BASICS OF BOSE-EINSTEIN CONDENSATION THEORY

Y. Castin, LKB - ENS, Paris

OUTLINE

- atoms: waves and particles
- the Bose law
- when the Bose gas becomes degenerate
- how to reach Bose-Einstein condensation
- atomic interactions and Gross-Pitaevskii equation

ATOMS: WAVES AND PARTICLES

Analogy with optics:

Object	optics	atomic physics
field	$\mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t)$	$\phi(\mathbf{r},t)$
equation of motion	$\left(\Delta - \frac{1}{c^2}\partial_t^2\right)\mathbf{B} = 0$	$i\hbar\partial_t\phi = -\frac{\hbar^2}{2m}\Delta\phi$
particle	photon	atom
energy	$\hbar\omega$	$\frac{1}{2}mv^2$
momentum	$\hbar\mathbf{k}$	$\mathbf{p} = m\mathbf{v}$
wavelength	$\lambda = \frac{2\pi}{k} = \frac{h}{\hbar k}$	$\lambda = \frac{h}{p}$
dispersion relation	$\omega = ck$	$\omega = \frac{\hbar k^2}{2m}$

Values in an ordinary gas:

- equipartition of energy:

$$\frac{1}{2}m\langle v_x^2\rangle = \frac{1}{2}k_B T$$

- sodium atoms at 300 K:

$$\Delta v_x = 300 \ \ \mathrm{m/s}$$

$$\lambda = 5 \times 10^{-11} \ \ \mathrm{m}$$

With Sisyphus cooling:

$$\lambda \sim 1 \ \mu\mathrm{m}.$$

ATOMIC MODES IN A BOX

Energy levels of an atom in a box:

- **periodic boundary conditions:**

$$\begin{aligned}\phi(x+L,y,z) &= \phi(x,y+L,z) = \phi(x,y,z+L)\\ &= \phi(x,y,z).\end{aligned}$$

- **quantisation of wavevectors:**

$$\phi(x,y,z) \propto e^{i(k_x x + k_y y + k_z z)}$$

$$k_\alpha = \frac{2\pi}{L} q_\alpha$$

- **quantisation of energy:**

$$\epsilon_{\mathrm{k}} = \frac{h^2}{2mL^2}\left(q_x^2 + q_y^2 + q_z^2\right)$$

THE BOSE LAW

Indistinguishable particles in quantum theory are:

- bosons:

$$P_\sigma|\psi\rangle_B = |\psi\rangle_B$$

- or fermions:

$$P_\sigma|\psi\rangle_F = \epsilon(\sigma)|\psi\rangle_F.$$

Configuration defined by a set of occupation numbers $\{n_\alpha\}$

Example: two spin 1/2 particles of opposite spin:

$$|\psi\rangle_B \propto |+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle$$

$$|\psi\rangle_F \propto |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle$$

$$|+\rangle \otimes |-\rangle \quad \text{meaningless}$$

Thermodynamics of the ideal Bose gas:

$$\mathrm{Proba}(\{n_\alpha\}) = \frac{1}{\Xi} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

where $\beta = 1/(k_B T)$ and μ is the chemical potential.

Bose law for the occupation number:

$$\langle n_\alpha \rangle = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}$$

so that

$$-\infty < \mu < \epsilon_0.$$

Lower limit for μ is non-degenerate regime:

$$\langle n_{\vec{k}} \rangle \simeq \rho \lambda^3 e^{-\beta \hbar^2 k^2 / 2m}$$

in a large box, where

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$$

is the thermal de Broglie wavelength.

The coherence length of the gas is $\sim \lambda$.

WHEN THE BOSE GAS BECOMES DEGENERATE

$$\rho\lambda^3 \gg 1$$

Saturation of excited state population:

$$N' \equiv \sum_{\alpha \neq 0} \langle n_\alpha \rangle < \sum_{\alpha \neq 0} \frac{1}{e^{\beta(\epsilon_\alpha - \epsilon_0)} - 1} \equiv N'_{\max}$$

For a large cubic box

$$L \gg \lambda, \qquad i.e. \qquad k_B T \gg \frac{h^2}{2mL^2},$$

$$N'_{\max} = \sum_{\vec{k} \neq \vec{0}} \frac{1}{e^{\beta\hbar^2 k^2/2m} - 1} \simeq 2.612 \frac{L^3}{\lambda^3}$$

If $N > N'_{\max} \ldots$

$\ldots$ there are at least $N - N'_{\max}$ atoms in the ground mode of the box.

