BEYOND ZERO-RANGE CLOSE TO UNITARY LIMIT

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OUTLINE OF THE TALK

- **Finite range corrections in the fermionic gas:**
  - **generalized relations** *à la Tan*
  - **zero-temperature damping (collapse) of the breathing mode**
- **How are Efimovian trimers born ?**
  - **analytics on a narrow Feshbach resonance for the $2+1$ fermionic problem**

FINITE RANGE CORRECTIONS

IN THE FERMIONIC GAS

CONTEXT AND MOTIVATION

Real spin-1/2 fermions do not have a contact interaction:

- interaction potential $V(r)$ with short range (here for simplicity, compact support of radius b)
- repulsive hard-core part (to avoid collapse) and negative parts so that s-wave interaction is resonant:

$$\text{scattering length } a, \quad |a| \gg b$$

- cold and dilute gas: $b \ll \frac{1}{k_F}, \lambda_{\mathrm{dB}} = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2}$

How does this differ from the Wigner-Bethe-Peierls zero-range model?

$$\psi(\mathrm{r}_1,\ldots,\mathrm{r}_N) \underset{r_{ij}\to 0}{=} \left(\frac{1}{r_{ij}}-\frac{1}{a}\right) A_{ij}\left(\mathrm{R}_{ij},(\mathrm{r}_k)_{k\neq i,j}\right)+O(r_{ij})$$

FINITE $1/a$ CORRECTIONS TO UNITARY LIMIT

General relations for the zero-range model:

- Tan relation (generalizing a Lieb relation to 3D):

$$\frac{dE}{d(-1/a)} = \frac{\hbar^2}{4\pi m} \sum_{i<j} \langle A_{ij} | A_{ij} \rangle$$

- Can be proved using Hellmann-Feynman theorem (as Lieb did in 1D) for a lattice model
- The same derivative (contact) appears in short range pair distribution function, or tail of momentum distribution by properties of the Fourier transform, as in 1D [Olshanii, Dunjko (2003)]
- Experimental checks, in particular by Deborah Jin.
- Morality: The unitary solution contains in itself information on finite $1/a$ corrections.

FINITE RANGE CORRECTIONS TO ZERO-RANGE LIMIT

- Leading correction is universal: depends only on effective range r_e, not on details of $V(r)$
- The zero-range solution $(\forall 1/a)$ contains in itself information on finite range corrections (Werner, Castin, 2012):

$$\left(\frac{\partial E}{\partial r_e}\right)_a \stackrel{3D}{=} 2\pi \sum_{i<j} \langle A_{ij} | [E - H_{ij}] | A_{ij} \rangle$$

with reduced Hamiltonian obtained by removing the relative particle ij:

$$H_{ij} = \left(H - \frac{p_{ij}^2}{m}\right)_{\vec{r}_i, \vec{r}_j \to \vec{R}_{ij}}$$

$$H_{ij} = -\frac{\hbar^2}{4m}\Delta_{\mathrm{R}_{ij}} - \frac{\hbar^2}{2m}\sum_{k \neq i,j} \Delta_{\mathrm{r}_k} + 2U(\mathrm{R}_{ij}) + \sum_{k \neq i,j} U(\mathrm{r}_k)$$

EXPERIMENTALLY RELEVANT CONSEQUENCES

1. Pair distribution function at short distances:

$$\bar{g}^{(2)}_{\uparrow\downarrow}(\vec{r}) = \frac{m}{4\pi\hbar^2}\left[\frac{\partial E}{\partial(-1/a)}\left(\frac{1}{r}-\frac{1}{a}\right)^2 - 2\frac{\partial E}{\partial r_e} + O(r)\right]$$

- This improves Tan's result [where $O(1)$ is neglected].

2. Equation of state of homogeneous gas can be deduced from the density profile of the trapped gas [Nascimbène, Navon, Jiang, Chevy, Salomon (2010)]

- Ku, Sommer, Cheuk, Zwierlein (2012): precise measurement of ξ on lithium [$\mu(T=0)=\xi E_F$ for the unpolarized unitary gas]
- Universality of finite-range correction + Bo Gao formula giving $r_e(R_*, a_{\rm bg}, C_6)$ + Carlson's Monte Carlo calculations $\longrightarrow$ impact on ξ is $\approx 0.5\%$.

