One-body and two-body correlation function of a Fermi gas at $T = 0$

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Consider a Fermi gas of N non-interacting particles confined into a D-dimensional box of size L with periodic boundary conditions. We assume the system to be at zero temperature ($T = 0$) and the fermions to be all in the same spin state $|+\rangle$. We define $\varphi_1, \ldots, \varphi_N$ to be the N single-particle wavefunctions of lowest energy.

1 Slater determinant

Write the complete state vector of the N fermions. What is the maximal momentum p_F accessible to the fermions in $3D$? And in $1D$?

2 One-body observables

Let the operator $\mathcal{B}$ be an observable involving only one-body operators:

$$\mathcal{B} = \sum_{i=1}^{N} B(i), \tag{1}$$

where $B(i) \equiv \mathrm{Id}(1)\mathrm{Id}(2)\ldots B(i)\ldots \mathrm{Id}(N)$ acts only on the state of the i-th particle.

a) Calculate the expectation value $\langle \mathcal{B} \rangle$ of the operator $\mathcal{B}$ as a function of $\varphi_1, \ldots, \varphi_N$.

b) We define the one-body density matrix $\hat{\rho}(1)$ such that for any operator $\mathcal{B}$ one has:

$$\langle \mathcal{B} \rangle = \mathrm{Tr}_1[\hat{\rho}(1)B(1)] = \mathrm{Tr}_1[B(1)\hat{\rho}(1)]. \tag{2}$$

Write the explicit expression of $\hat{\rho}(1)$ as a function of $\varphi_1, \ldots, \varphi_N$.

c) Use the above results to evaluate the spatial density of the fermions in the gas, $\rho(\mathbf{r}) = \rho_0$.

d) Consider the following operator

$$G(i) = |i : \mathbf{r}\rangle\langle i : \mathbf{r}'|, \tag{3}$$

acting as G on the i-th particle, while being the identity $\mathrm{Id}(j)$ for all $j \neq i$. In the thermodynamic limit, calculate explicitly the single-particle correlation function

$$g^{(1)}(\mathbf{r}, \mathbf{r}') = \langle \sum_{i=1}^{N} G(i) \rangle, \tag{4}$$

first in $1D$, then in $3D$.

3 Two-body observables

Let the operator $\mathcal{B}$ be an observable involving only two-body operators:

$$\mathcal{B} = \sum_{i=1}^{N} \sum_{j \neq i} B(i, j), \tag{5}$$

where $B(i, j) \equiv \mathrm{Id}(1)\mathrm{Id}(2) \ldots B(i) \ldots B(j) \ldots \mathrm{Id}(N)$ acts as B only on the states of the particle i and particle j.

a) Calculate the expectation value $\langle \mathcal{B} \rangle$ of the observable $\mathcal{B}$ as a function of $\varphi_1, \ldots, \varphi_N$.

b) We define the two-body density matrix $\hat{\rho}(1, 2)$ such that for any operator $\mathcal{B}$ one has:

$$\langle \mathcal{B} \rangle = \mathrm{Tr}_{1,2}[\hat{\rho}(1, 2)B(1, 2)] = \mathrm{Tr}_{1,2}[B(1, 2)\hat{\rho}(1, 2)]. \tag{6}$$

Express $\hat{\rho}(1, 2)$ as a function of $\varphi_1, \ldots, \varphi_N$.

c) Show that the trace of $\hat{\rho}(1, 2)$ restricted to the subspace of particle 2 gives $\hat{\rho}(1)$ up to a multiplicative factor.

d) Consider the following operator:

$$G(i, j) = |i : \mathbf{r}\rangle\langle i : \mathbf{r}| \otimes |j : \mathbf{r}'\rangle\langle j : \mathbf{r}'|. \tag{7}$$

Apply the above results to evaluate the spatial density of pairs in the gas, namely the two-body correlation function $g^{(2)}(\mathbf{r}, \mathbf{r}')$.

e) Express $g^{(2)}$ as a function of $g^{(1)}$.

f) In the thermodynamic limit, calculate explicitly $g^{(2)}(\mathbf{r}, \mathbf{r}')$ in $1D$ than in $3D$. We will write the result as follows:

$$g^{(2)}(\mathbf{r}, \mathbf{r}') = \rho_0^2[1 - \phi^2(\mathbf{r} - \mathbf{r}')]. \tag{8}$$

4 Application in $1D$: Fluctuation of the number of particles in a spatial interval of length X

Consider a $1D$ system and a spatial interval of length X along x as represented on Fig. 1 We are interested in the counting statistics of the number of particles in the interval X, which is related to the $g^{(2)}(x-x')$ function calculated above.

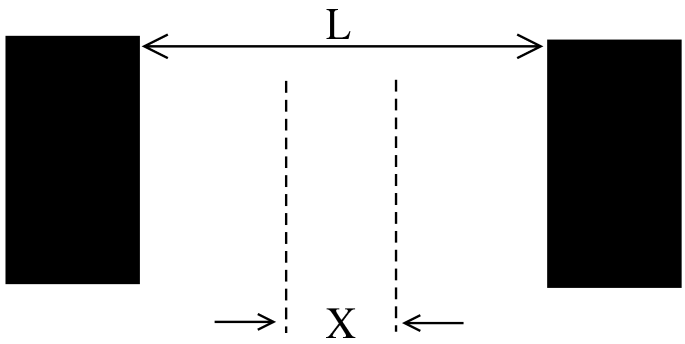

Figure 1: A one-dimensional system (1D) where an interval X is considered.

a) Let N_X be the operator that counts the number of particles inside the interval $[0, X]$. Show that $\langle N_X^2 \rangle = \langle N_X \rangle + \int_0^X dx \int_0^X dx' g^{(2)}(x-x')$.

b) Show that:

$$\int_0^X dx \int_0^X dx' g^{(2)}(x-x') = 2\left[X \int_0^X dx\, g^{(2)}(x) - \int_0^X dx\, x\, g^{(2)}(x)\right]. \tag{9}$$

For this purpose we can derive this relation (9) with respect to X.

c) In the limit where $k_F X \gg 1$ calculate the variance ΔN_X^2, given the following useful relations:

$$\int_0^x dt \frac{\sin^2 t}{t^2} = \frac{\pi}{2} - \frac{1}{2x} + O\left(\frac{1}{x}\right) \quad \text{for} \quad x \to \infty \tag{10}$$

$$\int_0^x dt \frac{\sin^2 t}{t} \simeq \frac{1}{2}\ln(2x) + \frac{1}{2}\gamma + O(1) \quad \text{for} \quad x \to \infty, \tag{11}$$

where $\gamma = 0.577\ldots$ is Euler's constant. Comment on the result.

d) In the regime of non-zero temperature we have an approximate result for $T \ll T_F$, calculated by Efetov et Larkin (Sov. Phys. JETP **42** (1976), 390):

$$g^{(2)}(x) \simeq \rho_0^2 - \left(k_F \frac{T}{2T_F} \frac{\sin(k_F x)}{\sinh(\pi T k_F x / 2T_F)}\right)^2. \tag{12}$$

Find the condition on T/T_F and on $k_F x$ such that $g^{(2)}(x)$ is close to its value at $T = 0$.

e) At $k_F X = 100$, calculate $\langle N_X \rangle$, ΔN_X^2 and $\Delta N_X^2 / \langle N_X \rangle$.

Evaluate the upper bound of T/T_F such that our equations are valid for this specific value of $k_F X$.