Some integrable models in the KPZ universality class

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KPZ universality class

KPZ universality class

- 1986: **Kardar, Parisi and Zhang** study the random growth of rough interfaces. They propose a continuous model: KPZ equation.
- Interface described by a height function $h(t,x)$, which satisfies the SPDE

$$\partial_t h = \underbrace{\partial_x^2 h}_{\text{local smoothing mechanism}} + \overbrace{(\partial_x h)^2}^{\text{radial growth}} + \underbrace{\dot{\mathscr{W}}}_{\text{uncorrelated noise}},$$

  where $\dot{\mathscr{W}}$ is a white noise. [KPZ86] made scaling predictions and claimed universality.
- KPZ equation is ill-posed (Bertini-Giacomin 1997, Hairer 2011).

Another approach of KPZ universality class

- Focus on **discrete** models ⇒ No issues with regularity and ill-posedness.
- Focus on **integrable** probabilistic systems ⇒ Exact formulas ⇒ Precise understanding of models & limit theorems.

Motivations

Real-World

KPZ models

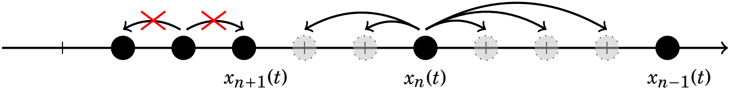
- Front propagation: bacteria colonies, tumoral cells, flame in random media, turbulence in liquid crystals, etc.
- deposition of material: coffee stains, snow...

Mathematical

- **Universality** to prove.
- **Integrability** to understand.
- **Challenge:** Systems are simple to describe but difficult to study. (ex: TASEP, ballistic deposition)

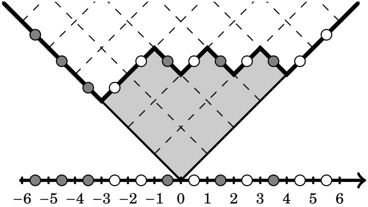
Two types of models in the KPZ class

From interfaces to exclusion processes



Description of the system

- Coordinates $x_n(t)$,
- Configuration encodes a **height function** $h(t,x)$ via Rost's mapping.



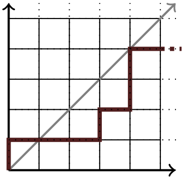
Positive temperature analogue

Directed polymers in 1+1 dim.

Measure Q_n on directed lattice paths π.

- Disorder: edge weights w_e.
- Energy of a path $H(\pi) = \sum_{e \in \pi} w_e$.
- For inverse temperature β

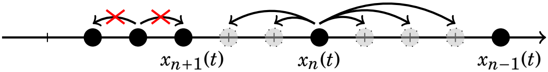
$$Q_n(\pi) = \frac{1}{Z_n} \exp\big(-\beta H(\pi)\big).$$



Apparently different models are related

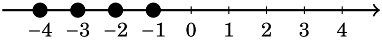
- When $\beta \to \infty$ the measure concentrates on the minimal $H(\pi)$: Geodesics in directed last/first passage percolation.
- Height function of exclusion process = Border of percolation cluster in directed last-passage percolation

Focus on exclusion processes



Exclusion processes in the KPZ class

Step initial data $x_n(0) = -n$:



Main assumptions

- Local dynamics.
- Translation invariant stationary measures μ_ρ are labelled by the average density of particles ρ.
- $j(\rho)$, flux of particles at equilibrium, is such that $j''(\rho) \neq 0$.

Macroscopic density profile

$$\rho(x,\tau) := \lim_{t\to\infty} \mathbb{P}(\text{There is a particle at site } xt \text{ at time } t\tau)$$

satisfies the conservation equation

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} j(\rho(x,t)) = 0,$$

with $\rho(x,0) = \mathbb{1}_{x<0}$ corresponding to step initial condition.

KPZ scaling theory : Heuristics

Tracy-Widom type limit theorem (Open)

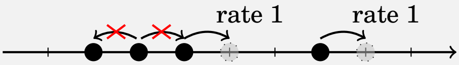
For any density ρ, for $n/t = \kappa(\rho)$,

$$\frac{x_n(t) - \pi(\rho)t}{\sigma(\rho)\cdot t^{1/3}} \xrightarrow[t\to\infty]{} \mathscr{L}_{TW},$$

where $\mathscr{L}_{TW}$ is the Tracy-Widom law from the fluctuations of the largest eigenvalue of Gaussian Unitary Ensemble.

