COHERENCE PROPERTIES OF A BOSE-EINSTEIN CONDENSATE

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OUTLINE

- Description of the problem
- Framework: Bogoliubov theory
- Spatial coherence
- Temporal coherence
  - N fluctuates
  - N fixed, E fluctuates: Canonical ensemble
  - N fixed, E fixed: Microcanonical ensemble

DESCRIPTION OF THE PROBLEM

A single-spin state Bose gas prepared at equilibrium:

- Spatially homogeneous, periodic boundary conditions.
- Prepared with N atoms, in the regime $T \ll T_c$ of an almost pure condensate.
- Interactions with a s-wave scattering length $a > 0$.
- Weakly interacting regime $(\rho a^3)^{1/2} \ll 1$.
- The gas is totally isolated in its evolution.

Spatial coherence of the gas:

- Determined by the measured first-order coherence function, $g_1(\mathbf{r}) = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{0})\rangle$ (Esslinger, Bloch, Hänsch, 2000).
- Expected: In thermodynamic limit, g_1 tends to condensate density $\rho_0 > 0$ at infinity.
- This is long-range order.

Coherence time of the condensate:

- Defined as the decay time of the measurable condensate mode coherence function, $\langle a_0^\dagger(t) a_0(0)\rangle$, where a_0 is the annihilation operator in mode $\mathbf{k} = 0$.
- At zero temperature, no decay, $\langle a_0^\dagger(t) a_0(0)\rangle \sim \langle N_0\rangle e^{i\mu_0 t/\hbar}$, coherence time is infinite (Beliaev, 1958).
- What happens at finite temperature $T > 0$? To our knowledge, the problem was still open in 1995.
- One expects infinite coherence time in thermodynamic limit.
- For finite size: By analogy with laser, one expects finite coherence time due to condensate phase diffusion.

FRAMEWORK: BOGOLIUBOV THEORY

Bogoliubov theory

- **Lattice model Hamiltonian:**

$$H = \sum_{\mathbf{r}} b^3 \left[\hat{\psi}^\dagger h_0 \hat{\psi} + \frac{g_0}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right]$$

- **Spatially homogeneous case:** $h_0 = -\frac{\hbar^2}{2m}\Delta_{\mathbf{r}}$.
- **Bare coupling constant** $g_0^{-1} = g^{-1} - \int_{\mathrm{FBZ}} \frac{d^3k}{(2\pi)^3} \frac{m}{\hbar^2 k^2}$, $g = 4\pi\hbar^2 a/m$. **Gives** $g_0 = g/(1 - C_3 a/b)$. **Here** $0 < a \ll b$.
- **Expansion of Hamiltonian around pure condensate:**

$$\hat{\psi}(\mathbf{r}) = \phi(\mathbf{r})\hat{a}_0 + \hat{\psi}_\perp(\mathbf{r})$$

with $\phi(\mathbf{r}) = 1/L^{3/2}$. **Key point: Eliminate amplitude** $\hat{a}_0$ **in condensate mode:**

$$\hat{n}_0 = \hat{N} - \hat{N}_\perp$$

with $\hat{n}_0 = \hat{a}_0^\dagger \hat{a}_0$ **and** $\hat{N}_\perp = \sum_{\mathbf{r}} b^3 \hat{\psi}_\perp^\dagger \hat{\psi}_\perp$.

Elimination of the condensate phase

- Modulus-phase representation (Girardeau, Arnowitt, 1959):
$$\hat{a}_0 = e^{i\hat{\theta}}\hat{n}_0^{1/2}$$
with hermitian operator $\hat{\theta}$, $[\hat{n}_0, \hat{\theta}] = i$.
- Cf. position $\hat{x}$ and momentum $\hat{p}$ operator of a particle:
$$[\hat{x}, \hat{p}] = i\hbar \Longrightarrow e^{i\hat{p}a/\hbar}|x\rangle = |x - a\rangle$$
$$[\hat{n}_0, \hat{\theta}] = i \Longrightarrow e^{i\hat{\theta}}|n_0 : \phi\rangle = |n_0 - 1 : \phi\rangle$$
then $\hat{a}_0$ has the right matrix elements.
- This fails when the condensate mode is empty:
$$e^{i\hat{\theta}}|0 : \phi\rangle \stackrel{?!}{=} |-1 : \phi\rangle$$
- Redefinition of non-condensed field (Castin, Dum; Gardiner, 1996) ; remains bosonic, but conserves $\hat{N}$:
$$\hat{\Lambda}(\mathrm{r}) = e^{-i\hat{\theta}}\hat{\psi}_{\perp}(\mathrm{r})$$

- Expansion of $\boldsymbol{H}$ to second order in $\hat{\psi}_{\perp}$:

$$H_{\text{Bog}} = \frac{g_0 N^2}{2L^3} + \sum_{\mathbf{r}} b^3 \left[\hat{\Lambda}^{\dagger}(h_0 - \mu_0)\hat{\Lambda} + \mu_0 \left(\frac{1}{2}\hat{\Lambda}^2 + \frac{1}{2}\hat{\Lambda}^{\dagger 2} + 2\hat{\Lambda}^{\dagger}\hat{\Lambda} \right) \right]$$

- Formally grand canonical for non-condensed modes, with chemical potential $\mu_0 = g_0\rho$.
- Elastic interaction $C - NC$: Hartree-Fock

$$C, 0 + NC, \mathrm{k} \longrightarrow C, 0 + NC, \mathrm{k}$$

- Inelastic interaction $C - NC$: Landau superfluidity

$$C, 0 + C, 0 \longrightarrow NC, \mathrm{k} + NC, -\mathrm{k}$$

Not forbidden by energy conservation.

Normal form for the Hamiltonian:

- $\boldsymbol{H}_{\mathrm{Bog}}$ quadratic, hence linear equations of motion:

$$i\hbar\partial_t \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} = \begin{pmatrix} h_0+\mu_0 & \mu_0 \\ -\mu_0 & -(h_0+\mu_0) \end{pmatrix} \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} \equiv \mathcal{L} \begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix}$$

- Expansion on eigenmodes of eigenenergies $\pm\epsilon_k$:

$$\begin{pmatrix} \Lambda \\ \Lambda^\dagger \end{pmatrix} = \sum_{\mathbf{k}\neq 0} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{L^{d/2}} \begin{pmatrix} U_k \\ V_k \end{pmatrix} \hat{b}_{\mathbf{k}} + \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{L^{d/2}} \begin{pmatrix} V_k \\ U_k \end{pmatrix} \hat{b}_{\mathbf{k}}^\dagger$$

because exchanging Λ and $\Lambda^\dagger$ equivalent to time reversal.

