COHERENCE PROPERTIES OF A BOSE-EINSTEIN CONDENSATE

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Outline

- Description of the problem
- Framework: Bogoliubov theory
- Spatial coherence
- Temporal coherence
  - N fluctuates
  - N fixed, E fluctuates: Canonical ensemble
  - N fixed, E fixed: Microcanonical ensemble

DESCRIPTION OF THE PROBLEM

A single-spin state Bose gas prepared at equilibrium:

- **Spatially homogeneous, periodic boundary conditions.**
- **Prepared with N atoms, in well-Bose-condensed regime $T \ll T_c$.**
- **Interactions with a s-wave scattering length $a > 0$.**
- **Weakly interacting regime $(\rho a^3)^{1/2} \ll 1$.**
- **The gas is totally isolated in its evolution.**

Spatial coherence of the gas:

- **Determined by the measured first-order coherence function, $g_1(\mathbf{r}) = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(0)\rangle$ (Esslinger, Bloch, Hänsch, 2000).**
- **Expected: In thermodynamic limit, g_1 tends to condensate density $\rho_0 > 0$ at infinity.**
- **This is long-range order.**

Coherence time of the condensate:

- Defined as the decay time of the measurable condensate mode coherence function, $\langle a_0^\dagger(t) a_0(0) \rangle$, where a_0 is the annihilation operator in mode $\mathbf{k} = 0$.
- At zero temperature, no decay, $\langle a_0^\dagger(t) a_0(0) \rangle \sim \langle N_0 \rangle e^{i\mu_0 t/\hbar}$, coherence time is infinite (Beliaev, 1958).
- What happens at finite temperature $T > 0$?
- One expects infinite coherence time in thermodynamic limit.

FRAMEWORK: BOGOLIUBOV THEORY

Bogoliubov theory

- **Lattice model Hamiltonian:**

$$H = \sum_{\mathbf{r}} b^3 \left[\hat{\psi}^\dagger h_0 \hat{\psi} + \frac{g_0}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

- **Spatially homogeneous case:** $h_0 = -\frac{\hbar^2}{2m} \Delta_{\mathbf{r}}$.
- **Bare coupling constant** $g_0 = g/(1 - C_3 a/b)$, $g = 4\pi\hbar^2 a/m$. **Here Born regime** $0 < a \ll b$.
- **Expansion of Hamiltonian around pure condensate:**

$$\hat{\psi}(\mathbf{r}) = \phi(\mathbf{r}) \hat{a}_0 + \hat{\psi}_\perp(\mathbf{r})$$

  with $\phi(\mathbf{r}) = 1/L^{3/2}$.
- **Key point: Eliminate amplitude** $\hat{a}_0$ **in condensate mode:**

$$\hat{n}_0 = \hat{N} - \hat{N}_\perp$$

  with $\hat{n}_0 = \hat{a}_0^\dagger \hat{a}_0$ **and** $\hat{N}_\perp = \sum_{\mathbf{r}} b^3 \hat{\psi}_\perp^\dagger \hat{\psi}_\perp$.

Elimination of the condensate phase

- Modulus-phase representation (Girardeau, Arnowitt, 1959):

$$\hat{a}_0 = e^{i\hat{\theta}}\hat{n}_0^{1/2}$$

with hermitian operator $\hat{\theta}$, $[\hat{n}_0, \hat{\theta}] = i$.

- Cf. position $\hat{x}$ and momentum $\hat{p}$ operator of a particle:

$$[\hat{x}, \hat{p}] = i\hbar \Longrightarrow e^{i\hat{p}a/\hbar}|x\rangle = |x - a\rangle$$

$$[\hat{n}_0, \hat{\theta}] = i \Longrightarrow e^{i\hat{\theta}}|n_0 : \phi\rangle = |n_0 - 1 : \phi\rangle$$

then $\hat{a}_0$ has the right matrix elements.

- This gets crazy when the condensate mode is empty:

$$e^{i\hat{\theta}}|0 : \phi\rangle \stackrel{?!}{=} |-1 : \phi\rangle$$

- Redefinition of non-condensed field (Castin, Dum; Gardiner, 1996) ; remains bosonic, but conserves $\hat{N}$:

$$\hat{\Lambda}(\mathbf{r}) = e^{-i\hat{\theta}}\hat{\psi}_\perp(\mathbf{r})$$

- Expansion of H to second order in $\hat{\psi}_\perp$:

$$H_{\mathrm{Bog}} = \frac{g_0 N^2}{2L^3} + \sum_{\mathbf{r}} b^3 \left[\hat{\Lambda}^\dagger (h_0 - \mu_0) \hat{\Lambda} + \mu_0 \left(\frac{1}{2} \hat{\Lambda}^2 + \frac{1}{2} \hat{\Lambda}^{\dagger 2} + 2 \hat{\Lambda}^\dagger \hat{\Lambda} \right) \right]$$

- Formally grand canonical for non-condensed modes, with chemical potential $\mu_0 = g_0 \rho$.
- Elastic interaction $C - NC$: Hartree-Fock

$$C, 0 + NC, \mathrm{k} \longrightarrow C, 0 + NC, \mathrm{k}$$

- Inelastic interaction $C - NC$: Landau superfluidity

$$C, 0 + C, 0 \longrightarrow NC, \mathrm{k} + NC, -\mathrm{k}$$

Not forbidden by energy conservation.

