LIMIT OF SPIN SQUEEZING IN FINITE TEMPERATURE BOSE-EINSTEIN CONDENSATES

Yvan Castin, Alice Sinatra
LKB, Ecole normale supérieure (Paris, France)
Emilia Witkowska
Polish Academy of Science (Warsaw, Poland)

OUTLINE OF THE LECTURE

- **Introduction and physical motivation**
- **An exactly solvable toy model**
- **The multimode case**
- **Comparison to numerical simulations**

INTRODUCTION AND PHYSICAL MOTIVATION

MOTIVATION FOR SPIN SQUEEZING

What are the atomic condensates good for ?

- **Main application of cold atoms: Atomic clocks (Salomon, Clairon, 1999)**
- **With interactions, prepare quantum correlations that are useful for metrology**
- **They can increase the signal-to-noise ratio in atomic clocks for a given interrogation time (Kitagawa, Ueda, 1993; Wineland, 1994)**
- **Proof-of-principle experiments with condensates (Oberthaler, 2008; Treutlein, 2010): A gain of a factor $\mathbf{3}$ on the signal-to-noise ratio**

ATOMIC CLOCKS IN BRIEF

What an atomic clock does:

- Measures the transition frequency ω_{ab} of two-level atoms
- Formally, a two-level atom is a spin $1/2$
- Collective spin $\mathbf{S} = \sum_{i=1}^{N} \mathbf{S}_i$, free Hamiltonian:
$$H_0 = \hbar\omega_{ab} S_z$$
- At time 0, prepare the collective spin along x. At time τ, measurement of the spin precession angle $\omega_{ab}\tau$ gives transition frequency ω_{ab} (Ramsey method).

Transverse quantum fluctuations: $\Delta S_y \Delta S_z \geq \frac{1}{2}|\langle S_x\rangle|$

- Standard quantum limit: All spins along x, $\langle S_x\rangle = N/2$:
$$\Delta S_y^{\mathrm{st}} = \Delta S_z^{\mathrm{st}} = \sqrt{N}/2 \longrightarrow \Delta\omega_{ab} = \frac{1}{N^{1/2}\tau}$$
- This is larger than technical noise in good clocks

ONE CAN GAIN WITH SPIN SQUEEZED STATES

- With respect to standard quantum limit, one can in principle reduce a lot ΔS_y, at the expense of increasing ΔS_z
- ... and of decreasing $|\langle S_x \rangle|$
- Gain $1/\xi$ on the Ramsey signal-to-noise ratio:

$$\xi^2 = \frac{N \Delta S^2_{\perp,\mathrm{min}}}{|\langle \mathrm{S} \rangle|^2} < 1 \longrightarrow \Delta\omega_{ab} = \frac{\xi}{N^{1/2}\tau}$$

Kitagawa-Ueda spin squeezing: $H = \hbar\omega_{ab} S_z + \hbar\chi S_z^2$

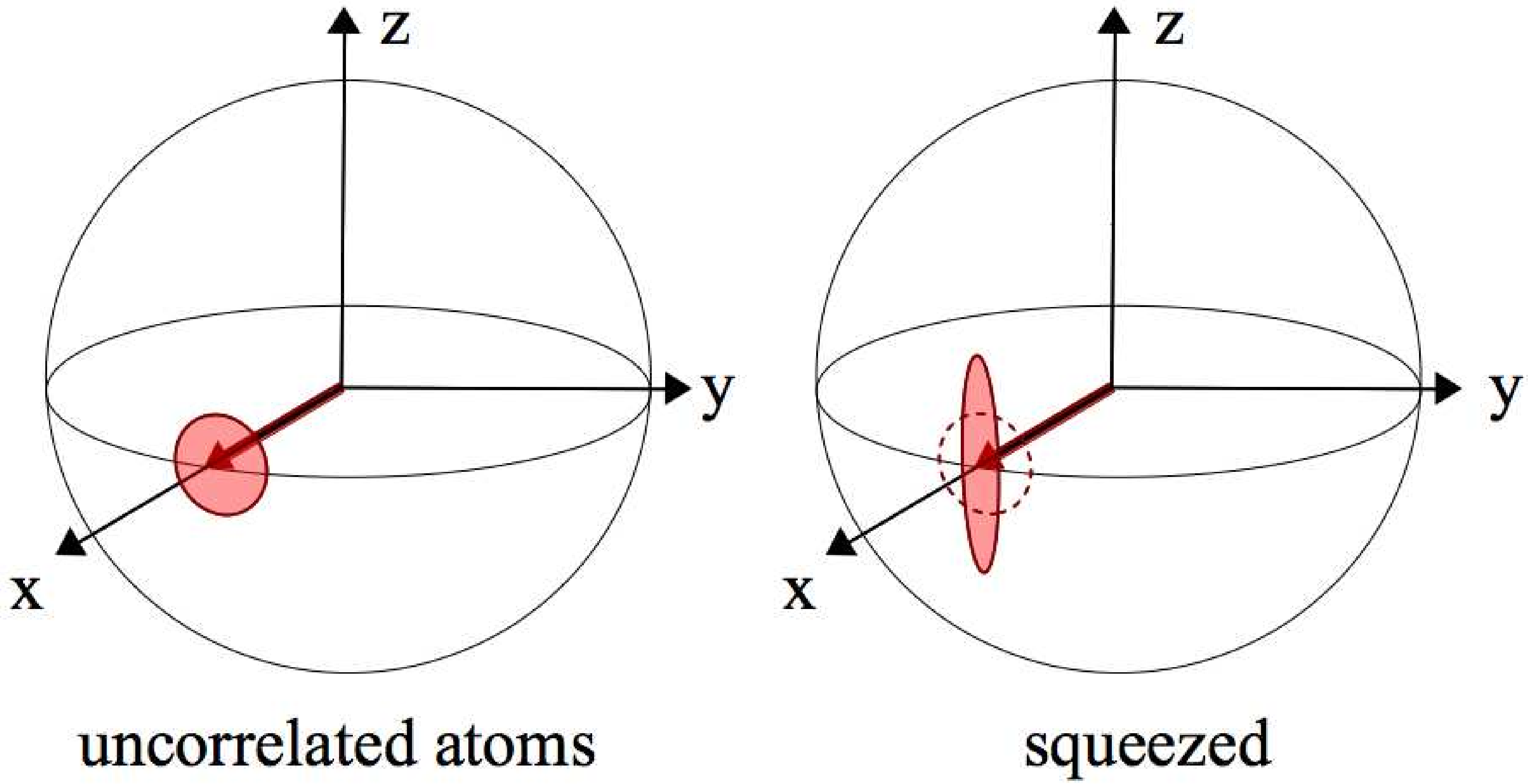
- Spin-dependent Larmor frequency: Evolution turns the fluctuation circle into a tilted ellipse. At best time:

$$\xi^2_{\mathrm{min}} \underset{N\to\infty}{\sim} \frac{3^{2/3}}{2N^{2/3}}$$

- Realisable with two-mode condensates (Cirac, 2001):

$$S_x + iS_y = a^\dagger b, \qquad S_z = (a^\dagger a - b^\dagger b)/2, \qquad \chi = \frac{g}{\hbar V}$$

ON THE BLOCH SPHERE



In practice, squeezed axis is tilted (rotation required):

$$\Delta S^2_{\perp,\min} = \frac{1}{2}\left[\langle S_y^2\rangle + \langle S_z^2\rangle - |\langle (S_y + iS_z)^2\rangle|\right]$$

WHAT HAPPENS IN REAL LIFE ?

There is decoherence in atomic gases:

- Mainly due to the coexistence of one-body and three-body losses
- Yun Li, Castin, Sinatra, 2008 : Loss events lead to a random dephasing among the two modes

$$\xi^2_{\min} \underset{N\to\infty}{\longrightarrow} \left(\frac{5\sqrt{3}}{28\pi}\frac{m}{\hbar a}\right)^{2/3}\left(\frac{7}{2}K_1K_3\right)^{1/3}$$

An atomic gas is a multimode system:

- At finite temperature, the non-condensed modes constitute a dephasing environment for the condensate mode
- Cf. previous lecture: This leads to phase spreading of the condensate and a finite coherence time
- What is the effect on spin squeezing ?

AN EXACTLY SOLVABLE TOY MODEL

A TWO-MODE MODEL WITH A RANDOM ELEMENT

$$H = \hbar\omega_{ab} S_z + \hbar\chi(S_z^2 + D S_z) \quad \text{[Minguzzi, 2011]}$$

- D is a Gaussian random real variable of zero mean
- To mimick a random potential (having opposite effects on a and b) with uniform variance and short range correlations, take scaling in thermodynamic limit:
$$\frac{\langle D^2 \rangle}{N} = \epsilon = \text{constant}$$
- D is time independent, but it varies randomly from one experimental realisation to the other
- The model is integrable, S_z being a constant of motion:
$$i\dot{a} = \frac{\chi}{2}\left(2S_z + D + \frac{1}{2}\right) a$$

MAIN RESULTS OF THE TOY MODEL

- **Best squeezing is in thermodynamic limit and finite:**

$$\xi^2_{\rm min} \overset{\text{lim.therm.}}{=} \frac{\epsilon}{1+\epsilon} + \frac{1}{N^{1/2}}\frac{2\epsilon^{1/2}}{(1+\epsilon)^{3/2}} + O(1/N)$$

- **The corresponding optimal time is divergent:**

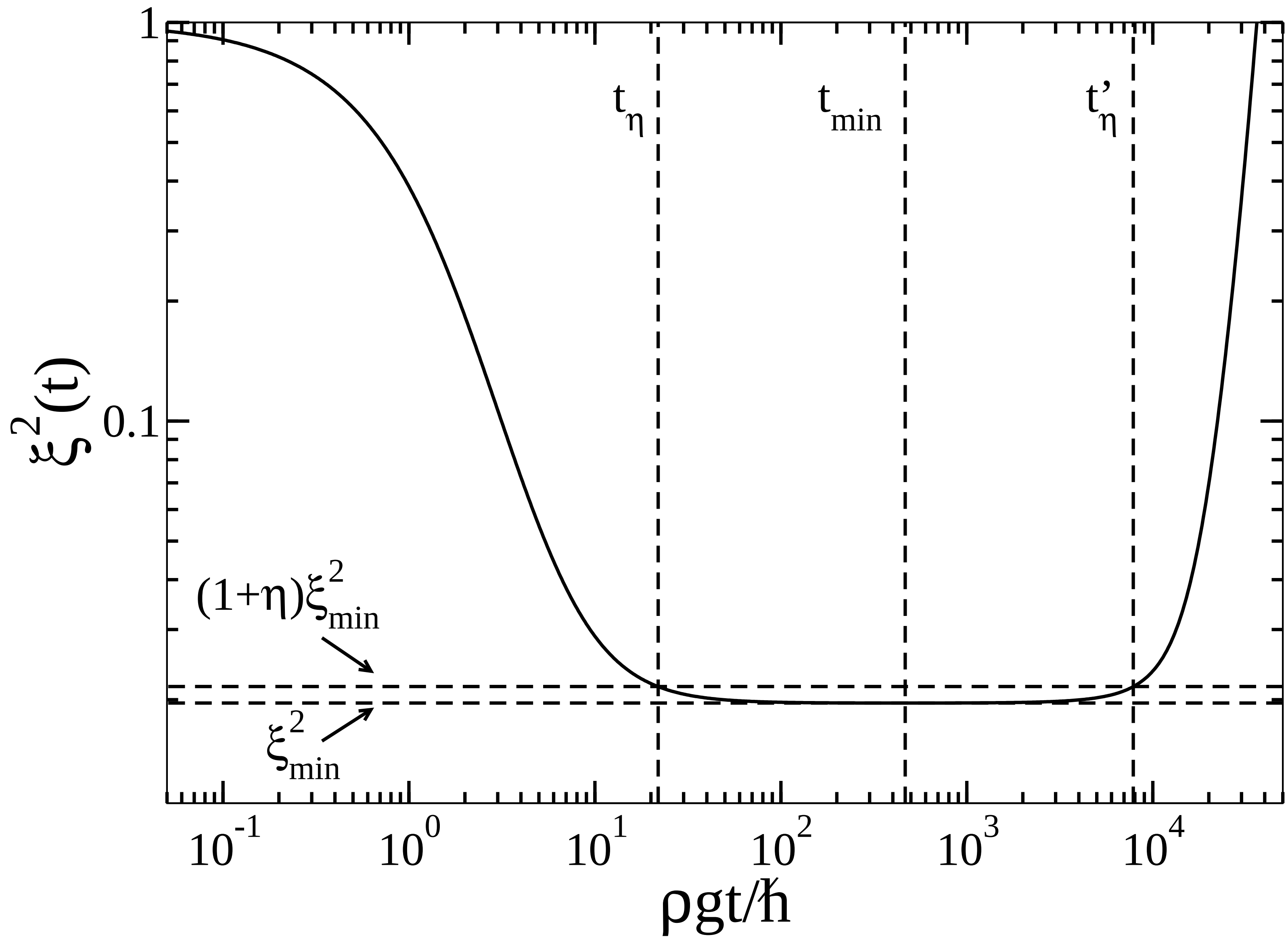
$$\frac{\rho g t_{\rm min}}{\hbar} \overset{\text{lim.therm.}}{\sim} \left[\frac{N}{\epsilon(1+\epsilon)^3}\right]^{1/4}$$