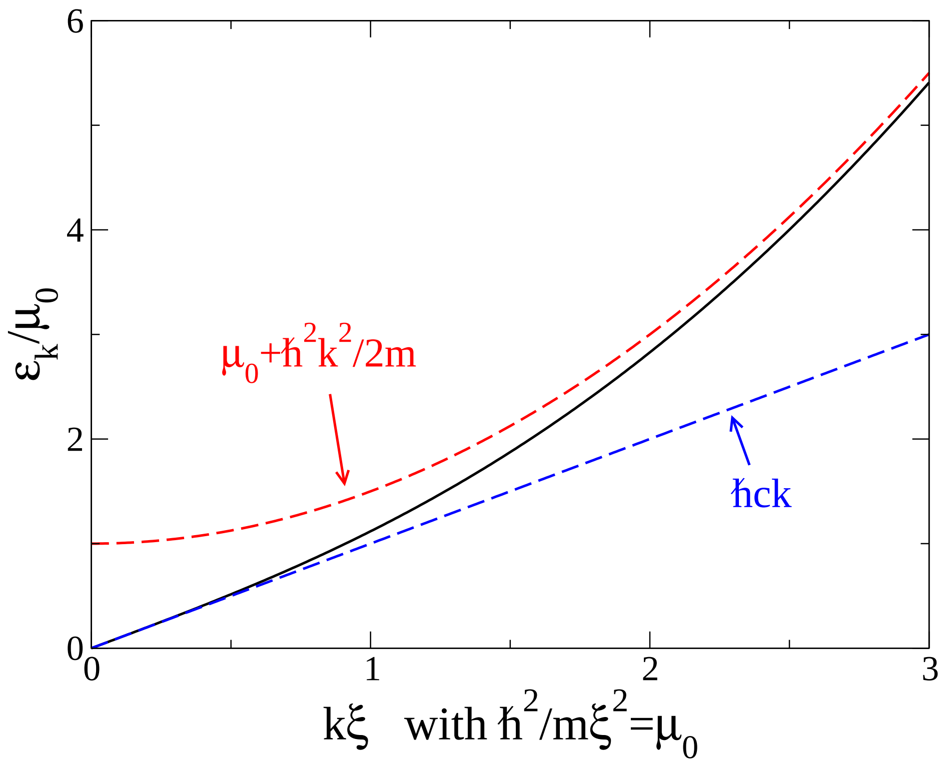
- Bosonic commutation relations for $U_k^2 - V_k^2 = 1$:

$$U_k + V_k = \frac{1}{U_k - V_k} = \left(\frac{\hbar^2 k^2/2m}{2\mu_0 + \hbar^2 k^2/2m}\right)^{1/4}$$

- **A grand-canonical ideal gas of bosonic quasi-particles:**

$$\boldsymbol{H}_{\mathrm{Bog}} = \boldsymbol{E}_0 + \sum_{\mathrm{k}\neq 0} \boldsymbol{\epsilon}_{\boldsymbol{k}} \hat{\boldsymbol{b}}_{\boldsymbol{k}}^{\dagger} \hat{\boldsymbol{b}}_{\mathrm{k}} \quad \text{with} \quad \boldsymbol{\epsilon}_{\boldsymbol{k}} = \left[\frac{\hbar^2 \boldsymbol{k}^2}{2m} \left(\frac{\hbar^2 \boldsymbol{k}^2}{2m} + 2\mu_0 \right) \right]^{1/2}$$

Bogoliubov spectrum



SPATIAL COHERENCE

Consistency check

In thermodynamic limit:

- **Non-condensed fraction:**

$$\frac{\langle N_\perp\rangle}{N} = \frac{\langle \hat{\Lambda}^\dagger \hat{\Lambda}\rangle}{\rho} = \frac{1}{\rho}\int \frac{d^3k}{(2\pi)^3}\left[\frac{U_k^2+V_k^2}{e^{\beta\epsilon_k}-1}+V_k^2\right]$$

- **No ultraviolet** $(k\to\infty)$ **divergence:** $V_k^2 = O(1/k^4)$
- **No infrared** $(k\to 0)$ **divergence:** $U_k^2, V_k^2 = O(1/k)$.
- **Small for** $T \ll T_c$ **and** $(\rho a^3)^{1/2} \ll 1$.
- **First order coherence function** $g_1(\mathbf{r}) = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(0)\rangle$:

$$g_1(\mathbf{r}) = \rho - \int \frac{d^3k}{(2\pi)^3}(1-\cos\mathbf{k}\cdot\mathbf{r})\left[\frac{U_k^2+V_k^2}{e^{\beta\epsilon_k}-1}+V_k^2\right]$$

tends to the condensate density for $r\to\infty$.

In lower dimensions:

- In 2D for $T > 0$ and in 1D $\forall T$, the non-condensed fraction has infrared divergence. No BEC in thermodynamic limit (Mermin, Wagner, 1966; Hohenberg, 1967).
- Quasi-condensate (weak density fluctuations, weak phase gradients) (Popov, 1972). One can save the idea of Bogoliubov by applying it to a modulus-phase representation of the field operator $\hat{\psi}$.
- $g_1^{\mathrm{Bog}}(\mathrm{r}) \to -\infty$ at infinity, but remarkably (Mora, Castin, 2003):

$$g_1^{\mathrm{QC}}(\mathrm{r}) = \rho \exp\left[\frac{g_1^{\mathrm{Bog}}(\mathrm{r})}{\rho} - 1\right].$$

so that logarithmic divergence of $g_1^{\mathrm{Bog}}(\mathrm{r})$ turned into power-law decay of $g_1^{\mathrm{QC}}(\mathrm{r})$.

TEMPORAL COHERENCE

GENERAL CONSIDERATIONS

- If weak fluctuations of $\hat{n}_0$:

$$\langle a_0^\dagger(t) a_0(0)\rangle \simeq \langle \hat{n}_0\rangle \langle e^{-i[\hat{\theta}(t)-\hat{\theta}(0)]}\rangle$$

- If phase change $\hat{\theta}(t) - \hat{\theta}(0)$ has Gaussian distribution:

$$\left|\langle a_0^\dagger(t) a_0(0)\rangle\right| \simeq \langle \hat{n}_0\rangle e^{-\mathrm{Var}\,[\hat{\theta}(t)-\hat{\theta}(0)]/2}$$

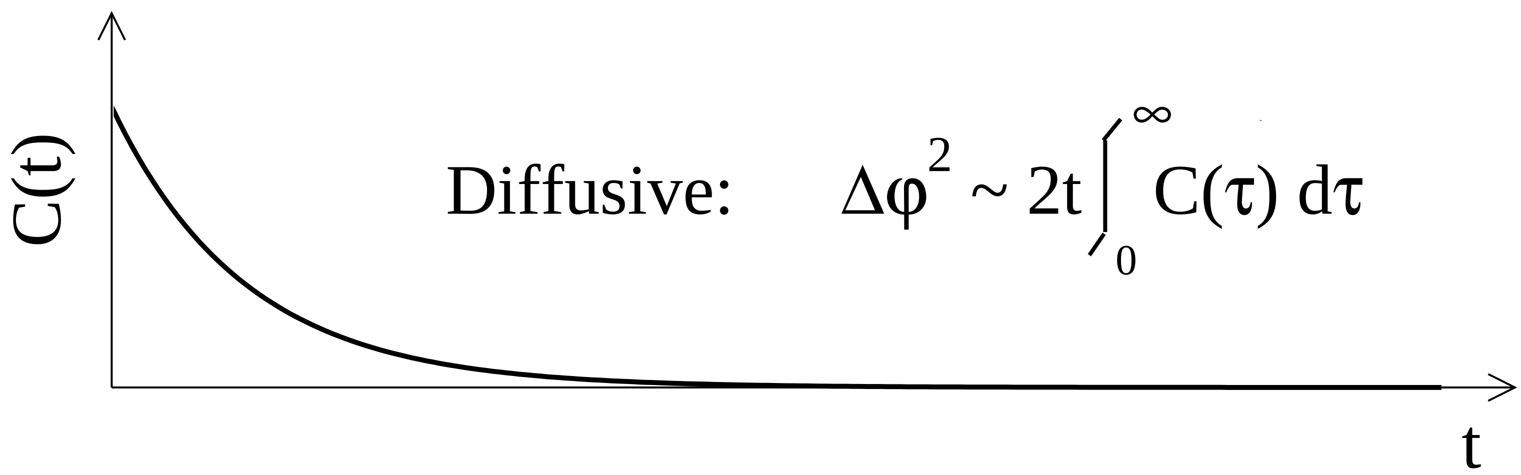
- In terms of correlation function $C(t) = \langle\dot{\theta}(t)\dot{\theta}(0)\rangle - \langle\dot{\theta}\rangle^2$:

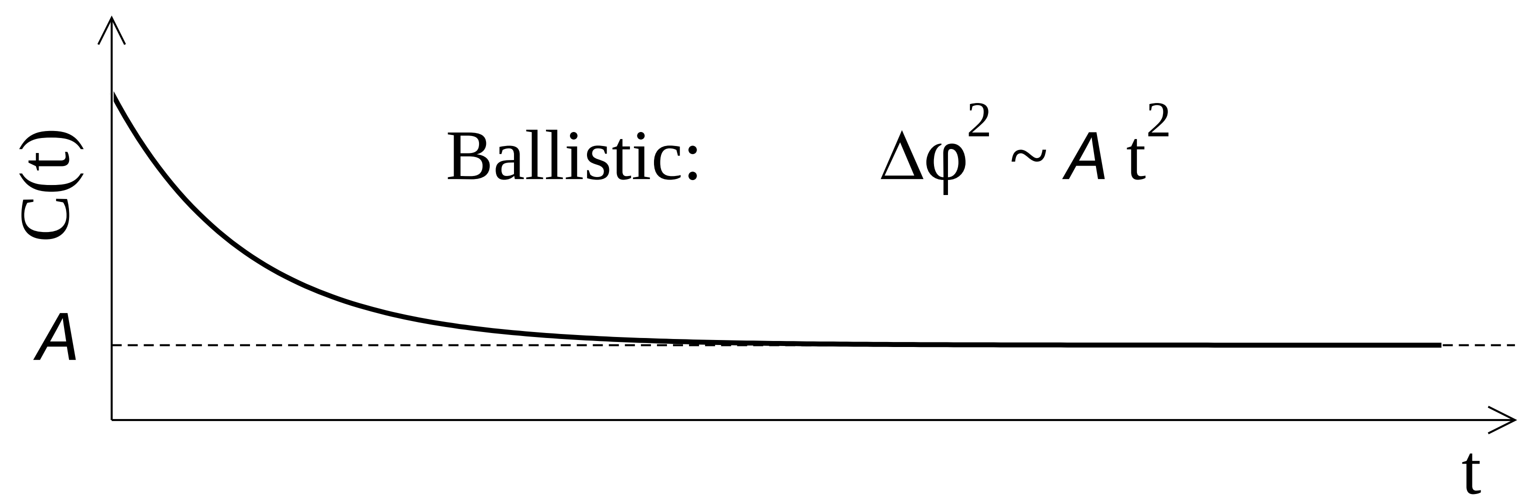
$$\mathrm{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] = 2t\int_0^t d\tau\, C(\tau) - 2\int_0^t d\tau\, \tau C(\tau)$$

ballistic regime	diffusive regime
$\lim_{\tau\to+\infty} C(\tau) \neq 0$	$C(\tau) \underset{\tau\to+\infty}{=} o(1/\tau)$
$\mathrm{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] \sim At^2$	$\mathrm{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] \sim 2Dt$

TWO CASES DEPENDING ON $C(t \to +\infty)$

$$\varphi = \hat{\theta}(t) - \hat{\theta}(0)$$





GENERAL CONSIDERATIONS (2)

Previous studies at $T > 0$:

- Zoller, Gardiner (1998), Graham (1998-2000): Diffusive.
- Contradicted by Kuklov, Birman (2000): Ballistic.
- Sinatra, Witkowska, Castin (2006-): Clarification and quantitative studies.

Two key actors:

- Bogoliubov procedure eliminating the condensate mode from the Hamiltonian:

$$H = E_0(N) + \sum_{\mathbf{k}\neq 0} \epsilon_k \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + H_3 + \ldots$$

  where ϵ_k is the Bogoliubov spectrum. Hamiltonian H_3 is cubic in field $\hat{\Lambda}$. It breaks integrability and plays central role in condensate dephasing (Beliaev-Landau processes):

$$H_3 = g_0 \rho^{1/2} \sum_{\mathbf{r}} b^3 \hat{\Lambda}^\dagger (\hat{\Lambda} + \hat{\Lambda}^\dagger) \hat{\Lambda}$$

Quasi-particle resonant interactions à la Beliaev $\hat{b}^\dagger \hat{b}^\dagger \hat{b}$ and à la Landau $\hat{b}^\dagger \hat{b} \hat{b}$: finite lifetime, kinetic equations on mean quasi-particle occupation numbers

- Time derivative of condensate phase operator:

$$\boxed{\dot{\theta} \equiv \frac{1}{i\hbar}[\theta, H] \simeq -\mu_{T=0}(N)/\hbar - \frac{g_0}{\hbar L^3} \sum_{\mathbf{k}\neq 0} (U_k + V_k)^2 \hat{n}_{\mathbf{k}}}$$

with $\hat{n}_{\mathbf{k}} = \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}$. This contradicts Graham, 1998 and 2000. Keep in mind useful "magic" relation:

$$\frac{g_0}{L^3}(U_k + V_k)^2 = \partial_N \epsilon_{\mathbf{k}}$$

Case of a pure condensate

- One-mode model, with $\hat{n}_0 = \hat{N}$: $H_{\text{one mode}} = \frac{g}{2L^3}\hat{N}^2$
- Evolution of the condensate phase:

$$\dot{\theta}(t) = \frac{1}{i\hbar}[\hat{\theta}, H_{\text{one mode}}] = -\frac{g\hat{N}}{\hbar L^3} = -\mu(\hat{N})/\hbar$$

- No phase spreading if fixed N.
- Ballistic spreading if N fluctuates (Sols, 1994; Walls, 1996; Lewenstein, 1996; Castin, Dalibard, 1997)

$$\text{Var}\,[\hat{\theta}(t) - \hat{\theta}(0)] = (t/\hbar)^2 \left(\frac{d\mu}{dN}\right)^2 \text{Var}\,\hat{N}$$

- Experiments: Seen not for $\langle a_0^\dagger(t) a_0 \rangle$ but for $\langle a_0^\dagger(t) b_0(t) \rangle$ by interfering two condensats with common $t = 0$ phase [Bloch, Hänsch (2002); Pritchard, Ketterle (2006); Reichel, 2010.]