Normal form for the Hamiltonian:

- $\boldsymbol{H}_{\mathrm{Bog}}$ quadratic, hence linear equations of motion:

$$i\hbar\partial_t \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} = \begin{pmatrix} h_0 + \mu_0 & \mu_0 \\ -\mu_0 & -(h_0 + \mu_0) \end{pmatrix} \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} \equiv \mathcal{L} \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix}$$

- $\mathcal{L}$ "hermitian" for scalar product of signature $(1, -1)$.
- Expansion on eigenmodes of eigenenergies $\pm\epsilon_k$:

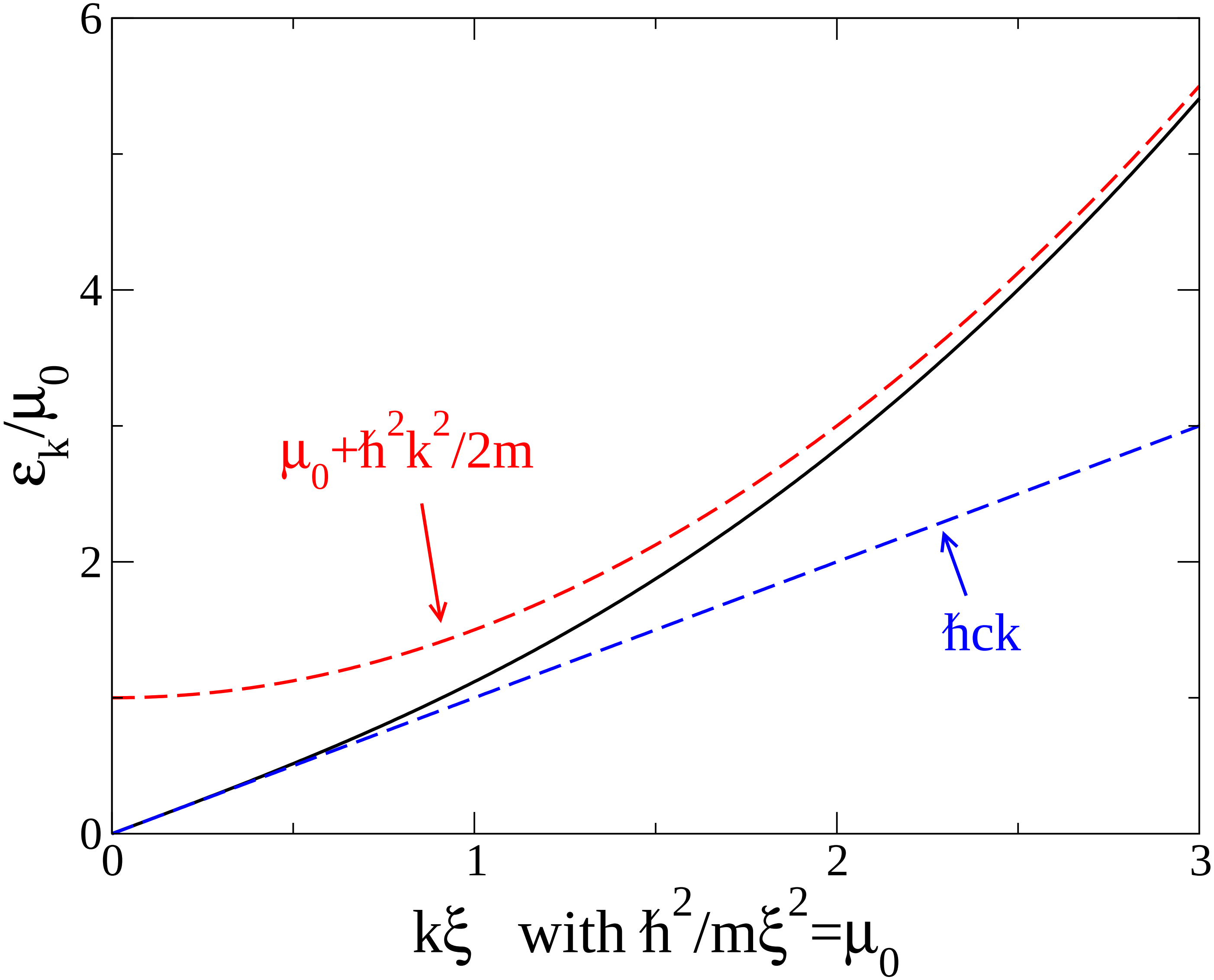
$$\begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} = \sum_{\mathrm{k}\neq 0} \frac{e^{i\mathrm{k}\cdot\mathrm{r}}}{L^{d/2}} \begin{pmatrix} U_k \\ V_k \end{pmatrix} \hat{b}_{\mathrm{k}} + \frac{e^{-i\mathrm{k}\cdot\mathrm{r}}}{L^{d/2}} \begin{pmatrix} V_k \\ U_k \end{pmatrix} \hat{b}_{\mathrm{k}}^\dagger$$

with $U_k^2 - V_k^2 = 1$, $U_k + V_k = \left(\frac{\hbar^2 k^2/2m}{2\mu_0 + \hbar^2 k^2/2m}\right)^{1/4}$.