A condensate forms if:

$$\rho\lambda^3 > 2.612\ldots \qquad \text{Einstein, 1925}$$

Totally counter-intuitive for Boltzmann statistics.

In a harmonic potential:

$$N'_{\max} \simeq 1.202 \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^3$$

where $\bar{\omega}$ is the geometric mean of the trap frequencies.

Even in a trap one has for $N = N'_{\max}$:

$$\rho(\vec{0}\,)\lambda^3 \simeq 2.612$$

Below T_c: condensate fraction

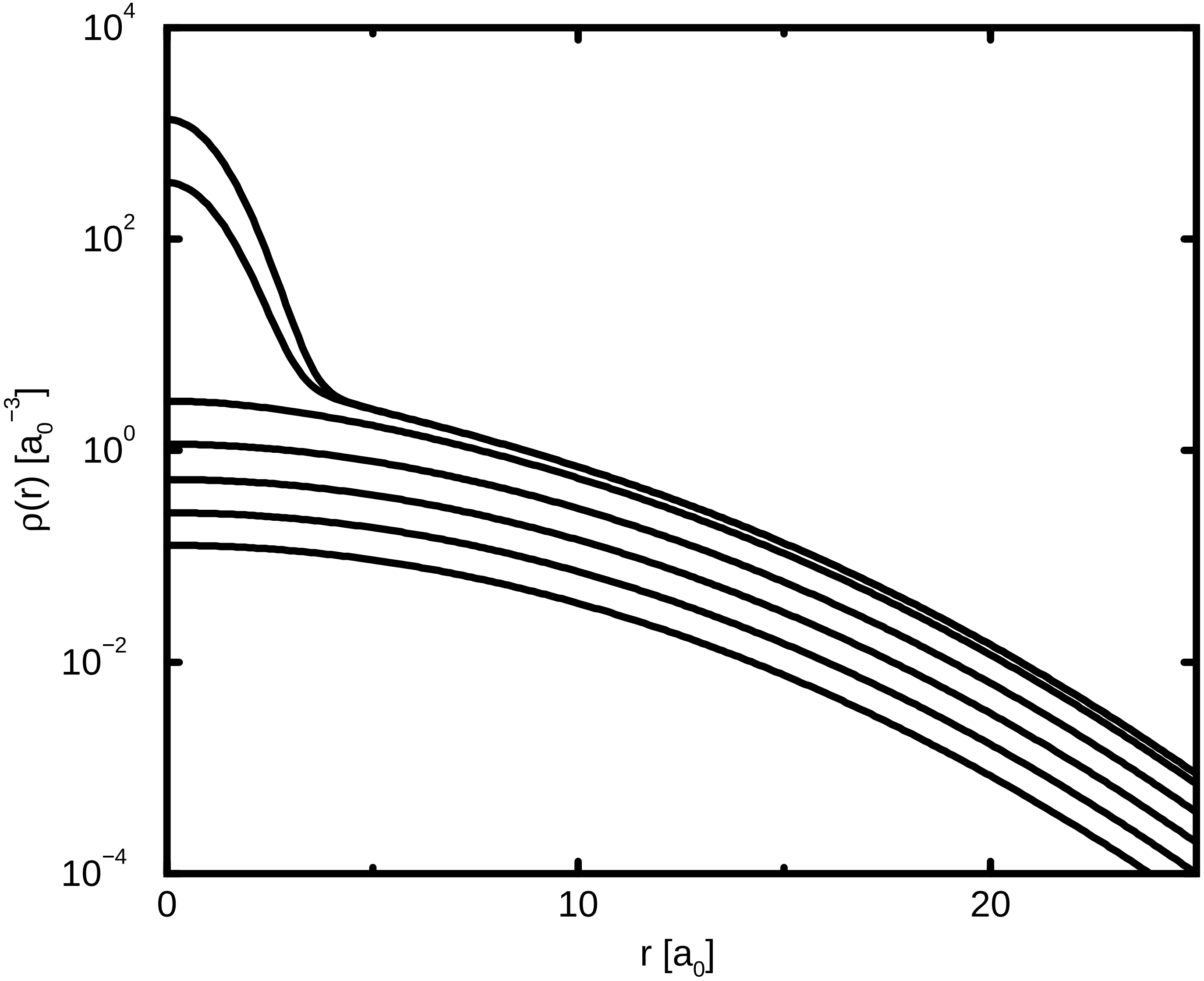
$$\begin{aligned} \frac{N_0}{N} \simeq \frac{N - N'_{\max}}{N} &\simeq 1 - \left(\frac{T}{T_c}\right)^{3/2} \qquad \text{box} \\ &\simeq 1 - \left(\frac{T}{T_c}\right)^{3} \qquad \text{harmonic trap} \end{aligned}$$