$T > 0$ gas prepared in the canonical ensemble

By analogy with previous case (Sinatra et al, 2007) :

- As N, the energy E is a constant of motion.
- Canonical ensemble = statistical mixture of eigenstates, $\mathrm{Var}\, E \neq 0$ but $\mathrm{Var}\, E \ll \bar{E}^2$ for a large system
- $\hat{\theta}(t) \sim -\mu_{\mathrm{mc}}(\hat{H})t/\hbar$ and weak fluctuations of $\hat{H}$:

$$\mathrm{Var}\left[\hat{\theta}(t) - \hat{\theta}(0)\right] \sim (t/\hbar)^2 \left[\frac{d\mu_{\mathrm{mc}}}{dE}(\bar{E})\right]^2 \mathrm{Var}\, E$$

From quantum ergodic theory (Sinatra et al, 2007) :

- **Time average:**

$$\langle\langle\dot{\theta}(t)\dot{\theta}(0)\rangle\rangle_t = \sum_\lambda \frac{e^{-\beta E_\lambda}}{Z}(\langle\Psi_\lambda|\dot{\theta}|\Psi_\lambda\rangle)^2$$

- **Deutsch (1991) : eigenstate thermalisation hypothesis. Mean value of observable $\hat{O}$ in one eigenstate Ψ_λ very close to microcanonical value:**

$$\langle\Psi_\lambda|\hat{O}|\Psi_\lambda\rangle \simeq \bar{O}_{\mathrm{mc}}(E = E_\lambda)$$

- **$\hat{O} = \dot{\theta}$ in Bogoliubov limit : $\boxed{\bar{\dot{\theta}}_{\mathrm{mc}} = -\mu_{\mathrm{mc}}/\hbar}$.**
- **Linearize around mean energy due to weak (relative) energy fluctuations:**

$$\mu_{\mathrm{mc}}(E_\lambda) \simeq \mu_{\mathrm{mc}}(\bar{E}) + (E_\lambda - \bar{E})\frac{d\mu_{\mathrm{mc}}}{dE}(\bar{E})$$

Implications of previous result (canonical ensemble)

- The correlation function $C(\tau)$ of $\dot{\theta}$ does not tend to zero when $\tau \to +\infty$. Neither does the one of $\hat{n}_0$.
- This qualitatively contradicts Zoller, Gardiner, Graham. In qualitative agreement with Kuklov, Birman.
- Ergodicity ensured by interactions (cf. H_3) among Bogoliubov quasi-particles.
- Approximating H with integrable H_{Bog}, as eventually done by Kuklov and Birman, gives incorrect coefficient of t^2.

A. Sinatra, Y. Castin, E. Witkowska, Phys. Rev. A 75, 033616 (2007)

Why failure of master equation method of Zoller-Gardiner ?

Setting $\hat{n}_{\mathrm{k}} \equiv \hat{b}^{\dagger}_{\mathrm{k}}\hat{b}_{\mathrm{k}} : C(t) = \sum_{\mathrm{k},\mathrm{k}'} A_{\mathrm{k}} A_{\mathrm{k}'} \langle \delta\hat{n}_{\mathrm{k}}(t)\delta\hat{n}_{\mathrm{k}'}(0)\rangle$

Master equation + quantum regression theorem:

- System = Bogoliubov modes k and k′. Other modes = reservoir. Born-Markov approximation:

$$\langle \delta\hat{n}_{\mathrm{k}}(t)\delta\hat{n}_{\mathrm{k}'}(0)\rangle = \delta_{\mathrm{k}\mathrm{k}'}\bar{n}_{\mathrm{k}}(1+\bar{n}_{\mathrm{k}})e^{-\Gamma_{\mathrm{k}} t}$$

so $C(t) \underset{t\to\infty}{\longrightarrow} 0$ and phase has diffusive spreading...

But reservoir not truly infinite:

- From ergodic theory:

$$\langle \delta\hat{n}_{\mathrm{k}}(t)\delta\hat{n}_{\mathrm{k}'}(0)\rangle \underset{t\to\infty}{\longrightarrow} \frac{\epsilon_{\mathrm{k}}\bar{n}_{\mathrm{k}}(\bar{n}_{\mathrm{k}}+1)\,\epsilon_{\mathrm{k}'}\bar{n}_{\mathrm{k}'}(\bar{n}_{\mathrm{k}'}+1)}{\sum_{\mathrm{q}\neq 0}\epsilon_{\mathrm{q}}^2\bar{n}_{\mathrm{q}}(1+\bar{n}_{\mathrm{q}})} \propto \frac{1}{V}$$

and double sum: $VC(t) \underset{t\to\infty}{\not\to} 0$ even in thermodynamic $V\to\infty$ limit.

Illustration with a classical field calculation

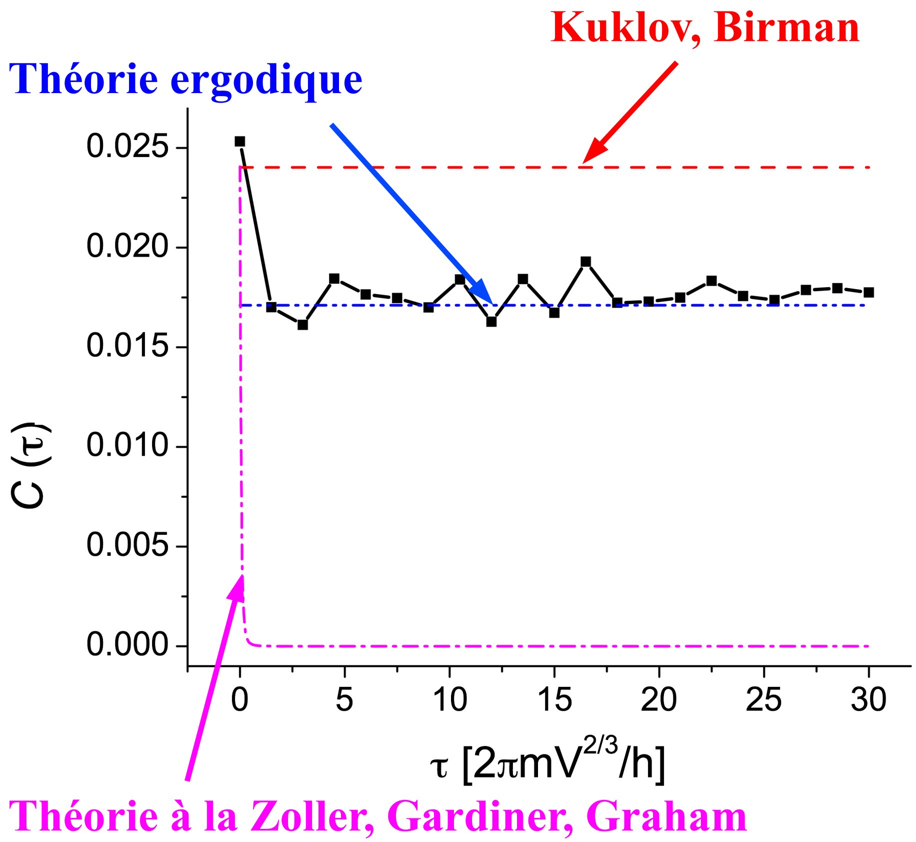


Figure 1: For a gas prepared in canonical ensemble, correlation function of $\dot{\boldsymbol{\theta}}$ for the classical field. The equation of motion is the non-linear Schrödinger equation. A. Sinatra, Y. Castin, E. Witkowska, Phys. Rev. A **75**, 033616 (2007).

Gas prepared in the microcanonical ensemble: phase diffusion

- The conserved quantities N, E do not fluctuate. One finds $C(\tau) \underset{\tau\to+\infty}{=} O(1/\tau^3)$ and $\mathrm{Var}\,[\hat{\theta}(t)-\hat{\theta}(0)] \sim 2Dt$.
- One needs the full dependence of $C(\tau)$ to get D.
- As we have seen, $C(\tau)$ can be deduced from all the $\langle \hat{n}_{\mathrm{k}}(\tau)\hat{n}_{\mathrm{k}'}(0)\rangle$.
- The gas is in a statistical mixture of Fock states quasi-particles $|\{n_{\mathrm{q}}\}\rangle$. One simply needs $\langle\{n_{\mathrm{q}}\}|\hat{n}_{\mathrm{k}}(\tau)|\{n_{\mathrm{q}}\}\rangle$.
- The evolution of the mean number of quasi-particles is given by quantum kinetic equations including the Beliaev-Landau processes due to H_3.

