Superfluidity and superconductivity. IHP, Paris, May 7 and 9, 2007

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Plan of the lecture-1

I. Bose superfluids.

1. Elementary excitations in quantum fluids.
2. Superfluidity of a Bose-liquid and Landau criterion.
3. Potentiality of flow. Landau-Feynman wave function. Bogoliubov conjecture and NDLRO.
4. Superfluid in rotation.
5. Dilute BEC gases.
6. BEC against superfluidity: interference against hydrodynamics.
7. Superfluidity in 2D. Berezinskii-Kosterlitz-Thouless transition.

Plan of lecture-2

II. Superfluidity of fermions and superconductivity.

1.Normal Fermi-liquid. Elementary excitation and Landau-Luttinger theorem.
2. Pairing and gap.
3. Pairing and NDLRO.
4. Gauge invariance and superconductivity. London equations.
5. Magnetic flux quantization.
6. Ginzburg-Landau equation and Abrikosov vortices.

Principal postulate of the condensed matter theory: Weakly excited states of an ordered body can be presented in the terms of quasiparticles (elementary excitations).

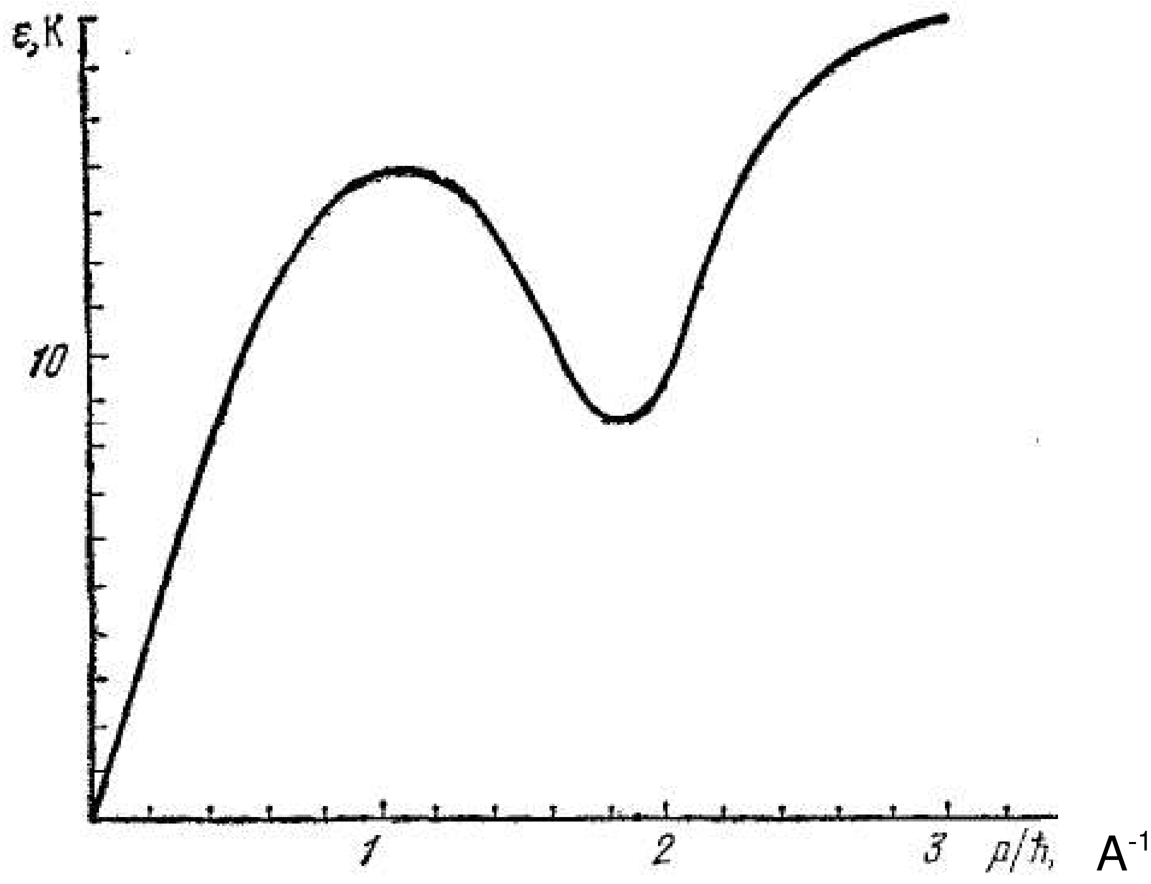
Let us one transfer to a body energy ϵ and momentum $\mathbf{p}$. Let $\epsilon(\mathbf{p})$ is minimal possible value of ϵ at given $\mathbf{p}$.

$\epsilon(\mathbf{p})$ is the energy of elementary excitation.

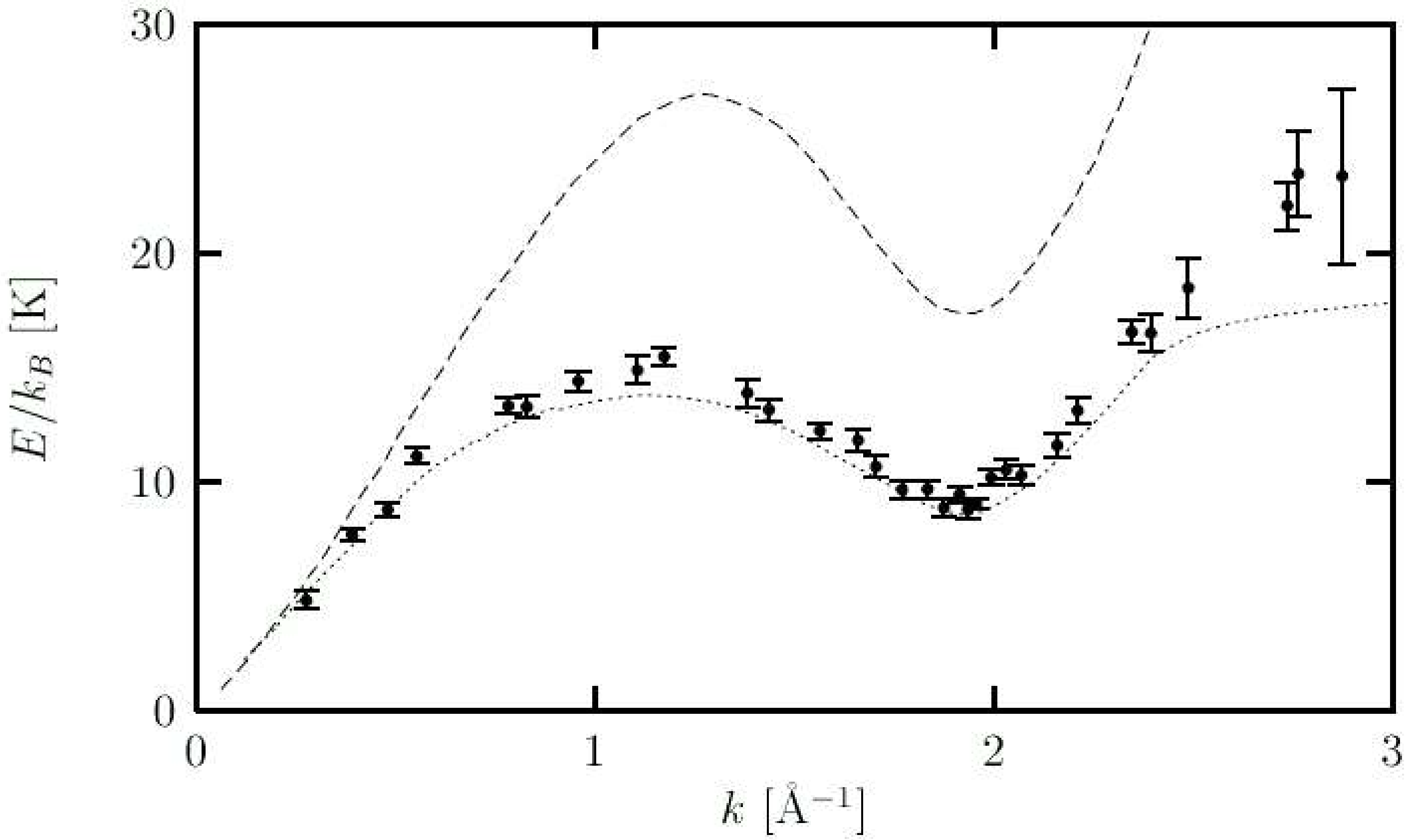
Distrubution function $\mathrm{N}(\mathbf{p})$ is complete characteristic of state of the body.

Bose and Fermi statistics.

Energy spectrum of liquid ^{4}He



Measured spectrum of ^{4}He



Decay of excitations

Decay of an excitation into two $\vee$ several is the only damping mechanism at $T = 0$

$$\epsilon(\boldsymbol{p}) = \sum_a \epsilon(\boldsymbol{p}_a)$$

$$\boldsymbol{p} = \sum_a \boldsymbol{p}_{\boldsymbol{a}}$$

Example : Beliaev (1958) damping at $p \to 0$:

$$\operatorname{Im} \epsilon \sim p^{5.}$$

Landau explanation of superfluidity

Dissipation means creation of elementary excitations.

Let us the fluid flowing along a capillary at velocity $\mathbf{v}_s$.

If an excitation with momentum $\mathbf{p}$ appears, its energy in the frame of liquid is $E_0+\epsilon(\boldsymbol{p})$

and in the laboratory frame

$$E_0+\epsilon(\boldsymbol{p})+\boldsymbol{p}\cdot\boldsymbol{v}+Mv^2/2.$$

The dissipation of energy means that

$$\epsilon(\boldsymbol{p})+\boldsymbol{p}\cdot\boldsymbol{v}<0.$$

This is impossible if

$$v<v_c\equiv\min\epsilon\frac{(\boldsymbol{p})}{\boldsymbol{p}}.$$

If $\mathbf{v}_c\neq 0$ the fluid is superfluid!

But: To be superfluid, the body must be fluid!

Normal part

According to Landau, gas of elementary excitations at $T \neq 0$ forms a normal fluid ''solved'' in the superfluid.

The equilibrium the distribution function of excitations is:

$$N(\boldsymbol{p}) = \frac{1}{\exp\left[\frac{\epsilon(\boldsymbol{p}) + \boldsymbol{p}\cdot(\boldsymbol{v}_s - \boldsymbol{v}_n)}{T}\right] - 1}.$$

$$\boldsymbol{j} = \rho_s \boldsymbol{v}_s + \rho_n \boldsymbol{v}_n$$

The density of normal part ρ_n is:

$$\rho_n = -\frac{1}{3}\int \frac{dN(\epsilon)}{d\epsilon} p^2 \frac{d\boldsymbol{p}}{(2\pi\hbar)^3}, \rho_s = \rho - \rho_n.$$

This is the principal equation of Landau's theory.

$$\rho_n(T=0) = 0, \ \ \rho_s(T=0) = \rho.$$

Potentiality of superfluid flow

In an ordinary liquid:

$$\frac{\partial}{\partial t}\mathbf{curl}\ \boldsymbol{v} = \frac{\eta}{mn}\Delta\left[\mathbf{curl}\ \boldsymbol{v}\right]$$

Viscosity $\eta = 0$:

$$\frac{\partial}{\partial t}\mathbf{curl}\ \boldsymbol{v}_s = 0$$

$$\mathbf{curl}\ \boldsymbol{v}_s = \boldsymbol{const} \equiv 0$$

Landau's equation of superfluid dynamics

Variables:

$$\rho(\boldsymbol{r},t),\boldsymbol{v}_s(\boldsymbol{r},t),N(\boldsymbol{p},\boldsymbol{r},t)\quad .$$

Equations:

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\left(\rho\boldsymbol{v}_s+\int\boldsymbol{p}N\frac{d\boldsymbol{p}}{(2\pi\hbar)^3}\right)=0,$$

$$\frac{\partial\boldsymbol{v}_s}{\partial t}+\mathrm{grad}\left(\frac{v_s{}^2}{2}+\mu_0+\int\frac{\partial\epsilon(\boldsymbol{p})}{\partial\rho}N\frac{d\boldsymbol{p}}{(2\pi\hbar)^3}\right)=0,$$

$$\frac{\partial N}{\partial t}+\frac{\partial N}{\partial\boldsymbol{r}}\frac{\partial\tilde{\epsilon}}{\partial\boldsymbol{p}}-\frac{\partial N}{\partial\boldsymbol{p}}\frac{\partial\tilde{\epsilon}}{\partial\boldsymbol{r}}=\mathrm{St}[N].$$

$$\tilde{\epsilon}(\boldsymbol{p},\boldsymbol{r},t)=\epsilon[p,\rho(\boldsymbol{r},t)]+\boldsymbol{p}\cdot\boldsymbol{v}_s(\boldsymbol{r},t)$$

Landau (and Feynman) wave function

$$\Phi(\boldsymbol{r}_{1,}\boldsymbol{r}_{2.}....\boldsymbol{r}_{N}) = \exp[i\sum_{i=1}^{N} S(\boldsymbol{r}_i)]\Phi_{gr}(\boldsymbol{r}_{1,}\boldsymbol{r}_{2.}....\boldsymbol{r}_{N}) \quad (1)$$

$$\Phi_{gr} - \textbf{many} - \textbf{body ground state wave function}\,(T=0)$$

$$\text{Mass flux}: \boldsymbol{j}(\boldsymbol{r}) = \hbar\,|\Phi_{gr}(\boldsymbol{r},\boldsymbol{r},.....\boldsymbol{r})|^2\,\nabla S(\boldsymbol{r}) \equiv n\,\boldsymbol{v}_s$$

$$\boldsymbol{v}_s = \frac{\hbar}{m}\nabla S(\boldsymbol{r})$$

$$\text{Equation}\,(1)\,\text{is exact for}\;\boldsymbol{v}_s = const\,,\;\text{when}\;S = (m/\hbar)\,\boldsymbol{v}_s\cdot\boldsymbol{r}\,.$$

$$\textbf{curl}\,\boldsymbol{v}_s = 0$$

But : Why this is wrong for fermions ? At T > $\mathbf{T_c}$?

Bose-Einstein condensation

A. Einstein, 1925:

$$k_B T_C = 3.3 \frac{\hbar^2}{m} n^{2/3} \quad , n = \frac{N}{V}$$

Momentum distribution:

$$T > T_C: \quad N_p = \frac{1}{e^{(\varepsilon - \mu)/k_B T} - 1}$$

$$T < T_C: \quad N_p = \frac{1}{e^{\varepsilon / k_B T} - 1} + N_0 \, \delta(\boldsymbol{p}), \; N_0 = N\left[1 - \left(T/T_C\right)^{3/2}\right]$$

Two classical limits of QM

1. Classical body: $m \to \infty$.

2. Classical electromagnetic waves:

Number of photons $N_{\text{ph}} = \dfrac{E}{\hbar\omega} \to \infty$.

