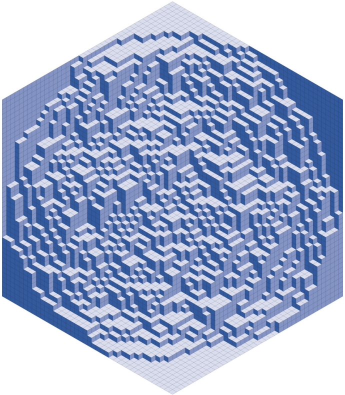
The picture can be seen as a surface.



The configuration can be encoded via an ensemble of non-intersecting paths.

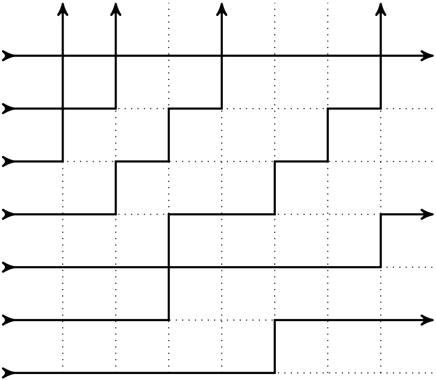
Arctic circle

- At large scale, disordered regions and frozen regions are delimited by a circle, the arctic circle [Cohn-Larsen-Propp 1998].
- Fluctuations of the boundary of the frozen region around the arctic circle are Tracy-Widom GUE distributed on the $n^{1/3}$ scale [Baik-Kriecherbauer-McLaughlin-Miller 2007, Petrov 2012].
- It corresponds to the extremal paths in the non-intersecting paths interpretation. Further, the second highest path converges to the second eigenvalue of the Airy process, and so on.

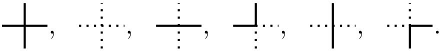


Stochastic six-vertex model

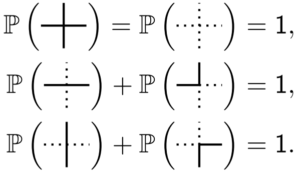
Consider collections of up right paths in the first-quadrant of $\mathbb{Z}^2$, starting from each point of the vertical axis.

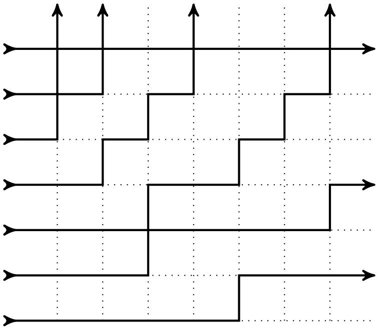


At each vertex, there are **six** types of configurations,

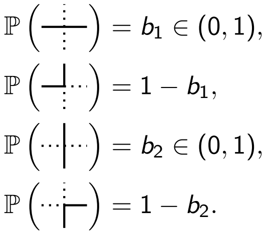


We assign weights to each conf. They are **stochastic** [Gwa-Spohn 1992] in the sense that at every vertex, if we fix the configuration of incoming paths, the weights define a probability distribution on outgoing configurations.

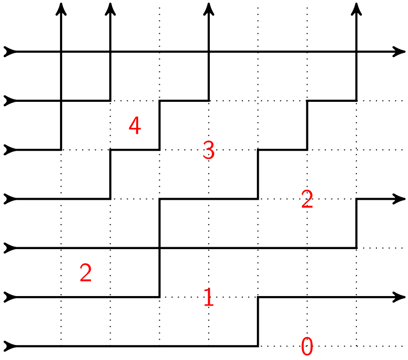




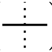
Let $0 < b_2 < b_1 < 1$. We set

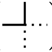


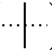
Associate a height function $h(x, y)$ (numbers in red on the picture). The bottom right part has height 0, and each time we cross a path upward or leftward, height increases by 1.

