KPZ scaling theory for integrable exclusion processes

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Introduction

What is KPZ?

Kardar, Parisi, Zhang, in 1986, study the random growth of rough interfaces. Propose a SPDE to describe the height $h(t,x)$ of the interface

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \dot{\mathcal{W}},$$

where $\dot{\mathcal{W}}$ is a white noise. They made scaling predictions and claimed universality.

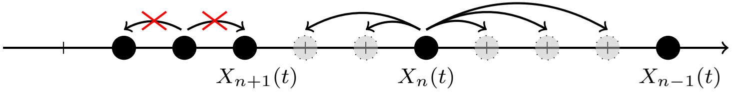
In this talk

- We focus on *exactly solvable* discrete random models.

$\hookrightarrow$ more precisely exclusion processes.

- We start from the most simple initial condition and study different dynamics.

Exclusion process



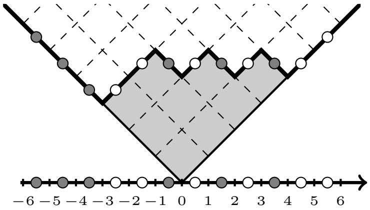
Description of the system

- Coordinates $X_n(t)$,
- Current (integrated)

$$N_x(t) = \#\{n \mid X_n(t) \geqslant x\},$$

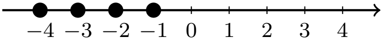
- $\Big($ Height function via Rost's mapping,

$$h(x,t) = x + 2N_x(t).\Big)$$



Limit theorems : Heuristics

Step initial data $x_n(0) = -n$:



Law of large numbers

One expects: for n and t going to infinity with $n/t = \kappa$,

$$\frac{X_n(t)}{t} \xrightarrow[t\to\infty]{a.s.} \pi(\kappa).$$

Tracy-Widom Central limit theorem

For models in the KPZ universality class, one expects

$$\frac{X_n(t) - \pi(\kappa)t}{\sigma(\kappa)\cdot t^{1/3}} \underset{t\to\infty}{\Longrightarrow} \mathcal{L}_{TW},$$

where $\mathcal{L}_{TW}$ is the Tracy-Widom law from the fluctuations of the largest eigenvalue of Gaussian Unitary Ensemble.

KPZ scaling theory : Heuristics

KPZ scaling theory (Krug, Meakin, Halpin-Healy 1992) constitutes an educated guess to predict the value of the constants $\pi(\kappa)$ and $\sigma(\kappa)$ arising in the limit theorems.

Assumptions

- Dynamics are local and space homogeneous.
- Translation invariant stationary measures μ_ρ are labelled by the average density of particles $\rho = \lim_{a\to\infty} \frac{\#\text{ part. between } -a \text{ and } a}{2a+1}$.
- The function $j(\rho) := \mathbb{E}^{\mu_\rho}\left[\frac{\mathrm{d}}{\mathrm{dt}} N_0(t)\right]$ is such that $j''(\rho) \neq 0$.

Macroscopic density profile

Let $\rho(x,\tau) = \lim_{t\to\infty} \mathbb{P}\,(\text{There is a particle at site } xt \text{ at time } t\tau)$ be the macroscopic density profile. It satisfies the conservation equation

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} j(\rho(x,t)) = 0,$$

with $\rho(x,0) = \mathbb{1}_{x<0}$ for step initial condition.

We choose $n/t = \kappa(\rho)$ such that $X_n(t)$ has a local environment given by μ_ρ. We expect $\frac{X_n(t)}{t} \longrightarrow \pi(\rho)$. If $\bar{\rho}(x,t)$ solves the conservation PDE, then $\bar{\rho}(\pi(\rho), 1) = \rho$.