From quantum electrodynamics to classical Maxwell equations

Commutation relations for the vector – potential :

$$[\hat{A}_i, \frac{\partial \hat{A}'_k}{\partial t}] \sim \delta(\boldsymbol{r} - \boldsymbol{r}'), \quad \hat{A}_i(\boldsymbol{r}, t) \rightarrow A_i(\boldsymbol{r}, t)$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r}, t) \rightarrow \boldsymbol{E}(\boldsymbol{r}, t), \ \hat{\boldsymbol{B}}\left(\boldsymbol{r}, t\right) \rightarrow \boldsymbol{B}\left(\boldsymbol{r}, t\right)$$

The Maxwell equations : $\quad \mathbf{curl}\ \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$ etc .

Bogoliubov conjecture. Bose-Einstein Condensation - classical limit for de Broglie waves

$$[\hat{\Psi}(\boldsymbol{r},t),\hat{\Psi}^{+}(\boldsymbol{r}',t)]=\delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$\hat{\Psi}\sim\sqrt{n}\,,\quad \hat{\Psi}\left(\boldsymbol{r},t\right)\rightarrow\Psi(\boldsymbol{r},t)$$

In an uniform condensate $\hat{\Psi}\rightarrow\sqrt{n}$.

N. N. Bogoliubov, 1947.

General definition of BEC

$$\hat{\Psi}(\boldsymbol{r},t)=\hat{\Psi}'(\boldsymbol{r},t)+\Psi_0(\boldsymbol{r},t)$$

One-body density matrix:

$$n^{(1)}(\boldsymbol{r},\boldsymbol{r}',t)=<\hat{\Psi}^{+}(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r}',t)>$$

$$=<\hat{\Psi}'^{+}(\boldsymbol{r},t)\hat{\Psi}'(\boldsymbol{r}',t)>+\Psi_0^{*}(\boldsymbol{r},t)\Psi_0(\boldsymbol{r}',t)$$

$$\rightarrow\Psi_0^{*}(\boldsymbol{r},t)\Psi_0(\boldsymbol{r}',t)\ \text{ at }\ |\boldsymbol{r}-\boldsymbol{r}'|\rightarrow\infty$$

L. Landau, 1951, O. Penrose, 1951.

$$\Psi_0(\boldsymbol{r}',t)=\sqrt{n_0}\ \text{ for ground state.}$$

Another definition:

$$<\Phi_{gr}(N-1)|\hat{\Psi}|\Phi_{gr}(N)>=\sqrt{n_0}.$$

BEC and superfluidity

$$\Psi_0(\boldsymbol{r}, t) = \sqrt{n_0}\, e^{iS}$$

$$\boldsymbol{j}_{\mathbf{0}} = m n_0 \boldsymbol{v}_{\mathbf{0}}, \quad \boldsymbol{v}_{\mathbf{0}} = \frac{\hbar}{m} \nabla S \equiv \boldsymbol{v}_s,$$

But: $\boldsymbol{j} = \rho_s \boldsymbol{v}_s ; n_0 \neq n \neq \rho_s / m$

At $T = 0$ in ^{4}He

$m n_0 / \rho \approx 10\,\%$, but $\rho_s / \rho = 1$.

Rotation. Vortex lines

Ordinary liquid:

$$v_r = \Omega r, \quad \textbf{curl } \boldsymbol{v} = 2\boldsymbol{\Omega} \neq 0$$

$$\text{Superfluid: } \Psi_0 = \sqrt{n_0}\, e^{iS}, \quad \boldsymbol{v}_s = \frac{\hbar}{m} \nabla S,$$

m is the real mass, not the effective one!

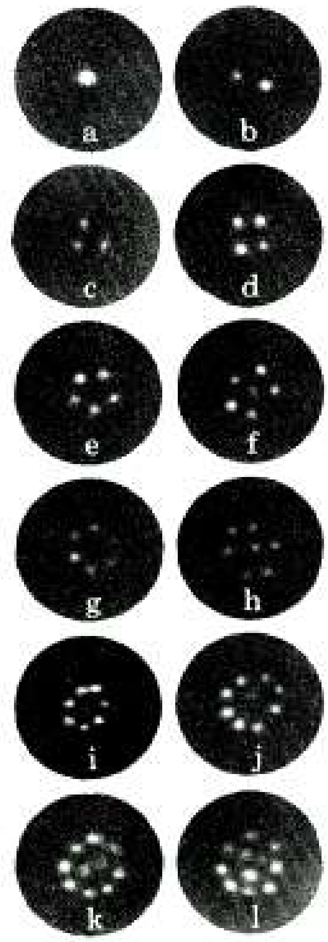
$$\oint \boldsymbol{v}_s \cdot d\boldsymbol{r} = \frac{\hbar}{m} \delta S = \frac{\hbar}{m} 2\pi l, \quad l = 0, \pm 1, \pm 2, \ldots$$

$$v_r = \frac{\hbar}{m} \frac{l}{r}$$

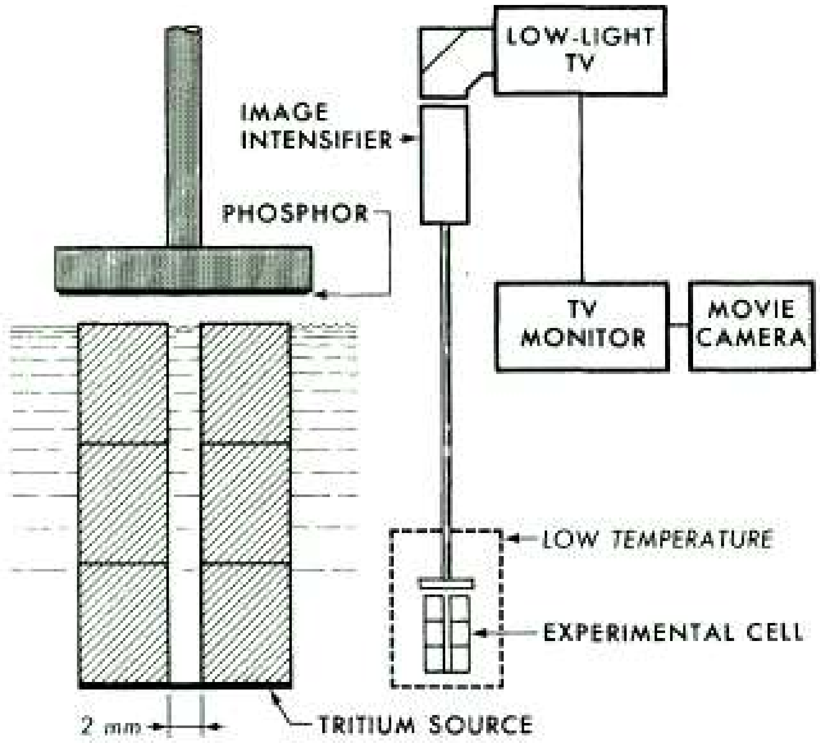
Vortex line. L. Onsager, 1949; R. Feynman, 1954.

Quantized vortices in rotating superfluid ^{4}He

Yarmchuk, Gordon and Packard, 1979.



Observation of quantized vortices in rotating ^4He



Yarmchuk, Gordon and Packard, 1979.

Critical angular velocity

The energy in rotating frame

$$E_{rot} = E - \boldsymbol{J} \cdot \Omega$$

must be minimal.

$$\epsilon_v = \pi L \rho_s \frac{\hbar^2}{m^2} \ln \frac{R}{\xi}, J = \pi L \rho_s \frac{\hbar}{m} R^2$$

First votex appears at

$$\Omega > \Omega_C = \frac{\hbar}{mR^2} \ln \frac{R}{\xi}.$$

Macroscopical flow at ground state.

For fast rotation $\Omega \gg \Omega_C$ distribution of vortices

simulates solid-like rotation: $< \mathbf{curl}\, \boldsymbol{v}_s > = 2\Omega$.

Number of vortices per area is $n_{vor} = m\Omega / \pi\hbar$.

The simplest way to check quantization rule!

“Second quantization” of $\Psi_0(\boldsymbol{r},t)$

$$\Psi_0(\boldsymbol{r},t) \approx \sqrt{n_0}\left[1 + iS(\boldsymbol{r},t)\right]$$

Landau quantum hydrodynamics approximation:

$$S(\boldsymbol{r},t) \rightarrow \hat{S}(\boldsymbol{r},t)$$

$$= \frac{im}{\sqrt{V}} \sum_{\boldsymbol{k}} \left(\frac{c}{\hbar \rho k}\right)^{1/2} \left(\hat{b}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{b}_{\boldsymbol{k}}^{+} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

at $k \rightarrow 0$, $\hat{b}_{\boldsymbol{k}}$ − operators of annihilation of phonons .

RESULTS

Momentum distribution at $k \to 0$:

$$T = 0 : N(\boldsymbol{p}) \to \frac{n_0 mc}{2np};$$

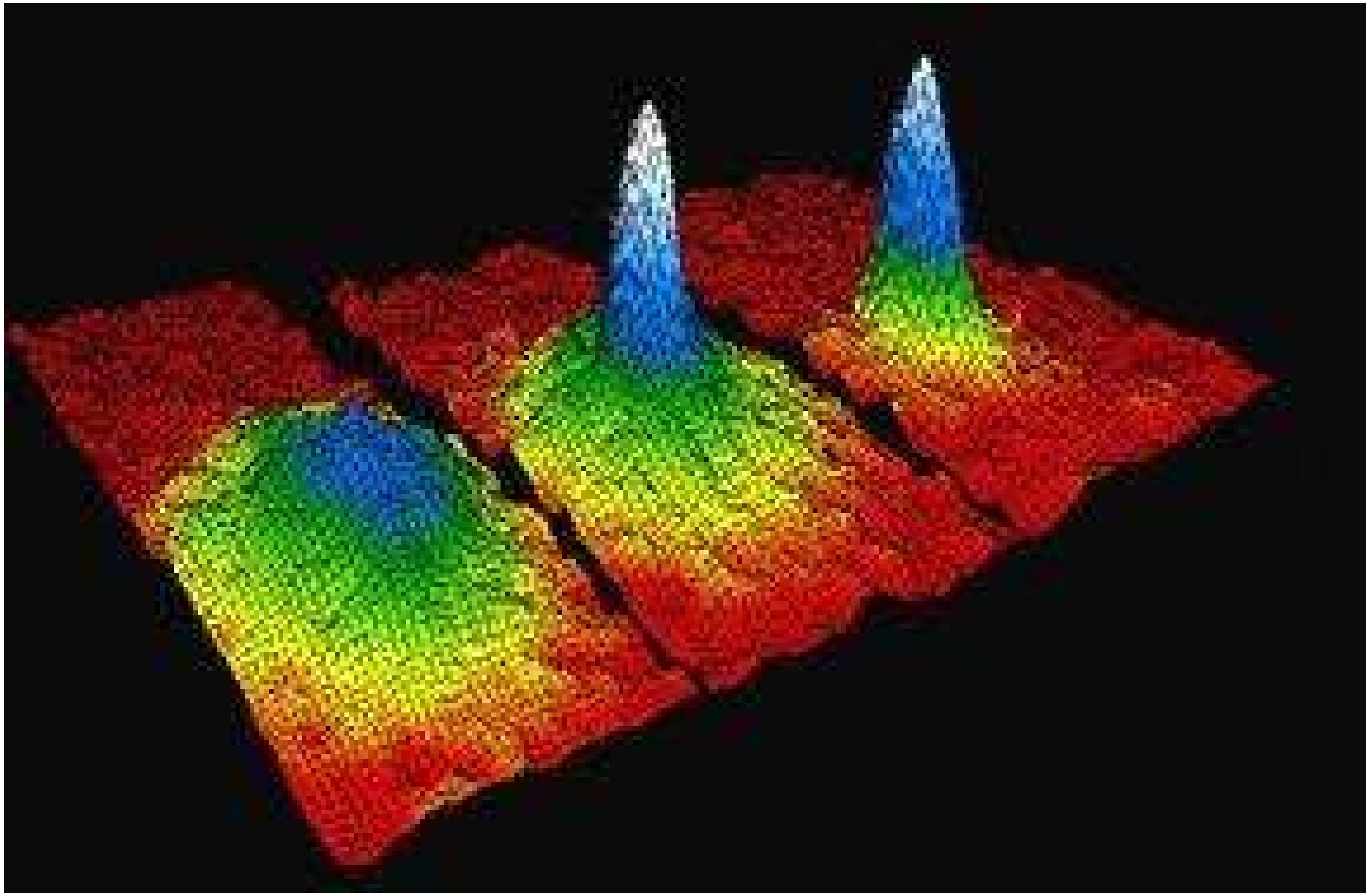
$$T \neq 0 : N(\boldsymbol{p}) \to \frac{n_0 mT}{np^2}.$$

Temperature dependence of condensate density:

$$n_0(T) \approx n_0(T=0) - \alpha T^2.$$

In comparison: $\rho_s(T) \approx \rho(0) - \beta T^4.$

Weakly-interacting Bose gas



Observation of BEC: From Cornell, 1996.

Bogoliubov theory

$$\hat{\Psi}(\boldsymbol{r},t)=\sqrt{n_0}+\hat{\Psi}'(\boldsymbol{r},t),\ \hat{\Psi}'\ll\sqrt{N_0}$$

First approximation.

$$\Psi\to\sqrt{n},\quad E^{(1)}_{gr}=N^2 g/2V,\quad c=\sqrt{gn/m}$$

$$g=\frac{4\pi\hbar^2}{m}a\ ,\quad a \text{ is } s\text{-wave scattering length.}$$

Second approximation.

