CONDENSATS DE BOSE-EINSTEIN ET LASERS À ATOMES BOSE-EINSTEIN CONDENSATES AND ATOM LASERS

Wave functions, relative phase and interference for atomic Bose–Einstein condensates

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Abstract. In this article, we present a tutorial discussion of the coherence properties of Bose–Einstein condensates. We use a formalism which is similar to the one used in quantum optics. We describe within the variational approximation the question of the relative phase of two condensates. To evaluate the structure factors of a condensate, we briefly review the Bogolubov approach and describe light scattering off a condensate using the linear response formalism. Finally, we study the effect of atomic interactions on the condensate's dynamics. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Bose–Einstein condensates / matter waves / coherence properties / Gross–Pitaevskii equation / Bogolubov theory / atomic interactions

Fonctions d'onde, phase relative et interférences pour des condensats de Bose-Einstein atomiques

Résumé. Dans cet article, nous présentons une discussion pédagogique des propriétés de cohérence des condensats de Bose–Einstein, en utilisant un formalisme similaire à celui de l'optique quantique. Dans le cadre de l'approximation variationnelle, nous discutons le problème de la phase relative de deux condensats. Ensuite, afin d'évaluer les facteurs de structure d'un condensat, nous présentons rapidement l'approche de Bogolubov et décrivons la diffusion de lumière par un condensat en utilisant la théorie de la réponse linéaire. Enfin, nous étudions l'effet des interactions atomiques sur la dynamique du condensat. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

condensats de Bose–Einstein / ondes de matière / propriétés de cohérence / équation de Gross–Pitaevskii / théorie de Bogolubov / interactions atomiques

1. Introduction

In quantum mechanics, the wave function $\psi(\mathbf{r})$ of a single particle lies in the 3D-space. If the particle is in a linear superposition of two states described respectively by $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$, we have $\psi(\mathbf{r}) = c_1\psi_1(\mathbf{r}) + c_2\psi_2(\mathbf{r})$. Then the probability $|\psi(\mathbf{r})|^2$ to find it in \mathbf{r} contains crossed terms $c_1c_2^*\psi_1(\mathbf{r})\psi_2^*(\mathbf{r}) + c.c.$ leading

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to interference effects. The situation for a N-particle system is more complicated as the general wave function $\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is defined in a 3N-D-space and is generally non-factorizable. This means that one cannot generally associate 3D waves to a system of N particles because of their entanglement. Considering the particular case of Bose–Einstein condensates, however, the description can be simplified by introducing a *macroscopic matter wave* in the 3D-space, with which the question of the relative phase and of the interference between two condensates can be investigated.

In the usual symmetry breaking approach, the 'macroscopic' matter wave is simply the mean value $\psi(\mathbf{r}) = \langle \hat{\psi}(\mathbf{r}) \rangle$ of the quantum field operator. To have $\psi(\mathbf{r}) \neq 0$, one needs the quantum state of the system to be a linear superposition of states with different values of the total number N of bosons. In other words, the density operator $\hat{\rho}$ has some off-diagonal elements, which violate the super-selection rules (see Section 2.2.3).

In this article, we aim at presenting a tutorial discussion of coherence properties of Bose–Einstein condensates by using more traditional approaches inspired by quantum optics. In particular, they involve neither symmetry breaking nor coherences $\langle N | \hat{\rho} | N' \rangle$ with $N \neq N'$. In the first part (Section 2), we introduce a formalism which is similar to second quantization in quantum optics and we recall a few results about correlation functions.

In the second part (Section 3), we present the variational approach leading to a description of the condensate in terms of a 'macroscopic' matter wave. We then focus on the question of the phase of a macroscopic matter wave: can one introduce simple quantum states describing two condensates with a well defined relative phase θ ? Can one introduce two interfering 3D-matter waves? What is the variable conjugate to θ ?

The following part (Section 4) consists of a brief introduction to Bogolubov theory where we study the simple case of a homogeneous condensate. Section 5 is devoted to the analysis of the probing of a condensate by light or particle scattering. In particular, we calculate the dynamic and static structure factors of a condensate both in the variational and Bogolubov approaches. We finally discuss in Section 6 the major difference between light and matter waves, namely the effect of atomic interactions.

A more detailed study of these questions has been given as a *Cours du Collège de France*. The lecture notes (in French) are available [1].

2. A few basic results on correlation functions

2.1. Correlation functions for quantum optical fields

In this section, we recall some results concerning correlation functions in quantum optics. For the sake of simplicity, we ignore the vector character of the field.