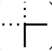


$0 < b_2 < b_1 < 1.$

$\mathbb{P}\Big($  $\Big) = b_1,$

$\mathbb{P}\Big($  $\Big) = 1 - b_1,$

$\mathbb{P}\Big($  $\Big) = b_2,$

$\mathbb{P}\Big($  $\Big) = 1 - b_2.$

Stochastic six-vertex model

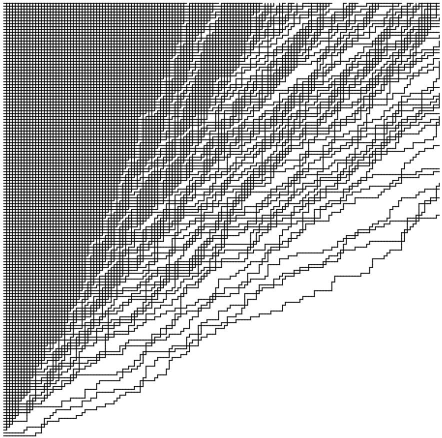
The first path has slope $\frac{1-b_1}{1-b_2}$. By symmetry, disorder is located in a cone between slopes $\frac{1-b_1}{1-b_2}$ and $\frac{1-b_2}{1-b_1}$.

Theorem (Borodin-Corwin-Gorin 2014)

For $\frac{1-b_1}{1-b_2} < \frac{x}{y} < \frac{1-b_2}{1-b_1}$,

$$h(xt, yt) \approx ct + c' t^{1/3} \chi,$$

where χ *is* F_{GUE} *distributed.*



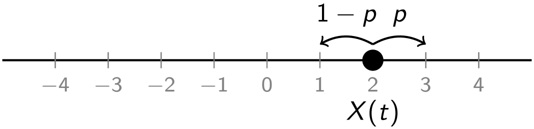
Simulation by [Leo Petrov, https ://lpetrov.cc/research/gallery/]

Integrable probability

- **Integrable probability** is an area of research at the interface between probability theory, mathematical physics, combinatorics and representation theory. It refers to the study of probabilistic models that are exactly solvable.
- The notion of **exact solvability** or **integrability** is somewhat vague. A model is called exactly solvable when observables of interest can be computed by a formula involving well-known functions (rational functions, exp, sin, Γ), so that the complexity of the formula does not increase as parameters go to ∞.

A simple example

Consider the simplest particle system.



- We can associate a difference operator

$$\Delta f(x) = p\ f(x-1) + (1-p) f(x+1),$$

so that $\psi_t(x) := \mathbb{P}(X(t) = x)$ solves $\psi_{t+1}(x) = \Delta \psi_t(x)$.

- For any $z \in \mathbb{C} \setminus \{0\}$, functions $x \mapsto z^x$ are eigenfunctions. We may consider the Fourier transform

$$\hat{f}(z) = \sum_{x \in \mathbb{Z}} f(x) z^x,$$

which can be inverted via

$$f(x) = \frac{1}{2i\pi} \oint_{|z|=1} \hat{f}(z) \frac{\mathrm{d}z}{z^{1+x}}.$$

(isometry between $\ell^2(\mathbb{Z})$ and $\mathbb{L}^2(\mathbb{T}, \frac{\mathrm{d}z}{z})$)

Recall $\psi_0(x) = \mathbb{P}(X(0) = x)$, and consider the identity

$$\psi_0(x) = \frac{1}{2i\pi} \oint_{|z|=1} \widehat{\psi_0}(z) \frac{\mathrm{d}z}{z^{1+x}}.$$

Acting t times with Δ on both sides, we obtain (recall $\psi_{t+1}(x) = \Delta\psi_t(x)$)

$$\mathbb{P}(X(t) = x) = \frac{1}{2i\pi} \oint_{|z|=1} \left(\underbrace{pz + (1-p)z^{-1}}_{\textit{eigenvalue}} \right)^t \underbrace{\sum_{y \in z} z^y \mathbb{P}(X(0) = y)}_{\widehat{\psi_0}(z)} \frac{\mathrm{d}z}{z^{1+x}}.$$

The last expression can be analyzed asymptotically using standard techniques of asymptotic analysis for contour integrals.