Keeping terms up to the second order in Ψ' in the Hamiltonian. The energy spectrum:

$$\epsilon(p)=\left[c^2p^2+(p^2/2m)^2\right]^{1/2},$$

$$p\to 0:\epsilon(p)\approx cp;\quad p\to\infty:\epsilon(p)\approx p^2/2m.$$

For $T\gg mc^2$ Tisza theory is approximately correct !

$$E^{(2)}_{gr}/E^{(1)}_{gr}\sim(na^3)^{1/2},\ E^{(2)}\sim\int^{\sim mc}(pc/2)\,d\,\boldsymbol{p}.$$

Non-uniform gas

$$\hat{\Psi}(\boldsymbol{r},t) \rightarrow \Psi_0(\boldsymbol{r},t).$$

Mean-field GP equation for Ψ_0:

$$i\hbar \frac{\partial}{\partial t}\Psi_0 = -\frac{\hbar^2}{2m}\Delta\Psi_0 + g\Psi_0|\Psi_0|^2 + U(\boldsymbol{r})\Psi_0$$

E. P. Gross, 1961; L. P. Pitaevskii, 1961.

$$g = \frac{4\pi\hbar^2}{m}a, \quad a \text{ is } s\text{-wave scattering length.}$$

Plays role of the Maxwell equations for the classical matter weaves.

Why Maxwell equations do not contain *h* ?

Energy-momentum relation for photons:

$$\epsilon = cp$$

Transition from particles to waves:

$$\epsilon\,,\,\boldsymbol{p} \rightarrow \omega\,,\,\boldsymbol{k}$$

$$\epsilon = \hbar\,\omega\,,\;\; p = \hbar\,k$$

Frequency - wave vector relation:

$$\omega = ck.$$

Does not contain $\hbar$.

Why GP equation contains *h*? From particles to classical matter waves

Energy-momentum relation for atoms:

$$\epsilon = \frac{p^2}{2m}.$$

Transition from particles to waves:

$$\epsilon,\ p \rightarrow \omega,\ k\ :$$

$$\epsilon = \hbar\omega,\ p = \hbar k.$$

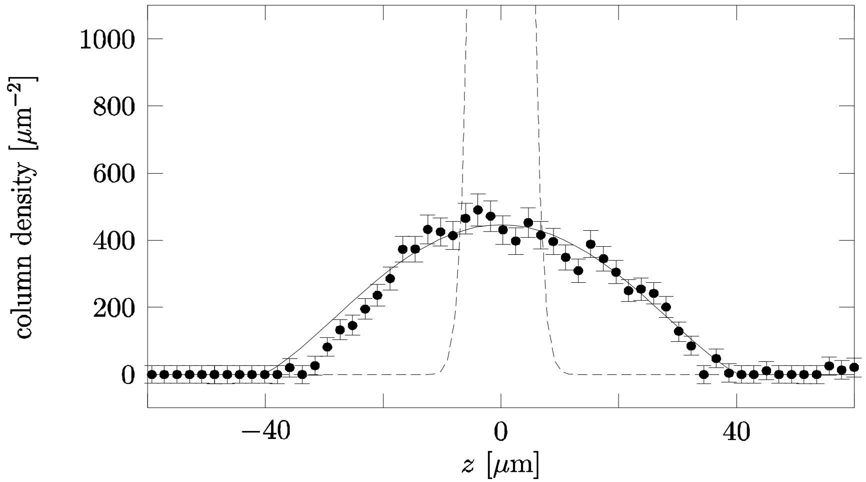
Frequency − wave vector relation

$$\omega = \frac{\hbar k^2}{2m}.$$

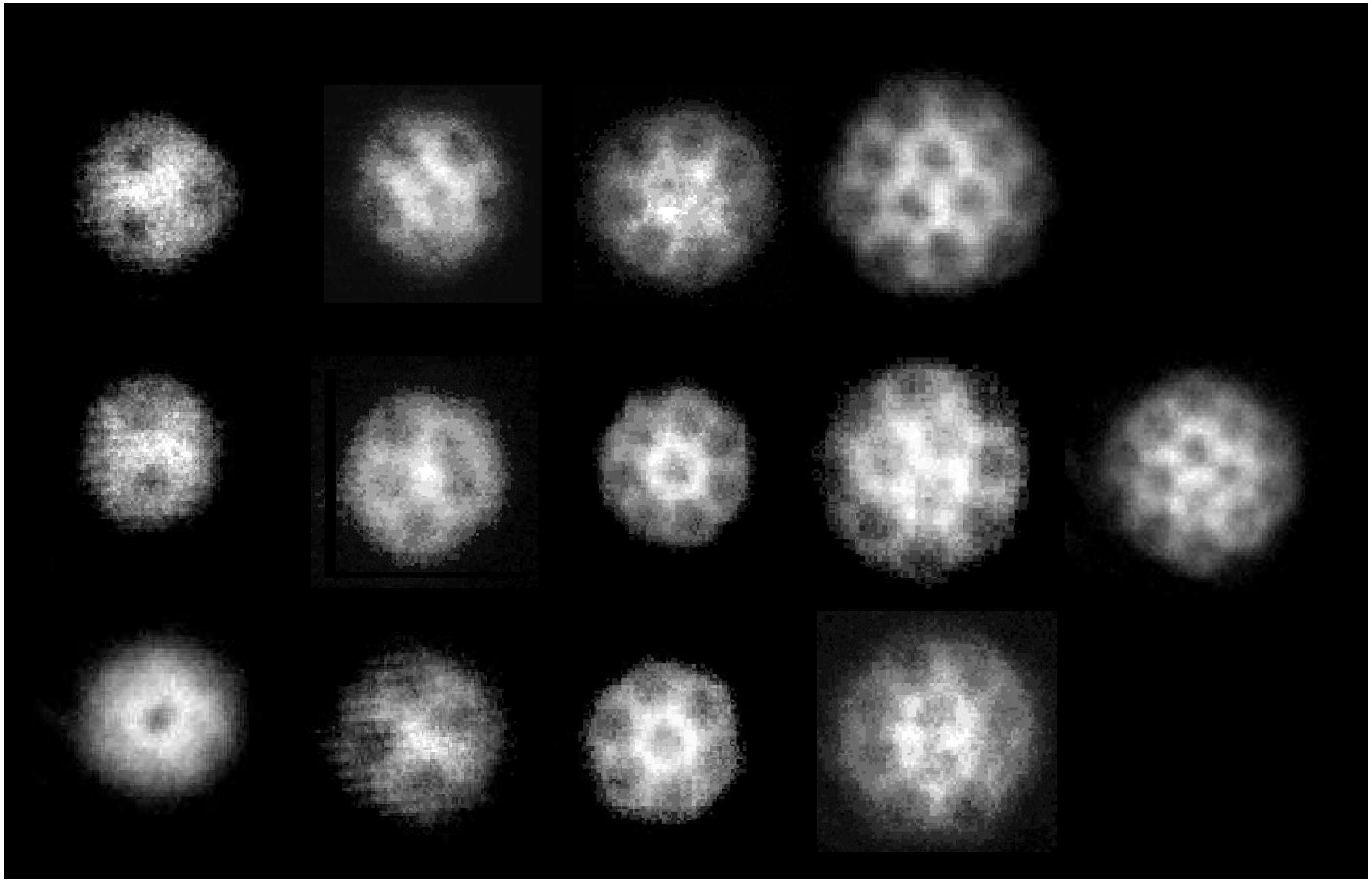
This relation contains $\hbar$.

Equation for the classical function $\Psi_0\left(\boldsymbol{r}, t\right)$ contains $\hbar$.

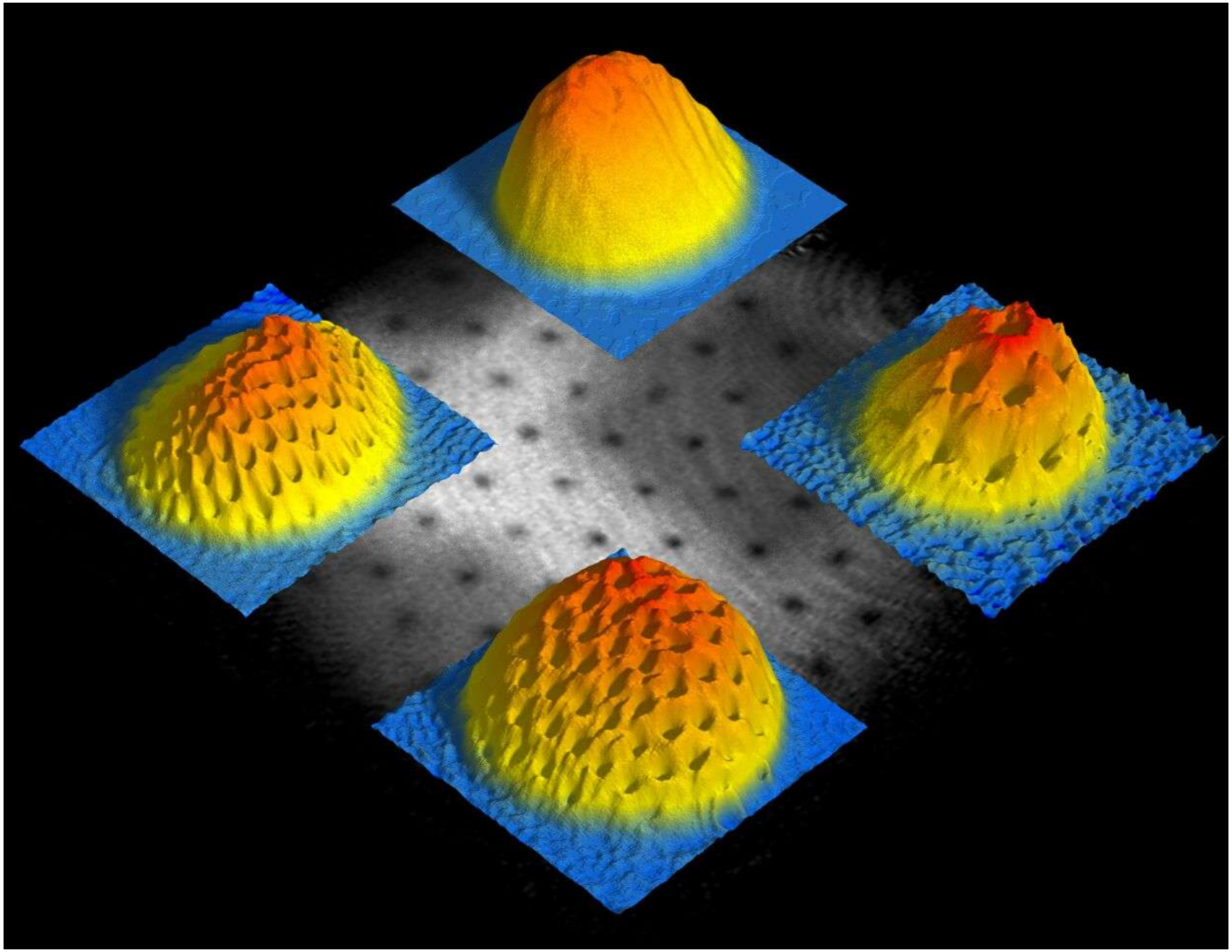
Density distribution of BEC in a trap



Rotation of BEC. From Chevy et al., 2001.

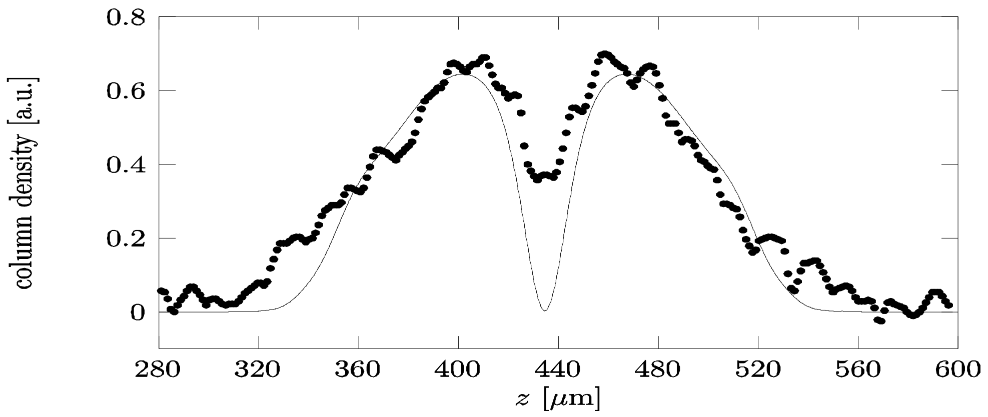


From Ketterle et al., 2001.



Structure of vortex line

$$|\Psi| = f(r/\xi), \; n = |\Psi|^2, \; \xi = \hbar/\sqrt{2n_\infty g m} = \hbar/(\sqrt{2}mc).$$



Experiment: Madison et al. (2000), Theory: Dalfovo and Modugno, 2000.

Zero shear viscosity

$$\oint \boldsymbol{v}_s \cdot d\,\boldsymbol{r} = 2\pi \frac{\hbar}{m}$$

$$\mathbf{curl}\ \boldsymbol{v}_s = 2\pi \frac{\hbar}{m}\,\delta^2\left(\boldsymbol{r}\right)$$

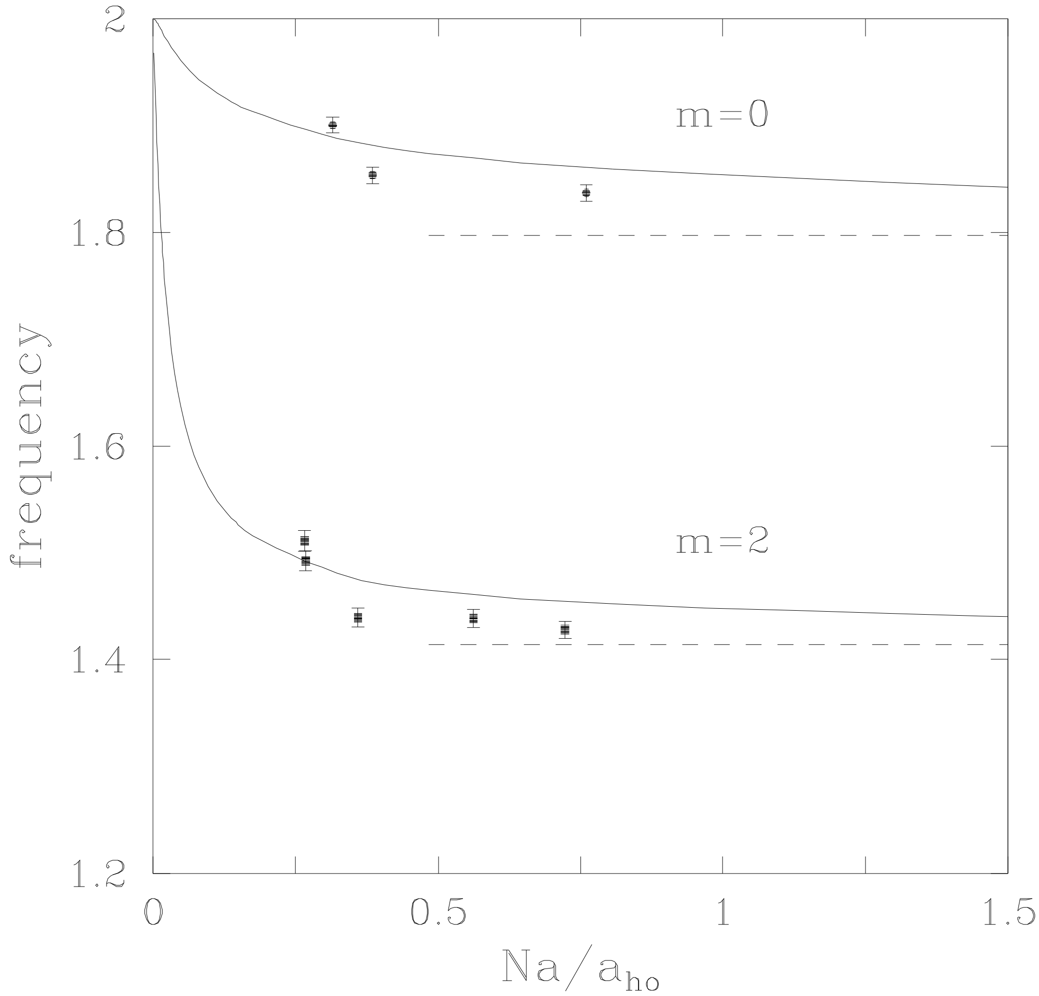
In an ordinary liquid:

$$\frac{\partial}{\partial t}\mathbf{curl}\ \boldsymbol{v} = \frac{\eta}{mn}\,\Delta\left(\mathbf{curl}\ \boldsymbol{v}\right)$$

Existence of the vortex lines means that

$$\eta \equiv 0\,.$$

Collective oscillations. Stringari prediction and experiment



Jin et al., 1996.