- A grand-canonical ideal gas of bosonic quasi-particles:

$$H_{\mathrm{Bog}} = E_0 + \sum_{\mathrm{k}\neq 0} \epsilon_k \hat{b}_k^\dagger \hat{b}_{\mathrm{k}} \quad \text{with} \quad \epsilon_k = \left[\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2\mu_0\right)\right]^{1/2}$$

Bogoliubov spectrum



SPATIAL COHERENCE

Consistency check

In thermodynamic limit:

- Non-condensed fraction:

$$\frac{\langle N_\perp \rangle}{N} = \frac{\langle \hat{\Lambda}^\dagger \hat{\Lambda} \rangle}{\rho} = \frac{1}{\rho} \int \frac{d^3k}{(2\pi)^3} \left[\frac{U_k^2 + V_k^2}{e^{\beta \epsilon_k} - 1} + V_k^2 \right]$$

- No ultraviolet ($k \to \infty$) divergence: $V_k^2 = O(1/k^4)$
- No infrared ($k \to 0$) divergence: $U_k^2, V_k^2 = O(1/k)$.
- Small for $T \ll T_c$ and $(\rho a^3)^{1/2} \ll 1$.
- First order coherence function $g_1(\mathrm{r}) = \langle \hat{\psi}^\dagger(\mathrm{r}) \hat{\psi}(0) \rangle$:

$$g_1(\mathrm{r}) = \rho - \int \frac{d^3k}{(2\pi)^3} (1 - \cos \mathrm{k} \cdot \mathrm{r}) \left[\frac{U_k^2 + V_k^2}{e^{\beta \epsilon_k} - 1} + V_k^2 \right]$$

tends to the condensate density for $r \to \infty$.

In lower dimensions:

- In 2D for $T > 0$ and in 1D $\forall T$, the non-condensed fraction has infrared divergence. No BEC in thermodynamic limit (Mermin, Wagner, 1966; Hohenberg, 1967).
- Quasi-condensate (weak density fluctuations, weak phase gradients) (Popov, 1972). One can save the idea of Bogoliubov by applying it to a modulus-phase representation of the field operator $\hat{\psi}$.
- $g_1^{\rm Bog}(\mathbf{r}) \to -\infty$ at infinity, but remarkably (Mora, Castin, 2003):

$$g_1^{\rm QC}(\mathbf{r}) = \rho \exp\left[\frac{g_1^{\rm Bog}(\mathbf{r})}{\rho} - 1\right].$$

TEMPORAL COHERENCE

GENERAL CONSIDERATIONS

- If weak fluctuations of $\hat{n}_0$:

$$\langle a_0^\dagger(t) a_0(0) \rangle \simeq \langle \hat{n}_0 \rangle \langle e^{-i[\hat{\theta}(t)-\hat{\theta}(0)]} \rangle$$

- If phase change $\hat{\theta}(t) - \hat{\theta}(0)$ has Gaussian distribution:

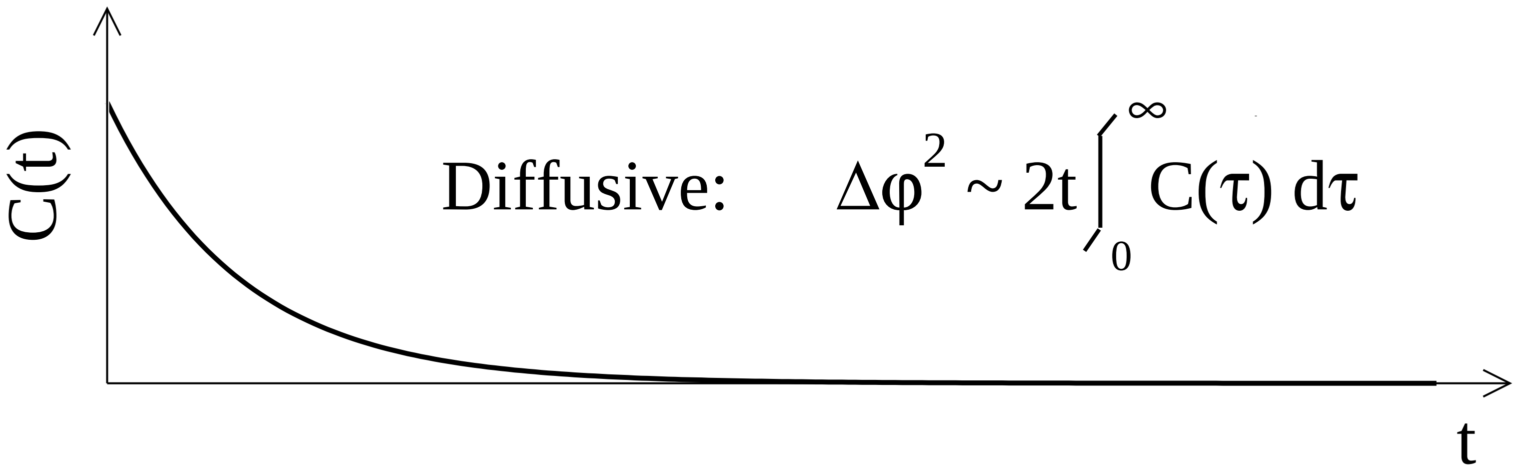
$$\left| \langle a_0^\dagger(t) a_0(0) \rangle \right| \simeq \langle \hat{n}_0 \rangle e^{-\mathrm{Var}\,[\hat{\theta}(t)-\hat{\theta}(0)]/2}$$