More complicated cases

- For more complicated systems, eigenfunctions will not be as simple as z^x but typically functions of many variables $z_1, z_2, z_3, \ldots$
- If we have now several particles jumping on $\mathbb{Z}$ with at most one particle per site and $p = 1$, all of this goes through, but the functions $x \mapsto z^x$ are replaced by so-called Grothendieck polynomials

$$G_{\vec{x}}(z_1, \ldots, z_n) = \frac{\det\left(z_i^{x_j}(1 - 1/z_i)^{1-j}\right)_{i,j=1}^{n}}{\det\left(z_i^{n-j}\right)_{i,j=1}^{n}}.$$

Sources of integrability

- Integrability is often rooted in properties of certain families of **symmetric functions** with many remarkable properties (orthogonality, summation/integral identities, eigenfunctions of certain operators). They can sometimes be interpreted as multivariate Fourier bases.
  Ex : Schur functions.

- In Physics, (quantum) integrability often comes from solutions of a **Yang-Baxter equation** (some sort of commutation relation between operators having a nice pictorial interpretation...)

A large part of modern representation theory is about dealing with those structures.

Summary

- Under mild assumptions, random growth models are in the **Kardar-Parisi-Zhang universality class**, meaning that the interface fluctuates in the $t^{1/3}$ scale with spatial decorrelation in the $t^{2/3}$ scale, and Tracy-Widom fluctuations (for certain initial conditions). All these models are conjectured to converge to a scale invariant universal process called the KPZ fixed point.
- Besides random growth, there are many more type of models in the KPZ class and many more conjectured to be.
- In order to study phenomena occurring in this universality class and refine its scope, one studies **integrable models** such as the corner growth model (TASEP).

Part II : The KPZ fixed point and sample covariance matrices

1. Sample covariance matrices and the Baik-Ben Arous-Péché phase transition
2. Sample covariance matrices and TASEP
3. The KPZ fixed point

Sample covariance matrices

Consider m independent **complex** Gaussian vectors $y_1, \ldots, y_m$ in $\mathbb{R}^n$ with covariance matrix Σ and mean $\vec{\mu}$. Let $\overline{Y} = \frac{1}{m}\sum_{i=1}^{m}$ be the sample mean. Form the matrix

$$X = \left(y_1 - \overline{Y}, \ldots, y_m - \overline{Y}\right)$$

and define the sample covariance matrix

$$S = \frac{1}{m} X X^*.$$

Denote

$$\ell_1 > \cdots > \ell_n > 0$$

the eigenvalues of Σ and

$$\lambda_1 > \cdots > \lambda_n > 0$$

the eigenvalues of S.

Asymptotics : null case

Let m, n go to infinity in such a way that $\frac{m}{n} \to \gamma^2 \geqslant 1$, and assume $\Sigma = I$. The eigenvalue density converges to

$$\frac{\gamma^2}{2\pi x}\sqrt{(b-x)(x-a)}$$

for some explicit a, b. Almost surely

$$\lambda_1 \xrightarrow[n,m\to\infty]{} \left(1+\gamma^{-1}\right)^2.$$

Further,

$$\lim_{m\to\infty} \mathbb{P}\left(\frac{\lambda_1 - \left(1+\gamma^{-1}\right)^2}{cst \cdot m^{-2/3}} \leqslant x\right) = F_{\mathrm{GUE}}(x).$$

BBP transition

Q : What happens if $\Sigma \neq I$? For some r, fix $\ell_{r+1} = \cdots = \ell_n = 1$.

Subcritical case : If $\ell_1 < 1 + \gamma^{-1}$, then noting happens.

Supercritical case : If $\ell_1 > \cdots > \ell_r > 1 + \gamma^{-1}$, then we have r outliers in the spectrum of S. The fluctuations of $\lambda_1, \ldots, \lambda_r$ occur on the $\sqrt{m}$ scale and are distributed as a $k \times k$ GUE, where k it the multiplicity of ℓ_1 in Σ.

Theorem (Baik-Ben-Arous-Péché 2004)

Assume that for all $1 \leqslant i \leqslant r$, we scale the population eigenvalues close to the critical threshold as $\ell_i = 1 + \gamma^{-1} + cst \cdot m^{-1/3} u_i$. Then, for all $x \in \mathbb{R}$,

$$\lim_{m \to \infty} \mathbb{P}\left(\frac{\lambda_1 - \left(1 + \gamma^{-1}\right)^2}{cst \cdot m^{-2/3}} \leqslant x \right) = F_{\mathrm{BBP},\vec{u}}(x),$$