GP-equation and hydrodynamics

$$\Psi = \sqrt{n}\, e^{iS}, \quad n = |\Psi|^2, \quad \boldsymbol{v}_s = (\hbar/m)\nabla S.$$

$$\frac{\partial n}{\partial t} + \operatorname{div}\left(n\,\boldsymbol{v}_s\right) = 0,$$

$$\hbar\frac{\partial S}{\partial t} + \left(\frac{1}{2}mv_s^2 + gn + U - \frac{\hbar^2}{2m\sqrt{n}}\Delta\sqrt{n}\right) = 0.$$

Last term is small if characteristic length is larger then healing length:

$$R \gg \xi = \hbar/\sqrt{2ngm}$$

Accuracy of GPE description

Static GPE has been **rigorously** derived (Lieb et al., 2000 and later).

Even if the exsistence of Bose−Einstein condensation

for interacting systems never was proved ?!

GPE coresponds to the **first** Bogolibov approximation.

What is the accuracy ? Order of the magnitude of the correction:

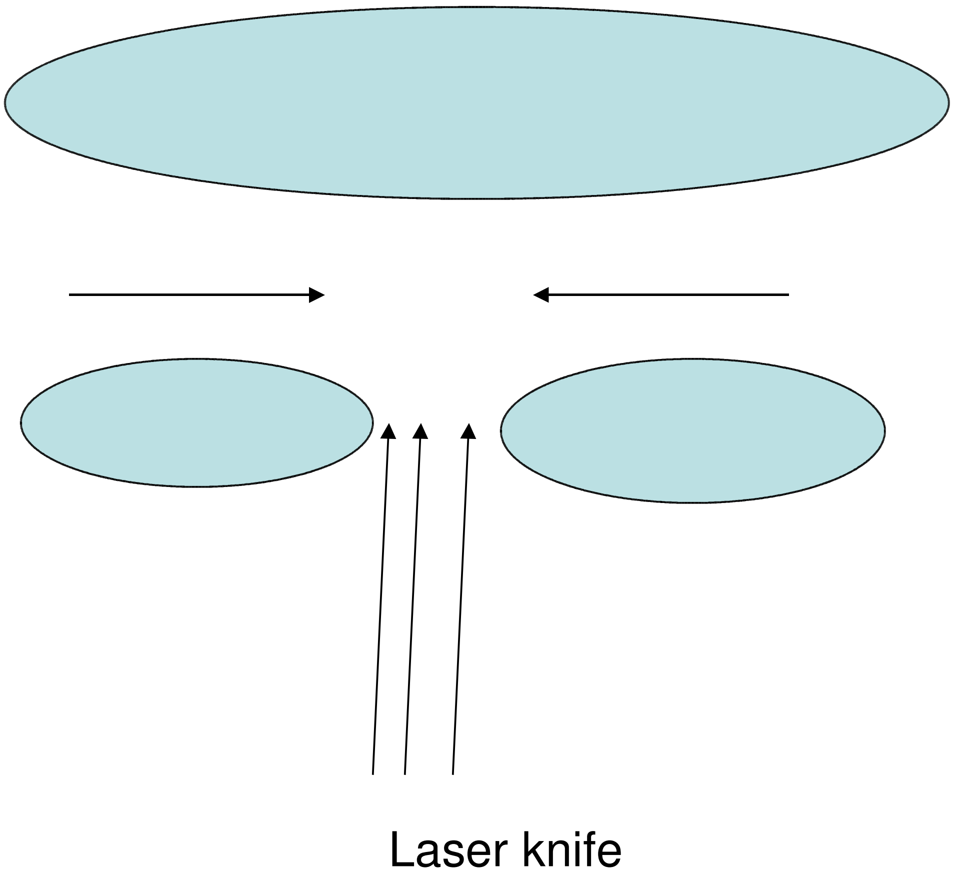
$$\frac{\delta\omega}{\omega} \sim [n(0)a^3]^{1/2} \sim \frac{1}{N}\left(\frac{Na}{a_{ho}}\right)^{6/5} \sim \frac{1}{N}\left(\frac{R}{\xi}\right)^3$$

Two different thermodynamic limits.

Lieb "GP" thermodynamic limit: $N \to \infty$, $R/\xi = \mathrm{const}$.

Usual thermodynamic limit: $N \to \infty$, $N/V = \mathrm{const}$.

Interference experiment



Phase, flow and interference

$$\Psi(\boldsymbol{r},t)=\sqrt{n}\,e^{iS}$$

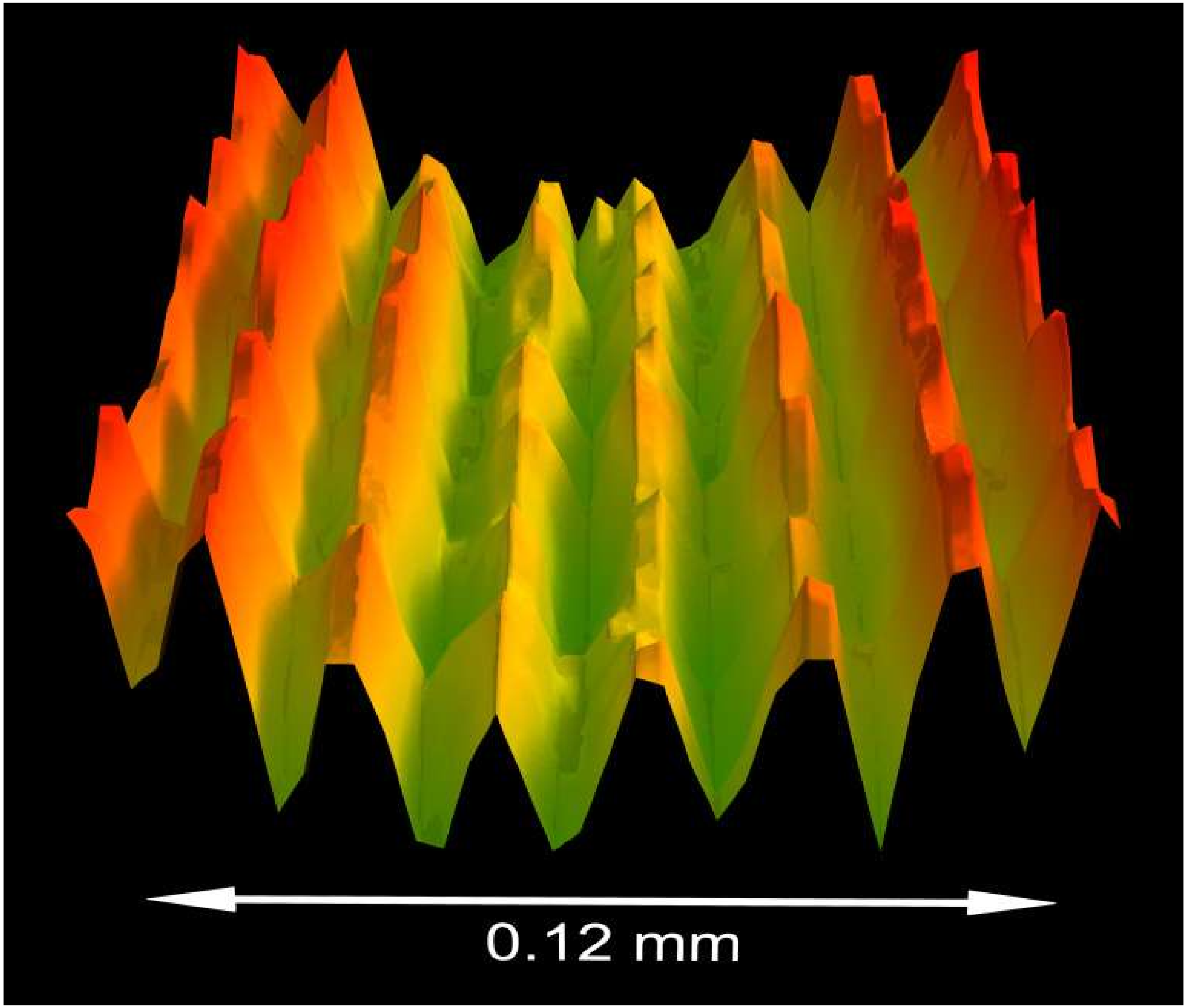
In absence of interaction:

$$\boldsymbol{v}_s(\boldsymbol{r},t)=\frac{\hbar}{m}\nabla S=\frac{\boldsymbol{r}}{t},\ S=\frac{mr^2}{2\hbar t}$$

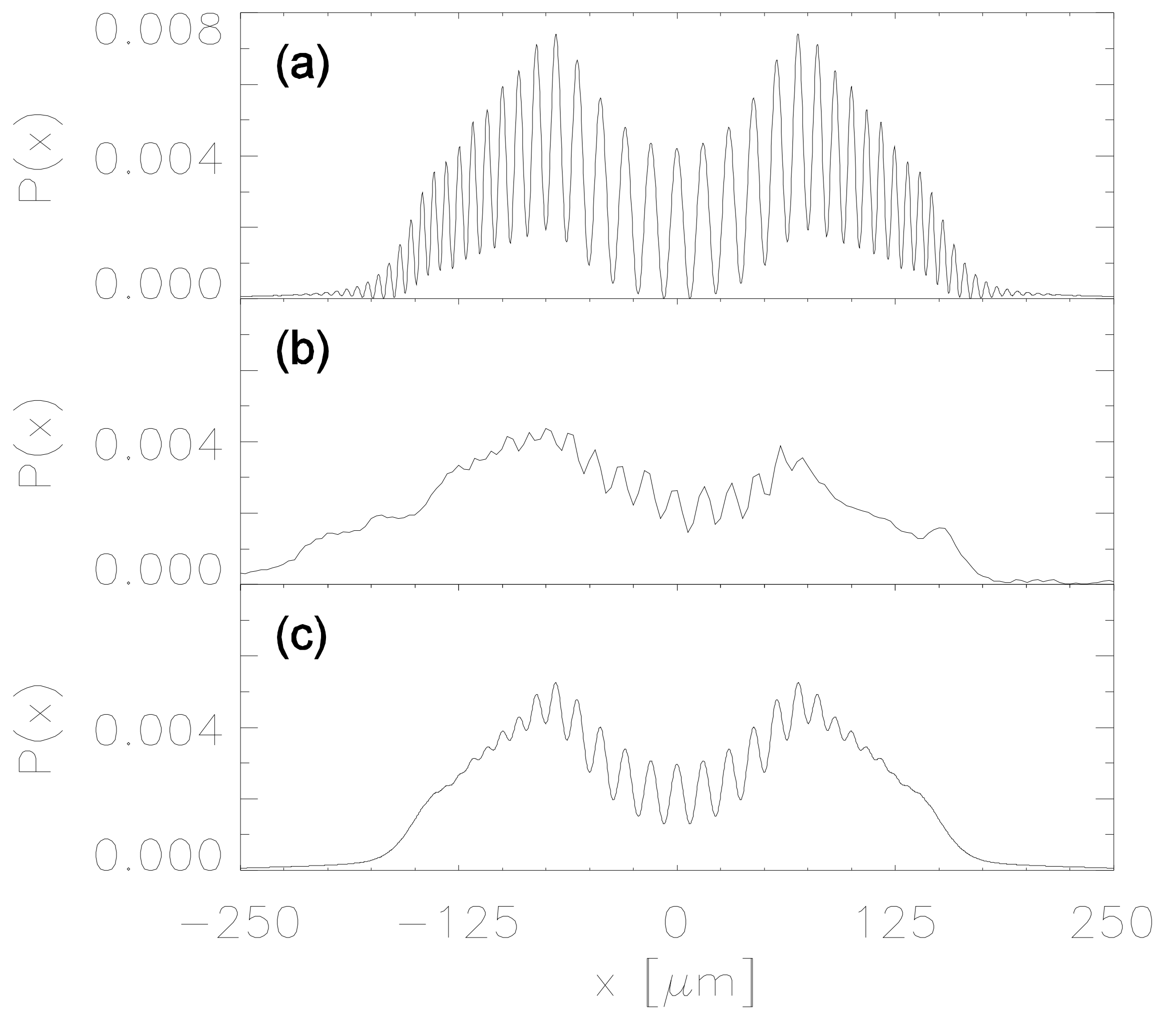
$$n=\lfloor \Psi_1+\Psi_2 \rfloor^2=n_1+n_2+2\sqrt{n_1 n_2}\cos\left(S_1-S_2\right)$$

$$n(\boldsymbol{r},t)=n_1(\boldsymbol{r},t)+n_2(\boldsymbol{r},t)$$

$$+2\sqrt{n_1(\boldsymbol{r},t)\,n_2(\boldsymbol{r},t)}\cos\left(\frac{md}{\hbar t}z+\phi_0\right)$$