- **In practice, close-to-best-squeezing time:**

$$\xi^2(t_\eta) = (1+\eta)\xi^2_{\rm min} \Longrightarrow \frac{\rho g t_\eta}{\hbar} \overset{\text{lim.therm.}}{\longrightarrow} \frac{1-\eta\epsilon}{(1+\epsilon)(\eta\epsilon)^{1/2}}$$

taking for example $\eta = 0.1$

$\xi^2(t)$ FOR THE TOY MODEL



FAST DERIVATION IN THERMODYNAMIC LIMIT

Modulus-phase representation for strongly occupied modes:

$$a = e^{i\theta_a} N_a^{1/2}, \quad [N_a, \theta_a] = i$$

- Much before the phase collapse time, $\rho g t/\hbar \ll N^{1/2}$:

$$S_x \simeq \frac{N}{2}, \quad S_y \simeq -\frac{N}{2}(\theta_a - \theta_b), \quad S_z = \frac{N_a - N_b}{2}$$

- From the Heisenberg equations for the phase operators:

$$(\theta_a - \theta_b)(t) = (\theta_a - \theta_b)(0) - \chi t[2S_z + D]$$

- Taking first the large N then the large time limit:

$$S_y \sim \frac{\rho g t}{\hbar}\sigma_y, \quad \sigma_y = S_z + \frac{1}{2}D$$

$$\xi^2_{\min} \to \frac{\langle \sigma_y^2 \rangle \langle S_z^2 \rangle - \langle \sigma_y S_z \rangle^2}{\langle \sigma_y^2 \rangle \langle S_z^2 \rangle} = \frac{\langle D^2 \rangle}{N + \langle D^2 \rangle} = \frac{\epsilon}{1+\epsilon}$$

- Also close-to-best-squeezing time can be recovered

THE MULTIMODE CASE

EXPERIMENTAL SEQUENCE

- Start with $\boldsymbol{N}$ atoms in internal state $\boldsymbol{a}$ at thermal equilibrium, no atoms in internal state $\boldsymbol{b}$
- Spatially homogeneous system, periodic boundary conditions
- At time $\mathbf{0}$, a $\boldsymbol{\pi/2}$ pulse to prepare spin $\boldsymbol{N/2}$ along $\boldsymbol{x}$:

$$\hat{\psi}_a(0^+) = \frac{1}{\sqrt{2}}\left[\hat{\psi}_a(0^-) - \hat{\psi}_b(0^-)\right]$$

- Evolution with interactions among each internal state: $g_{aa} = g_{bb} = g$. No crossed interactions: $g_{ab} = 0$
- Mean spin remains aligned along $\boldsymbol{x}$: one can still use

$$\Delta S^2_{\perp,\mathrm{min}} = \frac{1}{2}\left[\langle S_y^2\rangle + \langle S_z^2\rangle - |\langle (S_y + iS_z)^2\rangle|\right]$$

NUMBER-CONSERVING BOGOLIUBOV THEORY

- Modulus-phase representation for a_0, b_0
- Number conserving non-condensed fields:

$$\hat{\Lambda}_a(\mathrm{r}) = e^{-i\theta_a}\hat{\psi}_{a\perp}(\mathrm{r}) = \sum_{\mathrm{k}\neq 0}(U_k b_{a\mathrm{k}} + V_k b^\dagger_{a-\mathrm{k}})\frac{e^{i\mathrm{k}\cdot\mathrm{r}}}{V^{1/2}}$$

- The quasiparticle annihilation operators evolve with phase factors $e^{-i\epsilon_\mathrm{k}t/\hbar}$
- Evolution of phase operators (previous lecture):

$$(\theta_a - \theta_b)(t) = (\theta_a - \theta_b)(0^+) - \frac{gt}{\hbar V}[2S_z + \mathcal{D}]$$

$$\mathcal{D} \underset{\text{approx.}}{\overset{\text{secular}}{\simeq}} \sum_{\mathrm{k}\neq 0}(U_k + V_k)^2(\hat{n}_{a\mathrm{k}} - \hat{n}_{b\mathrm{k}})$$

where $\hat{n}_{\sigma\mathrm{k}}$ is a number of Bogoliubov quasi-particles.