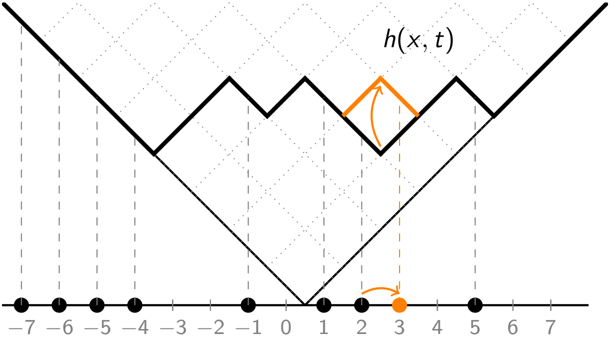
where

$$F_{\mathrm{BBP},\vec{u}}(x) = \det(I - K_{\mathrm{BBP},\vec{u}})_{L^2(x,\infty)},$$

and $K_{\mathrm{BBP},\vec{u}}$ is a deformation of the Airy kernel.

TASEP

- Consider an interface $h(x,t)$, and add unit boxes at rate 1 in every valley.



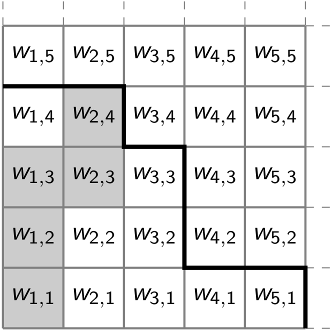
Last passage percolation

- Let $w_{i,j}$ be independent exponential variables with parameter $a_i + b_j$. Define

$$L(n,m) = \max_{\pi} \sum_{(i,j)\in\pi} w_{i,j},$$

  where the sum runs over up-right paths $\pi : (1,1) \to (n,m)$.

- The border of the set of n, m such that $L(n,m) \leqslant t$ defines an interface growth model.



- When $a_i + b_j \equiv 1$, the model is equivalent to TASEP.

An identity

Let X be a $m \times n$ complex random matrix with Gaussian entries $X_{i,j} \sim \mathcal{N}\left(\frac{1}{a_i+b_j}\right)$. Let λ_1 be the largest eigenvalue of XX^*.

Theorem (Borodin-Péché, 2007)

For all $x \in \mathbb{R}$,

$$\mathbb{P}\left(\lambda_1 \leqslant x\right) = \mathbb{P}\left(L(n,m) \leqslant x\right).$$

It can be shown (using IZHC and RSK) that both quantities equal

$$\prod_{i,j}(a_i + b_j) \int_{x \geqslant x_1 \geqslant \cdots \geqslant x_n} S_{\vec{x}}(\vec{a}) S_{\vec{x}}(\vec{b}),$$

where $S_{\vec{x}}(\vec{a})$ is a continuous Schur function

$$S_{\vec{x}}(\vec{a}) = \frac{\det\left(e^{-x_i a_i}\right)_{i,j=1}^{n}}{\det\left(x_i^{n-j}\right)_{i,j=1}^{n}}.$$

BBP transition in LPP

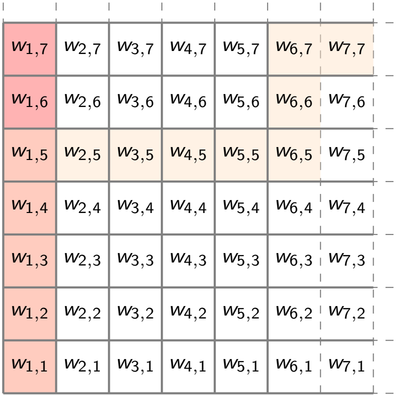
- Let $w_{i,j}$ be independent exponential variables with parameter $a_i + b_j$. Define

$$L(n,m) = \max_{\pi} \sum_{(i,j)\in\pi} w_{i,j},$$

  where the sum runs over up-right paths $\pi : (1,1) \to (n,m)$.
- Set $a_i = b_j \equiv 1$ except

$$a_1 = 2^{-1/3} n^{-1/3} u_1$$

  (so that weights on the first column are larger).