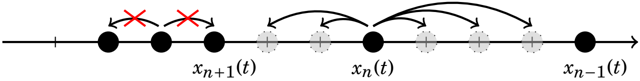
History

- LLN : hydrodynamic theory.
- **KPZ scaling theory** (Krug, Meakin, Halpin-Healy 1992) predict the form of $\sigma(\rho)$ and $t^{1/3}$.
- $\mathscr{L}_{TW}$ has been expected since Johansson's 2000 landmark work on TASEP.



Johansson's method works only for TASEP.

Universality ?



Question

Tracy-Widom limit theorem for general exclusion process?

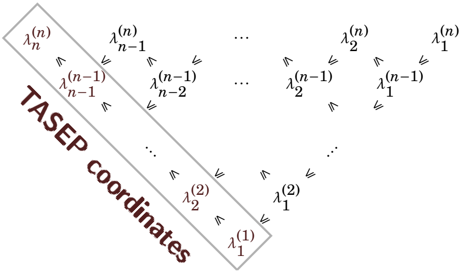
Partial answers

We will discuss:

- ASEP (Tracy-Widom 2008)
- q-TASEP and Macdonald processes (Borodin-Corwin 2011).
- An exactly solvable long-range exclusion process: The q-Hahn TASEP (Povolotsky 2013 / Corwin 2014).
- q-Hahn asymmetric exclusion process (B.-Corwin 2015)

Sources of integrability

- Integrability of TASEP understood via Schur process. Measures on interlacing arrays with nice properties.



- Integrability of ASEP (as shown by Tracy-Widom 2008) is yet less clear.

Macdonald processes (Borodin-Corwin 2011)

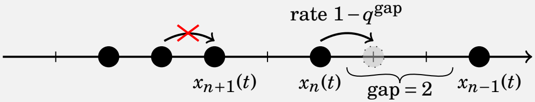
Measures on interlacing arrays in terms of Macdonald symmetric functions. Generalizes Schur process.

The q-TASEP

There exist families of Markov dynamics on interlacing arrays, such that the push-forward of Macdonald process is a Macdonald process with updated parameters.

Definition of q-TASEP

Fix $q \in (0,1)$



Theorem (Borodin-Corwin 2011)

For a certain Macdonald process (q-Whittaker, pure-gamma specialization), we have

$$\lambda_n^{(n)} = x_n(t) + n.$$

$$\mathbb{E}\left[q^{k\lambda_n^{(n)}}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2i\pi)^k} \oint \dots \oint \prod_{1 \leqslant A < B \leqslant k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{g(qz_j, \gamma)}{g(z_j, \gamma)} \frac{\mathrm{d}z_j}{z_j},$$

where g is an explicit (simple) function.

Asymptotics of the q-TASEP

Translation invariant stationary measures are known (Andjel 1982).

Theorem (Ferrari-Vető 2013, B. 2014)

At any density $\rho \in (0,1)$*, for* $n/t = \kappa(\rho)$

$$\frac{x_n(t) - \pi(\rho)}{\sigma(\rho)\, t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathscr{L}_{TW}.$$

KPZ scaling theory is verified.

Asymptotic analysis

- The c.d.f. of Tracy-Widom GUE law is a Fredholm determinant.
- Fredholm determinant formula (Borodin-Corwin-Ferrari) for the law of $x_n(t)$.
- **Saddle-point analysis** of a Fredholm determinant. Implies careful study of a particular holomorphic function involving q-digamma functions.

Slow particles : heuristic approach

Question

What happens if some particles are slower ? Say, for example, that the first particle jumps at rate β.