WITHIN A $SO(2,1)$ LADDER

- Reminder of ladder structure (unitary gas in an isotropic harmonic trap):

$$E_q = E_{\mathrm{CoM}} + (s+1+2q)\hbar\omega, \quad q \in \mathbb{N}$$

  with q quantum number for undamped breathing mode

- general N-body problem unsolved: $\partial E/\partial r_e$ unknown

- Separability of ψ in hyperspherical coordinates implies separability of A_{ij}, with known hyperradial part, and with unknown hyperangular part common to a ladder. This leads to explicit expressions (in terms of s and q and the Γ function) for

$$\frac{\partial E_q/\partial r_e}{\partial E_0/\partial r_e} \quad \text{and} \quad \frac{\partial E_q/\partial(1/a)}{\partial E_0/\partial(1/a)}$$

  [(Werner, Castin, 2012). See also Moroz (2012).]

Large N, unpolarized case:

- One evaluates the derivatives of E_0 using the local density approximation.
- Corrections to E_q linear in q: change of breathing frequency 2ω. Agrees with superfluid hydrodynamics (Bulgac, Bertsch)
- Corrections to E_q quadratic in q: collapse (zero-temperature damping) of breathing mode:

$$1/t_{\text{collapse}} = \frac{|\delta\omega|}{N^{2/3}} \left| \frac{C_1}{k_F a} + C_2 k_F r_e \right|$$

  where the mode was excited by the abrupt trap frequency change $\delta\omega$, and $C_1 \simeq 0.21$, $C_2 \simeq 0.048$.
- There is a revival of the breathing mode. At half the revival time, a Schrödinger cat state [Yurke, Stoler (1986)].

SVISTUNOV'S QUESTION

Boris Svistunov (Amherst):

- Relevant parameters close to the zero-range limit are the ones that appear in the deviation of the two-body T-matrix from its zero-energy limit:

$$\langle \mathrm{k}_1, \mathrm{k}_2 | T(E + i0^+) | \mathrm{k}_3, \mathrm{k}_4 \rangle, \quad k_1, k_2, k_3, k_4, E \longrightarrow 0$$

- For the on-shell T-matrix, this is the effective range r_e.
- For the off-shell T-matrix $(E_3 + E_4 \neq E_1 + E_2)$, another effective range ρ_e appears.
- Why does this ρ_e not show up in the first finite-range correction to the energy ?

Answer: low-energy s-wave T-matrix [Gibson (1972)]:

$$\frac{t_0(k,k';E)}{t_0(E)} - 1 \underset{k,k',E\to 0}{\sim} \left(\frac{2mE}{\hbar^2} - k^2 - k'^2\right)\frac{1}{2}\rho_e^2$$

$$\frac{1}{2}\rho_e^2 = \int_0^{+\infty} dr\, r[(1-r/a) - r\chi_0(r)]$$

where χ_0 is the zero-energy scattering wavefunction.

- Reminiscent of the Landau-Smorodinski formula for r_e (1944):

$$\frac{1}{2}r_e = \int_0^{+\infty} dr\left[(1-r/a)^2 - r^2\chi_0^2(r)\right]$$

- If $V(r)$ has no deeply bound states, one can show that

$$0 \le r_e \le 2b \qquad 0 \le \rho_e^2 \le b^2$$

and the contribution of ρ_e to the energy correction is $O(b^2)$, negligible as compared to the one of r_e.

Warning! Case of lattice models (of spacing b):

- Galilean invariance is broken. T-matrix depends on relative wavenumber k and on center-of-mass wavevector $\mathbf{K}$ [Burovski, Prokof'ev, Svistunov, Troyer (2006)]:

$$\frac{1}{t(\mathbf{K},k)} = \frac{m}{4\pi\hbar^2}\left(\frac{1}{a} + ik - \frac{1}{2}r_e k^2 - \frac{1}{2}R_e K^2\right) + \ldots$$

- Then energy correction to first order in lattice spacing b:

$$\delta E = \frac{\partial E}{\partial r_e} r_e + \frac{\partial E}{\partial R_e} R_e$$

- One can zero both r_e et R_e with taylored dispersion relation [Werner, Castin (2012)]

DERIVATION OF $\partial E/\partial r_e$

- **D'après le théorème de Hellmann-Feynman :**

$$\frac{dE}{db} = \sum_{i=1}^{N_\uparrow} \sum_{j=N_\uparrow+1}^{N} \int d^3r_1 \ldots d^3r_N |\psi(\mathrm{r}_1, \ldots, \mathrm{r}_N)|^2 \partial_b V(r_{ij}; b)$$