BOGOLIUBOV RESULTS

Calculate with a double expansion technique:

- Up to first order in ϵ_{Bog}= non-condensed fraction
- To leading order in $1/N$ in thermodynamic limit:

$$\mathcal{D} \approx (N\epsilon_{\mathrm{Bog}})^{1/2}, \quad \theta_a - \theta_b \approx \frac{1}{N^{1/2}}$$

Central result:

$$\boxed{\xi^2(t) = \frac{1 - O(\epsilon_{\mathrm{Bog}})}{(\tau + \sqrt{1+\tau^2})^2} + \frac{2[\frac{\langle \mathcal{D}^2 \rangle}{N}\tau^2 + O(\epsilon_{\mathrm{Bog}})]}{(\tau + \sqrt{1+\tau^2})\sqrt{1+\tau^2}}}$$

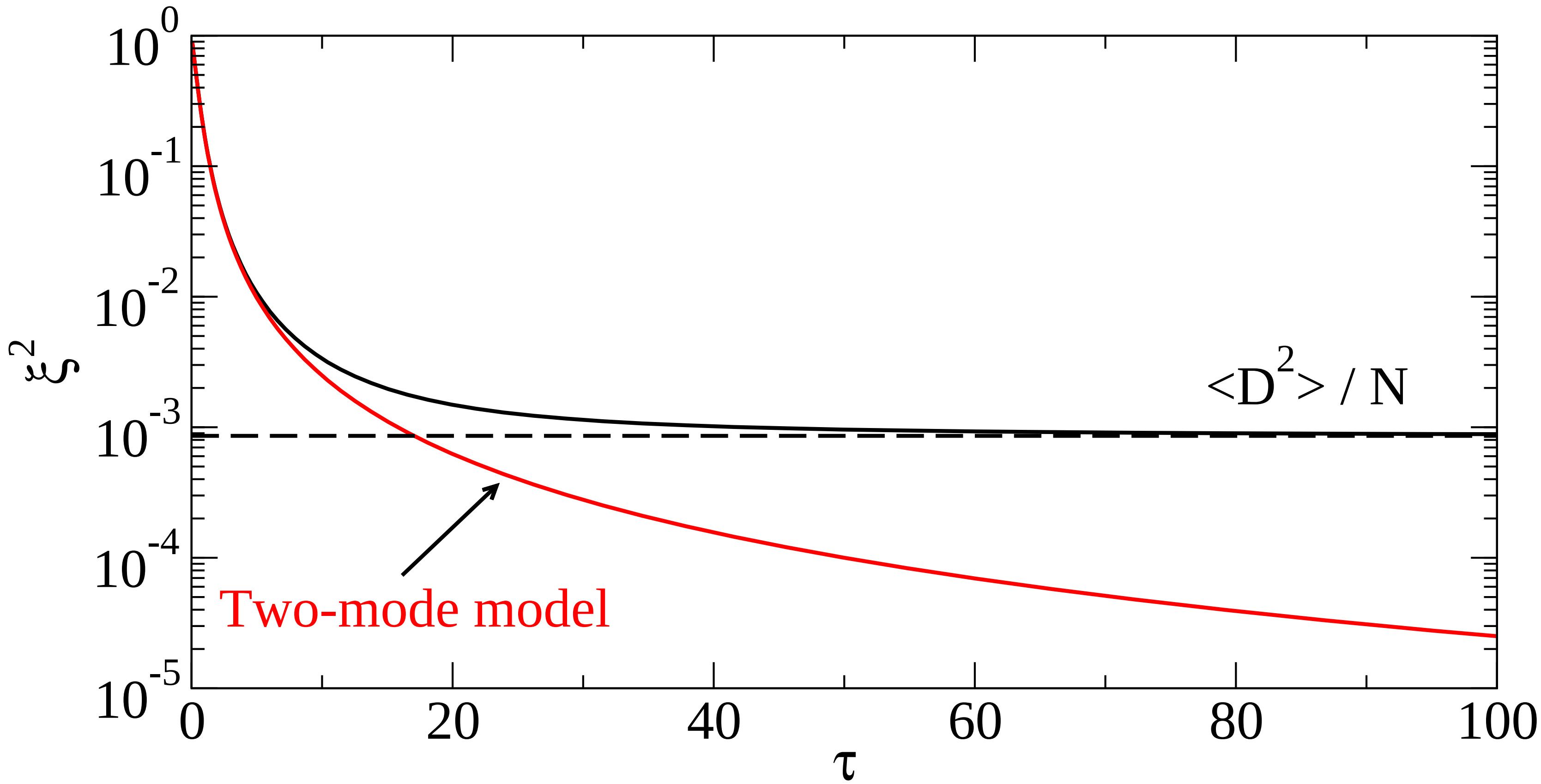
with renormalized time $\tau = [\rho g t/(2\hbar)][1 + O(\epsilon_{\mathrm{Bog}})]$

- First term [without $O(\epsilon_{\mathrm{Bog}})$] is Kitagawa-Ueda model
- Second term saturates to minimal squeezing:

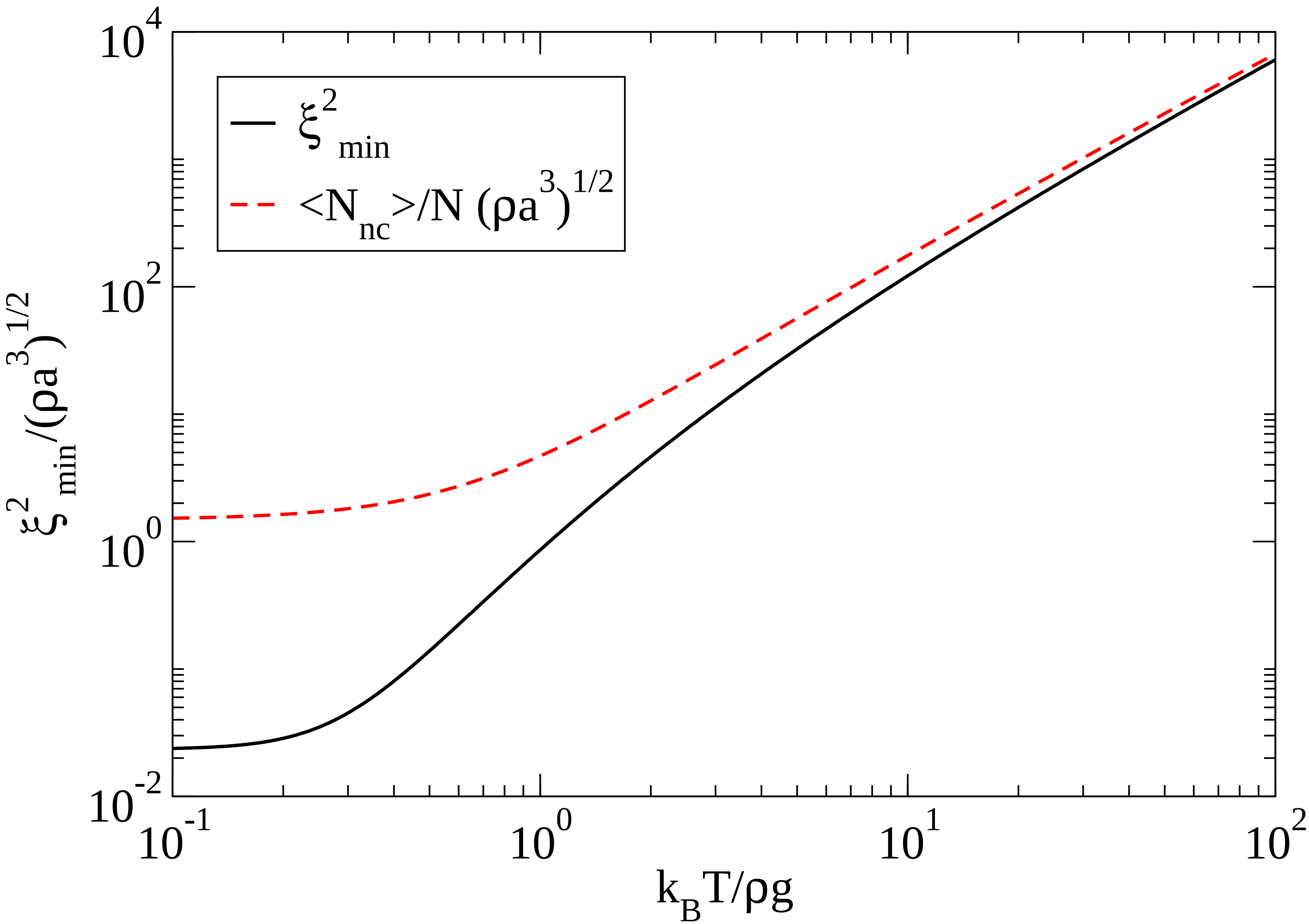
$$\xi^2_{\mathrm{min}} = \frac{\langle \mathcal{D}^2 \rangle}{N} = (\rho a^3)^{1/2} f(k_B T/\rho g)$$

$\xi^2(t)$ FOR BOGOLIUBOV THEORY

$$(\rho a^3)^{1/2} = 10^{-3}, k_B T/\rho g = 1$$



MINIMAL ξ^2 FOR BOGOLIUBOV THEORY



VALIDITY CONDITIONS

- System out-of-equilibrium after pulse
- Will thermalize, this is neglected in Bogoliubov theory
- Have the close-to-best-squeezing time