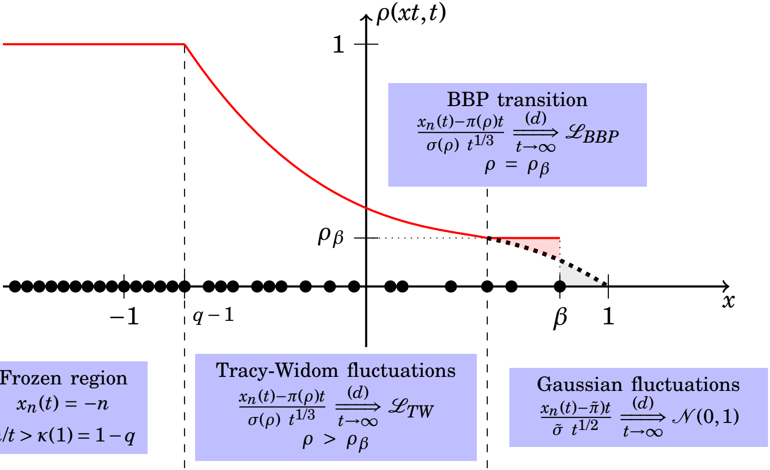
Heuristic remarks

- If $\beta \geqslant 1$ nothing happens.
- If $\beta < 1$ the first particle have speed β. Hence the next particles have speed at most β.
- In the usual q-TASEP, many particles have speed greater than β.
- **Consequence:** The particles that are in a region where the density is small will be slowed down by the first particle.

Theorem (B.)

One observes the BBP phase transition.

BBP phase transition



$\mathscr{L}_{BBP}$: extreme eigenvalues statistics of perturbed ensembles of Gaussian hermitian matrices. (Baik-Ben Arous-Péché, 2005).
Same result holds true for TASEP.

A long-range exclusion process

The q-Hahn distribution and binomial formula

For $q \in (0,1)$ and $0 \leqslant \nu \leqslant \mu \leqslant 1$, $(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1})$

$$\varphi_{q,\mu,\nu}(j|n) := \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{n-j}}{(\nu;q)_n} \begin{bmatrix} n \\ y \end{bmatrix}_q,$$

probability distribution on $\{0,1,\dots,n\}$.

Povolotsky 2013 / Rosengren 2000

If $YX = \alpha XX + \beta XY + \gamma YY$

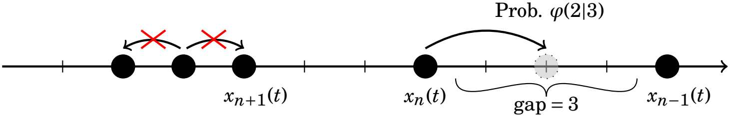
$$\big(pX + (1-p)Y\big)^n = \sum_{k=0}^{n} \varphi_{q,\mu,\nu}(j|n) X^k Y^{n-k}.$$

Gnedin-Olshanski 2009

Interpretation of $\varphi_{q,\mu,\nu}(j|n)$ as a probability in a q-deformation of Pólya's urn model.
Hence, q-Hahn distribution = q-Beta-Binomial.

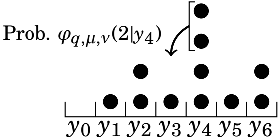
q-Hahn TASEP

Introduced by Povolotsky 2013. Discrete time process



Exclusion/Zero-range

- Coupling $x_k - x_{k+1} - 1 == y_k$
- Exclusion processes == Zero range processes
- Here, corresponding process called q-Hahn Boson



Some tools to study these systems

Markov duality

Definition

Two Markov processes $\vec{X}(t) \in \mathscr{X}$ and $\vec{Y}(t) \in \mathscr{Y}$ are said dual w.r.t $H : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ if for any initial data,

$$\mathbb{E}\left[H(\vec{X}(t), \vec{Y}(0))\right] = \mathbb{E}\left[H(\vec{X}(0), \vec{Y}(t))\right] \quad \Leftrightarrow \; L^X H(\vec{x}, \vec{y}) = L^Y H(\vec{x}, \vec{y})$$

Markov Duality (Corwin 2014 / B. 2014)

The q-Hahn TASEP and the q-Hahn Boson are dual w.r.t. $H(\vec{x}, \vec{y}) = \prod_{i=1}^{N} q^{y_i(x_i+i)}$.

$$\mathbb{E}[H(\vec{x}(t), \vec{y}(0))] = \mathbb{E}[H(\vec{x}(0), \vec{y}(t))].$$

It relies on a symmetry of the q-Hahn distribution: If $X \sim q\text{-Hahn}(x, q, \mu, \nu)$ and $Y \sim q\text{-Hahn}(y, q, \mu, \nu)$, then

$$\mathbb{E}\left[q^{yX}\right] = \mathbb{E}\left[q^{xY}\right].$$

Replica trick (rigorous variant)

- Method designed for q-TASEP (Borodin-Corwin-Sasamoto 2012). Works also for discrete q-TASEP (Borodin-Corwin 2013), q-Hahn TASEP (Corwin 2014), and the next processes.
- One wants to compute the law of $x_n(t)$. Here, the e_q-Laplace transform of $q^{x_n(t)}$,

$$\mathbb{E}\left[e_q\left(\zeta q^{x_n(t)}\right)\right] := \mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)};q)_\infty}\right]$$