$$\pi(\rho) = \frac{\partial j(\rho)}{\partial \rho}.$$

The function $\kappa(\rho)$ can then be calculated by integrating the density, and one finds for step initial condition

$$\kappa(\rho) = -\rho \frac{\partial j(\rho)}{\partial \rho} + j(\rho).$$

Magnitude of fluctuations

Let $\lambda = -j''(\rho)$ and $A = \sum_{j \in \mathbb{Z}} Cov_{\mu_\rho}(\eta_0, \eta_j)$ where $\eta_0, \eta_j \in \{0, 1\}$ are occupation variables at sites 0 and j. Then

$$\sigma(\rho) = \left(\frac{-\lambda A^2}{2\rho^3} \right)^{1/3}.$$

Integrated covariance A

Consider $X_i, i \in \mathbb{Z}$ a stationary sequence of mean zero r.v. Under some assumptions, $S_n/\sqrt{n}$ converges to a Gaussian of variance σ^2 where

$$\sigma^2 = \lim_{N\to\infty} \mathbb{E}\left[\frac{S_N^2}{N}\right] = \lim_{N\to\infty} \mathbb{E}\left[\frac{\left(\sum_{i=1}^N X_i\right)\left(\sum_{i=1}^N X_i\right)}{N}\right]$$

$$= \mathbb{E}\left[\sum_{i\in\mathbb{Z}} X_0 X_i\right] = \sum_{i\in\mathbb{Z}} Cov(X_0, X_i).$$

Product form invariant measures

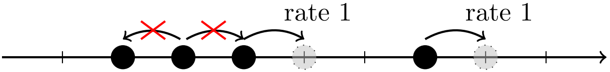
If $\mu_\alpha(gap = k) \propto \alpha^k/(g(1)\ldots g(k))$ for some positive increasing function g, then

$$A = -\alpha\rho\frac{\mathrm{d}\rho}{\mathrm{d}\alpha}$$

where $\rho(\alpha)$ is the density of particles under law μ_α.

Example: TASEP

Description of the dynamics



Properties

One finds that the invariant measures are such that each site is occupied independently with probability ρ.
This yields $j(\rho) = \rho(1-\rho)$, $\pi(\rho) = 1 - 2\rho$ and $\kappa(\rho) = \rho^2$, so that $\pi = 1 - 2\sqrt{\kappa}$. One finds $\sigma(\rho) = \left(\frac{(1-\rho)^2}{\rho}\right)^{1/3}$.

Theorem (Johansson 2000)

For $n/t = \kappa \in (0,1)$,

$$\frac{X_n(t) - (1-2\sqrt{\kappa})t}{\sigma(\rho) t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathcal{L}_{TW}.$$

A brief introduction to q-analogues I

Newton binomial formula:

$$(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}.$$

If $YX = qXY$, one can a priori write

$$(X+Y)^n = \sum_{k=0}^{n} C_n^k(q) X^k Y^{n-k}.$$

Definitions

- q-deformed integer $[n]_q := 1 + q + \cdots + q^{n-1}$.
- q-deformed factorial $n!_q := [n]_q[n-1]_q \dots [1]_q$.
- q-Pochhammer symbol: $(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1})$.

Then the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q (n-k)!_q} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = C_n^k(q)$$

A brief introduction to q-analogues II

Fix $0 < q < 1$ for the rest of the talk.

Definition

The q-exponential is defined by

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{k_q!}$$

Then we have the identity

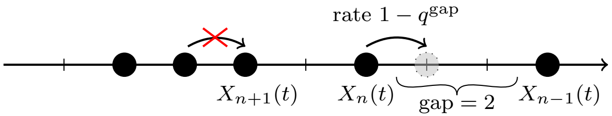
$$e_q(x) = \sum_{k=0}^{\infty} \frac{(x(1-q))^k}{(q;q)_k} = \frac{1}{(x(1-q);q)_\infty}.$$

The q-Laplace transform of a random variable X is

$$\mathbb{E}\left[\frac{1}{(\zeta(1-q)X;q)_\infty}\right]$$

Definition of the q-TASEP

Introduced by Borodin and Corwin in the context of Macdonald processes (2011). Set $q \in (0,1)$.



Stationary measures

Translation invariant stationary measures are such that gaps are distributed according to q-geometric random variables:

$$\mathbb{P}(X_n - X_{n+1} - 1 = k) = \frac{\alpha^k}{(q;q)_k}(\alpha;q)_\infty,$$

for $\alpha \in (0,1)$.