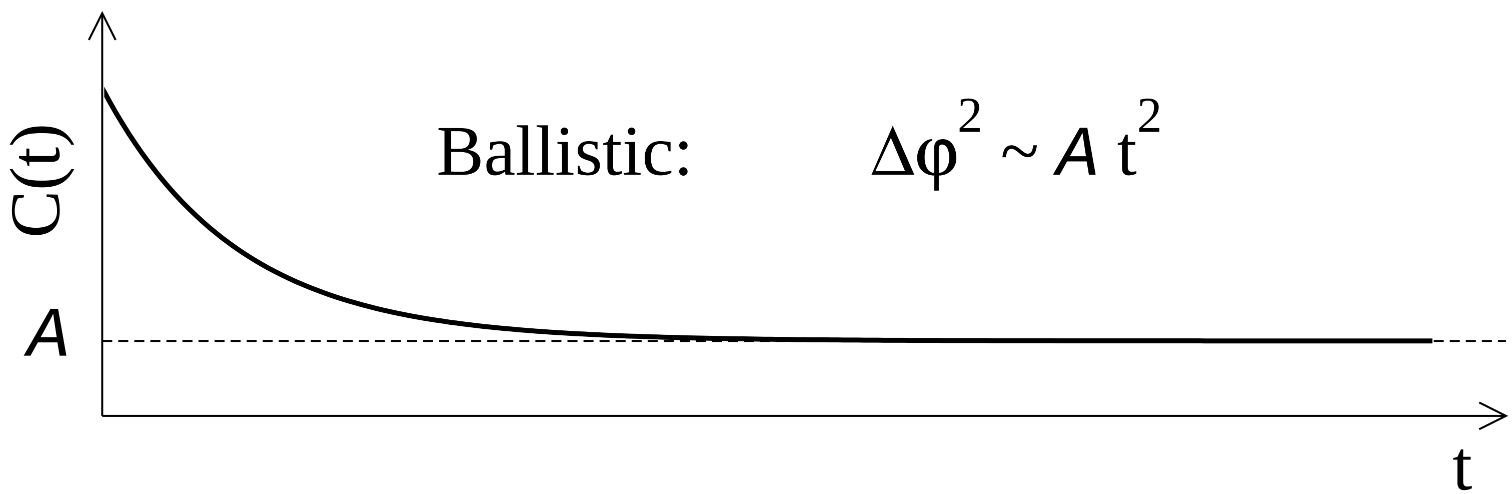
- In terms of correlation function $C(t) = \langle \dot{\theta}(t)\dot{\theta}(0) \rangle - \langle \dot{\theta} \rangle^2$:

$$\mathrm{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] = 2t \int_0^t d\tau\, C(\tau) - 2 \int_0^t d\tau\, \tau C(\tau)$$

ballistic regime	diffusive regime
$\lim_{\tau \to +\infty} C(\tau) \neq 0$	$C(\tau) \underset{\tau \to +\infty}{=} o(1/\tau)$
$\mathrm{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] \sim At^2$	$\mathrm{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] \sim 2Dt$

TWO CASES DEPENDING ON $C(t \to +\infty)$





GENERAL CONSIDERATIONS (2)

Previous studies at $T > 0$:

- **Zoller, Gardiner (1998), Graham (1998-2000): Diffusive.**
- **Contradicted by Kuklov, Birman (2000): Ballistic.**
- **Sinatra, Witkowska, Castin (2006-2009): Clarification and quantitative studies.**

Two key actors:

- **Bogoliubov procedure eliminating the condensate mode from the Hamiltonian:**

$$H = E_0(N) + \sum_{\mathbf{k}\neq 0} \epsilon_k \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + H_3 + \ldots$$

where ϵ_k is the Bogoliubov spectrum. Hamiltonian H_3 is cubic in field $\hat{\Lambda}$. It breaks integrability and plays central role in condensate dephasing (Beliaev-Landau pro-

cesses):

$$H_3 = g_0 \rho^{1/2} \sum_{\mathbf{r}} b^3 \hat{\Lambda}^+ (\hat{\Lambda} + \hat{\Lambda}^\dagger) \hat{\Lambda}$$

- Time derivative of condensate phase operator:

$$\dot{\theta} \equiv \frac{1}{i\hbar}[\theta, H] \simeq -\mu_{T=0}(N)/\hbar - \frac{g_0}{\hbar L^3} \sum_{\mathbf{k} \neq 0} (U_k + V_k)^2 \hat{n}_{\mathbf{k}}$$

with $\hat{n}_{\mathbf{k}} = \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}$. This contradicts Graham, 1998 and 2000.