- Using moments:
  1. Find a system of ODEs for $\mathbb{E}\left[\prod_i q^{y_i x_i(t)}\right]$ with unique solution. Using the duality, one writes Kolmogorov equations for the zero-range with k particles.
  2. Solve the system of equations using Bethe ansatz.
  3. Formula for $\mathbb{E}\left[q^{k x_n(t)}\right]$ for $k \in \mathbb{N}$ which characterize the law of $x_n(t)$.
  4. Take generating series.

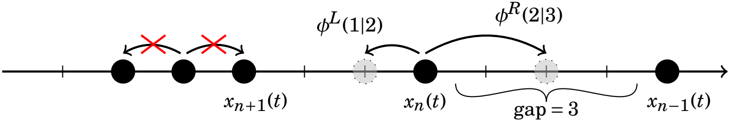
Asymmetric processes

Asymmetric q-Hahn exclusion process

Question

Is it possible to generalize the q-Hahn TASEP allowing jumps in both directions, preserving duality and Bethe ansatz solvability?

Continuous time process: (Corwin-B.)



Rates

Let $R, L \in \mathbb{R}_+$ be asymmetry parameters, with $R + L = 1$. We define

$$\phi^R_{q,\nu}(j|m) := R\,\frac{\nu^{j-1}}{[j]_q}\,\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\,\frac{(q;q)_m}{(q;q)_{m-j}} \simeq R \lim_{\mu\to\nu} \varphi_{q,\mu,\nu}(j|m)$$

$$\phi^L_{q,\nu}(j|m) := L\,\frac{1}{[j]_q}\,\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\,\frac{(q;q)_m}{(q;q)_{m-j}} \simeq L \lim_{\mu\to\nu} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m).$$

Previously mentioned methods still apply

Fredholm determinant

Theorem (B.-Corwin)

Fix $0 < q < 1$ *and* $0 \leqslant \nu < 1$. *For all* $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)};q)_\infty}\right] = \det(I + K_\zeta)_{\mathbb{L}^2(C)},$$

where $\det(I + K_\zeta)_{\mathbb{L}^2(C)}$ *is the Fredholm determinant of* K_ζ *defined by its integral kernel*

$$K_\zeta(w,w') = \frac{1}{2i\pi}\int_{1/2+i\mathbb{R}} \frac{\pi}{\sin(\pi s)}(-q^{-n}\zeta)^s \frac{g(w)}{g(q^s w)} \frac{\mathrm{d}s}{q^s w - w'}$$

with

$$g(w) = \left(\frac{(\nu w;q)_\infty}{(w;q)_\infty}\right)^n \exp\left((q-1)t\sum_{k=0}^{\infty} R\frac{wq^k}{1-\nu wq^k} - L\frac{wq^k}{1-wq^k}\right)\frac{1}{(\nu w;q)_\infty},$$

and the integration contour C *is a small circle around* 1.

A formal saddle-point analysis of above formula is consistent with KPZ scaling theory.

Snapshot of an intermediate moment formula

A moment formula similar with Macdonald processes moment formulas.

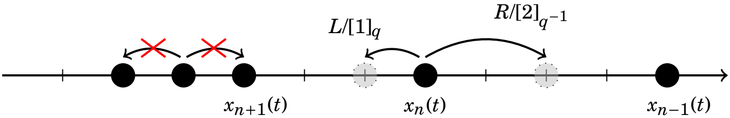
$$\mathbb{E}\left[\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint_{\gamma_1} \cdots \oint_{\gamma_k} \underbrace{\prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B}}_{\text{interaction term}}$$

$$\times \prod_{j=1}^{k} \left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j} \exp\left((q-1)t\left(\frac{R z_j}{1-\nu z_j} - \frac{L z_j}{1-z_j}\right)\right) \frac{dz_j}{z_j(1-\nu z_j)},$$

where the integration contours $\gamma_1, \dots, \gamma_k$ are nested in order to enclose all poles except 0 and $1/\nu$.

First Particle: non-universal behaviour

When $\nu = q$ the rates become much simpler. $[j]_q := (1-q^j)/(1-q)$



Multi-particle Asymmetric Diffusion Model (Sasamoto-Wadati 1998).