Main result

- For the system at equilibrium given by the stationary measure $\mu_\alpha(k) = \frac{\alpha^k}{(q;q)_k}(\alpha;q)_\infty$, the average density is given by

$$\rho_\alpha = \frac{1}{1+\mathbb{E}[gap]} = \frac{1}{1+\sum_{k=0}^{\infty}\frac{\alpha q^k}{1-\alpha q^k}}.$$

- The speed of a particle is $\mathbb{E}^{\mu_\alpha}\left[1-q^{gap}\right] = \alpha$.
- This implies that $j(\rho_\alpha) = \alpha\rho_\alpha$.
- This yields formulas for $\kappa(\rho_\alpha), \pi(\rho_\alpha)$ and $\sigma(\rho_\alpha)$ given by KPZ scaling theory. (involves q-deformed special functions)

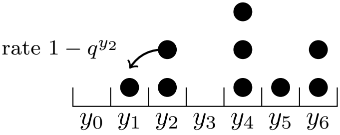
Theorem (Ferrari-Vető, 2013 / B. 2014)

For $\alpha \in (0,1)$, $n/t = \kappa(\alpha)$ *ranges in* $(0,1)$ *and*

$$\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha)\cdot t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathcal{L}_{TW}.$$

Exclusion process vs Zero Range

- Coupling $x_k - x_{k+1} - 1 \sim y_k$
- Exclusion processes $\leftrightarrow$ Zero range processes
- here, q-totally asymmetric zero range process, also called q-Boson model.



Definition

Two Markov processes $\vec{X}(t) \in \mathcal{X}$ and $\vec{Y}(t) \in \mathcal{Y}$ are said dual w.r.t $H : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ if for any initial data,

$$\mathbb{E}[H(\vec{X}(t), \vec{Y}(0))] = \mathbb{E}[H(\vec{X}(0), \vec{Y}(t))] \quad \Leftrightarrow \quad L^X H(\vec{x}, \vec{y}) = L^Y H(\vec{x}, \vec{y})$$

Proposition (Borodin-Corwin-Sasamoto, 2012)

A direct calculation shows that for $H(\vec{x}, \vec{y}) = \prod_{i=0}^{N} q^{(x_i+i)y_i}$,

$$L^{q-\text{TASEP}} H = L^{q-\text{Boson}} H.$$

Remark

The duality is useful if H characterizes enough the law of the process. Here $\mathbb{E}[H(\vec{X}(t), \vec{y})]$ are mixed moments of the variables $q^{X_i(t)}$

What one can do with duality?

We compute the probability distribution function of $X_n(t)$ (cf Borodin-Corwin-Sasmoto 2012).

1. Find a closed system of ODEs for $\mathbb{E}\left[\prod_i q^{y_i X_i(t)}\right]$. Using the duality, one writes Kolmogorov equations for a q-Boson with k particles.
2. Solve the system of equations using Bethe ansatz.
3. It yields formulas for $\mathbb{E}\left[q^{kX_n(t)}\right]$ for $k \in \mathbb{N}$ which characterize the law of $X_n(t)$.
4. Take generating function to express the q-Laplace transform $\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)};q)_\infty}.\right]$.
5. Can be inverted to find the probability distribution function.

Fredholm determinant representation

Theorem (Borodin-Corwin, 2011)

Fix $0 < q < 1$. *For all* $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, *if* $X_n(t)$ *are coordinates of particles of the q-TASEP with step initial data,*

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)}; q)_\infty}\right] = \det(I + K_\zeta)_{\mathbb{L}^2(C)},$$

where $\det(I + K_\zeta)_{\mathbb{L}^2(C)}$ *is the Fredholm determinant of* K_ζ *defined by its integral kernel*

$$K_\zeta(w, w') = \frac{1}{2i\pi} \int_{1/2+i\mathbb{R}} \frac{\pi}{\sin(\pi s)} (-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{\mathrm{ds}}{q^s w - w'}$$

with

$$g(w) = \left(\frac{w}{(w; q)_\infty}\right)^n e^{-tw},$$

and the integration contour C *is a small circle around* 1.