$$\frac{\rho g t_\eta}{\hbar} \simeq \frac{1}{\eta^{1/2}\xi_{\min}}$$

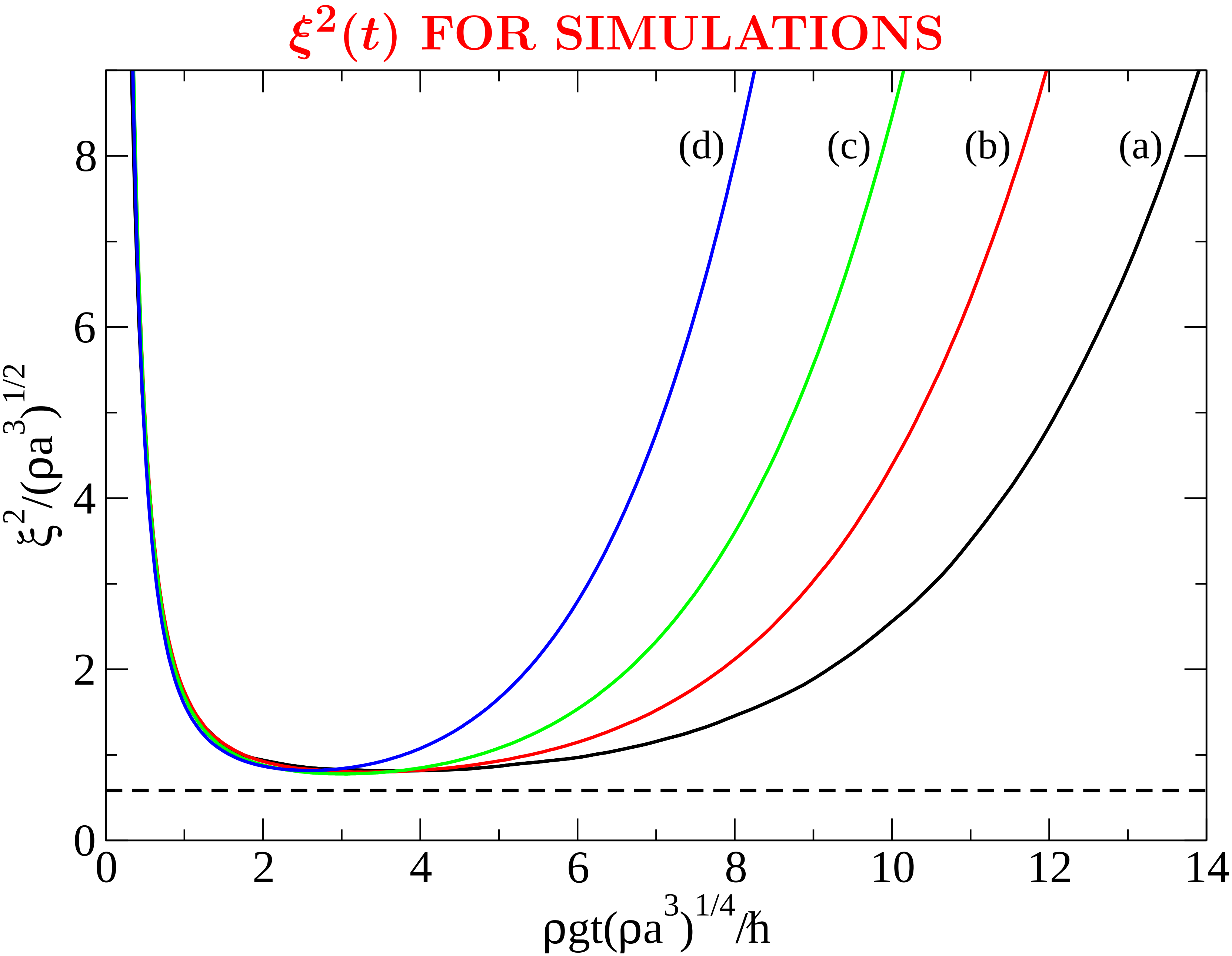
smaller than thermalisation time, estimated by Beliaev-Landau damping rates of modes of energy $k_B T$ or ρg:

$$\frac{\rho g t_{\mathrm{therm}}}{\hbar} \propto \frac{1}{(\rho a^3)^{1/2}}$$

- Validity condition satisifed in weakly interacting limit:

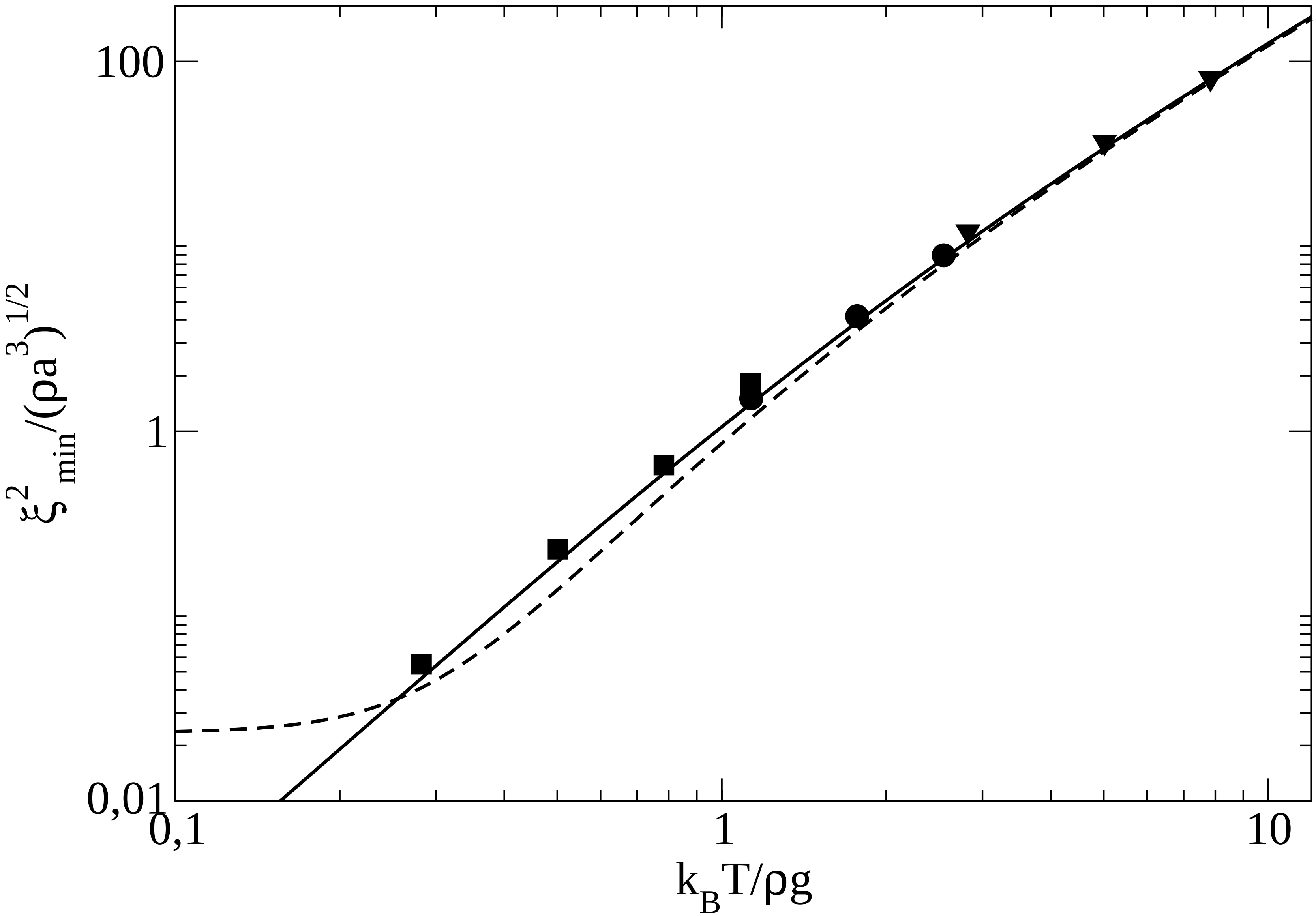
$$\frac{t_\eta}{t_{\mathrm{therm}}} \propto (\rho a^3)^{1/4} \ll 1$$