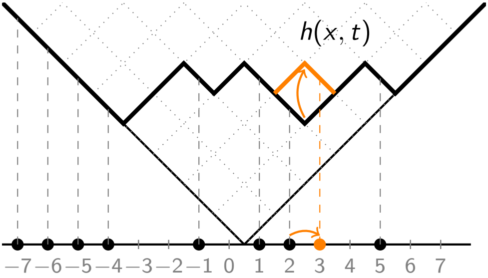


Then,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L(n,n) - 4n}{2^{1/3} n^{1/3}} \leqslant x \right) = F_{\mathrm{BBP},u_1}(x).$$

KPZ fixed point

Let us go back to the interface $h(x,t)$ defined by TASEP.



Definition (Matetski-Quastel-Remenik 2016)

The KPZ fixed point $\mathcal{H}(t,x)$ is a Markov process on upper-semi continuous functions which is the $\varepsilon \to 0$ limit of

$$H_\varepsilon(x,t) := \frac{h(\varepsilon^{-1}x, \varepsilon^{-3/2}t) - (t/2)\varepsilon^{-3/2}}{(t/2)^{1/3}\varepsilon^{-1/2}}.$$

KPZ fixed point and the Airy sheet

There exist a process $\mathcal{A}(x, y)$ [Dauvergne-Ortmann-Virág 2018] such that for any initial condition $\mathcal{H}_0(x)$, the distribution of the KPZ fixed point at time 1 is given by

$$\mathcal{H}(t, y) = \sup_x \left\{ \mathcal{H}_0(x) + \mathcal{A}(x, y) \right\}.$$

In order to know the distribution of the KPZ fixed point,

$$\mathbb{P} \left(\forall y \in \mathbb{R}, \mathcal{H}(t, y) \leqslant g(y) \right).$$

we need to understand the distribution of

$$\sup_{x,y} \left\{ f(x) + \mathcal{A}(x, y) + g(y) \right\},$$

for all functions f, g.

BBP transition and the Airy sheet

- Let $w_{i,j}$ be independent exponential variables with parameter $a_i + b_j$. Set $a_i = b_j \equiv 1$ except

$$a_1 = 2^{-1/3} n^{-1/3} u_1$$

(so that weights on the first column are larger).

$w_{1,7}$	$w_{2,7}$	$w_{3,7}$	$w_{4,7}$	$w_{5,7}$	$w_{6,7}$	$w_{7,7}$
$w_{1,6}$	$w_{2,6}$	$w_{3,6}$	$w_{4,6}$	$w_{5,6}$	$w_{6,6}$	$w_{7,6}$
$w_{1,5}$	$w_{2,5}$	$w_{3,5}$	$w_{4,5}$	$w_{5,5}$	$w_{6,5}$	$w_{7,5}$
$w_{1,4}$	$w_{2,4}$	$w_{3,4}$	$w_{4,4}$	$w_{5,4}$	$w_{6,4}$	$w_{7,4}$
$w_{1,3}$	$w_{2,3}$	$w_{3,3}$	$w_{4,3}$	$w_{5,3}$	$w_{6,3}$	$w_{7,3}$
$w_{1,2}$	$w_{2,2}$	$w_{3,2}$	$w_{4,2}$	$w_{5,2}$	$w_{6,2}$	$w_{7,2}$
$w_{1,1}$	$w_{2,1}$	$w_{3,1}$	$w_{4,1}$	$w_{5,1}$	$w_{6,1}$	$w_{7,1}$

$$\frac{L(n,n) - 4n}{2^{1/3} n^{1/3}} \underset{n\to\infty}{\Longrightarrow} \sup\{\mathcal{B}_1(x) + u_1 x + \mathcal{A}(x,0)\}.$$

The large scale limit is the KPZ fixed point with Brownian initial data

BBP transition and the Airy sheet II

- Let $w_{i,j}$ be independent exponential variables with parameter $a_i + b_j$. Set $a_i = b_j \equiv 1$ except