Asymptotic analysis I

- One expects $X_n(t) \sim \pi(\alpha)t + t^{1/3}\sigma(\alpha)\chi_{TW}$ where χ_{TW} is a Tracy-Widom distributed random variable.
- The function $x \mapsto 1/(-q^x;q)_\infty$ have limits 0 in $-\infty$ and 1 in $+\infty$. If one sets $\zeta = -q^{-\pi(\alpha)t - t^{1/3}\sigma(\alpha)x}$ for $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)};q)_\infty}\right] = \lim_{t\to\infty} \mathbb{P}\left(\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha)t^{1/3}} \leqslant x\right).$$

- One needs to prove that

$$\lim_{t\to\infty} \det(I + K_\zeta) = F_{\mathrm{TW}}(x),$$

where F_{TW} is the distribution function of a Tracy-Widom r.v.

Asymptotic analysis II

Fredholm Determinant

$$\det(I+K)_{\mathbb{L}^2(C)} = 1+\sum_{n=1}^{\infty} \frac{1}{n!} \int_C \dots \int_C \det\left(K(w_i, w_j)\right)_{1\leqslant i,j\leqslant n} \mathrm{d}w_1 \dots \mathrm{d}w_j.$$

Fredholm determinant representation of $F_{\mathrm{TW}}(x)$

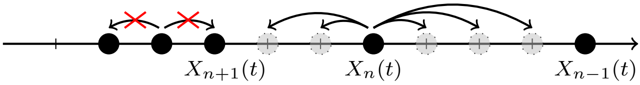
$F_{\mathrm{TW}}(x) = \det(I + K_{\mathrm{Ai}})_{\mathbb{L}^2(\Gamma)}$ where

$$K_{\mathrm{Ai}}(w, w') = \frac{1}{2i\pi} \int_{\Xi} \mathrm{d}z \frac{e^{z^3/3-zx}}{e^{w^3/3-wx}} \frac{1}{z-w} \frac{1}{z-w'},$$

where Γ and Ξ are some flexible contours.

Idea of the proof

One applies Laplace's method (saddle point analysis) on each n-fold integral in the Fredholm determinant series expansion.



Question

Can we prove a Tracy-Widom central limit theorem for the most general exclusion process?

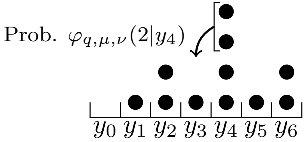
Partial answers

- CLT for ASEP (Asymmetric simple exclusion process) (Tracy-Widom 2008).
- Discrete time version of (q)-TASEP. (Borodin-Corwin 2013).
- Many other partial answers in the literature, namely proving fluctuation exponents under hypotheses.
- Exactly solvable long-range exclusion process: The q-Hahn TASEP (Povolotsky 2013 / Corwin 2014).

The q-Hahn process

q-Hahn Boson process

Discrete-time Markov chain. Particles live on N sites. From a site occupied by y particles, $j \leqslant y$ particles move to the left with probability $\varphi(j|y)$.
Introduced by Povolotsky 2013



q-Hahn distribution

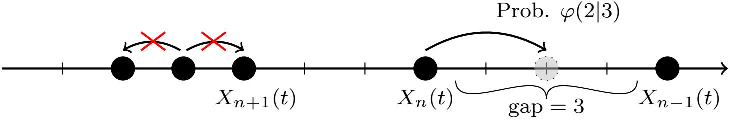
For $0 < q < 1$ and $0 \leqslant \nu \leqslant \mu \leqslant 1$,

$$\varphi_{q,\mu,\nu}(j|y) := \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{y-j}}{(\nu;q)_y} \begin{bmatrix} y \\ j \end{bmatrix}_q,$$

defines a probability distribution on $\{0,1,\dots,y\}$. (This is also the weight function for the q-Hahn orthogonal polynomials)

Duality with q-Hahn TASEP

The q-Hahn process can be described by an exclusion process:



Markov Duality (Corwin 2014, B. 2014)

The q-Hahn TASEP and the q-Hahn Boson are dual w.r.t. $H(\vec{x},\vec{y}) = \prod_{i=1}^{N} q^{y_i(x_i+i)}$.

$$\mathbb{E}\left[H(\vec{X}(t),\vec{Y}(0))\right] = \mathbb{E}\left[H(\vec{X}(0),\vec{Y}(t))\right].$$

It relies on a symmetry of the q-Hahn distribution:

$$\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) q^{jy} = \sum_{j=0}^{y} \varphi_{q,\mu,\nu}(j|y) q^{jm}.$$