Realistic examples:

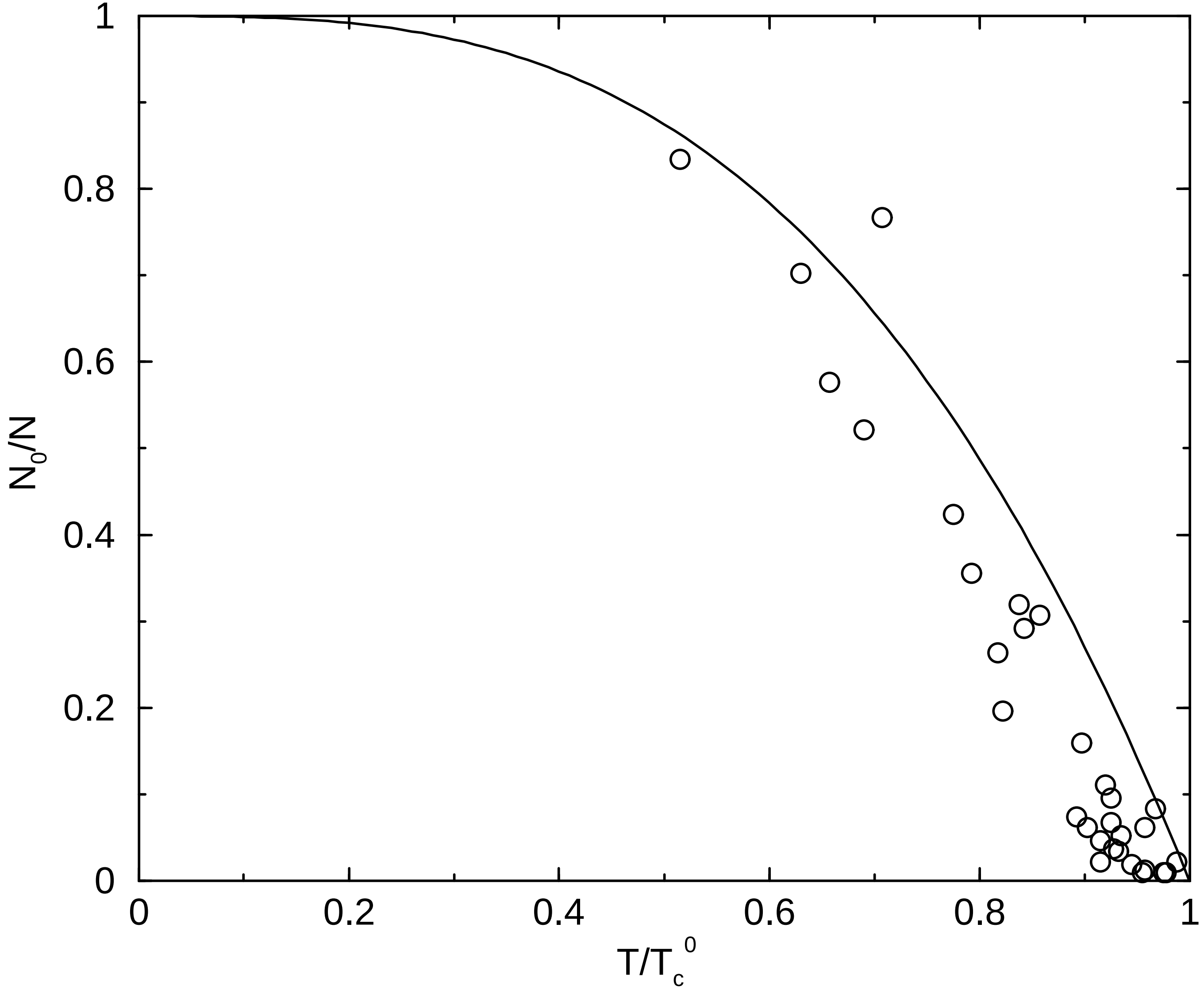
$$T/T_c = 1/2 \qquad \text{everyday}$$

$$T/T_c = 1/4 \qquad \text{the good days}$$

BEC in position space: $k_B T = 20\hbar\omega$ $N = 500$ to 32000



Results of JILA:



HOW TO REACH BOSE-EINSTEIN CONDENSATION

The problem of solidification:

- **For air with pressure 1 atm:**

$$T_c \simeq 0.4K$$

  but then one expects a solid phase.

- **He^4 does not solidify. Experiences superfluid transition at $\sim$ 2K**
  but is a liquid, not a gas (condensate fraction < 0.1).

- **Only polarized hydrogen is gaseous at 1 atm, 0 K.**

Low density route: use of metastability

- **2-body elastic collisions ensure thermalisation:**

$$\gamma_{\text{elas}} \propto \rho$$

- **3-body collisions form molecules:**

$$\gamma_{\text{inel}} \propto \rho^2$$

  but are much slower at low density!

- **the obtained condensate is metastable.**

- **Price to pay: ultralow temperatures**

$$T_c \sim 40nK \quad to \quad 1\mu K.$$

How to cool ?

- laser cooling alone not yet succeeded:

$$\lambda \sim \lambda_{\text{opt}} = \frac{2\pi}{k_L}$$

  and bad effects of light when $\rho \lambda_{\text{opt}}^3 \sim 1$.

- forced evaporative cooling: remove atoms in high energy tails, let gas rethermalize, and so on