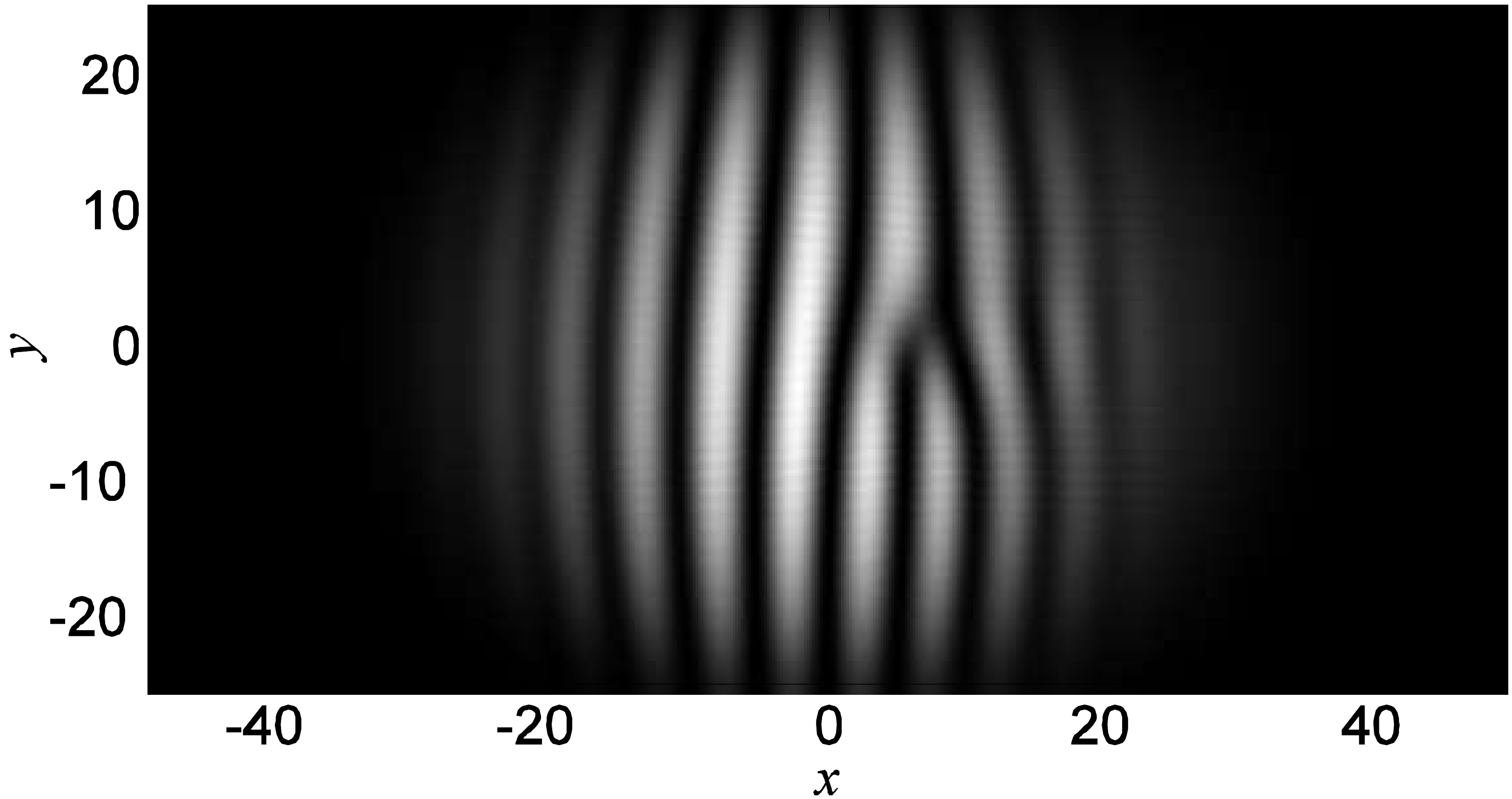


Interference fringes: Andrews et al., 1997.



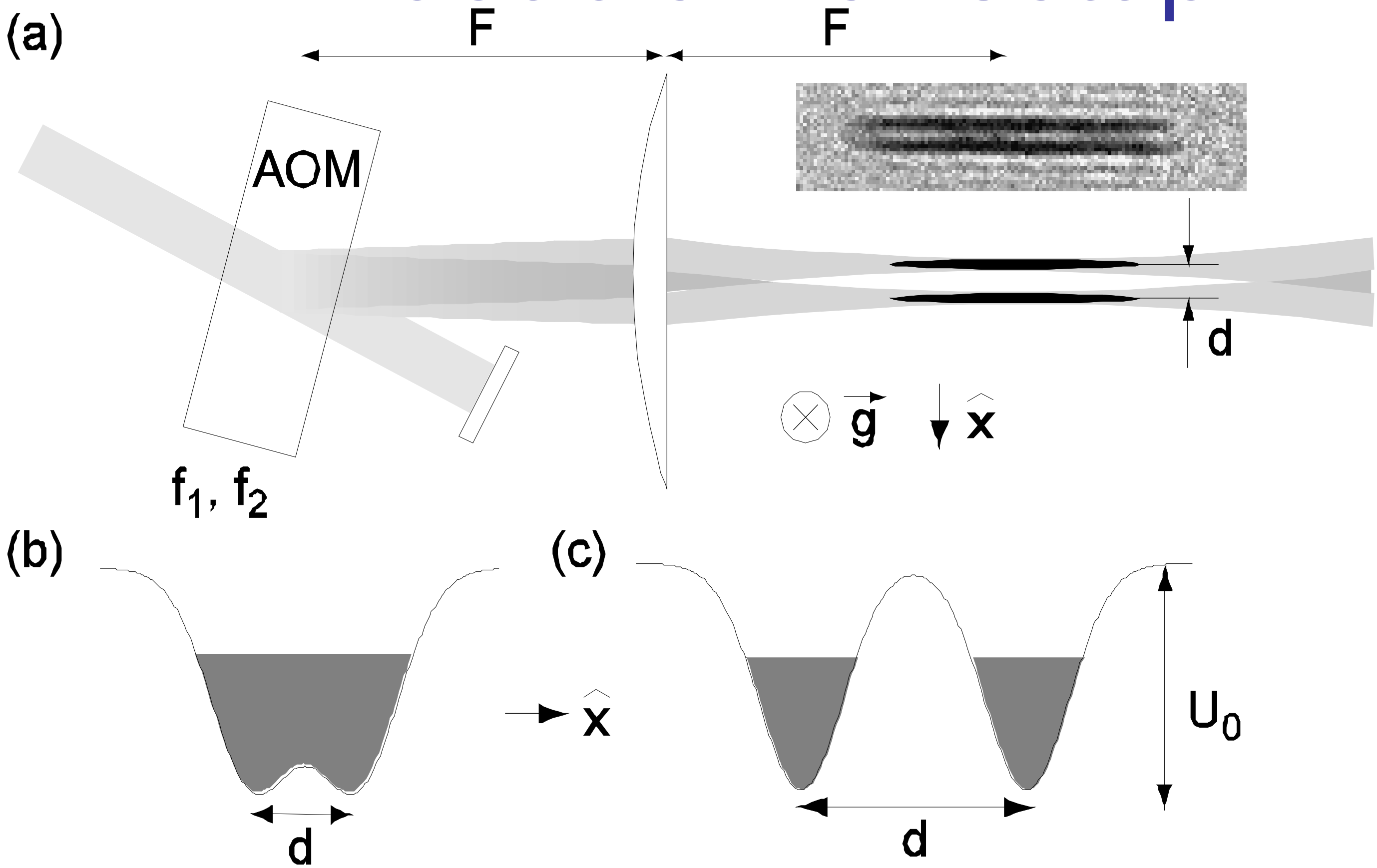
Exp.: Andrews et al., 1997; Theor.: Rohrl et al., 1997.

Interference at the presence of a vortex line



Bolda and Walls, 1998.

MIT double-well setup



(a) Setup for double-well potential. (b), (c): d=6, 13mcm
Y. Shin, M. Saba, T. Pasquini, W. Ketterle, D. Pritchard, and A. Leanhard, PRL **92**, 050405 (2004).

Measurement of the phase difference.

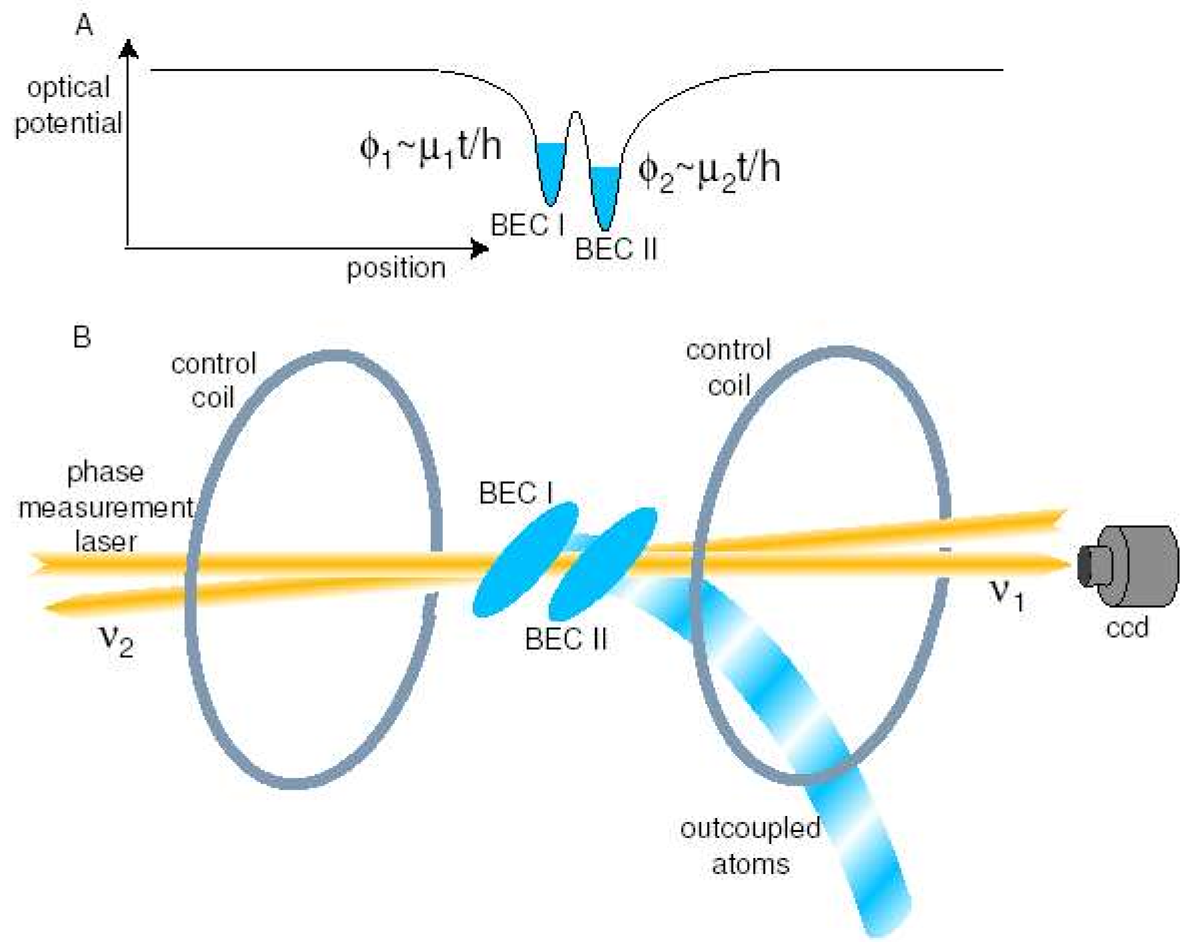
Interference in momentum space.

Momentum distribution of two identical condensates in a double-well trap:

$$n(p_x) = 2\left[1 + \cos\left(\frac{p_z d}{\hbar} + \Phi\right)\right] n_a(p_x)$$

Measurement can be almost non-destructive!

L. Pitaevskii and S. Stringari, 1999.



Setup for continuous phase measurement. M. Saba, T. Pasquini, C. Sanner, Y. Shin, W. Ketterle, and D. Pritchard, Science (2005).