Almost the same Fredholm determinant

Theorem (Corwin 2014)

Fix $0 < q < 1$ *and* $0 \leqslant \nu \leqslant \mu \leqslant 1$*. For all* $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)}; q)_\infty}\right] = \det(I + K_\zeta)_{\mathbb{L}^2(C)},$$

where $\det(I + K_\zeta)_{\mathbb{L}^2(C)}$ *is the Fredholm determinant of* K_ζ *defined by its integral kernel*

$$K_\zeta(w, w') = \frac{1}{2i\pi} \int_{1/2+i\mathbb{R}} \frac{\pi}{\sin(\pi s)} (-q^{-n}\zeta)^s \frac{g(w)}{g(q^s w)} \frac{\mathrm{d}s}{q^s w - w'}$$

with

$$g(w) = \left(\frac{(\nu w; q)_\infty}{(w; q)_\infty}\right)^n \left(\frac{(\mu w; q)_\infty}{(\nu w; q)_\infty}\right)^t \frac{1}{(\nu w; q)_\infty},$$

and the integration contour C *is a small circle around* 1*.*

Degenerations

- $\nu = 0$: Corresponds to a discrete-time q-TASEP : Geometric q-TASEP.
- If $\nu = 0$ and scaling $\mu = (1-q)\epsilon$ and rescaling time by $\tau = \epsilon^{-1}t$, one recovers the q-TASEP.
- Many other degenerations

Translation invariant stationary measures

$$\mu_\alpha(\text{gap} = k) = \alpha^k \frac{(\nu;q)_k}{(q;q)_k} \frac{(\alpha;q)_\infty}{(\alpha\nu;q)_\infty}.$$

Theorem (Vető (2014))

Under some restrictions on the range of parameters q, μ and ν, and for $\alpha > 2q/(1+q)$,

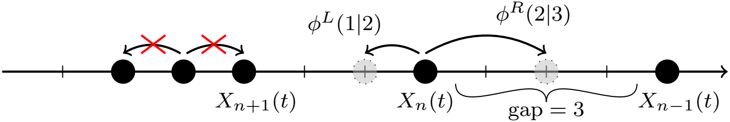
$$\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha) \cdot t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathcal{L}_{TW}.$$

Two sided q-Hahn process

Question

Is it possible to generalize the processes allowing jumps in both directions, preserving duality and Bethe ansatz solvability?

Continuous time process:



Rates

Let $R, L \in \mathbb{R}_+$ be asymmetry parameters, with $R + L = 1$. We define

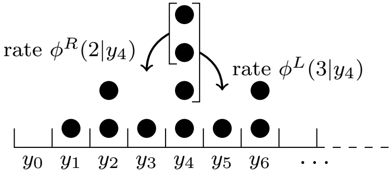
$$\phi^R_{q,\nu}(j|m) := R\frac{\nu^{j-1}}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}} \simeq R\lim_{\mu\to\nu}\varphi_{q,\mu,\nu}(j|m)$$

$$\phi^L_{q,\nu}(j|m) := L\frac{1}{[j]_q}\frac{(\nu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_{m-j}} \simeq L\lim_{\mu\to\nu}\varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m).$$

Duality

Two-sided q-Hahn Boson

- Sites indexed by $\mathbb{N}$.
- For each $j, j' \leqslant y_i$, j particles move to site $i-1$ with rate $\phi^R_{q,\nu}(j|y_i)$ and j' particles move to site $i+1$ with rate $\phi^L_{q,\nu}(j'|y_i)$.



Duality

For any initial conditions $\vec{X}(0)$ being a finite perturbation of the step, and $\vec{Y}(0)$ with a finite number of particles,

$$\mathbb{E}\left[\prod_{i=1}^{\infty} q^{Y_i(0)(X_i(t)+i)}\right] = \mathbb{E}\left[\prod_{i=1}^{\infty} q^{Y_i(t)(X_i(0)+i)}\right].$$

Fredholm determinant

Theorem (B.-Corwin (in prep.))