The quantum kinetic equations

$$\dot{n}_{\mathrm{q}} = -\frac{g^2\rho}{\hbar\pi^2}\int d^3\mathrm{k}\Big\{\left[n_{\mathrm{q}}n_{\mathrm{k}} - n_{\mathrm{q+k}}(1+n_{\mathrm{k}}+n_{\mathrm{q}})\right]\left(\mathcal{A}_{k,q}^{|\mathrm{q+k}|}\right)^2$$

$$\times\delta(\epsilon_q+\epsilon_k-\epsilon_{|\mathrm{q+k}|})\Big\}$$

$$-\frac{g^2\rho}{2\hbar\pi^2}\int d^3\mathrm{k}\Big\{\left[n_{\mathrm{q}}(1+n_{\mathrm{k}}+n_{\mathrm{q-k}}) - n_{\mathrm{k}}n_{\mathrm{q-k}}\right]\left(\mathcal{A}_{k,|\mathrm{q-k}|}^{q}\right)^2$$

$$\times\delta(\epsilon_k+\epsilon_{|\mathrm{q-k}|}-\epsilon_q)\Big\}$$

with the Beliaev-Landau coupling amplitudes:

$$\mathcal{A}_{k,k'}^{q} = U_qU_kU_{k'} + V_qV_kV_{k'} + (U_q+V_q)(V_kU_{k'}+U_kV_{k'})\,.$$

E. M. Lifshitz, L. P. Pitaevskii "Physical Kinetics", Landau and Lifshitz Course of Theoretical Physics vol. 10, chap. VII, Pergamon Press (1981)

Diffusion coefficient of the condensate phase

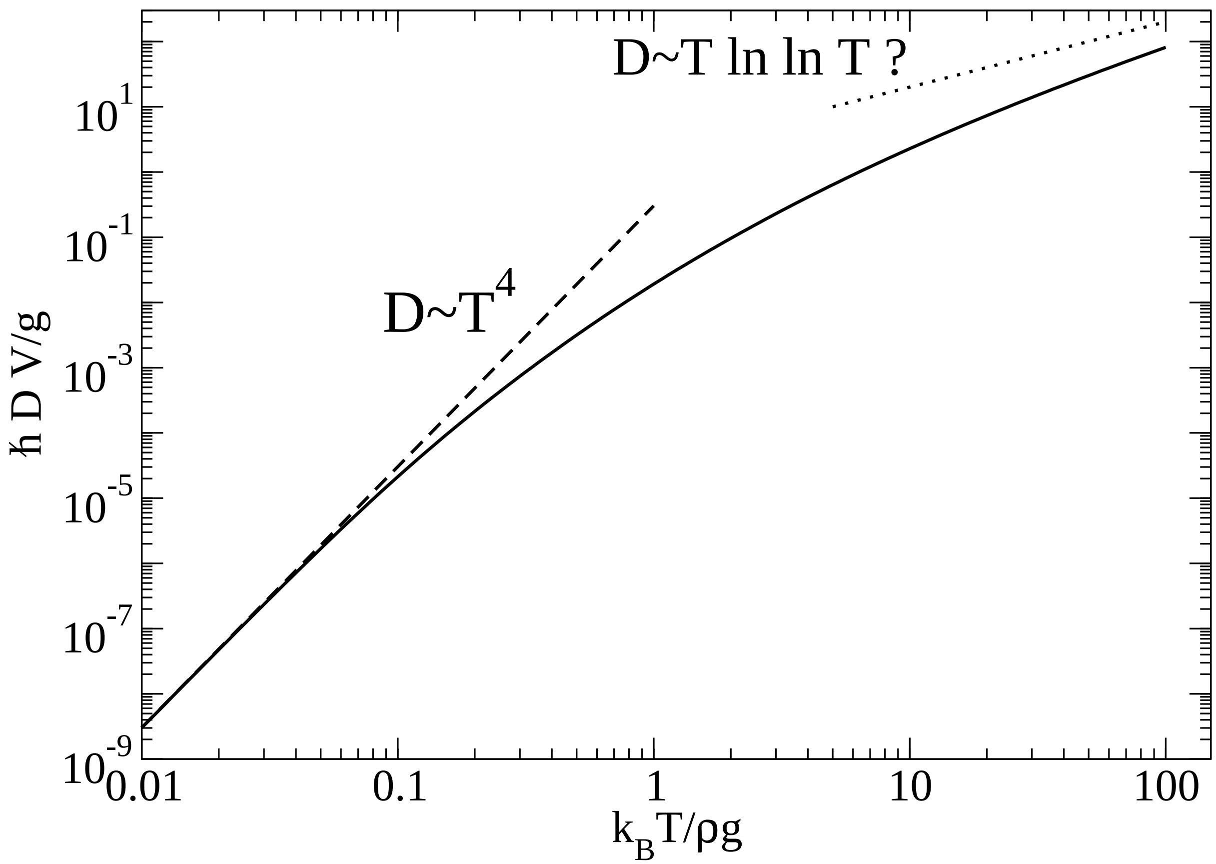


Figure 2: Universal result in Bogoliubov limit (weakly interacting, $T \ll T_c$).

A. Sinatra, Y. Castin, E. Witkowska, Phys. Rev. A 80, 033614 (2009). T^4 law reminiscent of normal fraction.

Summary of results for the phase spreading

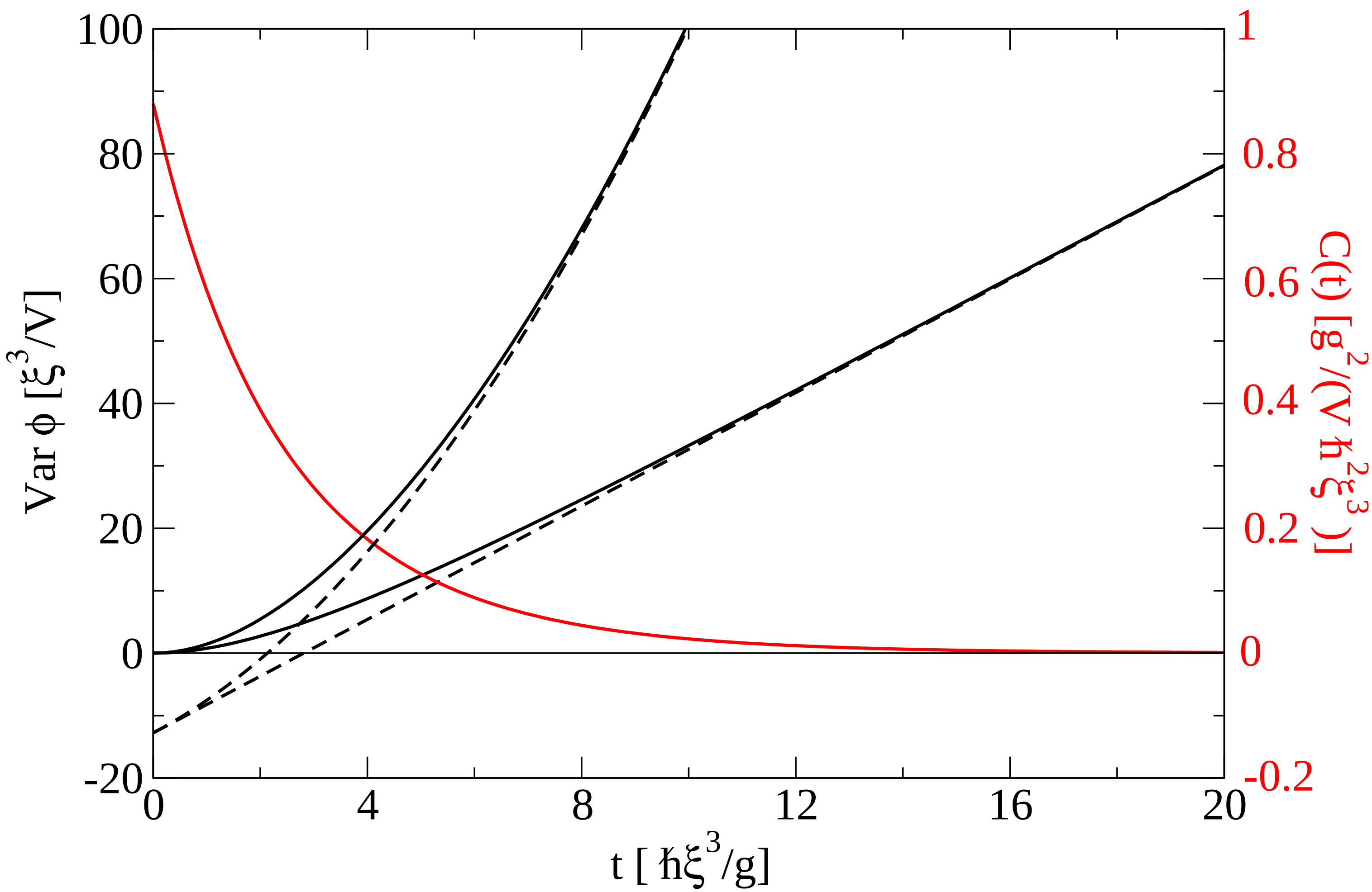
$$\mathrm{Var}\,[\theta(t)-\theta(0)] \underset{t\to+\infty}{=} \mathrm{Var}\,(E)\,\left[\frac{d\mu_{\mathrm{mc}}}{\hbar dE}(\bar{E})\right]^2 t^2+2Dt+c+O(\frac{1}{t})$$