Interference against hydrodynamics

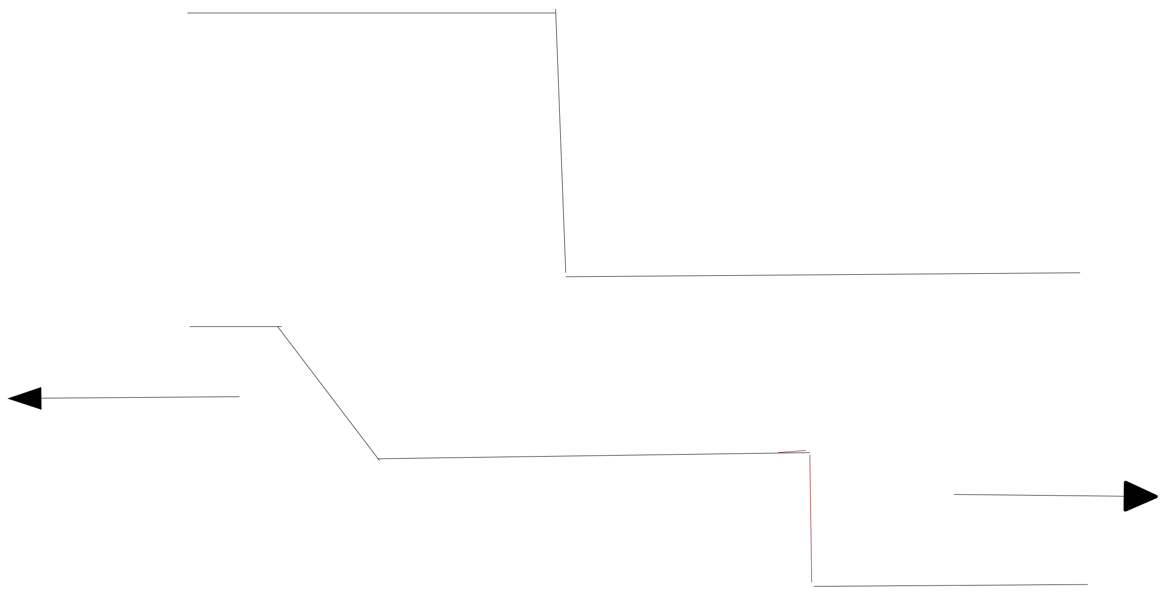
$$\mathrm{GP} \rightarrow$$

$$\partial_t v + \nabla\left(v^2/2 + \mu\right) = 0$$

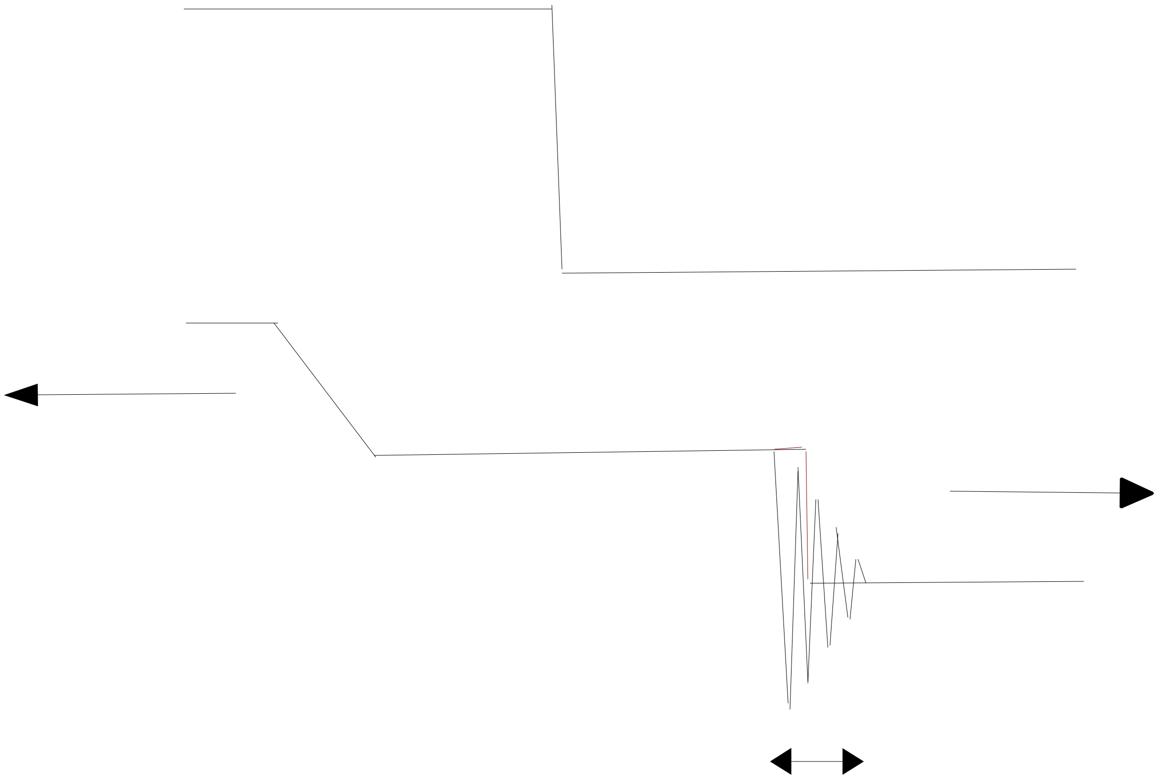
$$\partial_t n + \nabla \cdot \left(nv\right) = 0$$

Have no solution as a rule!

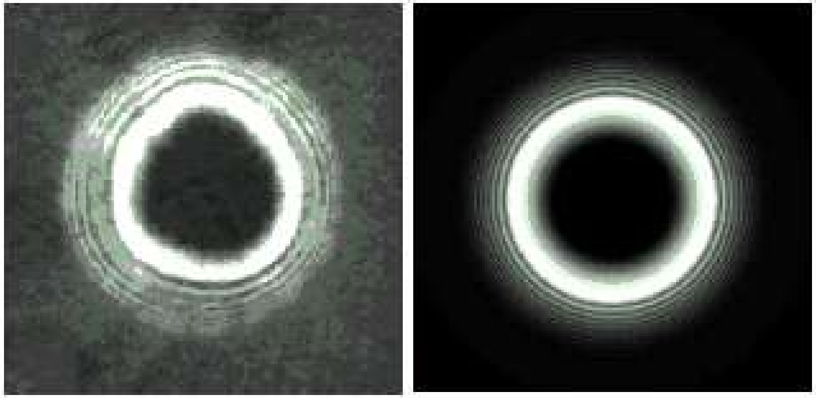
Decay of initial discontinuity in normal liquid. Shock wave.



Decay of initial discontinuity in normal liquid. Collisionless shock wave.



Collisionless shock wave in expanding BEC. Experiment and theory.



Hoefer et al., 2006.

Mixture of two superfluids

Two superfluids:

$$m_{1,} \rho_{1,} \Psi_{01}; m_{2,} \rho_{2}, \Psi_{02}:$$

$$\boldsymbol{v}_{\boldsymbol{s1}} = \frac{\hbar}{m_1} \nabla S_1; \boldsymbol{v}_{\boldsymbol{s2}} = \frac{\hbar}{m_1} \nabla S_2.$$

$$\boldsymbol{j}_{\boldsymbol{s1}} = \rho_{s11} \boldsymbol{v}_{\boldsymbol{s1}} + \rho_{s12} \boldsymbol{v}_{\boldsymbol{s2}}; \boldsymbol{j}_{\boldsymbol{s2}} = \rho_{s22} \boldsymbol{v}_{\boldsymbol{s2}} + \rho_{s12} \boldsymbol{v}_{\boldsymbol{s1}}$$

A. Andreev, E. Bashkin.

Two kinds of vortices.

Vortex in the first superfluid:

$$v_{s1} = \frac{\hbar}{m_1} \frac{1}{r}, v_{s2} = 0,$$

$$j_{s1} = \rho_{s11} \frac{\hbar}{m_1} \frac{1}{r}, j_{s2} = \rho_{s12} \frac{\hbar}{m_1} \frac{1}{r}.$$

Vortex in the second superfluid – analogously.

2D Bose-liquid

$$\text{One}-\text{body density matrix.}$$

$$|\boldsymbol{r}-\boldsymbol{r}'|\to\infty: \quad n^{(1)}(\boldsymbol{r},\boldsymbol{r}',t)$$

$$=<\hat{\Psi}^{+}(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r}',t)>\sim|\boldsymbol{r}-\boldsymbol{r}'|^{-\alpha T}.$$

$$\text{Algebraic Non}-\text{Diagonal Long Range Order.}$$

$$\text{Non}-\text{gauge}-\text{invariant} \quad \text{field}:$$

$$-\lambda[\hat{\Psi}^{+}(\boldsymbol{r},t)+\hat{\Psi}(\boldsymbol{r},t)]$$

$$\lambda\sim[<\hat{\Psi}>]^{\beta/T}$$

$$\text{It is enough very small field to create} <\hat{\Psi}> .$$

Berezinskii-Kosterlitz-Thouless phase transition (1972-74)

Vortices in 2 D superfluid as elementary excitations.

Energy of a vortex: $\epsilon_v = n_v \pi L \rho_s \frac{\hbar^2}{m^2} \ln \frac{R}{\xi}$, L is the film thickness.

$$\text{Entropy of a vortex}: s_v = \ln \frac{R^2}{\xi^2} = 2 \ln \frac{R}{\xi}.$$

Creation of vortixes is possible if the contribution of a vortex in free energy

$$f_v = \epsilon_v - T s_v < 0,$$

or, at $R \to \infty$, if

$$T > T_{\text{BKT}} = [L \rho_s (T_{\text{BKT}})] \frac{\pi \hbar^2}{2 m^2}.$$

This is a new kind of phase transitions.

Plan of lecture-2

II. Superfluidity of fermions and superconductivity.

1.Normal Fermi-liquid. Elementary excitation and Landau-Luttinger theorem.
2. Pairing and gap.
3. Pairing and NDLRO.
4. Gauge invariance and superconductivity. London equations.
5. Magnetic flux quantization.
6. Ginzburg-Landau equation and Abrikosov vortices.

Normal Fermi liquid

$$N_a(\boldsymbol{p}) = \frac{1}{\exp\left[\frac{\epsilon(\boldsymbol{p})_a - \mu}{T}\right] + 1}.$$

Transition to the particle − hole picture

$$p > p_F : \epsilon(\boldsymbol{p}) = \epsilon(\boldsymbol{p})_a - \mu \approx v_F(p - p_F),\, N(\boldsymbol{p}) = N_a(\boldsymbol{p})$$

$$p < p_F : \epsilon(\boldsymbol{p}) = -\epsilon(\boldsymbol{p})_a + \mu \approx v_F(p_F - p),\, N(\boldsymbol{p}) = 1 - N_a(\boldsymbol{p})$$

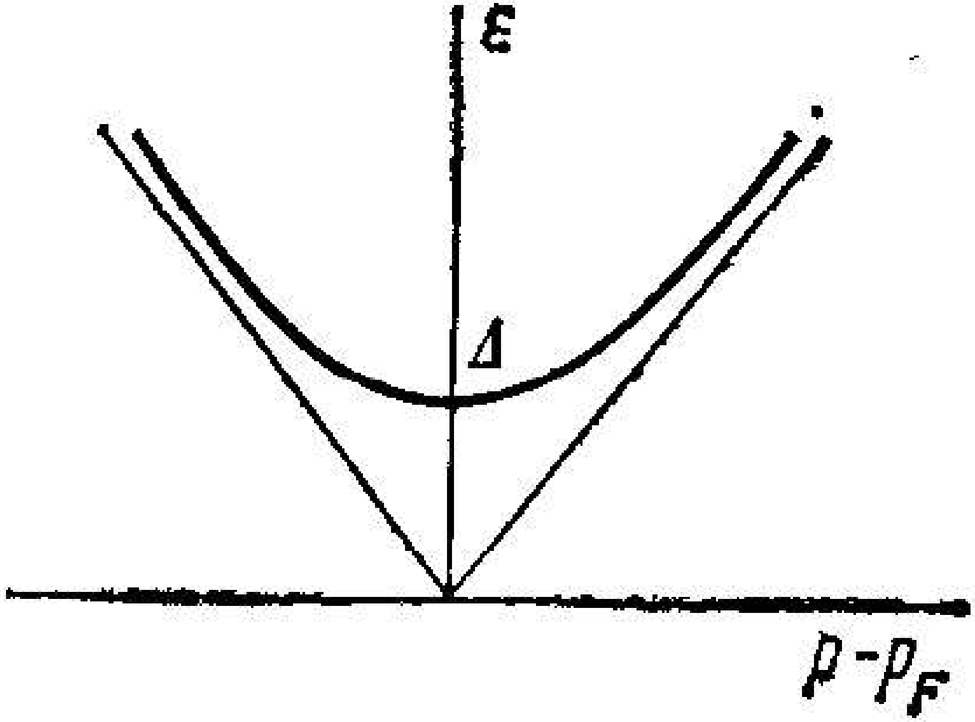
$$1 - [e^x + 1]^{-1} = [e^{-x} + 1]^{-1}$$

$$N(\boldsymbol{p}) = \frac{1}{\exp\left[\frac{v_F|p - p_F|}{T}\right] + 1}.$$

For **any** interaction : $\frac{4\pi}{3} 2 \frac{p_F^3}{(2\pi\hbar)^3} = n$

Landau − Luttinger theorem .

Energy spectrum of Fermi-liquid.



Landau criterion of superfluidity for Fermi liquid.

$$N(\boldsymbol{p}) = \frac{1}{\exp\left[\frac{\epsilon(p) + \boldsymbol{p}\cdot\boldsymbol{v}_s}{T}\right] + 1}.$$

Let $T \to 0$.

If $v_s < \min \epsilon(p)/p$, $N(\boldsymbol{p}) \to 0, \rho_n \to 0.$

If $v_s > \min \epsilon(p)/p$, $N(\boldsymbol{p}) \to 1$, for $\epsilon(p) + \boldsymbol{p}\cdot\boldsymbol{v}_s < 0.$

Comparison with bosons:

$$N_B(\boldsymbol{p}) = \frac{1}{\exp\left[\frac{\epsilon(p) + \boldsymbol{p}\cdot\boldsymbol{v}_s}{T}\right] - 1}.$$

If $\epsilon(p) + \boldsymbol{p}\cdot\boldsymbol{v}_s < 0$, $N_B(\boldsymbol{p}) < 0.$

Thermodynamic equilibrium is impossible!

NDLRO in Fermi superfluid

Two-body density matrix:

$$n^{(2)}(\boldsymbol{r},\boldsymbol{r}',t) = < \hat{\Psi}^{+}_{\uparrow}(\boldsymbol{r},t)\hat{\Psi}^{+}_{\downarrow}(\boldsymbol{r},t)\hat{\Psi}_{\downarrow}(\boldsymbol{r}',t)\hat{\Psi}_{\uparrow}(\boldsymbol{r}',t) >$$

$$\to \Psi^{*}_{0}(\boldsymbol{r},t)\Psi_{0}(\boldsymbol{r}',t) \quad \text{at} \quad |\boldsymbol{r}-\boldsymbol{r}'| \to \infty$$

$$\Psi_{0}(\boldsymbol{r},t) = < \hat{\Psi}_{\uparrow}(\boldsymbol{r},t)\hat{\Psi}_{\downarrow}(\boldsymbol{r},t) >$$

$$\Psi_{0}(\boldsymbol{r},t) = |\Psi_{0}(\boldsymbol{r},t)| e^{iS(\boldsymbol{r},t)}$$

$$\boldsymbol{j}_{s} = \rho_{s}\boldsymbol{v}_{s}, \quad \boldsymbol{v}_{s} = \frac{\hbar}{\boldsymbol{2}m}\nabla S$$

Of course, $\rho_{s} \neq m|\Psi_{0}(\boldsymbol{r},t)|^{2}$

Consequences of NDLRO

1. Gap is a result of particle – hole hybrodization.

Bogoliubov transformation (1958)

mixes $\hat{a}$ and $\hat{a}^{+}$ operators.

