THE UNITARY GAS: SYMMETRY PROPERTIES AND APPLICATIONS

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Outline

- What is the unitary gas ?
- Simple facts from scaling invariance
- Time-dependent solution in a trap
- Separability in hyperspherical coordinates
- The 4-body Efimov effect

WHAT IS THE UNITARY GAS ?

Definition of the unitary gas

- Particles with s-wave binary interaction. Two-body scattering amplitude

$$\phi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f_k \frac{e^{ikr}}{r}$$

- For a unitary gas, $f_k = -1/(ik) \quad \forall k$. "Maximally" interacting: Unitarity of S matrix imposes $|f_k| \leq 1/k$.

- In real experiments with magnetic Feshbach resonance (Thomas, Salomon, Jin, Ketterle, Grimm, ...) :

$$-\frac{1}{f_k} = \frac{1}{a} + ik - \frac{1}{2}k^2 r_e + O(k^4 b^3)$$

so have "infinite" scattering length a and "zero" ranges:

$$k_{\text{typ}}|a| \gg 1, k_{\text{typ}}|r_e| \ll 1, k_{\text{typ}} b \ll 1.$$

- All these two-body conditions are only necessary.

The zero-range Wigner-Bethe-Peierls model

- Interactions are replaced by contact conditions.
- For $r_{ij} \to 0$ with fixed ij-centroid $\vec{C}_{ij} = (m_i\vec{r}_i + m_j\vec{r}_j)/(m_i + m_j)$ different from $\vec{r}_k, k \neq i, j$:

$$\psi(\vec{r}_1, \ldots, \vec{r}_N) = \left(\frac{1}{r_{ij}} - \frac{1}{a}\right) A_{ij}[\vec{C}_{ij}; (\vec{r}_k)_{k\neq i,j}] + O(r_{ij})$$

- Elsewhere, non interacting Schrödinger equation

$$E\psi(\vec{X}) = \left[-\frac{\hbar^2}{2m}\Delta_{\vec{X}} + \frac{1}{2}m\omega^2 X^2\right]\psi(\vec{X})$$

  with $\vec{X} = (\vec{r}_1, \ldots, \vec{r}_N)$.
- Exchange symmetry: Even for boson positions, odd for same-spin fermion positions.
- Unitary gas exists iff Hamiltonian is self-adjoint.

Exercising with the Bethe-Peierls model

Scattering state of two particles:

$$\phi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f_k \frac{e^{ikr}}{r}$$

- For $r > 0$ this is an eigenstate of the non-interacting problem.
- Contact condition in $r = 0$

$$\frac{f_k}{r} + (1 + ikf_k) + O(r) = \frac{A}{r} + O(r)$$

determines scattering amplitude f_k:

$$f_k = -\frac{1}{ik}$$

SIMPLE FACTS FROM SCALING INVARIANCE

Scaling invariance of contact conditions

$$\psi(\vec{X}) \underset{r_{ij}\to 0}{=} \frac{1}{r_{ij}} A_{ij}[\vec{C}_{ij};(\vec{r}_k)_{k\neq i,j}] + O(r_{ij})$$

- Domain of Hamiltonian is scaling invariant: If ψ obeys the contact conditions, so does ψ_λ with

$$\psi_\lambda(\vec{X}) \equiv \frac{1}{\lambda^{3N/2}}\psi(\vec{X}/\lambda)$$

- Consequences (also true for the ideal gas):

free space	box (periodic b.c.)	trap
$\forall N$, no bound states$^{(*)}$	$PV = 2E/3$ $^{(**)}$	virial theorem

$^{(*)}$ If ψ of eigenenergy E, ψ_λ of eigenenergy E/λ^2. Square integrable eigenfunctions (after center of mass removal) correspond to point-like spectrum, for selfadjoint H. $^{(**)}$ $F(N, V\lambda^3, T/\lambda^2) = F(N, V, T)/\lambda^2$, derivative in $\lambda = 1$ and Gibbs-Duhem $F = E - TS = \mu N - PV$.