Unusual phenomena

- The macroscopic density profile is discontinuous: antishock at the first particle.
- If $R > L$, particles have a net drift to the right, but because of very long range possible jumps on the left, particles are attracted when far.

Consequence for the first particle

$$\frac{x_1(t) - \pi\, t}{\sigma\, t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathscr{L}_{TW}.$$

(Very different than ASEP for which scaling is diffusive!)

Polymer limits

Consider the simple random walk X_t on $\mathbb{Z}$, starting from 0.

$$\mathbb{P}\big(X_{t+1} = X_t + 1\big) = \frac{\alpha}{\alpha+\beta}, \quad \mathbb{P}\big(X_{t+1} = X_t - 1\big) = \frac{\beta}{\alpha+\beta}.$$

The Central Limit Theorem says that

$$\frac{X_t - t\frac{\alpha-\beta}{\alpha+\beta}}{\sigma\sqrt{t}} \Longrightarrow \mathcal{N}(0,1).$$

Theorem (Cramér)

For $\frac{\alpha-\beta}{\alpha+\beta} < x < 1$,

$$\frac{\log\Big(\mathbb{P}(X_t > xt)\Big)}{t} \xrightarrow[t\to\infty]{} -I(x),$$

where $I(x)$ *is the Legendre transform of*

$$\lambda(z) := \log\Big(\mathbb{E}\big[e^{zX_1}\big]\Big) = \log\left(\frac{\alpha e^{z} + \beta e^{-z}}{\alpha+\beta}\right).$$

In random environment ?

Question

What can we say for a random walk in random environment ?

Consider simple random walk on $\mathbb{Z}$ in space-time i.i.d. environment:

$$\mathsf{P}\left(X_{t+1} = x+1 \middle| X_t = x\right) = B_{t,x}, \quad \mathsf{P}\left(X_{t+1} = x-1 \middle| X_t = x\right) = 1 - B_{t,x},$$

where $(B_{t,x})_{t,x}$ are i.i.d.

- $\mathbb{P}, \mathbb{E}$: law of the environment.
- P, E : law of the random walk, conditionally on the environment.

Central limit theorem and large deviation principle are still true, even conditionally on the environment.

Quenched large deviation principle

Theorem (Rassoul-Agha, Seppäläinen and Yilmaz, 2013)

Assume that $\log(B_{t,x})$ *have a finite third moment. Then, the limiting moment generating function*

$$\lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log\left(\mathsf{E}\left[e^{zX_t}\right]\right),$$

exists a.s., and

$$\frac{\log\left(\mathsf{P}(X_t > xt)\right)}{t} \xrightarrow[t\to\infty]{a.s.} -I(x).$$

where $I(x)$ *is the Legendre transform of* λ*.*

This result is more general (dimension, environment) and also holds for polymers.

An exactly solvable model: the Beta RWRE

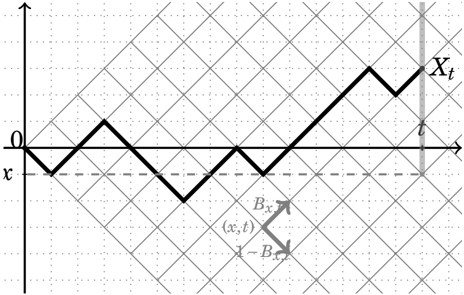
We assume that $(B_{t,x})$ follow the $Beta(\alpha, \beta)$ distribution.

$$\mathbb{P}\big(B \in [x, x+\mathrm{d}x]\big) = x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mathrm{d}x.$$

- **Exactly solvable** means that we can exactly compute the law of

  $$\mathsf{P}(X_t > xt)$$

  (and more).



For simplicity, assume $\alpha = \beta = 1$. (Uniform case)

Theorem (B.-Corwin)

The LDP rate function is

$$I(x) = 1 - \sqrt{1 - x^2}.$$

Fluctuations around the almost-sure LDP such that

$$\frac{\log\Big(\mathsf{P}\big(X_t > xt\big)\Big) + I(x)t}{\sigma(x)\cdot t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathcal{L}_{GUE},$$

with

$$\sigma(x)^3 = \frac{2I(x)^2}{1 - I(x)},$$

under the (technical) hypothesis that $x > 4/5$.

The theorem should extend to the general parameter case α, β and when x covers the full range of large deviation events (i.e. $x \in (0,1)$).