2.1.1. Quantum optical fields

Let us consider first a classical field $\mathcal{E}(\mathbf{r},t) = \mathcal{E}^+(\mathbf{r},t) + \mathcal{E}^-(\mathbf{r},t)$ (with $\mathcal{E}^- = (\mathcal{E}^+)^*$). Its positive frequency component \mathcal{E}^+ can be written as:

$$\mathcal{E}^{+}(\boldsymbol{r},t) = \sum_{i} \mathcal{E}_{i}(\boldsymbol{r})\alpha_{i}(t)$$
(1)

where $\{\mathcal{E}_i(\boldsymbol{r})\}\$ is a complete set of mode functions and $\alpha_i(t)$ is the normal variable associated with the mode *i*. In the absence of sources, the normal variables undergo a decoupled evolution $\alpha_i(t) = \alpha_i(0) e^{-i\omega_i t}$.

In the quantum description of an optical field [2], α_i and α_i^* become respectively the annihilation and creation operators \hat{a}_i and \hat{a}_i^+ of a photon in the mode *i*. They obey the commutation rule:

$$\left[\hat{a}_i, \hat{a}_j^+\right] = \delta_{ij} \tag{2}$$

Relative phase and interference

Using these operators, the quantum field operator $\widehat{E}^+(r)$ is written:

$$\widehat{E}^{+}(\mathbf{r}) = \sum_{i} \mathcal{E}_{i}(\mathbf{r}) \widehat{a}_{i}$$
(3)

Let us now consider a few useful examples of quantum radiation fields:

- A single mode field is a field with all the modes in the vacuum state (0 photon) except one mode (labeled mode 1). The field operator then writes $\hat{E}^+(\mathbf{r}) = \mathcal{E}_1(\mathbf{r})\hat{a}_1$. The most general state of the mode is $|\psi\rangle = \sum_{N_1} c_{N_1} |N_1\rangle$, $|N_1\rangle$ being the Fock state with N_1 photons in the mode 1.
- A two-mode field contains two modes (1 and 2) which are not empty. The field operator is written $\hat{E}^+(\mathbf{r}) = \mathcal{E}_1(\mathbf{r})\hat{a}_1 + \mathcal{E}_2(\mathbf{r})\hat{a}_2$ and any state $|\psi\rangle$ can be written as a linear superposition of two-mode Fock states $|N_1, N_2\rangle$: $|\psi\rangle = \sum_{N_1, N_2} c_{N_1 N_2} |N_1, N_2\rangle$.
- A coherent state |α₁⟩ of a single-mode field is an eigenstate of the annihilation operator â₁ with the complex eigenvalue α₁. Using Fock states |N₁⟩, one can write [2]:

$$|\alpha_1\rangle = e^{-|\alpha_1|^2/2} \sum_{N_1=0}^{\infty} \frac{\alpha_1^{N_1}}{\sqrt{N_1!}} |N_1\rangle$$
(4)

The probability $\mathcal{P}(N_1)$ to have N_1 photons in a coherent state $|\alpha_1\rangle$ obeys a Poisson law $\mathcal{P}(N_1) = e^{-|\alpha_1|^2} |\alpha_1|^{2N_1} / N_1!$ with a mean value $\langle \hat{N}_1 \rangle = |\alpha_1|^2$ and a variance $\Delta N_1^2 = |\alpha_1|^2 = \langle \hat{N}_1 \rangle$.

The coherent state $|\alpha_1\rangle$ of a single mode field is the quantum state closest to the classical radiation field $\mathcal{E}^+(\mathbf{r})$ described by the normal variable α_1 . For example:

$$\widehat{E}^{+}(\boldsymbol{r})|\alpha_{1}\rangle = \mathcal{E}_{1}(\boldsymbol{r})\widehat{a}_{1}|\alpha_{1}\rangle = \alpha_{1}\mathcal{E}_{1}(\boldsymbol{r})|\alpha_{1}\rangle = \mathcal{E}^{+}(\boldsymbol{r})|\alpha_{1}\rangle$$
(5)

- Coherent states of multimode fields $|\{\alpha_j\}\rangle$ are simply tensorial products of coherent states $|\alpha_i\rangle$ corresponding to individual modes of the field. Their properties are similar to those for single-mode fields, and in particular $\hat{E}^+(\mathbf{r})|\{\alpha_i\}\rangle = \mathcal{E}^+(\mathbf{r})|\{\alpha_i\}\rangle$.
- Let us consider a *statistical mixture of coherent states* of a single mode field, having the same $\langle N_1 \rangle = |\alpha_1|^2$ and a random phase equally distributed between 0 and 2π :

$$\hat{\rho} = \frac{1}{2\pi} \int_{0}^{2\pi} \mathrm{d}\phi_1 ||\alpha_1| \,\mathrm{e}^{\mathrm{i}\phi_1} \rangle \langle |\alpha_1| \,\mathrm{e}^{\mathrm{i}\phi_1} | \tag{6}$$