Virial theorem

- Particles trapped in general external potential $U(\mathrm{r})$:

$$H = H_{\mathrm{Laplace}} + \sum_{i=1}^{N} U(\mathrm{r}_i)$$

- Consider eigenstate ψ of energy E. Mean energy of ψ_λ:

$$E_\lambda = \frac{\langle H_{\mathrm{Laplace}} \rangle_\psi}{\lambda^2} + \langle \sum_{i=1}^{N} U(\lambda \mathrm{r}_i) \rangle_\psi$$

- Eigenstate is stationary point of mean energy: $\frac{d}{d\lambda} E_\lambda = 0$ in $\lambda = 1$. Gives energy from density (Thomas, 2008):

$$E = \sum_{i=1}^{N} \langle U(\mathrm{r}_i) + \frac{1}{2} \mathrm{r}_i \cdot \partial_{\mathrm{r}_i} U(\mathrm{r}_i) \rangle_\psi \underset{U \text{ harmonic}}{=} 2 E_{\mathrm{trap}}$$

- For hard walls $E = \frac{3}{2} PV$ [$-\partial_{\mathrm{r}} U$ = force due to the wall]

TIME-DEPENDENT SOLUTION IN A TRAP

IN A TIME-DEPENDENT TRAP

- At $t = 0$: static trap $U(\mathrm{r}) = m\omega^2 r^2/2$, system in eigenstate $\psi_0(\vec{X})$ of energy E.
- For $t > 0$, arbitrary time dependence of trap spring constant, $\omega(t)$. Known solution for ideal gas:

$$\psi(\vec{X}, t) = \frac{e^{-i\theta(t)}}{\lambda^{3N/2}(t)} \exp\left[\frac{im\dot{\lambda}}{2\hbar\lambda} X^2\right] \psi_0(\vec{X}/\lambda(t))$$

with $\ddot{\lambda} = \omega^2\lambda^{-3} - \omega^2(t)\lambda$ and $\dot{\theta} = E\lambda^{-2}/\hbar$.

- This is a gauge plus scaling transform.
- The gauge transform also preserves contact conditions:

$$r_i^2 + r_j^2 = 2C_{ij}^2 + \frac{1}{2} r_{ij}^2$$

so solution also applies to unitary gas!

Y. Castin, Comptes Rendus Physique 5, 407 (2004).

IN THE MACROSCOPIC LIMIT

$$\psi(\vec{X},t) = \frac{e^{-i\theta(t)}}{\lambda^{3N/2}} \exp\left[\frac{im\dot{\lambda}}{2\hbar\lambda}X^2\right] \psi_0(\vec{X}/\lambda)$$

density $\rho(\vec{r},t) = \rho_0(\vec{r}/\lambda)/\lambda^3$	velocity field $\vec{v}(\vec{r},t) = \vec{r}\dot{\lambda}/\lambda$
local temp. $T(\vec{r},t) = T/\lambda^2$	pressure $P(\vec{r},t) = P_0(\vec{r}/\lambda)/\lambda^5$
local entropy per particle	$s(\vec{r},t) = s_0(\vec{r}/\lambda)$

This has to solve the hydrodynamic equations for a normal gas. Entropy production equation:

$$\rho k_B T(\partial_t s + \vec{v}\cdot\vec{\nabla}s) = \vec{\nabla}\cdot(\kappa\nabla T) + \zeta(\vec{\nabla}\cdot\vec{v})^2$$
$$+\frac{\eta}{2}\sum_{i,j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}\right)^2$$

so the bulk viscosity is zero: $\zeta(\rho,T) = 0\ \forall T > T_c$. Reproduces the conformal invariance result of Son (2007).