Fredholm determinant

Theorem (B.- Corwin)

Let $u \in \mathbb{C} \setminus \mathbb{R}_+$, and t, x with the same parity. Then for any parameters $\alpha, \beta > 0$ one has

$$\mathbb{E}\left[e^{u\mathsf{P}(X_t > x)}\right] = \det(I + K_u)_{\mathbb{L}^2(C_0)}$$

where C_0 is a small positively oriented circle containing 0 but not $-\alpha-\beta$ nor -1, and $K_u : \mathbb{L}^2(C_0) \to \mathbb{L}^2(C_0)$ is defined by its integral kernel

$$K_u(w, w') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)} (-u)^s \frac{g(w)}{g(w+s)} \frac{\mathrm{d}s}{s + w - w'}$$

where

$$g(w) = \left(\frac{\Gamma(w)}{\Gamma(\alpha + w)}\right)^{(t-x)/2} \left(\frac{\Gamma(\alpha + \beta + w)}{\Gamma(\alpha + w)}\right)^{(t+x)/2} \Gamma(w).$$

Idea of the proof

Origin

- Let $(x_n(t))$ coordinates of the q-Hahn TASEP.
- Convergence as $q \to 1$
$$q^{x_n(t)} \overset{(d)}{\Longrightarrow} Z(t,n)$$
where $Z(t,n)$ partition function of a polymer model (or random average process) with Beta-distributed weights.
- $Z(t,n) = \mathsf{P}(X_t > x)$ with $x = t - 2n + 2$.

Similar method as for exclusion processes

1. Write the recurrence relation for $Z(t,n)$.
2. Evolution equation for $t \mapsto \mathbb{E}[Z(t,n_1)\dots Z(t,n_k)]$.
3. Solution via Bethe ansatz.
4. The moment generating series can be again written as a Fredholm determinant.

Extreme value statistics & Tracy-Widom

Consider $X_t^{(1)},\dots,X_t^{(N)}$ be random walks drawn independently in the same environment.

Fact

The order of the maximum of N i.i.d. random variables is the quantile or order $1-1/N$.

Relation LDP / extreme values

Second order corrections to the LDP have an interpretation in terms of second order fluctuations of the maximum of i.i.d. samples.

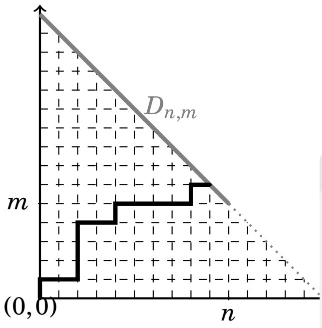
Corollary (B.-Corwin)

Set $N = e^{ct}$. Then, for $\alpha = \beta = 1$,

$$\frac{\max_{i=1,\dots,N}\left\{X_t^{(i)}\right\} - t\sqrt{1-(1-c)^2}}{d(c)\cdot t^{1/3}} \Longrightarrow \mathscr{L}_{GUE},$$

where $d(c)$ is an explicit function (proved under assumption $c > 2/5$).

Zero temperature limit



First passage-time $T(n,m)$ from $(0,0)$ to the half-line $D_{n,m}$ by

$$T(n,m) = \min_{\pi:(0,0)\to D_{n,m}} \sum_{e\in\pi} t_e$$

Passage times

For $(\xi_{i,j})$ i.i.d. Bernoulli and (E_e) i.i.d. Exponential,

$$t_e = \begin{cases} \xi_{i,j} E_e & \text{if } e \text{ is horizontal,} \\ (1-\xi_{i,j}) E_e & \text{if } e \text{ is vertical.} \end{cases}$$

Theorem (B.-Corwin)

For any $\kappa > a/b$ and parameters $a, b > 0$, there exist constants $\rho(\kappa)$ and $\tau(\kappa)$, s.t.

$$\frac{T(n,\kappa n) - \tau(\kappa) n}{\rho(\kappa) n^{1/3}} \Longrightarrow \mathscr{L}_{GUE}.$$

Outlook

Directions left for future work

- Asymptotic analysis: cover the whole range of parameters.
- Limits of the q-Hahn asymmetric exclusion process.
- Other types of scaling limits.
- Better understanding of the integrability of Beta RWRE / Bernoulli-Exponential FPP. Determinantal processes ?

More general open questions

- Universal fluctuations first particle.
- KPZ theory for RWRE.
- Better understanding Tracy-Widom distribution.

Thank you

Questions