Case of a pure condensate

- One-mode model, with $\hat{n}_0 = \hat{N}$: $H_{\text{one mode}} = \frac{g}{2L^3}\hat{N}^2$
- Evolution of the condensate phase:

$$\dot{\theta}(t) = \frac{1}{i\hbar}[\hat{\theta}, H_{\text{one mode}}] = -\frac{g\hat{N}}{\hbar L^3} = -\mu(\hat{N})/\hbar$$

- No phase spreading if fixed N.
- Ballistic spreading if N fluctuates (Sols, 1994; Walls, 1996; Lewenstein, 1996; Castin, Dalibard, 1997)

$$\text{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] = (t/\hbar)^2 \left(\frac{d\mu}{dN}\right)^2 \text{Var}\,\hat{N}$$

- Experiments: Seen not for $\langle a_0^\dagger(t) a_0\rangle$ but for $\langle a_0^\dagger(t) b_0(t)\rangle$ by interfering two condensats with common $t = 0$ phase: Bloch, Hänsch (2002); Pritchard, Ketterle (2006); Reichel, 2010.

$T > 0$ gas prepared in the canonical ensemble

Analogy with previous case (Sinatra et al, 2007) :

- As N, the energy E is a constant of motion.
- Canonical ensemble = statistical mixture of eigenstates, $\text{Var}\, E \neq 0$ but $\text{Var}\, E \ll \bar{E}^2$ for a large system
- $\hat{\theta}(t) \sim -\mu_{\text{mc}}(\hat{H})t/\hbar$ and weak fluctuations of $\hat{H}$:

$$\text{Var}\left[\hat{\theta}(t) - \hat{\theta}(0)\right] \sim (t/\hbar)^2 \left[\frac{d\mu_{\text{mc}}}{dE}(\bar{E})\right]^2 \text{Var}\, E$$

From quantum ergodic theory (Sinatra et al, 2007) :

- Deutsch (1991) : eigenstate thermalisation hypothesis. Mean value of observable $\hat{O}$ in one eigenstate Ψ_λ very close to microcanonical value:

$$\langle \Psi_\lambda | \hat{O} | \Psi_\lambda \rangle \simeq \bar{O}_{\text{mc}}(E = E_\lambda)$$

- $\hat{O} = \dot{\theta}$ in Bogoliubov limit : $\bar{\dot{\theta}}_{\text{mc}} = -\mu_{\text{mc}}/\hbar.$

Implications of previous result (canonical ensemble)

- The correlation function $C(\tau)$ of $\dot{\theta}$ does not tend to zero when $\tau \to +\infty$. Neither does the one of $\hat{n}_0$.
- This qualitatively contradicts Zoller, Gardiner, Graham. In qualitative agreement with Kuklov, Birman.
- Ergodicity ensured by interactions (cf. H_3) among Bogoliubov quasi-particles.
- Approximating H with integrable H_{Bog}, as eventually done by Kuklov and Birman, gives incorrect coefficient of t^2.