LADDER STRUCTURE OF THE SPECTRUM

- Infinitesimal change of ω for $0 < t < t_f$. For $t > t_f$:

$$\lambda(t) - 1 = \epsilon\, e^{-2i\omega t} + \epsilon^*\, e^{2i\omega t} + O(\epsilon^2)$$

so an udamped mode of frequency 2ω.

- Corresponding wavefunction change:

$$\psi(\vec{X},t) = \Big[e^{-iEt/\hbar} - \epsilon e^{-i(E+2\hbar\omega)t/\hbar} L_+ + \epsilon^* e^{-i(E-2\hbar\omega)t/\hbar} L_-\Big]\, \psi_0(\vec{X}) + O(\epsilon^2)$$

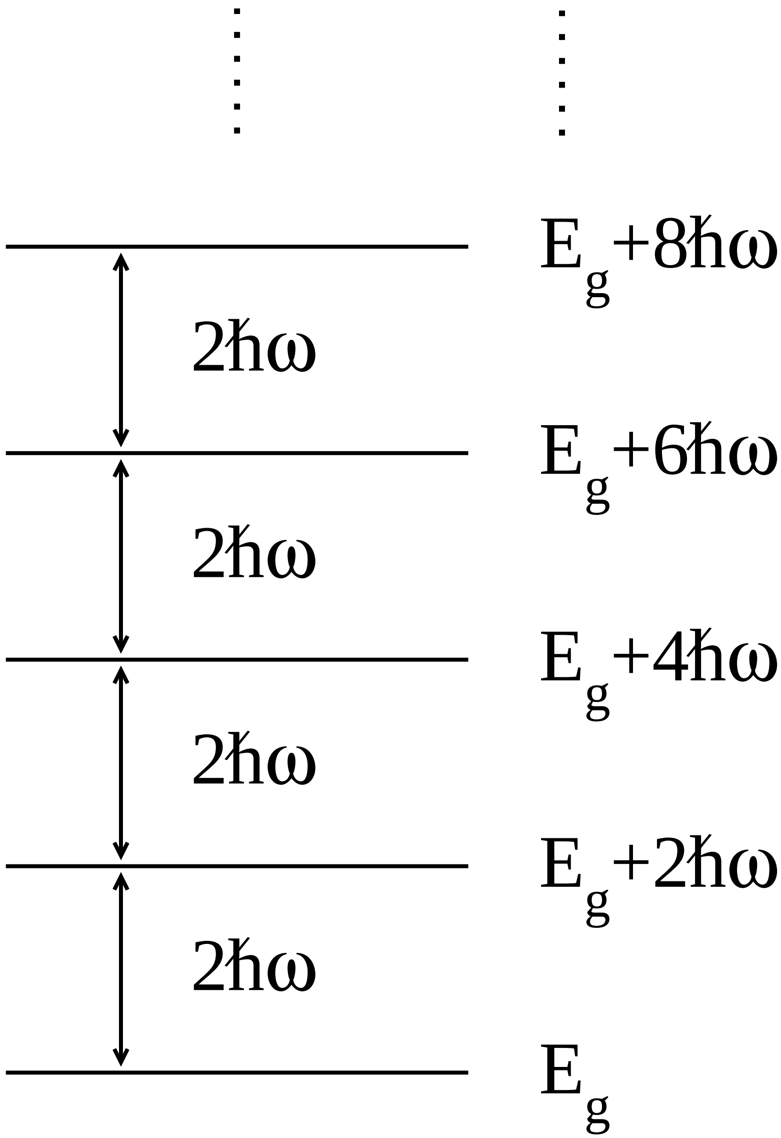
- Raising and lowering operators:

$$L_\pm = \pm i \left[\frac{3N}{2i} - i\vec{X}\cdot\partial_{\vec{X}}\right] + \frac{H}{\hbar\omega} - m\omega X^2/\hbar$$

(in red, generator of scaling transform)

- Spectrum=collection of semi-infinite ladders of step $2\hbar\omega$. $SO(2,1)$ hidden symmetry (Pitaevskii, Rosch, 1997).