- Existence of a t^2 term first in Kuklov, Birman, 2000.
- Coefficient of t^2 depends on the ensemble. First obtained with quantum ergodic theory (Sinatra, Castin, Witkowska, 2007) but also with quantum kinetic theory (from existence of undamped mode of linearized kinetic equations due to energy conservation). Interpretation:

$$\theta(t)-\theta(0) \underset{t\to+\infty}{\sim} -\mu(H)t/\hbar.$$

- Diffusion coefficient D is ensemble independent. $\hbar DL^3/g$ function of $k_BT/\rho g$ (Sinatra, Castin, Witkowska, 2009).
- Ensemble independent $c\neq 0$: $C_{\mathrm{mc}}(t)$ not a Dirac.

AN EXAMPLE FOR $k_B T = 10\rho g$



Our publications on the subject

- **A. Sinatra, Y. Castin, E. Witkowska, "Nondiffusive phase spreading of a Bose-Einstein condensate at finite temperature", Phys. Rev. A 75, 033616 (2007)**
- **A. Sinatra, Y. Castin, "Genuine phase diffusion of a Bose-Einstein condensate in the microcanonical ensemble: A classical field study", Phys. Rev. A 78, 053615 (2008)**
- **A. Sinatra, Y. Castin, E. Witkowska, "Coherence time of a Bose-Einstein condensate", Phys. Rev. A. 80, 033614 (2009)**
- **A. Sinatra, Y. Castin, "Spatial and temporal coherence of a Bose-condensed gas", in "Quantum Fluids: hot-topics and new trends", eds. Michele Modugno, Alberto Bramati (Springer, Berlin, 2012).**

LIMIT OF SPIN SQUEEZING IN FINITE TEMPERATURE BOSE-EINSTEIN CONDENSATES

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ATOMIC CLOCKS IN BRIEF

What an atomic clock does:

- Measures the transition frequency ω_{ab} of two-level atoms
- Formally, a two-level atom is a spin $1/2$
- Collective spin $\mathbf{S} = \sum_{i=1}^{N} \mathbf{S}_i$, free Hamiltonian:

$$H_0 = \hbar\omega_{ab} S_z$$

- At time 0, prepare the collective spin along x. At time τ, measurement of the spin precession angle $\omega_{ab}\tau$ gives transition frequency ω_{ab} (Ramsey method).

Transverse quantum fluctuations: $\Delta S_y \Delta S_z \geq \frac{1}{2}|\langle S_x \rangle|$

- Standard quantum limit: All spins along x, $\langle S_x \rangle = N/2$:

$$\Delta S_y^{\mathrm{st}} = \Delta S_z^{\mathrm{st}} = \sqrt{N}/2 \longrightarrow \Delta\omega_{ab} = \frac{1}{N^{1/2}\tau}$$

- This is larger than technical noise in good clocks

ONE CAN GAIN WITH SPIN SQUEEZED STATES

- Can reduce a lot ΔS_y, at the expense of increasing ΔS_z
- Gain $1/\xi$ on signal-to-noise ratio (Wineland, 1994):

$$\xi^2 = \frac{N \Delta S^2_{\perp,\min}}{\langle S_x \rangle^2} < 1 \longrightarrow \Delta\omega_{ab} = \frac{\xi}{N^{1/2}\tau}$$

Kitagawa-Ueda spin squeezing: $H = \hbar\omega_{ab} S_z + \hbar\chi S_z^2$

- Spin-dependent Larmor frequency: Evolution turns the fluctuation circle into a tilted ellipse. At best time:

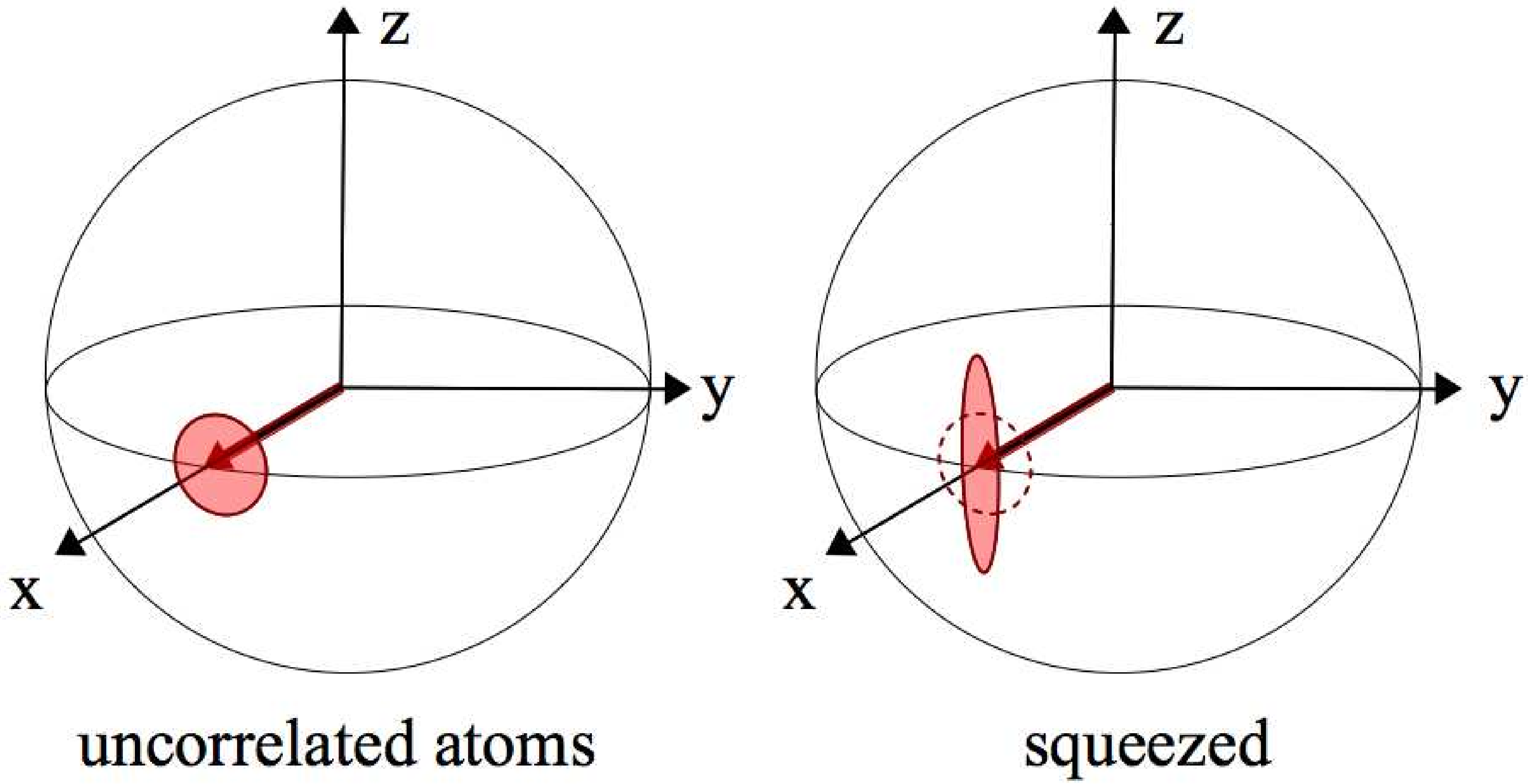
$$\xi^2_{\min} \underset{N\to\infty}{\sim} \frac{3^{2/3}}{2N^{2/3}}$$