2. Quantized vortex lines in Fermi superfluid.

$$\oint \boldsymbol{v}_s \cdot d\boldsymbol{r} = 2\pi \frac{\hbar}{2m}.$$

Number of vortices per area is now

$$n_{vor} = \frac{\mathbf{2}\, m\, \Omega}{\pi \hbar}.$$

Elementary excitations in superfluid Fermi-liquid

BOSONIC EXCITATIONS:

$$p \to 0, \epsilon(p) = cp, \quad c^2 = \frac{n}{m}\frac{\partial \mu}{\partial n}$$

$$C \sim T^3, \rho_n \sim T^4$$

FERMIONIC EXCITATIONS:

$$\epsilon(p) = \Delta + \frac{(p - p_0)^2}{2m^*}$$

$$C \sim \exp(-\Delta / T).$$

DIFFERENT LIMITS OF SUPERFLUIDITY OF FERMIONS

BCS LIMIT:

$$a<0, k_F|a| \ll 1$$

$$\epsilon(\boldsymbol{p}) = \sqrt{\Delta^2 + [v_F(p-p_F)]^2}$$

$$\Delta = g|\Psi_0|, \quad \text{L. P. Gorkov}, 1958.$$

$$\Delta(T=0) = C\epsilon_F \exp\left(-\frac{\pi}{2k_F|a|}\right)$$

$$\Delta(T=0) = 1.76\, T_C$$

$$T \to T_C, \quad \Delta \sim \sqrt{T_C - T}$$

$$\mu \approx \mu_{\mathrm{id}}.$$

BEC (molecular) limit

$$\textbf{MOLECULAR LIMIT:}$$

$$a > 0, k_F\, a \ll 1$$

$$E_{\mathrm{BIND}} = \frac{\hbar^2}{ma^2}$$

Dilute gas of molecules (dimers).

Dimer $-$ dimer scattering amplitude $a_{\mathrm{dd}} = 0.6\,a$,

dimer $-$ dimer relaxation rate $\alpha_{\mathrm{dd}} \sim a^{-2.55}$,

Petrov, Salomon and Shlyapnikov, 2004.

$$\mu = -E_{\mathrm{BIND}}/2 + \frac{2\pi\hbar^2}{m} a_{\mathrm{dd}} \left[1 + \frac{32}{3\sqrt{\pi}} \overset{???}{(na_{\mathrm{dd}}^3)^{1/2}} \right]$$

Unitarity limit

UNITARITY LIMIT

$$a \to \pm\infty \; \mu = (1+\beta)\mu_{\text{id}}, \; \beta \sim -0.58$$

$$c = \sqrt{1+\beta}\, v_{F\,\text{id}} / \sqrt{3}$$

$$T_C = \alpha E_F, \; \alpha \sim 0.23\,(\text{Bulgac } et\ al.), \sim 0.16\,(\text{Burovski } et\ al.)$$

$$\Delta(T=0) = \nu E_F, \; \nu \sim 0.5$$

Collective excitations at T=0

Isotropic harmonic trap

Politropic equation of state $\mu \sim n^{\gamma}$

Isotropic breathing oscillations

$$\omega = \sqrt{3\gamma + 2}\,\omega_{ho}$$

$$\text{BEC}: \gamma = 1, \omega = \sqrt{5}\,\omega_{ho}$$

$$\text{UNITARITY AND BCS}: \gamma = 2/3, \omega = 2\,\omega_{ho}$$

IN UNITARITY THIS RESULT IS VALID

AT FINITE TEMPERATURE

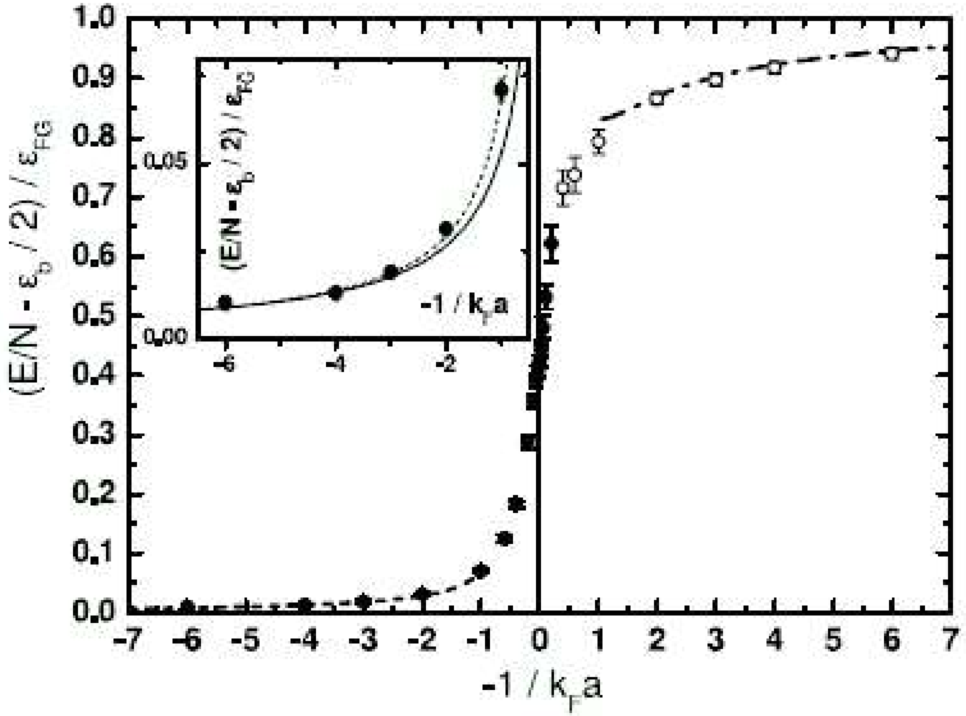
AND BEYOND HYDRODYNAMIC, (I. Castin, 2004).

$$I = \int r^2 n(\boldsymbol{r}, t)\, d\boldsymbol{r}, \quad \ddot{I} + 4\omega_{ho}^2 I = E.$$

A PROBLEM: IS THE BEYOND MEAN−FIELD TERM

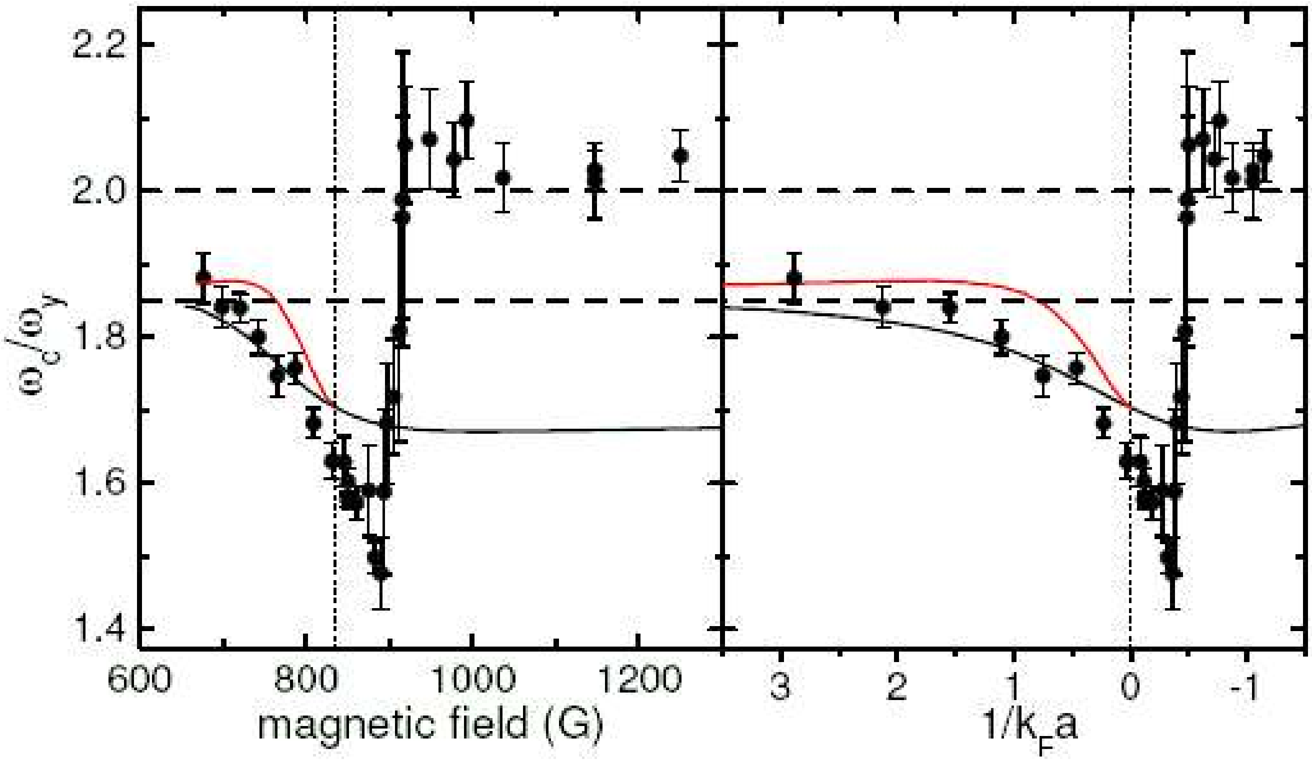
THE PRINCIPAL CORRECTION?

Energy per particle of Fermi gas at BEC-BCS crossover



Giorgini et al., 2004

Collective oscillations in BEC-BCS crossover



Exp. Altmeyer et. al., 2006, Theor. Astrakharchik et al., 2004.

file:/users/u2/pitaevsk/KettVort0505635.pdf - KPDF

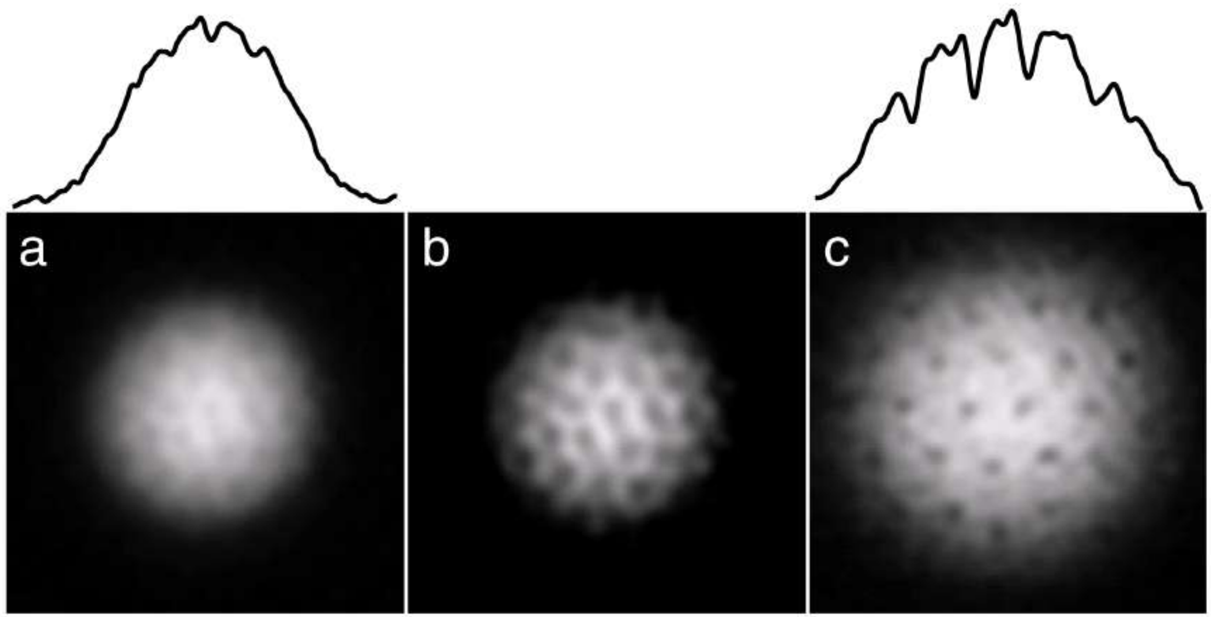
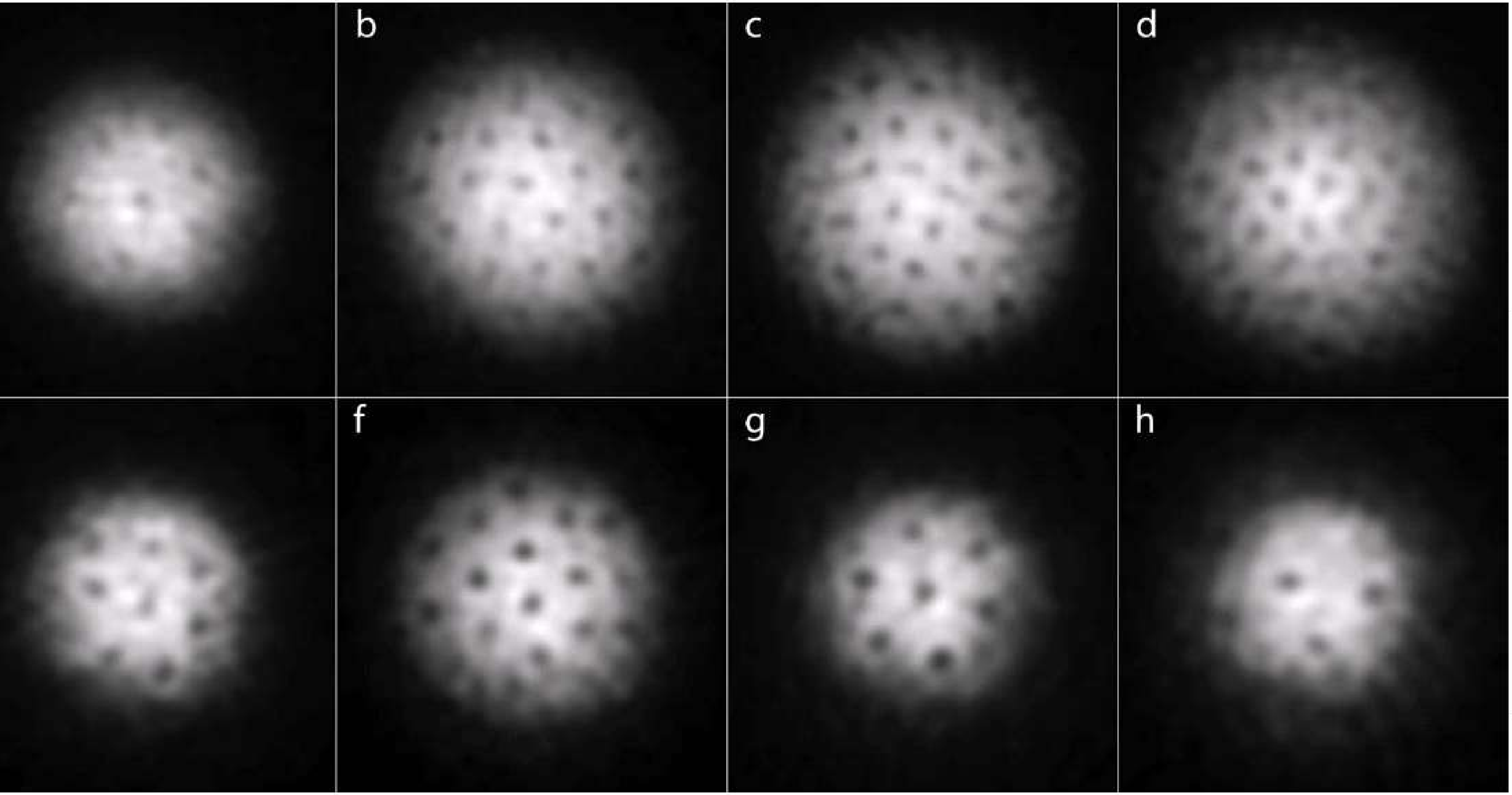


Fig. 1: Observation of a vortex lattice in a molecular condensate. (a) Fixed field. Stirring for 800 ms, followed by 400 ms of equilibration, and imaging after 12 ms time-of-flight all took place at 766 G. The vortex core depletion of the integrated density profile is barely 10%, as indicated by the 5- μm -wide cut on top. (b) Fourier-filter applied to (a) to accentuate the vortex contrast. Spatial frequencies with an absolute value of about the inverse vortex core size were enhanced by a factor of four. (c) Varying field. The vortex lattice was created at 766 G and imaged at 735 G following the procedure outlined in the text. The vortex core depletion is now about 35%. The field of view is $780\ \mu m \times 780\ \mu m$.