LADDER STRUCTURE OF THE SPECTRUM (2)



A USEFUL MAPPING

- Each energy ladder has a ground step of energy E_g, eigenfunction ψ_g.
- Integration of $L_-\psi_g = 0$ gives, with $\vec{X} = X\vec{n}$:

$$\psi_g(\vec{X}\,) = e^{-m\omega X^2/2\hbar}\, X^{E_g/(\hbar\omega)-3N/2} f(\vec{n})$$

- Limit $\omega \to 0$: mapping to zero energy free space solutions. N.B.: $E_g/(\hbar\omega)$ is a constant.
- Free space problem solved for $N = 3$ (Efimov, 1972)... so trapped case also solved (Werner, Castin, 2006).

SEPARABILITY IN HYPERSPHERICAL COORDINATES

SEPARABILITY IN HYPERSPHERICAL COORDINATES

Werner, Castin (2006)

- Use Jacobi coordinates to separate center of mass $\vec{C}$
- Hyperspherical coordinates (arbitrary masses m_i):

$$(\vec{r}_1, \ldots, \vec{r}_N) \leftrightarrow (\vec{C}, R, \vec{\Omega}\,)$$

with $3N-4$ hyperangles $\vec{\Omega}$ and the hyperradius

$$m_u R^2 = \sum_{i=1}^{N} m_i (\vec{r}_i - \vec{C}\,)^2$$

where m_u is a arbitrary mass unit.

- Hamiltonian is clearly separable:

$$H_{\text{internal}} = -\frac{\hbar^2}{2m_u}\left[\partial_R^2 + \frac{3N-4}{R}\partial_R + \frac{1}{R^2}\Delta_{\vec{\Omega}}\right] + \frac{1}{2}m_u\omega^2 R^2$$

Do the contact conditions preserve separability ?

- For free space $E = 0$, yes, due to scaling invariance:

$$\psi_{E=0} = R^{s_N - (3N-5)/2} \phi(\vec{\Omega}).$$

$E = 0$ Schrödinger's equation implies

$$\Delta_{\vec{\Omega}} \phi(\vec{\Omega}) = - \left[s_N^2 - \left(\frac{3N-5}{2} \right)^2 \right] \phi(\vec{\Omega})$$

with contact conditions. $s_N^2 \in$ discrete real set.

- For arbitrary E, Ansatz with $E = 0$ hyperrangular part obeys contact conditions $[R^2 = R^2(r_{ij} = 0) + O(r_{ij}^2)]$:

$$\psi = F(R) R^{-(3N-5)/2} \phi(\vec{\Omega})$$

- Schrödinger's equation for a fictitious particle in 2D:

$$EF(R) = -\frac{\hbar^2}{2m_u} \Delta_R^{2D} F(R) + \left[\frac{\hbar^2 s_N^2}{2m_u R^2} + \frac{1}{2} m_u \omega^2 R^2 \right] F(R)$$

SOLUTION OF HYPERRADIAL EQUATION $(N \geq 3)$

$$EF(R) = -\frac{\hbar^2}{2m_u}\Delta_R^{2D} F(R) + \left[\frac{\hbar^2 s^2}{2m_u R^2} + \frac{1}{2} m_u \omega^2 R^2\right] F(R)$$

- Which boundary condition for $F(R)$ in $R = 0$? Wigner-Bethe-Peierls does not say.
- Key point: particular solutions $\sim R^{\pm s}$ for $R \to 0$.