- Realisable with two-mode condensates (Cirac, 2001):

$$S_x + iS_y = a^\dagger b, \qquad S_z = (a^\dagger a - b^\dagger b)/2, \qquad \chi = \frac{g}{\hbar V}$$

- In the lab (Oberthaler, 2008; Treutlein, 2010): $1/\xi = 3$

ON THE BLOCH SPHERE



In practice, squeezed axis is tilted (rotation required):

$$\Delta S^2_{\perp,\mathrm{min}} = \frac{1}{2}\left[\langle S_y^2\rangle + \langle S_z^2\rangle - |\langle (S_y + iS_z)^2\rangle|\right]$$

WHAT HAPPENS IN REAL LIFE ?

An atomic gas is a multimode system:

- **At $T > 0$, the non-condensed modes constitute a dephasing environment for the condensate: phase spreading and a finite coherence time**
- **What is the effect on spin squeezing ?**

From Bogoliubov theory in brief:

- **Much before the phase collapse time, $\rho g t/\hbar \ll N^{1/2}$:**

$$\langle S_x \rangle \simeq \frac{N}{2}, \quad S_y \simeq -\frac{N}{2}(\theta_a - \theta_b), \quad S_z = \frac{N_a - N_b}{2}$$

- **Evolution of phase operators (previous lecture):**

$$(\hat{\theta}_a - \hat{\theta}_b)(t) = (\hat{\theta}_a - \hat{\theta}_b)(0^+) - \frac{gt}{\hbar V}[2S_z + D]$$

$$D = \sum_{\mathrm{k}\neq 0} (U_k + V_k)^2 (\hat{n}_{a\mathrm{k}} - \hat{n}_{b\mathrm{k}}) \quad [\hat{n}_{\sigma\mathrm{k}} = \text{quasi-particle nber}]$$

In thermodynamic limit:

$$\boxed{\xi^2(t) = \frac{1}{(\tau + \sqrt{1+\tau^2})^2} + \frac{\frac{2\langle D^2\rangle}{N}\tau^2}{(\tau + \sqrt{1+\tau^2})\sqrt{1+\tau^2}}}$$

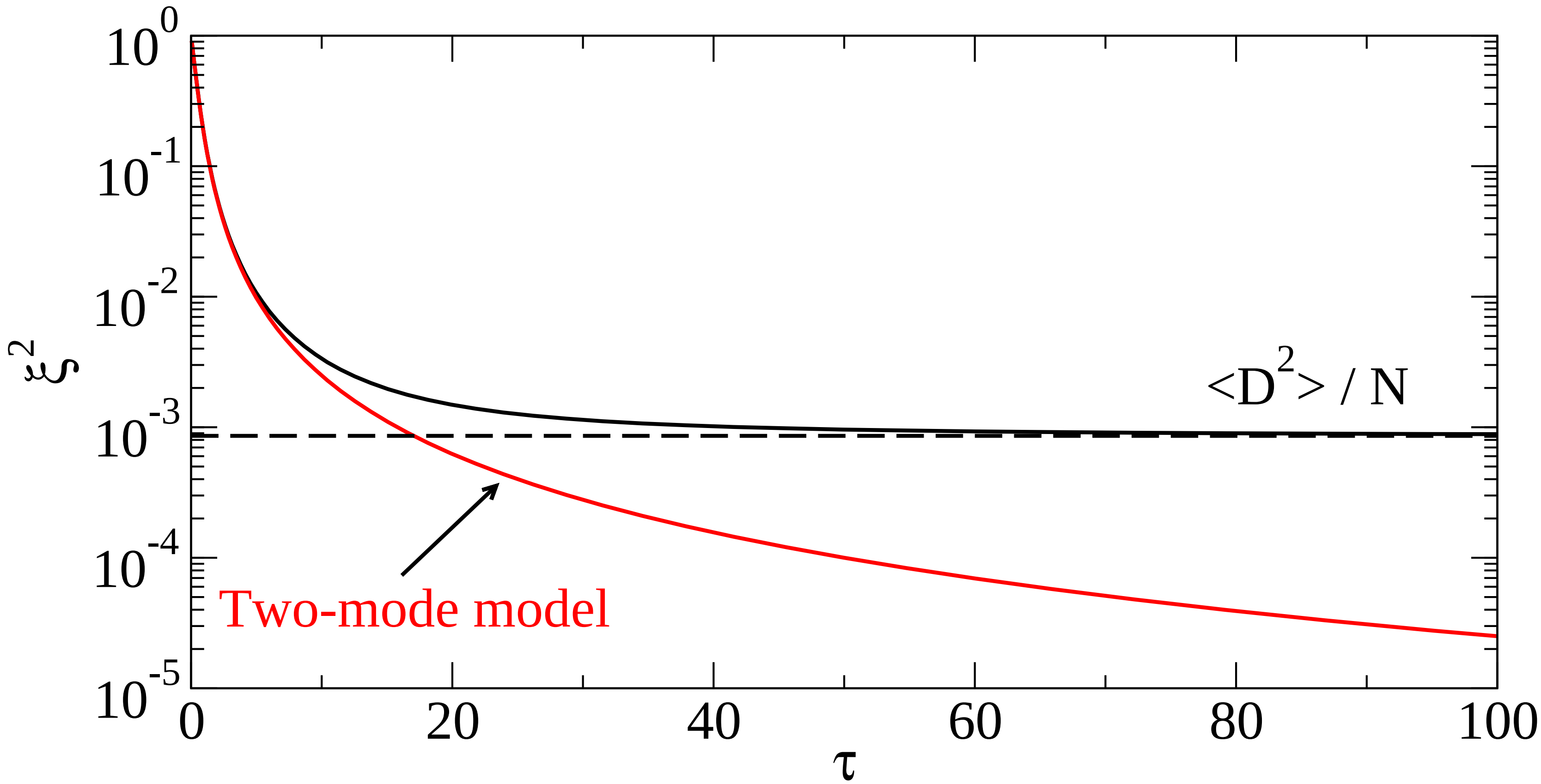
with reduced time $\tau = \rho g t/(2\hbar)$.