A. Sinatra, Y. Castin, E. Witkowska, Phys. Rev. A 75, 033616 (2007)

Illustration with a classical field calculation

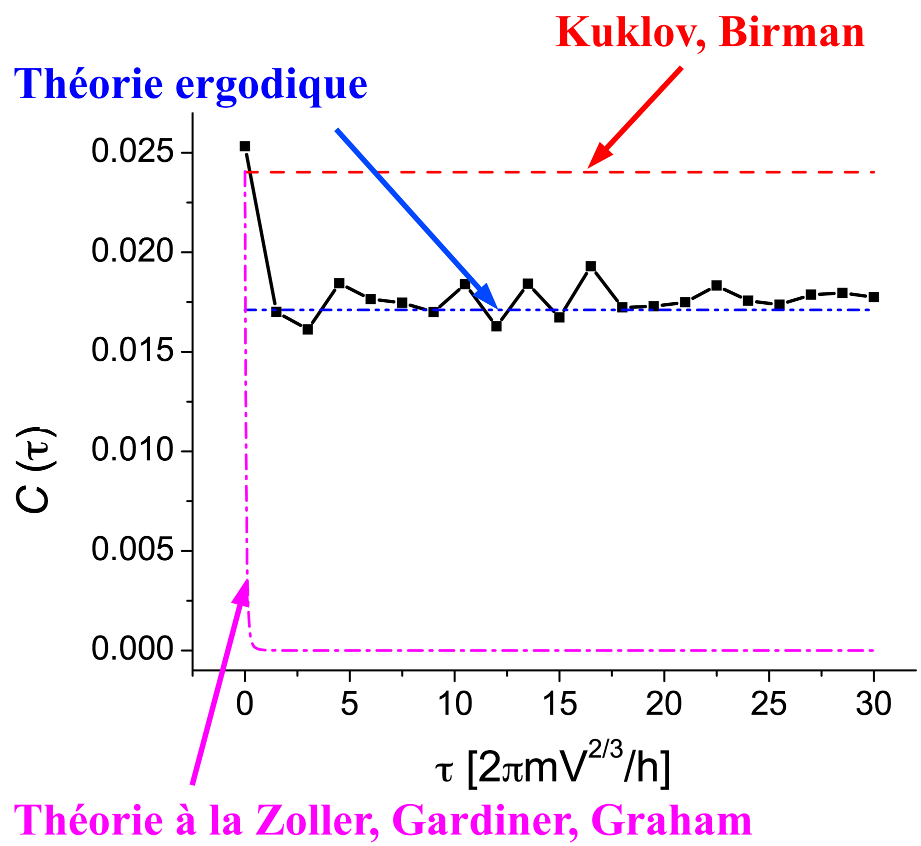


Figure 1: For a gas prepared in canonical ensemble, correlation function of $\dot{\boldsymbol{\theta}}$ for the classical field. The equation of motion is the non-linear Schrödinger equation. A. Sinatra, Y. Castin, E. Witkowska, Phys. Rev. A **75**, 033616 (2007).

Gas prepared in the microcanonical ensemble: phase diffusion

- The conserved quantities N, E do not fluctuate. One finds $C(\tau) \underset{\tau\to+\infty}{=} O(1/\tau^3)$ and $\mathrm{Var}\,[\hat{\theta}(t)-\hat{\theta}(0)] \sim 2Dt$.
- One needs the full dependence of $C(\tau)$ to get D.
- In the Bogoliubov limit, setting $\hat{n}_{\mathrm{k}} \equiv \hat{b}^{\dagger}_{\mathrm{k}}\hat{b}_{\mathrm{k}}$:

$$-\hbar\dot{\hat{\theta}}(\tau) \simeq \mu_{T=0}(\hat{N}) + \frac{g}{L^3}\sum_{\mathrm{k}\neq 0}(U_k+V_k)^2\hat{n}_{\mathrm{k}}(\tau)$$

  $C(\tau)$ can be deduced from all the $\langle \hat{n}_{\mathrm{k}}(\tau)\hat{n}_{\mathrm{k}'}(0)\rangle$.
- The gas is in a statistical mixture of Fock states quasi-particles $|\{n_{\mathrm{q}}\}\rangle$. One simply needs $\langle\{n_{\mathrm{q}}\}|\hat{n}_{\mathrm{k}}(\tau)|\{n_{\mathrm{q}}\}\rangle$.
- The evolution of the mean number of quasi-particles is given by quantum kinetic equations including the Beliaev-Landau processes due to H_3.

The quantum kinetic equations

$$\dot{n}_{\mathrm{q}} = -\frac{g^2\rho}{\hbar\pi^2}\int d^3\mathrm{k}\Big\{ \left[n_{\mathrm{q}}n_{\mathrm{k}} - n_{\mathrm{q+k}}(1+n_{\mathrm{k}}+n_{\mathrm{q}})\right]\left(\mathcal{A}_{k,q}^{|\mathrm{q+k}|}\right)^2$$

$$\times\delta(\epsilon_q+\epsilon_k-\epsilon_{|\mathrm{q+k}|})\Big\}$$

$$-\frac{g^2\rho}{2\hbar\pi^2}\int d^3\mathrm{k}\Big\{ \left[n_{\mathrm{q}}(1+n_{\mathrm{k}}+n_{\mathrm{q-k}}) - n_{\mathrm{k}}n_{\mathrm{q-k}}\right]\left(\mathcal{A}_{k,|\mathrm{q-k}|}^{q}\right)^2$$

$$\times\delta(\epsilon_k+\epsilon_{|\mathrm{q-k}|}-\epsilon_q)\Big\}$$

with the Beliaev-Landau coupling amplitudes:

$$\mathcal{A}_{k,k'}^{q} = U_qU_kU_{k'} + V_qV_kV_{k'} + (U_q+V_q)(V_kU_{k'}+U_kV_{k'})\,.$$

E. M. Lifshitz, L. P. Pitaevskii “Physical Kinetics”, Landau and Lifshitz Course of Theoretical Physics vol. 10, chap. VII, Pergamon Press (1981)