$$a_1 = 2^{-1/3} n^{-1/3} u_1,$$
$$a_2 = 2^{-1/3} n^{-1/3} u_2.$$

$w_{1,7}$	$w_{2,7}$	$w_{3,7}$	$w_{4,7}$	$w_{5,7}$	$w_{6,7}$	$w_{7,7}$
$w_{1,6}$	$w_{2,6}$	$w_{3,6}$	$w_{4,6}$	$w_{5,6}$	$w_{6,6}$	$w_{7,6}$
$w_{1,5}$	$w_{2,5}$	$w_{3,5}$	$w_{4,5}$	$w_{5,5}$	$w_{6,5}$	$w_{7,5}$
$w_{1,4}$	$w_{2,4}$	$w_{3,4}$	$w_{4,4}$	$w_{5,4}$	$w_{6,4}$	$w_{7,4}$
$w_{1,3}$	$w_{2,3}$	$w_{3,3}$	$w_{4,3}$	$w_{5,3}$	$w_{6,3}$	$w_{7,3}$
$w_{1,2}$	$w_{2,2}$	$w_{3,2}$	$w_{4,2}$	$w_{5,2}$	$w_{6,2}$	$w_{7,2}$
$w_{1,1}$	$w_{2,1}$	$w_{3,1}$	$w_{4,1}$	$w_{5,1}$	$w_{6,1}$	$w_{7,1}$

$$\frac{L(n,n) - 4n}{2^{1/3} n^{1/3}} \xrightarrow[n\to\infty]{} \sup_x \{\mathcal{B}_1 \circ \mathcal{B}_2(x) + \mathcal{A}(x,0)\}.$$

The large scale limit is the KPZ fixed point with initial data

$$\mathcal{B}_1 \circ \mathcal{B}_2 : x \mapsto \sup_z \{\mathcal{B}_1(z) + \mathcal{B}_2(x) - \mathcal{B}_2(z)\}$$

where $\mathcal{B}_1, \mathcal{B}_2$ are Brownian motions with drifts u_1, u_2.

BBP transition and the Airy sheet III

- More generally, the distribution of

$$\sup_x \{\mathcal{B}_1 \circ \mathcal{B}_2 \circ \cdots \circ \mathcal{B}_k(x) + \mathcal{A}(x, 0)\}$$

  is given by the BBP distribution $F_{\mathrm{BBP},\vec{u}}$.

- The distribution of

$$\sup_{x,y} \{\mathcal{B}_1 \circ \mathcal{B}_2 \circ \cdots \circ \mathcal{B}_k(x) + \mathcal{A}(x, y) + \mathcal{B}_{k+1} \circ \cdots \circ \mathcal{B}_{k+r}(y)\}$$

  is also given by the BBP distribution F_{BBP} with parameters $u_1, \ldots, u_{k+r}$.

- Recall that the KPZ fixed point is characterized by

$$\sup_{x,y} \{f(x) + \mathcal{A}(x, y) + g(y)\},$$

  for all functions f, g.

Virág's characterization theorem

Let $H_\varepsilon(t,x)$ be the height function of some model depending on a parameter ε, to which we associate a ε Airy sheet $\mathcal{A}_\varepsilon$.

Theorem (Virág 2020)

Assume that we have a family of compositions laws $\circ_\varepsilon$ and a family of "initial data" $B_u^\varepsilon(x)$ such that

1. *Symmetry property :* $B_{u_1}^\varepsilon \circ_\varepsilon B_{u_2}^\varepsilon = B_{u_2}^\varepsilon \circ_\varepsilon B_{u_1}^\varepsilon$ *and* $B_{u_1}^\varepsilon \circ_\varepsilon \mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon \circ_\varepsilon B_{u_1}^\varepsilon$.
2. *If* $h(0,x) = B_u^\varepsilon(x)$, *then* $h(t,x) - B_u^\varepsilon(x)$ *is tight in* ε.
3. B_u^ε *has drift* u *and sublinear fluctuations, uniformly as* $\varepsilon \to 0$.
4. B_u^ε *converges as* $\varepsilon \to 0$ *to a Brownian motion with drift* u.
5. *BBP asymptotics for the initial data*

$$B_{u_1}^\varepsilon \circ_\varepsilon \cdots \circ_\varepsilon B_{u_k}^\varepsilon$$

Then,

H_ε converges to the KPZ fixed point $\mathcal{H}$.

Summary

- The laws of the distribution of outliers in the BBP transition near the critical threshold appear in last passage percolation (and presumably all models in the KPZ class).
- These deformations of the Tracy-Widom distribution are not very relevant (yet) in statistics, but turn out to be fundamental in characterizations of the KPZ fixed point.