Fix $0 < q < 1$ *and* $0 \leqslant \nu < 1$. *For all* $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)}; q)_\infty}\right] = \det(I + K_\zeta)_{\mathbb{L}^2(C)},$$

where $\det(I + K_\zeta)_{\mathbb{L}^2(C)}$ *is the Fredholm determinant of* K_ζ *defined by its integral kernel*

$$K_\zeta(w, w') = \frac{1}{2i\pi} \int_{1/2 + i\mathbb{R}} \frac{\pi}{\sin(\pi s)} (-q^{-n}\zeta)^s \frac{g(w)}{g(q^s w)} \frac{\mathrm{ds}}{q^s w - w'}$$

with

$$g(w) = \left(\frac{(\nu w; q)_\infty}{(w; q)_\infty}\right)^n \exp\left((q-1)t \sum_{k=0}^{\infty} R \frac{w q^k}{1 - \nu w q^k} - L \frac{w q^k}{1 - w q^k}\right) \frac{1}{(\nu w; q)_\infty},$$

and the integration contour C *is a small circle around* 1.

Scaling theory

Translation invariant stationary measures

$$\mu_\alpha(\text{gap} = k) = \alpha^k \frac{(\nu; q)_k}{(q; q)_k} \frac{(\alpha; q)_\infty}{(\alpha\nu; q)_\infty},$$

same as for q-Hahn TASEP.

Model dependent constants

One can still find expressions for ρ as a function of α, and then $\kappa(\alpha)$, $\pi(\alpha)$ and $\sigma(\alpha)$. (involves q-deformed special functions).

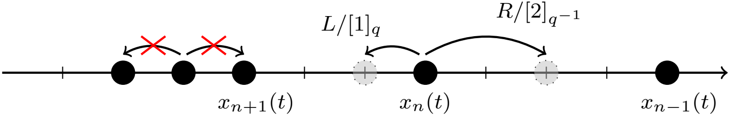
Tracy-Widom Central limit theorem

Fix $0 < q, \nu < 1$ and $R > L$. For all meaningful α, keeping $n/t = \kappa(\alpha)$ we expect

$$\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha) \cdot t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathcal{L}_{TW}.$$

Multi-particle Asymmetric Diffusion Model

When $\nu = q$ the rates no longer depend on the gap and become $R/[j]_{q^{-1}}$ and $L/[j]_q$.



- Introduced by Sasamoto and Wadati 1998, in the Boson formulation.
- Translation invariant stationary measures are products of i.i.d. Bernoulli.

Simulations

One can check the predictions of KPZ scaling theory (here only the LLN) with simulations:

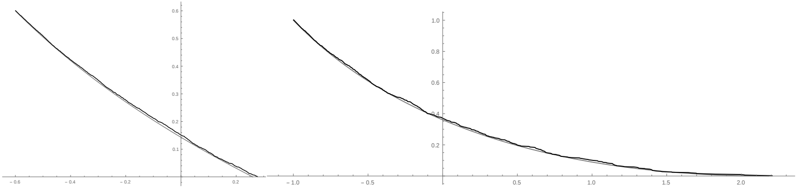


Figure : $N_{xt}(t)/t$ in function of x for $t = 1500$. Left: $R = 0.8$, Right: $R = 1$. $L = 1 - R$

Result

Theorem (B.-Corwin, in preparation)

Fix $0 < q < 1$ *and* $R > L$. *For* $\alpha \geqslant 2q/(1+q)$, *keeping* $n/t = \kappa(\alpha)$ *we have*

$$\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha) \cdot t^{1/3}} \xrightarrow[t\to\infty]{(d)} \mathcal{L}_{TW}.$$

The saddle point analysis is computationally difficult for $\alpha < 2q/(1+q)$.

Surprising phenomena

- The density profile has a discontinuity at the first particles.
- Because of long range jumps on the left, the position of the first particle does not satisfy a classical CLT but a Tracy-Widom CLT. (It is not the case for ASEP)

Conclusion

We have seen

- General expression of model-dependent constant for the renormalization theory of models in the KPZ universality class, in the context of exclusion processes.
- Exactly solvable examples : TASEP and the q deformed exclusion processes: q-TASEP and q-Hahn TASEP.
- Exact solvability of the q-Hahn process extends to two-sided jumps. Some degenerations were already known to be integrable.

Thank you for your attention