COMPARISON TO CLASSICAL FIELD SIMULATIONS

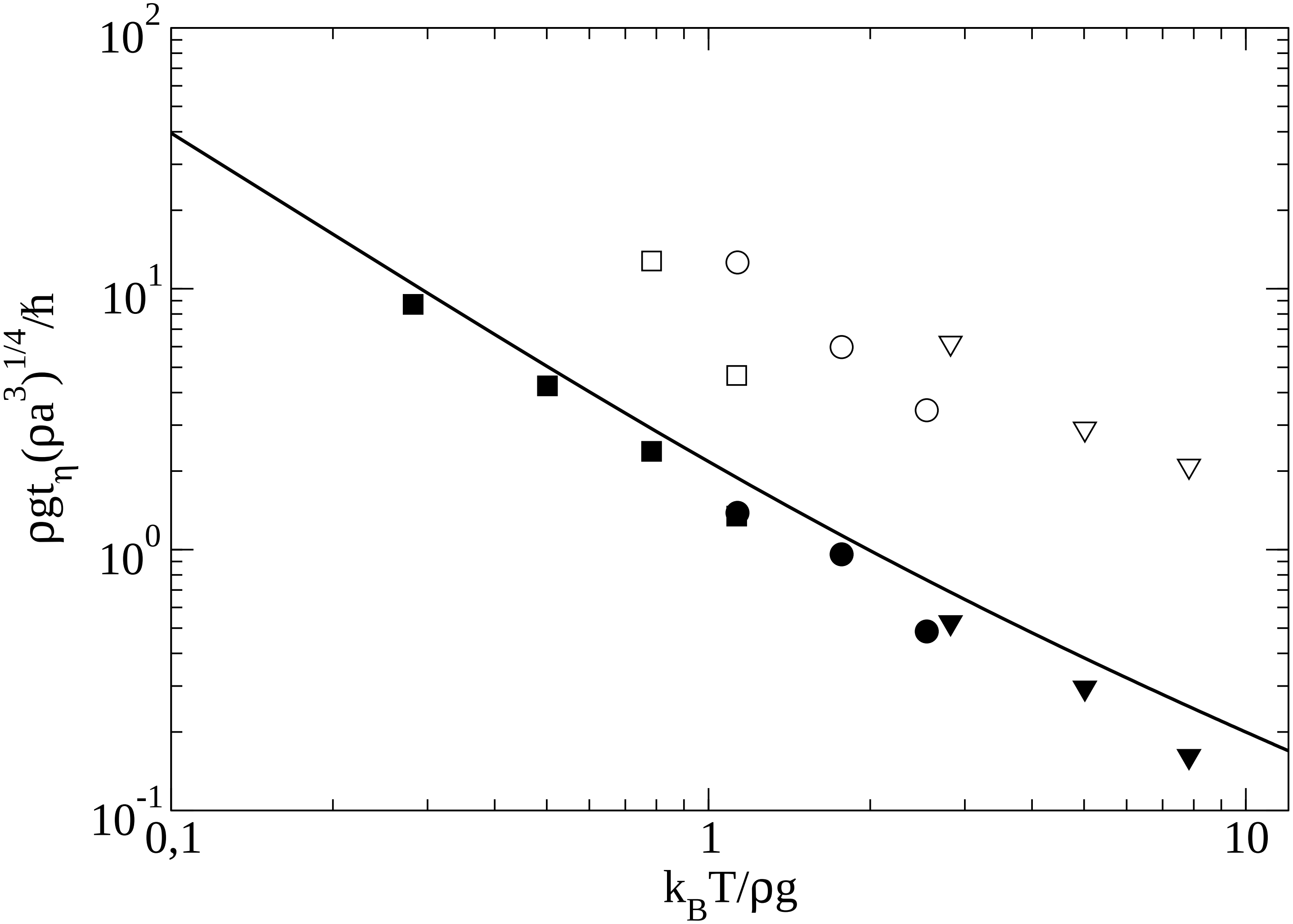


$\xi^2(t)$ FOR SIMULATIONS
(d)
(c)
(b)
(a)
0
2
4
6
8
0
2
4
6
8
10
12
14
$\xi^2/(\rho a^3)^{1/2}$
$\rho g t(\rho a^3)^{1/4}/\hbar$