- **Ansatz pour deux particules proches :**

$$\psi(\mathrm{r}_1, \ldots, \mathrm{r}_N) \simeq \chi(r_{ij}) A_{ij}(\mathrm{R}_{ij}, (\mathrm{r}_k)_{k\neq i,j})$$

avec χ invariant par rotation car interaction résonnante dans l'onde s seulement.

$$\frac{dE}{db} \simeq \sum_{i<j} \int d^3R_{ij} \prod_{k\neq i,j} \int d^3r_k \; A_{ij}^2(\mathrm{R}_{ij}, (\mathrm{r}_k)_{k\neq i,j})$$

$$\times \int d^3r_{ij} \; \chi^2(r_{ij}) \partial_b V(r_{ij}; b)$$

- On injecte l'ansatz dans l'équation de Schrödinger et on néglige le potentiel de piégeage :

$$\mathcal{E}\chi(r_{ij}) \simeq [-\frac{\hbar^2}{m}\Delta_{\mathrm{r}_{ij}} + V(r_{ij};b)]\chi(r_{ij})$$

$$\mathcal{E} = E - \frac{1}{A_{ij}(\mathrm{R}_{ij},(\mathrm{r}_k)_{k\neq i,j})} H_{ij} A_{ij}(\mathrm{R}_{ij},(\mathrm{r}_k)_{k\neq i,j}) \equiv \frac{\hbar^2 k^2}{m}$$

- Normalisation de χ tendant vers $r^{-1} - a^{-1}$ si $k \to 0$:

$$\chi(r) \underset{r\to\infty}{=} \frac{1}{f_k}\frac{\sin(kr)}{kr} + \frac{e^{ikr}}{r} \in \mathbb{R}$$

- Amplitude de diffusion d'après le théorème optique :

$$f_k = -\frac{1}{ik + u(k)} \quad \text{et} \quad u(k) \in \mathbb{R}$$

- **Développement à faible k :**

$$u(k) = \frac{1}{a} - \frac{1}{2}k^2 r_e + O(k^4)$$

- **Relation obtenue par l'astuce du wronskien :**

$$\int d^3r\ \chi^2(r)\partial_b V(r;b) = -\frac{4\pi\hbar^2}{m}\partial_b u(k) \simeq 2\pi\mathcal{E}\frac{dr_e}{db}$$

à reporter dans l'expression précédemment obtenue :

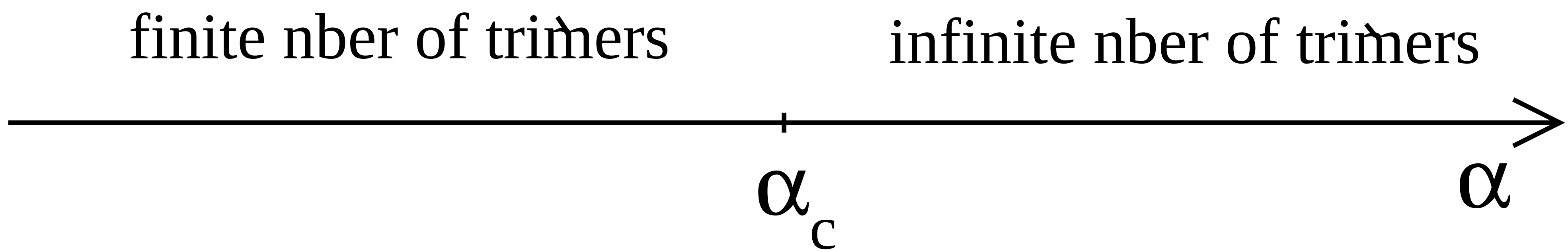
$$\frac{dE}{db} \simeq \sum_{i<j}\int d^3R_{ij} \prod_{k\neq i,j}\int d^3r_k\ A_{ij}^2(\mathrm{R}_{ij},(\mathrm{r}_k)_{k\neq i,j})$$

$$\times \int d^3r_{ij}\ \chi^2(r_{ij})\partial_b V(r_{ij};b)$$

HOW ARE EFIMOVIAN TRIMERS BORN ?

THE BIRTH OF EFIMOV TRIMERS

Most interesting case: a control parameter to switch on/off the Efimov effect [here, $1/a = 0$]:



Is it a phase transition ? Critical exponents ?