Diffusion coefficient of the condensate phase

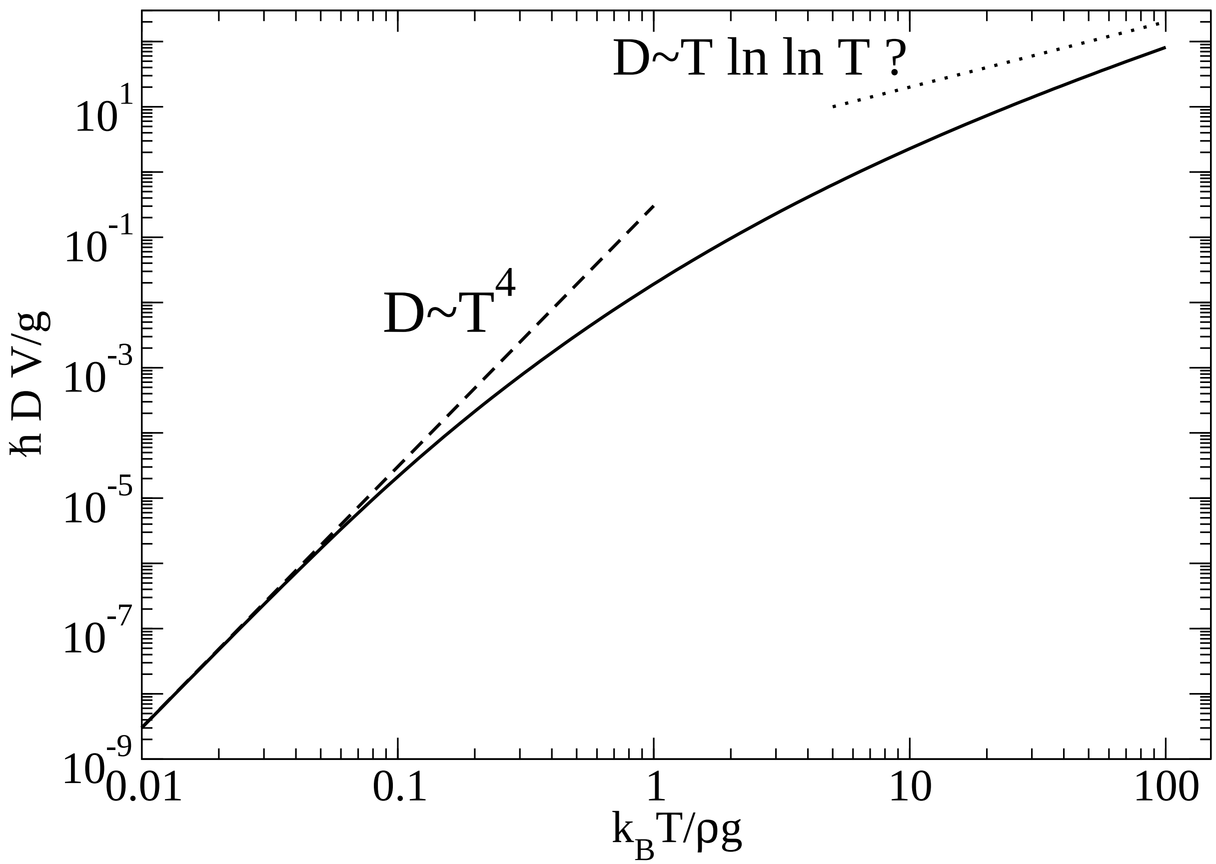


Figure 2: Universal result in Bogoliubov limit (weakly interacting, $T \ll T_c$).

A. Sinatra, Y. Castin, E. Witkowska, Phys. Rev. A 80, 033614 (2009)

Summary of results for the phase spreading

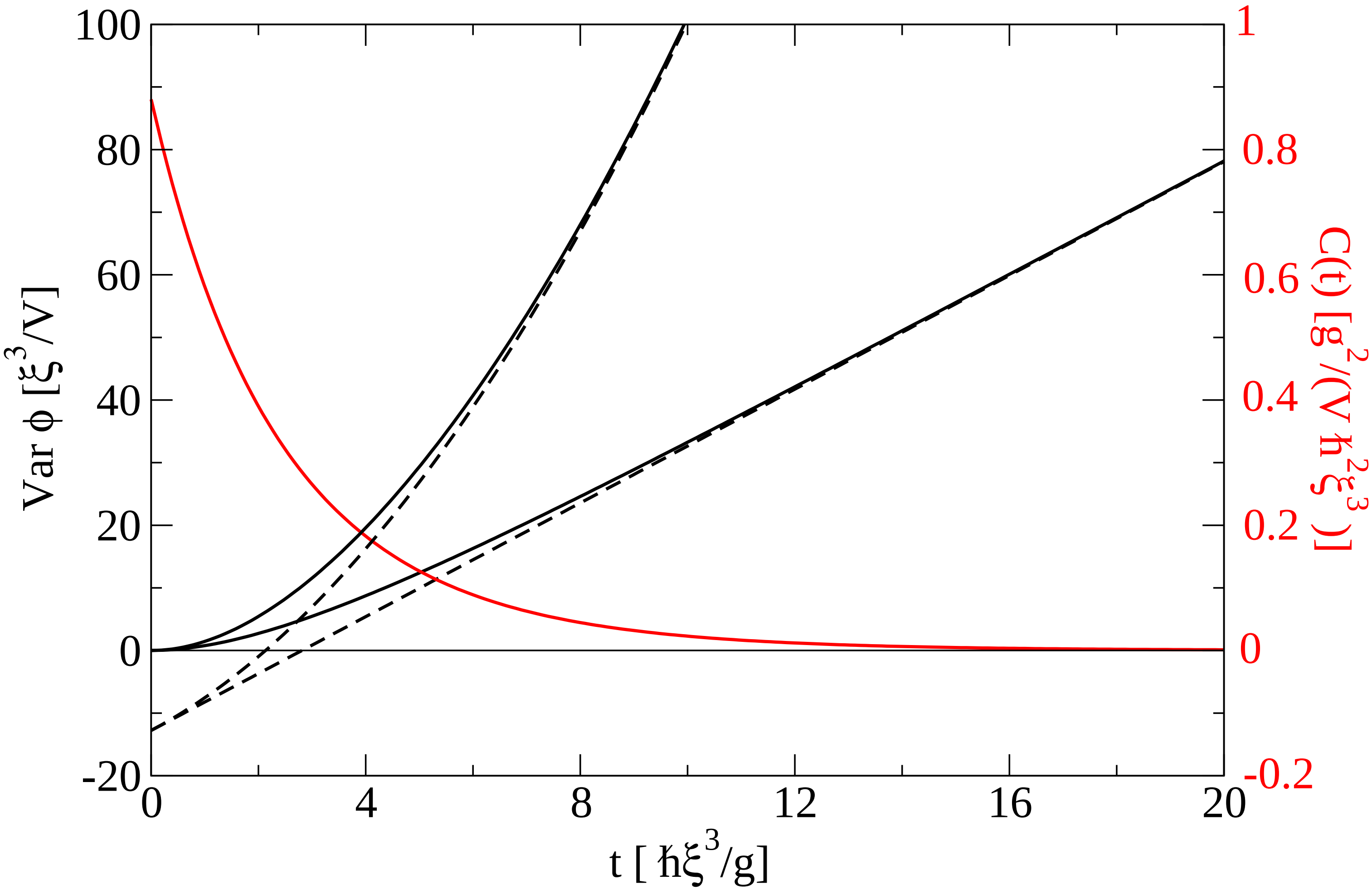
$$\mathrm{Var}\,[\theta(t)-\theta(0)] \underset{t\to+\infty}{=} \mathrm{Var}\,(E)\,\left[\frac{d\mu_{\rm mc}}{\hbar dE}(\bar{E})\right]^2 t^2+2Dt+c+O(\frac{1}{t})$$