MINIMAL ξ^2 FOR SIMULATIONS

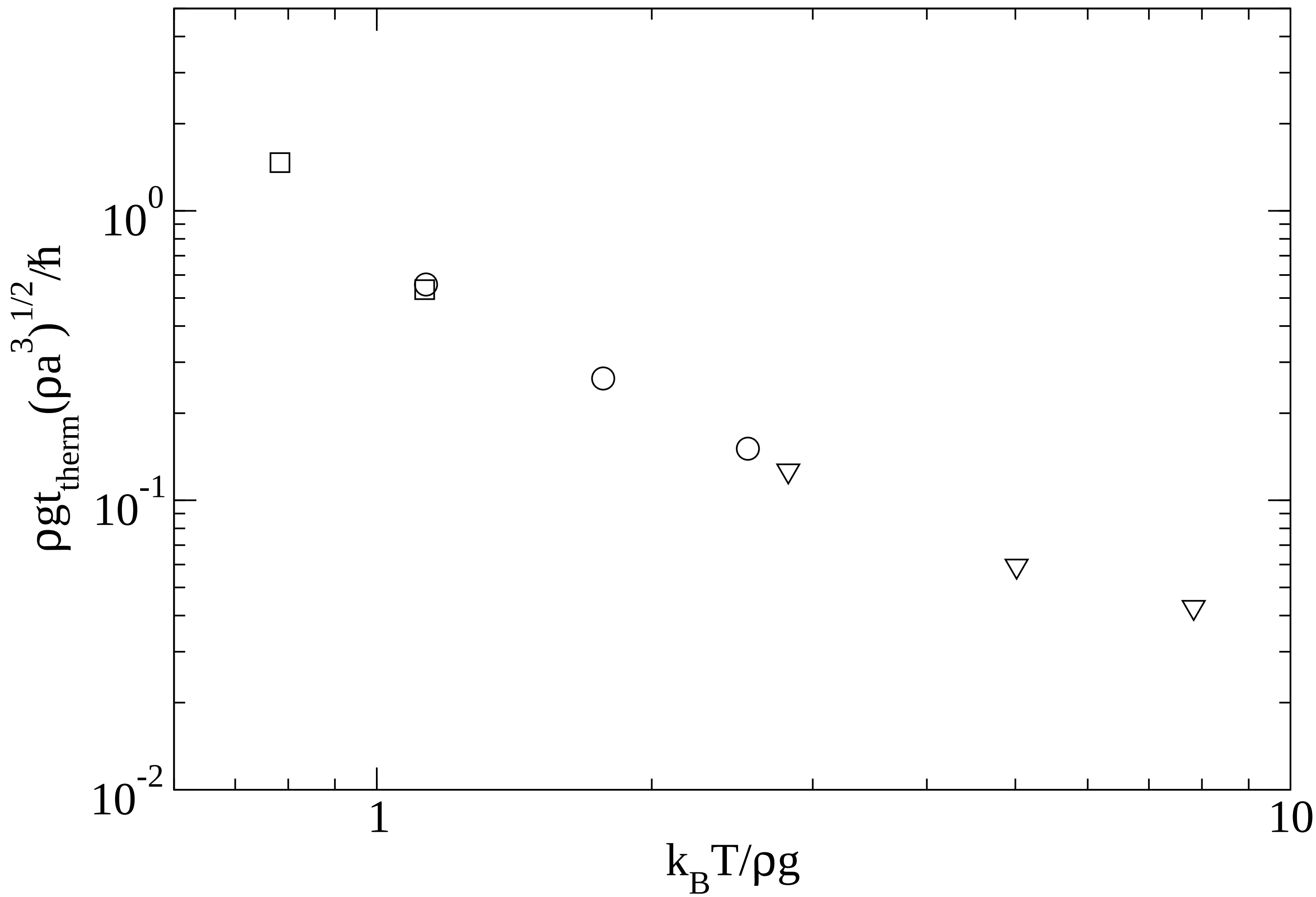


CLOSE-TO-BEST-SQUEEZING TIME FOR SIMULATIONS (filled symbols)



Empty symbols: thermalisation time.

THERMALISATION TIME FOR SIMULATIONS



Summary of results for spin squeezing:

- For atoms with two internal states a and b, apply a $\pi/2$ pulse on a condensate initially in a. Due to interactions, phase state transformed into spin squeezing state
- If injected in an atomic clock, statistical uncertainty on clock frequency after interrogation time τ:

$$\Delta\omega_{ab} = \frac{\Delta S_{\perp,\min}}{\langle S_x\rangle\tau} \equiv \frac{\xi}{N^{1/2}\tau}$$

- Spin dynamics is a phase dynamics: S_z =const, S_x ≈const,

$$S_y \propto \theta_a - \theta_b \propto (N_a - N_b + D)t\partial_N\mu/\hbar$$

where D due to multimode nature of the fields (random dephasing environment). Best squeezing in weakly interacting, thermodynamic limit does not vanish:

$$\xi^2_{\min} \simeq \frac{\langle D^2\rangle}{N}$$

- We have extended the theory to the harmonically trapped case: for $k_BT \simeq \mu \simeq 10\hbar\omega$ and $N = 10^6$, $\xi \approx 1/30$ vs $\xi \approx 1/100$ for Kitagawa-Ueda model.
- Reaching such squeezing levels requires reduction of technical noise in experiments.

Our publications on the subject

- Yun Li, Y. Castin, A. Sinatra, "Optimum spin-squeezing in Bose- Einstein condensates with particle losses", Phys. Rev. Lett. 100, 210401 (2008).
- A. Sinatra, E. Witkowska, J.-C. Dornstetter, Yun Li, Y. Castin, "Limit of Spin Squeezing in Finite Temperature Bose-Einstein Condensates", Phys. Rev. Lett. 107, 060404 (2011).
- A. Sinatra, J.-C. Dornstetter, Y. Castin, "Spin squeezing in Bose- Einstein condensates: Limits imposed by decoherence and non-zero temperature", Front. Phys. 7, 86 (2012).
- A. Sinatra, E. Witkowska, Y. Castin, "Spin squeezing in finite temperature Bose-Einstein condensates: Scaling

with the system size", Eur. Phys. J. Special Topics 203, 87 (2012).

- A. Sinatra, Y. Castin, E. Witkowska, "Limit of spin squeezing in trapped Bose-Einstein condensates", arXiv:1303.0299