- **For s_3 : critical exponent $1/2$, $|s_3| \propto (\alpha - \alpha_c)^{1/2}$**

$$\Lambda(s, \alpha) = 0, \quad s = 0 \text{ double root for } \alpha = \alpha_c$$

- **What about κ_3, i.e. global energy scale in trimer spectrum ? Efimov's theory gives the function $\Lambda(s, \alpha)$ but not κ_3...**

AN EXACTLY SOLVABLE MODEL

Here $2+1$ fermionic system. From zero range theory:

- Control parameter α: fermion-to-impurity mass ratio.
- Lecture 3: Efimov effect for $\alpha > \alpha_c = 13.6069\ldots$.
- Why ? Born-Oppenheimer picture for very light impurity: Effective long range attraction among fermions,

$$V_{\mathrm{BO}}(r) \propto -\frac{\hbar^2}{m_{\mathrm{impur}} r^2}$$

  that beats the necessarily odd-l centrifugal barrier

$$V_{\mathrm{fugal}}(r) = \frac{\hbar^2 l(l+1)}{m r^2}$$

  when $m_{\mathrm{impur}} \to 0$.
- For increasing α, one has successively apparition of infinite number of Efimov trimers with angular momentum $l=1, l=3, \ldots$

Resonant impurity-fermion interaction on a narrow Feshbach resonance:

- Apart from scattering length, interaction characterized by Feshbach length $R_* \gg$ true potential range b (Petrov)
- Large Feshbach length due to very weak coupling Λ between closed and open channel:

$$R_* = \frac{\pi\hbar^4/\mu^2}{\Lambda^2}$$

- Corresponds to scattering amplitude with $r_e = -2R_*$:

$$f_k = -\frac{1}{\frac{1}{a} + ik + k^2 R_*}$$

- The parameter κ_3 was then calculated exactly for three bosons [Gogolin, Mora, Egger (2008)]
- We have extended this calculation to the $2+1$ fermionic problem [Y. Castin, E. Tignone (2011)]

MAIN RESULT ($a^{-1} = 0$): NO NEW CRITICAL EXPONENT

For any fixed odd angular momentum l:

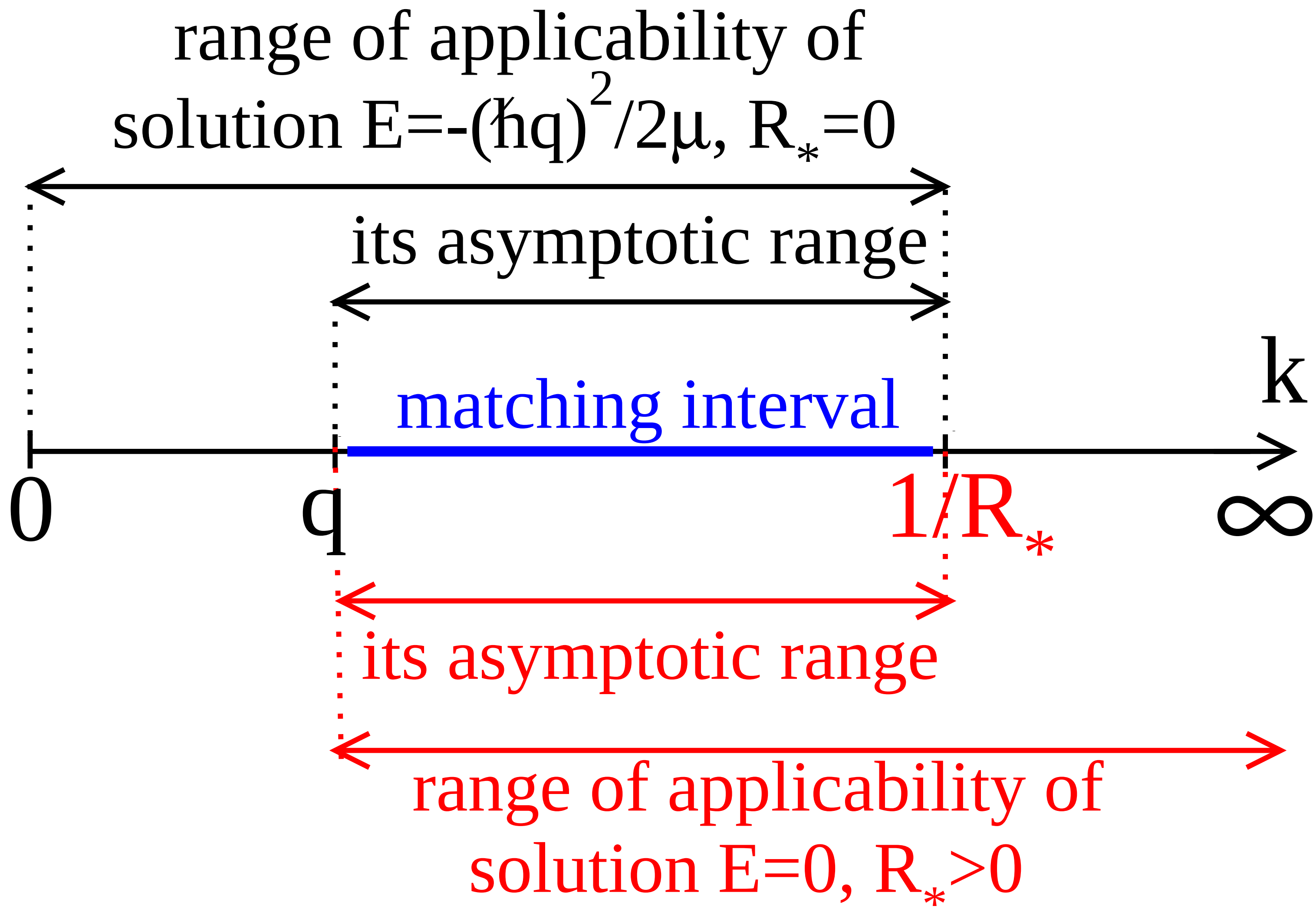
- For $\alpha < \alpha_c$, there is no trimer [if $l = 1$, Kartavtsev-Malykh trimers expected for large but finite a]
- For $\alpha > \alpha_c$, there is an infinite number of trimers. Ground state labeled with quantum number $n = 1$.
- If one tends from above to the critical mass ratio:

$$E_n \underset{\alpha \to \alpha_c^+}{\sim} -\frac{2\hbar^2}{\mu R_*^2} e^{2A} e^{-2\pi n/|s_3|}$$

$$A = 3\psi(1) - 2\psi(l+1) - \psi(l+2)$$
$$+ \sum_{k \geq 1} \psi(x_k) + \psi(1 + x_k) - \psi(l + 1 + 2k) - \psi(l + 2 + 2k)$$

with $\psi(x) = \Gamma'(x)/\Gamma(x)$ is digamma function, and x_k are the real positive roots of $\Lambda(x, \alpha_c) = 0$

A GLIMPSE ON THE METHOD



HOW TO GET ZERO ENERGY SOLUTION

$$0 = (1+kR_*)D(k) + \int_0^\infty \frac{dk'}{k\pi\cos\nu} D(k') \int_{-1}^{1} du \frac{P_l(u)k'^2}{k^2 + k'^2 + 2kk'u\sin\nu}$$

where mass angle $\nu = \arcsin\frac{\alpha}{1+\alpha}$.

- The red term is scaling invariant: Unchanged if function $D(k')$ is replaced by $D(\lambda k')$ and k replaced by $\lambda^{-1}k$.
- In $X = \ln(kR_*)$ variable, becomes translationally invariant $\longrightarrow$ perform a Fourier transform.
- Then multiplication by $kR_* = e^X$ gives a translation by i in Fourier space:

$$0 = F(S+i) + \Lambda(iS,\alpha)F(S)$$

that can be solved in terms of the Γ function $[\Gamma(z+1) = z\Gamma(z)]$

Our publications on the subject

- F.Werner, L. Tarruell, Y. Castin, Number of closed-channel molecules in the BEC-BCS crossover, Eur. Phys. J. B 68, 401 (2009).
- Y. Castin, F. Werner, Single-particle momentum distribution of an Efimov trimer, Phys. Rev. A 83, 063614 (2011).
- Y. Castin, E. Tignone, Trimers in the resonant (2+1)-fermion problem on a narrow Feshbach resonance: Crossover from Efimovian to hydrogenoid spectrum, Phys. Rev. A 84, 062704 (2011).
- F. Werner, Y. Castin, General relations for quantum gases in two and three dimensions: Two-component fermions, Phys. Rev. A 86, 013626 (2012), and “II. Bosons and mixtures”, Phys. Rev. A 86, 053633 (2012).