- Existence of a t^2 term first in Kuklov, Birman, 2000.
- Coefficient of t^2 depends on the ensemble. First obtained with quantum ergodic theory (Sinatra, Castin, Witkowska, 2007) but also with quantum kinetic theory (from existence of undamped mode of linearized kinetic equations due to energy conservation). Interpretation:

$$\theta(t)-\theta(0) \underset{t\to+\infty}{\sim} -\mu(H)t/\hbar.$$

- Diffusion coefficient D is ensemble independent. $\hbar D L^3/g$ function of $k_B T/\rho g$ (Sinatra, Castin, Witkowska, 2009).
- Ensemble independent $c \neq 0$: $C_{\rm mc}(t)$ not a Dirac.

AN EXAMPLE FOR $k_B T = 10\rho g$



Our publications on the subject

- A. Sinatra, Y. Castin, E. Witkowska, "Nondiffusive phase spreading of a Bose-Einstein condensate at finite temperature", Phys. Rev. A 75, 033616 (2007)
- A. Sinatra, Y. Castin, "Genuine phase diffusion of a Bose-Einstein condensate in the microcanonical ensemble: A classical field study", Phys. Rev. A 78, 053615 (2008)
- A. Sinatra, Y. Castin, E. Witkowska, "Coherence time of a Bose-Einstein condensate", Phys. Rev. A. 80, 033614 (2009)

More on kinetic theory

- For large system sizes, kinetic equations may be linearized around mean occupation numbers $\bar{n}_k$ (coarse graining argument).
- Collecting coefficients appearing in $\dot{\theta}$ in a vector $\vec{A}$,

$$A_{\mathrm{k}} \equiv \frac{g}{\hbar L^3}(U_k + V_k)^2$$

- Collecting the unknowns in a vector $\vec{x}\,(t)$,

$$x_{\mathrm{k}}(t) = \sum_{\mathrm{k}'\neq 0} A_{\mathrm{k}'}\langle \delta\hat{n}_{\mathrm{k}}(t)\delta\hat{n}_{\mathrm{k}'}(0)\rangle$$

- Then one solves

$$\dot{\vec{x}}(t) = M\vec{x}(t)$$

where M results from linearisation of the quantum kinetic equations around the mean occupation numbers.

The initial condition can be expressed analytically in canonical, microcanonical and more general ensembles.

- Then correlation function of the time derivative of the phase is

$$C(t) = \vec{A} \cdot \vec{x}(t)\,.$$

- Crucial point: M is not invertible because of energy conservation:

$$^{t}M\vec{\epsilon} = \vec{0}.$$

Zero frequency eigenvector of M is $\alpha_{\mathrm{k}} \propto d\bar{n}_k/dT$. Then splitting

$$\vec{x}(t) = \gamma\vec{\alpha} + \vec{X}\,(t)$$

with

$$\hbar\gamma = \mathrm{Var}\,(E)\frac{d\mu_{\mathrm{mc}}}{dE}(\bar{E}).$$

γ is time independent whereas $\vec{X}(t) \to 0$ at long times.