$s > 1$	$0 < s < 1$	$s \in i\mathbb{R}^{+*}$
$F \sim R^s$	$F \sim (qR)^s \pm (qR)^{-s}$	$F \sim \mathrm{Im}\left[(qR)^s\right]$
0 bound st.	one bound st. if $-$	∞ nber of bound st.
$E_n = (2n + s + 1)\hbar\omega, n \geq 0$	$E \propto -\frac{\hbar^2 q^2}{m_u}$: N-body resonance	$E_n \propto -\frac{\hbar^2 q^2}{m_u} e^{-2\pi n/\lvert s\rvert}$, $n \in \mathbb{Z}$: Efimov effect

THE 4-BODY EFIMOV EFFECT

THREE-BODY EFIMOV EFFECT

- Efimov (1971): Three bosons, $1/a = 0$, no dimer state. Then there exists an infinite number of trimer states, $E = 0$ accumulation point, geometric spectrum:

$$E_n^{(3)} \underset{n\to+\infty}{\sim} E_{\rm ref}^{(3)} e^{-2\pi n/|s_3|}$$

where purely imaginary $s_3 = i \times 1.00624$ solves transcendental equation, $E_{\rm ref}^{(3)}$ depends on microscopic details.

- Efimov (1973): Solution for three arbitrary particles, $1/a = 0$. E.g. Efimov trimers for two fermions (masse M, same spin state) and one impurity (masse m) if (Petrov, 2003)

$$\alpha \equiv \frac{M}{m} > \alpha_c(2;1) \simeq 13.607$$

with $s_3(\alpha) \in i\mathbb{R}^{+*}$ from known transcendental equation.

ARE THERE EFIMOVIAN TETRAMERS ?

$$E_n^{(4)} \underset{n\to+\infty}{\sim} E_{\text{ref}}^{(4)} e^{-2\pi n/|s_4|} \ ?$$

Negative results:

- **Amado, Greenwood (1973): "There is No Efimov effect for Four or More Particles". Explanation: Case of bosons, there exist trimers, tetramers decay.**
- **Hammer, Platter (2007), von Stecher, D'Incao, Greene (2009), Deltuva (2010): The four-boson problem (here $1/a = 0$) depends only on $E_{\text{ref}}^{(3)}$, no $E_{\text{ref}}^{(4)}$ to add.**
- **Key point: $N = 3$ Efimov effect breaks separability in hyperspherical coordinates for $N = 4$.**

Idea: Consider three fermions (M) and one impurity (m).

REMINDER: MAIN POINTS OF GENERAL THEORY

- To find N-body Efimov effect, one simply needs to calculate the exponents s_N, that is to solve the Wigner-Bethe-Peierls model at zero energy:

$$\psi_{E=0}(\vec{r}_1, \ldots, \vec{r}_N) = R^{s_N-(3N-5)/2}\phi(\vec{\Omega})$$

- The N-body Efimov effect takes place if and only if one of the s_N^2 is < 0.
- General theory OK if $\Delta_{\vec{\Omega}}$ self-adjoint: no n-body Efimov effect $\forall n \leq N-1$.

THE $3+1$ FERMIONIC PROBLEM
(Castin, Mora, Pricoupenko, 2010)

- Three fermions (mass M, same spin state) and one impurity (mass m)
- General theory OK for a mass ratio

$$\alpha \equiv \frac{M}{m} < \alpha_c(2;1) \simeq 13.607$$

- Calculate $E=0$ solution in momentum space. An integral equation for Fourier transform of A_{ij}:

$$0 = \left[\frac{1+2\alpha}{(1+\alpha)^2}(k_1^2+k_2^2) + \frac{2\alpha}{(1+\alpha)^2}\vec{k}_1\cdot\vec{k}_2\right]^{1/2} D(\vec{k}_1,\vec{k}_2)$$
$$+\int \frac{d^3k_3}{2\pi^2}\frac{D(\vec{k}_1,\vec{k}_3)+D(\vec{k}_3,\vec{k}_2)}{k_1^2+k_2^2+k_3^2+\frac{2\alpha}{1+\alpha}(\vec{k}_1\cdot\vec{k}_2+\vec{k}_1\cdot\vec{k}_3+\vec{k}_2\cdot\vec{k}_3)}$$

- D has to obey fermionic symmetry.