- **First term is Kitagawa-Ueda model.**
- **Second term saturates to minimal squeezing:**

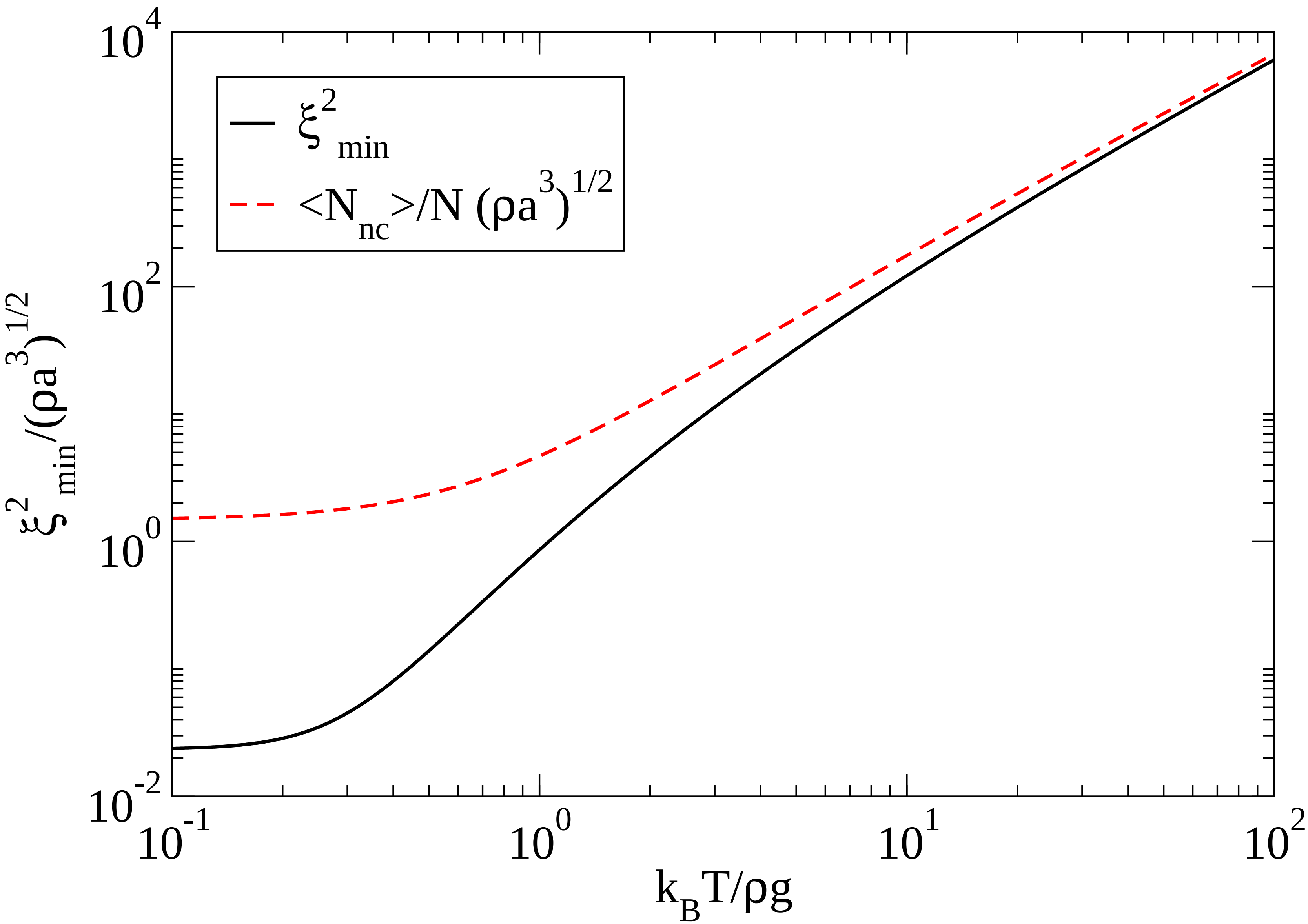
$$\xi^2_{\min} = \frac{\langle D^2\rangle}{N} = (\rho a^3)^{1/2} f(k_B T/\rho g)$$

$\xi^2(t)$ FOR BOGOLIUBOV THEORY

$$(\rho a^3)^{1/2} = 10^{-3}, k_B T/\rho g = 1$$



MINIMAL ξ^2 FOR BOGOLIUBOV THEORY



VALIDITY CONDITIONS

- System out-of-equilibrium after pulse
- Will thermalize, this is neglected in Bogoliubov theory
- Have the close-to-best-squeezing time

$$\frac{\rho g t_\eta}{\hbar} \simeq \frac{1}{\eta^{1/2}\xi_{\mathrm{min}}}$$

smaller than thermalisation time, estimated by Beliaev-Landau damping rates of modes of energy $k_B T$ or ρg:

$$\frac{\rho g t_{\mathrm{therm}}}{\hbar} \propto \frac{1}{(\rho a^3)^{1/2}}$$

- Validity condition satisifed in weakly interacting limit:

$$\frac{t_\eta}{t_{\mathrm{therm}}} \propto (\rho a^3)^{1/4} \ll 1$$

Summary of results for spin squeezing:

- For atoms with two internal states a and b, apply a $\pi/2$ pulse on a condensate initially in a. Due to interactions, phase state transformed into spin squeezing state
- If injected in an atomic clock, statistical uncertainty on clock frequency after interrogation time τ:

$$\Delta\omega_{ab} = \frac{\Delta S_{\perp,\mathrm{min}}}{\langle S_x\rangle \tau} \equiv \frac{\xi}{N^{1/2}\tau}$$

- Spin dynamics is a phase dynamics: S_z =const, S_x ≈const,

$$S_y \propto \theta_a - \theta_b \propto (N_a - N_b + D)t\partial_N \mu/\hbar$$

where D due to multimode nature of the fields (random dephasing environment). Best squeezing in weakly interacting, thermodynamic limit does not vanish:

$$\xi^2_{\mathrm{min}} \simeq \frac{\langle D^2\rangle}{N}$$

Our publications on the subject

- **Yun Li, Y. Castin, A. Sinatra, "Optimum spin-squeezing in Bose- Einstein condensates with particle losses", Phys. Rev. Lett. 100, 210401 (2008).**
- **A. Sinatra, E. Witkowska, J.-C. Dornstetter, Yun Li, Y. Castin, "Limit of Spin Squeezing in Finite Temperature Bose-Einstein Condensates", Phys. Rev. Lett. 107, 060404 (2011).**
- **A. Sinatra, J.-C. Dornstetter, Y. Castin, "Spin squeezing in Bose- Einstein condensates: Limits imposed by decoherence and non-zero temperature", Front. Phys. 7, 86 (2012).**
- **A. Sinatra, E. Witkowska, Y. Castin, "Spin squeezing in finite temperature Bose-Einstein condensates: Scaling with the system size", Eur. Phys. J. Special Topics 203, 87 (2012).**
- **A. Sinatra, Y. Castin, E. Witkowska, "Limit of spin squeezing in trapped Bose-Einstein condensates", EPL 102, 40001 (2013)**