Zwierlein et al., 2005.



. 2: Vortices in a strongly interacting gas of fermionic atoms on the BEC- and the BCS-side of
Feshbach resonance. At the given field, the cloud of lithium atoms was stirred for 300 ms
to 500 ms (b-h) followed by an equilibration time of 500 ms. After 2 ms of ballistic
pansion, the magnetic field was ramped to 735 G for imaging (see text for details). The
gnetic fields were (a) 740 G, (b) 766 G, (c) 792 G, (d) 812 G, (e) 833 G, (f) 843 G, (g) 853 G
l (h) 863 G. The field of view of each image is $880\ \mu m \times 880\ \mu m$.

Structure of vortex line

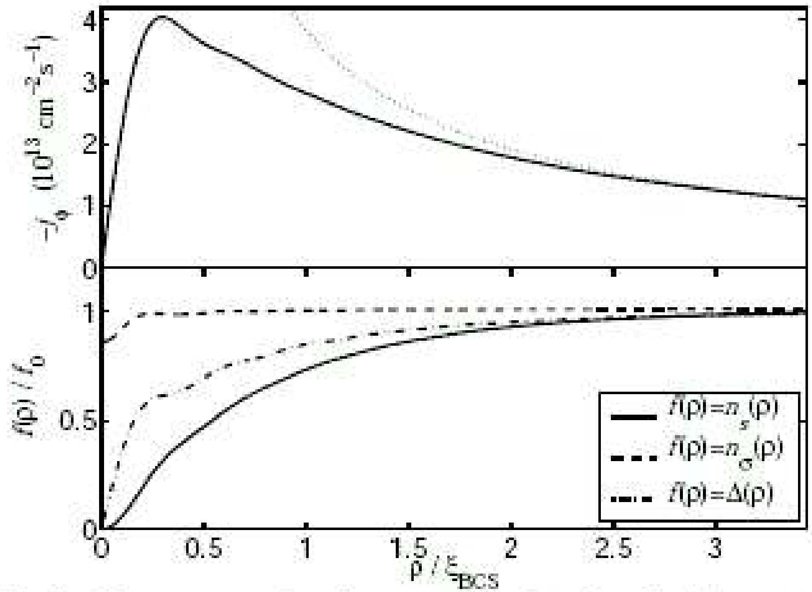


FIG. 4. Upper panel: the current density $\mathbf{j}_s(\rho) = j_\phi(\rho)\mathbf{e}_\phi$ (solid line) and its asymptotic form (dotted line). Lower panel: the density $n_\sigma(\rho)$, the superfluid density $n_s(\rho)$ and the gap function $\Delta(\rho)$ normalized to their theoretical bulk values. In both panels $k_F|a| = 0.59$.

Nygaard et al., 2003.

Superconductivity as superfluidity of charged fermions

Gauge transformation for charged particles:

$$\boldsymbol{A}(\boldsymbol{r}) \to A(\boldsymbol{r}) + \nabla f(\boldsymbol{r}),\ \hat{\Psi} \to \Psi \exp\left(\frac{ie}{\hbar c} f\right).$$

V. Fock, 1927, F. London, 1927.

$$\boldsymbol{S}(\boldsymbol{r}) \to S(\boldsymbol{r}) + \frac{2e}{\hbar c} f(\boldsymbol{r}).$$

Equation $\boldsymbol{v}_s = (\hbar/\mathbf{2}m)\nabla S$

must be corrected to satisfy the invariance:

$$\boldsymbol{v}_s = \frac{\hbar}{\mathbf{2}m}\left(\nabla S - \frac{2e}{\hbar c}\boldsymbol{A}\right).$$

Superconducting current:

$$\boldsymbol{j}_s = n_s \frac{e\hbar}{\mathbf{2}m}\left(\nabla S - \frac{2e}{\hbar c}\boldsymbol{A}\right).$$

Londons equation

Take **curl**:

$$\mathbf{curl}\,\boldsymbol{A} = \boldsymbol{B} = -\frac{e^2 n_s}{mc}\,\mathbf{curl}\,\boldsymbol{j}_s$$

F. London, H. London, 1935.

Maxwell equations:

$$\mathbf{curl}\,\boldsymbol{B} = -\frac{4\pi}{c}\,\boldsymbol{j}\,,\ \mathrm{div}\,\boldsymbol{B} = 0$$

$$\delta^2 \Delta \boldsymbol{B} = \boldsymbol{B}\,,\ \delta^2 = \frac{mc^2}{4\pi e^2 n_s}.$$

$$\boldsymbol{B}(x) = B(x=0)\,e^{-x/\delta}$$

Flux quantization

Supervcoducting ring:

$$\oint \boldsymbol{A}\cdot d\,\boldsymbol{r} = \int \mathbf{curl}\,\boldsymbol{A}\cdot d\,\boldsymbol{f} = \int \boldsymbol{B}\cdot d\,\boldsymbol{f} = \Phi$$

I n the body of the ring:

$$\boldsymbol{j}_s = 0 : \boldsymbol{A} = \frac{\hbar c}{2e}\nabla S,$$

$$\oint \boldsymbol{A}\cdot d\,\boldsymbol{r} = \frac{\hbar c}{2e}\oint \nabla S\cdot d\,\boldsymbol{r} = \frac{\hbar c}{2e}\delta S = \Phi_0 l,$$

$$l = 0, \pm 1, \pm 2, \ldots \Phi_0 = \frac{\pi \hbar c}{|e|}$$

F. London, 1950.

Ginzburg-Landau equations, 1950

$$\frac{1}{4m}\left(-i\hbar\nabla-\frac{2e}{c}\boldsymbol{A}\right)^2\psi-\alpha(T_C-T)\psi+b|\psi|^2\psi=0$$

$$\boldsymbol{j}_s=\frac{-ie\hbar}{2m}(\psi^*\nabla\psi-\psi\nabla\psi^*)-\frac{2e^2}{mc}|\psi|^2\boldsymbol{A}$$

$$\frac{\partial\psi}{\partial t}=\boldsymbol{???}$$

$$\omega\sim(T_C-T),\ \Delta\sim\sqrt{(T_C-T)}$$

Equation cannot be local in t.

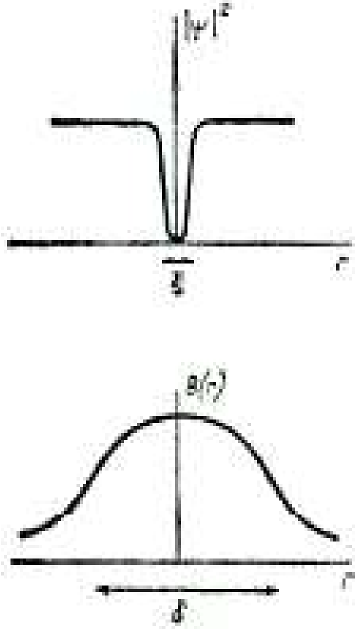
$$\text{GL PARAMETER}:\kappa=\frac{mcb^{1/2}}{(2\pi)^{1/2}\hbar|e|}$$

$\kappa<1/\sqrt{2}$ – Superconductors of the first order.

$\kappa>1/\sqrt{2}$ – Superconductors of the second order.

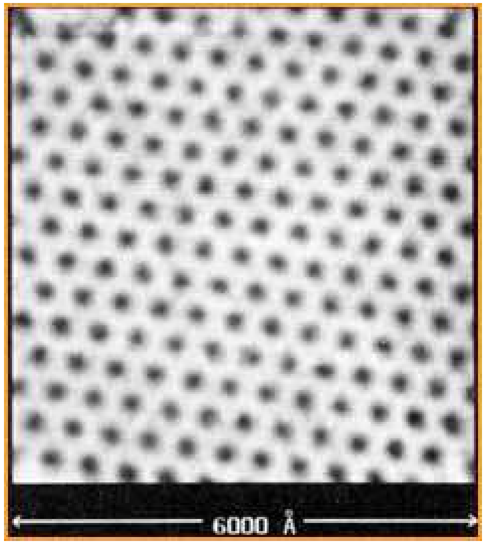
Abrikosov vortex lines

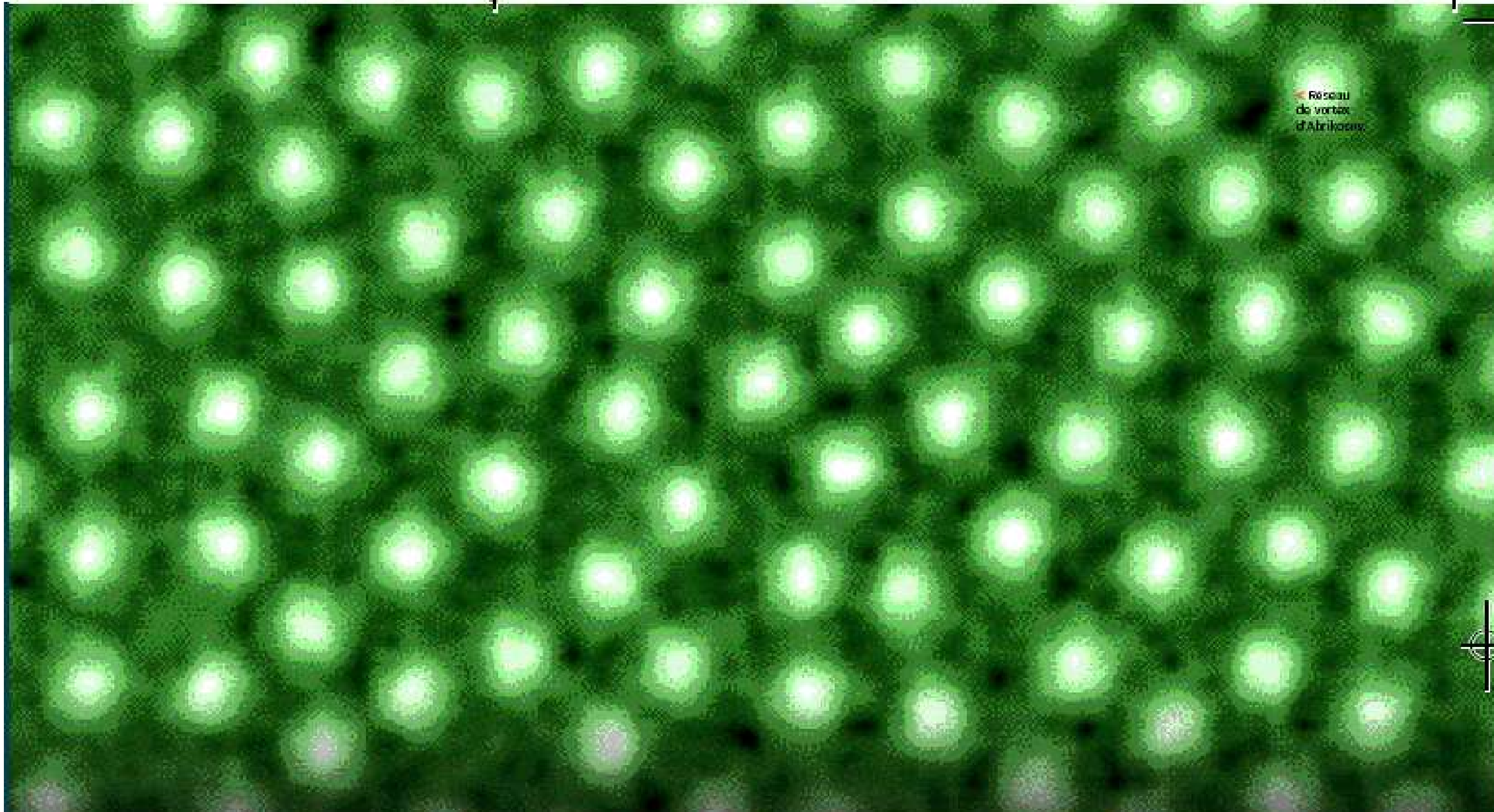
$$\Phi = \Phi_0$$



$$\delta / \xi = \kappa \gg 1/\sqrt{2}$$

Abrikosov lattice





< Réseau de vortex d'Abrikosov

"CETTE IMAGE REPRÉSENTE DES SUPRACONDUCTEURS DE TYPE II DONT J'AVAIS PRÉDIT L'EXISTENCE IL Y A 50 ANS"

ALEXEI ABRIKOSOV

Colauréat du prix Nobel de physique, en 2003, pour ses travaux dans le domaine de la théorie des supraconducteurs et superfluides.

Cette image, appelée "réseau de vortex d'Abrikosov", représente des supraconducteurs de type II dans un champ magnétique. Il y a presque cinquante ans, j'ai prédit l'existence de ces nouveaux supraconducteurs et j'ai développé une théorie sur leurs propriétés magnétiques. Cette théorie a été confirmée plus tard par de nombreuses expériences. Mais cela reste un phénomène quantique très difficile à expliquer. A l'époque, j'avais trente ans et je vivais en Russie. Les expérimentateurs ne croyaient pas en ma théorie. Ce sont des Français, en particulier Michel Cribier, qui ont réalisé les premières expériences convaincantes en utilisant la diffraction à neutrons. La structure en réseau a été confirmée. Ils étaient heureux et moi aussi : comme je l'avais prédit, il existait bien de tels supraconducteurs. Et à ce jour, tous les supraconducteurs découverts depuis les années 60 ont été des supraconducteurs de type II.