Spontaneous emission and reabsorption: Study of a simple model

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Consider a scalar model for the electromagnetic radiation in a one-dimensional box of size $[0, L]$ with periodic boundary conditions. We restrict to positive wave vectors $k > 0$; physically this corresponds to a situation where the radiation propagates only in one direction. The dispersion relation for the free field is given by $\omega = ck$. A two-level atom is placed at position $x = 0$ and is coupled to the radiation.

The Hamiltonian of the system can be written in the form $H = H_0 + V$ with

$$H_0 = \sum_n \hbar\omega_n \hat{a}_n^\dagger \hat{a}_n + \hbar\omega_0 |e\rangle\langle e| \qquad \text{and} \qquad V = v \sum_n \left(|e\rangle\langle g| \hat{a}_n + \hat{a}_n^\dagger |g\rangle\langle e| \right) . \tag{1}$$

The operators $\hat{a}_n$ and $\hat{a}_n^\dagger$ annihilate and create a photon in the mode n, $\hbar\omega_n = \hbar c k_n = \hbar c(2\pi n/L)$. We write the following relations :

$$\hbar\omega_n = n\delta \qquad \text{with} \qquad \delta = \frac{2\pi\hbar c}{L} . \tag{2}$$

The two-level atom transition is at resonance with a photon of angular frequency ω_0 and wave vector k_0, and we will use a cutoff on the coupling v at an energy $E = 2\hbar\omega_0 = 2E_0$ such that the photonic states with $k > 2k_0$ are not coupled to the atom. During the whole exercise we will restrict to states with zero or one photon.

1. Calculate $PG(z)P$, where $G(z)$ is the resolvent of the Hamiltonian H and P projects on the state $|e; 0\rangle$, in the limit where $E_0 \to \infty$ with $z - E_0$ fixed. We will use

$$\tilde{z} = z - E_0 \, ; \quad \tilde{n} = n - n_0 \quad \tilde{E} = E - E_0 \tag{3}$$

and the identity

$$\sum_{k=-\infty}^{+\infty} \frac{1}{z-k} = \pi \,\text{cotan}(\pi z) \tag{4}$$

with z being a non-integer complex number.

2. Discuss graphically the equation satisfied by the eigenvalues $\tilde{E}_m$ of H and show that there is only one eigenvalue in each of the intervals $]n\delta, (n+1)\delta[$.

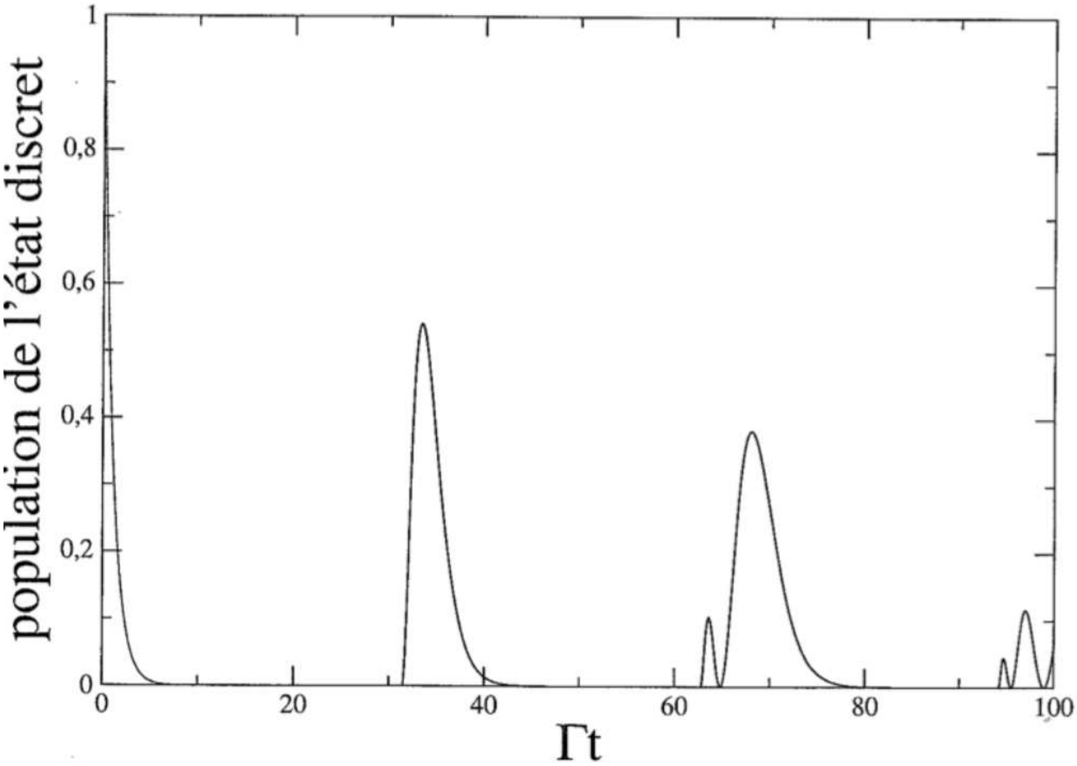

FIG. 1 – A discrete state coupled to an infinite "stairs" of states. Parameters : $\delta = 0.2\hbar\Gamma$.

3. Determine the probability amplitude at time $t > 0$ for the system to remain in the initial state $|\psi(0)\rangle = |e; 0\rangle$ in the form of a sum over m.
4. Explain qualitatively the presence of revivals in the probability of finding the system in the initial state by using an approximate property of the spectrum of H. Use the non-perturbed spectrum of H to obtain an estimated time where the first revival takes place.
5. Take the limit $\delta, v \to 0$ with constant v^2/δ, replace the sum with an integral, and interpret the result. We will use :

$$\tilde{v} = \frac{v^2}{\delta} \qquad \text{and} \qquad \Gamma = \frac{2\pi}{\hbar}\tilde{v}\,. \tag{5}$$

6. Show that one can obtain the same result by taking the limit of a continuous spectrum directly in $PG(z)P$. We will calculate $PG(z)P$ in the continuous limit for $\mathrm{Im}(z) > 0$ and we will perform an analytical continuation to the lower half-plane.