REDUCTION OF THE INTEGRAL EQUATION

Rotational invariance:

- D is the $m_l = 0$ component of a spinor of spin l:

$$\vec{D}(\vec{k}_1, \vec{k}_2) = {}^t\rho\, \vec{D}(\mathcal{R}\vec{k}_1, \mathcal{R}\vec{k}_2)$$

- Clever choice of the rotation matrix $\mathcal{R}$:

$$\vec{D}(\vec{k}_1, \vec{k}_2) = {}^t\rho \quad \underbrace{\vec{D}[k_1\vec{e}_x, k_2(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y)]}_{2l+1 \text{ unknown functions } f_{m_l}^{(l)}(k_1,k_2,\theta)}$$

Scaling invariance for $E = 0$:

$$f_{m_l}^{(l)}(k_1, k_2, \theta) = (k_1^2 + k_2^2)^{-(s_4+7/2)/2}(\cosh x)^{3/2}\Phi_{m_l}^{(l)}(x, \theta)$$

with $x = \ln(k_2/k_1)$.

The integral equation gives $M_{s_4}^{(l)}[\vec{\Phi}^{(l)}] = 0$.

$$\boxed{s_4 \text{ allowed} \Longleftrightarrow M_{s_4}^{(l)} \text{ has a zero eigenvalue}}$$