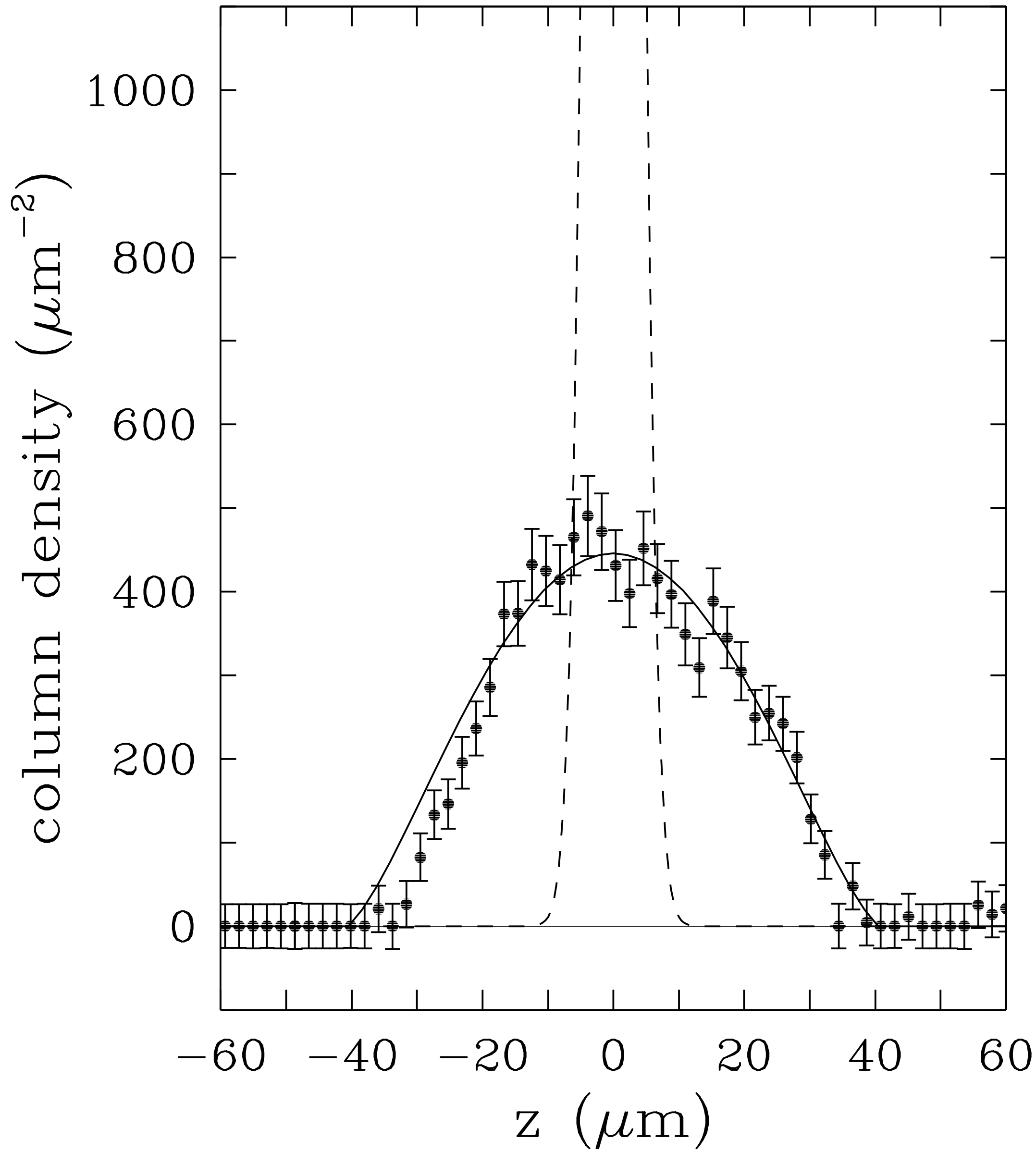
- efficient if

$$\frac{\gamma_{\text{elas}}}{\gamma_{\text{loss}}} > 100.$$

EFFECT OF ATOMIC INTERACTIONS

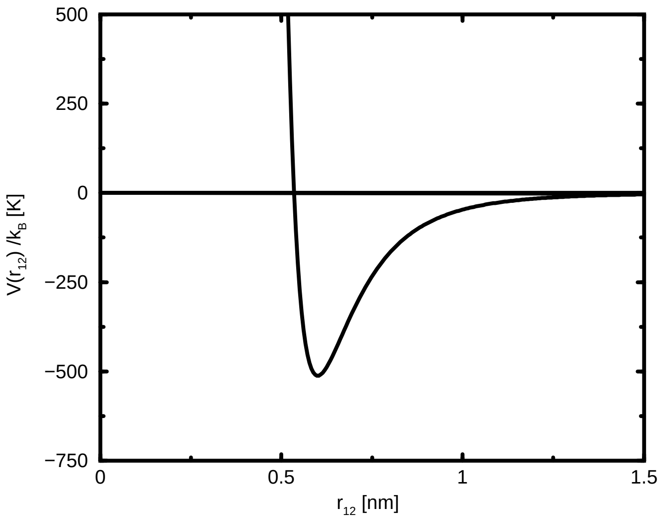
- **Not ideal Bose gas!**

- **Coherence length!!**



How to characterize the interaction potential?

by its r dependence:



by its scattering length:

$$-\frac{\hbar^2}{m}\Delta\phi(r) = 0$$

$$\phi(r) = C_0 + C_1/r$$

$$\propto 1 - \frac{a}{r}$$

Typical values

$$a = 50 \text{ nm } (^{87}\mathrm{Rb}) \quad a = -1.5 \text{ nm } (^{7}\mathrm{Li})$$

but a can be tuned.

THE GROSS-PITAEVSKII EQUATION

$$i\hbar\partial_t\phi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\Delta + U(\vec{r}) + \mathrm{g}N_0|\phi(\vec{r},t)|^2 - \mu\right]\phi(\vec{r},t)$$

Comes from mean field for model interaction potential

$$V(\vec{r}) = \mathrm{g}\delta(\vec{r})\partial_{\mathrm{r}}(\mathrm{r}\,\cdot)$$

with coupling constant $\mathrm{g} = \dfrac{4\pi\hbar^2}{m}\mathrm{a}$.

Explains almost everything, including superfluidity.