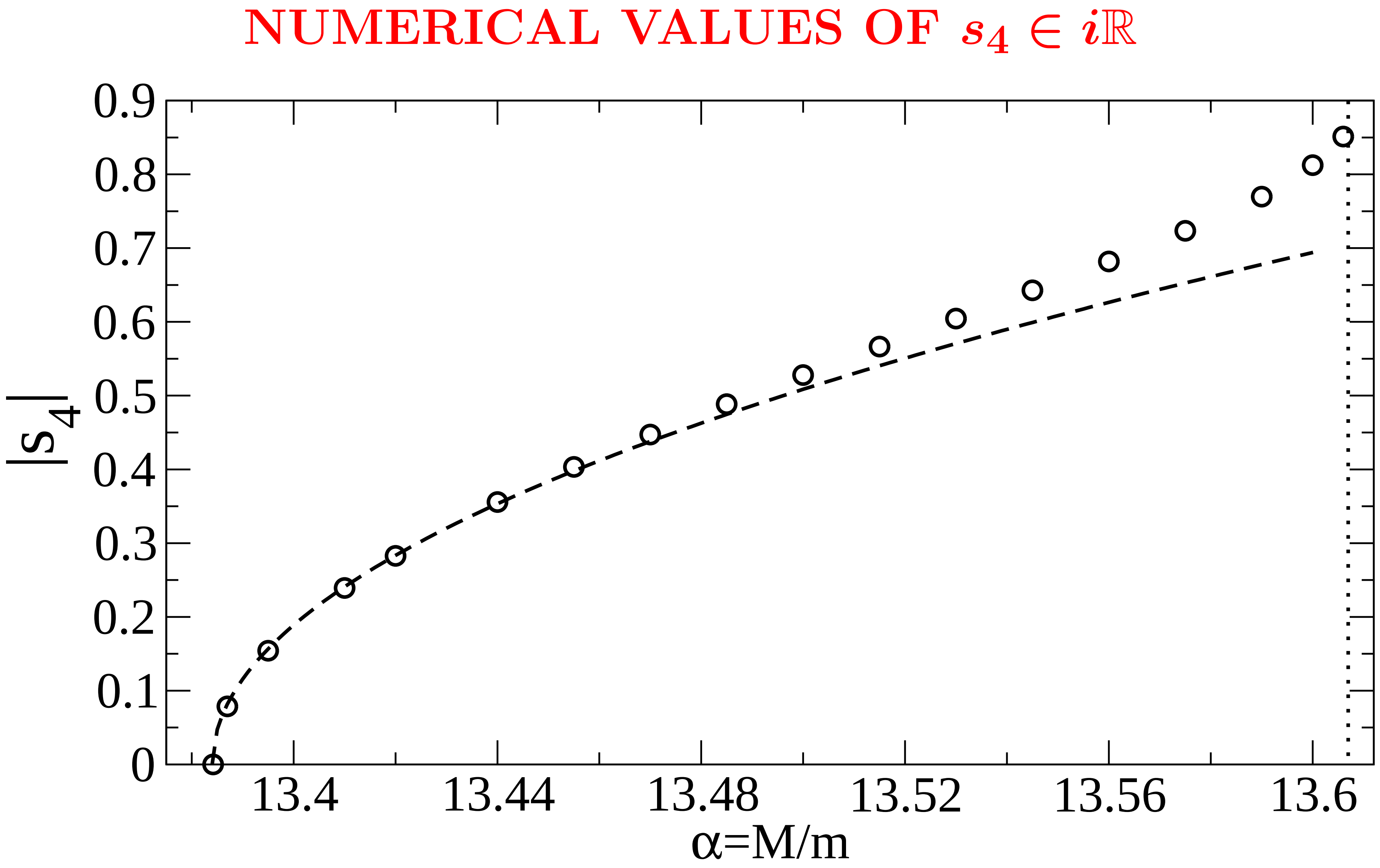
RESULTS

- Numerical exploration up to $l = 10$
- Four-body Efimov effect obtained for a single s_4, in channel $l = 1$ with even parity:

$$D(\vec{k}_1, \vec{k}_2) = \vec{e}_z \cdot \frac{\vec{k}_1 \times \vec{k}_2}{||\vec{k}_1 \times \vec{k}_2||} \, f_0^{(1)}(k_1, k_2, \theta)$$

in the interval of mass ratio

$$\alpha_c(3;1) \simeq 13.384 < \alpha < \alpha_c(2;1) \simeq 13.607$$



NUMERICAL VALUES OF $s_4 \in i\mathbb{R}$
|s4|
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
13.4
13.44
13.48
13.52
13.56
13.6
α=M/m

EXPERIMENTAL ASPECTS

- Large scattering length with magnetic Feshbach resonance (Grimm, 2006; Hulet, 2009)
- Radio-frequency spectroscopy of trimers (Jochim, 2010)
- Remaining issue: Narrow interval of mass ratio.

Solution 1: The right mixture

- ^{41}Ca and ^{3}He* have mass ratio $\alpha \simeq 13.58 \in [13.384, 13.607]$
- A priori, $|s_4| \simeq 0.75$ large enough to see two tetramer states
- ^{41}Ca has same radioactivity as ^{239}Pu (half-life 10^5 years)

Solution 2: Mass tuning

- ^{40}K and ^{3}He* have slightly-off mass ratio $\alpha \simeq 13.25$
- Use optical lattice to tune effective mass (Petrov, Shlyapnikov, 2007)

MINLOS'S THEOREM (1995)

Theorem: *In the* $n+1$ *fermionic problem, the Wigner-Bethe-Peierls Hamiltonian is self-adjoint and bounded from below iff*

$$(n-1)\frac{2\alpha(1+1/\alpha)^3}{\pi\sqrt{1+2\alpha}}\int_0^{\mathrm{asin}\frac{\alpha}{1+\alpha}} dt\, t\sin t < 1.$$

- We expect that "not bounded from below" is equivalent to "with Efimov effect".
- Case $n=3$: $\alpha_c^{\mathrm{Minlos}} \simeq 5.29$ totally differs from ours...
- Case $\alpha=1$: No stable unitary gas for $n>9$...
- Weak point: Proof not included in Minlos' paper.
- Recent proof: Teta, Finco (2010). But we have found a hole in the proof. We can still hope that the macroscopic $\alpha=1$ unitary gas is stable.