Calculating $\langle N | \hat{\rho} | N' \rangle$, one gets:

$$\langle N|\hat{\rho}|N'\rangle = \frac{1}{2\pi} \int_{0}^{2\pi} \mathrm{d}\phi_{1} \langle N||\alpha_{1}|e^{\mathrm{i}\phi_{1}}\rangle \langle |\alpha_{1}|e^{\mathrm{i}\phi_{1}}|N'\rangle = e^{-|\alpha_{1}|^{2}} \frac{|\alpha_{1}|^{N+N'}}{\sqrt{N!}\sqrt{N'!}} \frac{1}{2\pi} \int_{0}^{2\pi} \mathrm{d}\phi_{1} e^{\mathrm{i}(N-N')\phi_{1}} = e^{-|\alpha_{1}|^{2}} \frac{|\alpha_{1}|^{2N}}{N!} \delta_{NN'}$$
(7)

This shows that $\hat{\rho}$ appears equivalently as a statistical mixture $\hat{\rho} = \sum_{N=0}^{\infty} \mathcal{P}(N) |N\rangle \langle N|$ of Fock states $|N\rangle$ with the Poisson distribution $\mathcal{P}(N)$ for N.

2.1.2. Correlation functions

We now introduce the correlation functions of a light field. In fact, correlation functions are widely used in statistical physics, for they allow to study the statistical properties of a field much deeper than with only its mean value and variance.

The first order correlation function $G^{(1)}(\mathbf{r},\mathbf{r}') = \langle \widehat{E}^-(\mathbf{r})\widehat{E}^+(\mathbf{r}')\rangle$ describes spatial correlations between the fields in two different points \mathbf{r} and \mathbf{r}' . Its classical analogue corresponds to the visibility of the

interference fringes obtained with two fields originating from \mathbf{r} and \mathbf{r}' . Using the Heisenberg picture instead of the Schrödinger picture, one can also introduce $G^{(1)}(\mathbf{r} t, \mathbf{r}' t') = \langle \hat{E}^-(\mathbf{r} t) \hat{E}^+(\mathbf{r}' t') \rangle$ which describes the dynamics of the correlations. $G^{(1)}(\mathbf{r} t, \mathbf{r}' t')$ is simply the quantum generalization of the classical correlation function $\mathcal{G}^{(1)}(\mathbf{r} t, \mathbf{r}' t') = \overline{\mathcal{E}^-(\mathbf{r} t) \mathcal{E}^+(\mathbf{r}' t')}$.

The second order correlation function is written $G^{(2)}(\mathbf{r},\mathbf{r}') = \langle \hat{E}^-(\mathbf{r})\hat{E}^+(\mathbf{r}')\hat{E}^+(\mathbf{r}')\hat{E}^+(\mathbf{r})\rangle$. It corresponds to the probability of detecting one photon in \mathbf{r} and another one in \mathbf{r}' .

To the third order, $G^{(3)}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}'') = \langle \widehat{E}^-(\boldsymbol{r}) \widehat{E}^-(\boldsymbol{r}') \widehat{E}^+(\boldsymbol{r}'') \widehat{E}^+(\boldsymbol{r}') \widehat{E}^+(\boldsymbol{r}) \rangle$ is the probability to detect one photon in \boldsymbol{r} , a second one in \boldsymbol{r}' and a third one in \boldsymbol{r}'' . All higher order correlation functions are built similarly.

In some particular cases, the correlation functions have simple expressions:

- For a gaussian field (for instance a thermal field), all correlation functions can be expressed in terms of products of a certain number of G⁽¹⁾. More precisely, they are sums of products of all possible products of one Ê⁻ by one Ê⁺. For instance, G⁽²⁾(**r**, **r**') is the sum of the 2! = 2 terms \laple \begin{aligned} \begin{aligned} \mathcal{E}^{-}(\mathbf{r})\beta \beta^{-}(\mathbf{r})\beta \beta^{-}(\mathbf{r})\beta \beta^{-}(\mathbf{r})\beta \beta^{-}(\mathbf{r})\beta \beta^{-}(\mathbf{r}')\beta \beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(\mathbf{r}')\beta^{-}(
- For a *coherent state* $|\{\alpha_i\}\rangle$, we have seen that $\widehat{E}^+(\mathbf{r})|\{\alpha_i\}\rangle = \mathcal{E}^+(\mathbf{r})|\{\alpha_i\}\rangle$ where $\mathcal{E}^+(\mathbf{r}) = \sum_j \alpha_j \mathcal{E}_j(\mathbf{r})$ is the classical field described by the normal variables $\{\alpha_j\}$. Similarly, $\langle\{\alpha_i\}|\widehat{E}^-(\mathbf{r}) = \mathcal{E}^-(\mathbf{r})\langle\{\alpha_i\}|$ with $(\mathcal{E}^+)^* = \mathcal{E}^-$. It follows that:

$$G^{(1)}(\boldsymbol{r},\boldsymbol{r}') = \overline{\mathcal{E}^{-}(\boldsymbol{r})\mathcal{E}^{+}(\boldsymbol{r}')}$$
(8)

$$G^{(2)}(\boldsymbol{r},\boldsymbol{r}') = \overline{\mathcal{E}^{-}(\boldsymbol{r})\mathcal{E}^{-}(\boldsymbol{r}')\mathcal{E}^{+}(\boldsymbol{r}')\mathcal{E}^{+}(\boldsymbol{r})} = \overline{\mathcal{I}(\boldsymbol{r})\mathcal{I}(\boldsymbol{r}')}$$
(9)

$$G^{(3)}(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}'') = \overline{\mathcal{I}(\boldsymbol{r})\mathcal{I}(\boldsymbol{r}')\mathcal{I}(\boldsymbol{r}'')}$$
(10)

 $G^{(2)}(\mathbf{r},\mathbf{r}')$ is thus simply the correlation function of the intensity $\mathcal{I}(\mathbf{r}) = |\mathcal{E}^+(\mathbf{r})|^2$. Generally, all correlation functions of a coherent state $|\{\alpha_i\}\rangle$ are equal to the average value of the products of the corresponding classical fields \mathcal{E}^+ and \mathcal{E}^- .

It is sometimes useful to introduce normalized correlation functions. The first order normalized correlation function is written as:

$$g^{(1)}(\mathbf{r},\mathbf{r}') = \frac{G^{(1)}(\mathbf{r},\mathbf{r}')}{\sqrt{G^{(1)}(\mathbf{r},\mathbf{r})G^{(1)}(\mathbf{r}',\mathbf{r}')}}$$
(11)

One can see that $g^{(1)}(\mathbf{r}, \mathbf{r}) = 1$. Generally, $|g^{(1)}(\mathbf{r}, \mathbf{r}')|$ decreases from 1 to 0 when $|\mathbf{r} - \mathbf{r}'|$ increases from 0 to $+\infty$. The characteristic length over which the decrease occurs is the *coherence length* λ_c of the field.

Similarly, one can define the normalized second order correlation function as

$$g^{(2)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{G^{(2)}(\boldsymbol{r}, \boldsymbol{r}')}{G^{(1)}(\boldsymbol{r}, \boldsymbol{r})G^{(1)}(\boldsymbol{r}', \boldsymbol{r}')}$$
(12)

For a gaussian field, $g^{(2)}(\mathbf{r}, \mathbf{r}')$ is rewritten as $g^{(2)}(\mathbf{r}, \mathbf{r}') = 1 + |g^{(1)}(\mathbf{r}, \mathbf{r}')|^2$. $g^{(2)}(\mathbf{r}, \mathbf{r}')$ thus decreases from 2 to 1 when $|\mathbf{r} - \mathbf{r}'|$ increases from 0 to ∞ . This bunching effect is called the Hanbury Brown–Twiss effect [3] and was observed in the 1950's by Hanbury Brown and Twiss [4].

For a coherent state by contrast, $g^{(2)}(\mathbf{r}, \mathbf{r}') = 1$ whatever \mathbf{r} and \mathbf{r}' , which means that a coherent state exhibits no bunching effect.

The third order normalized correlation function is:

$$g^{(3)}(\mathbf{r},\mathbf{r}',\mathbf{r}'') = \frac{G^{(3)}(\mathbf{r},\mathbf{r}',\mathbf{r}'')}{G^{(1)}(\mathbf{r},\mathbf{r})G^{(1)}(\mathbf{r}',\mathbf{r}')G^{(1)}(\mathbf{r}'',\mathbf{r}'')}$$
(13)

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For a gaussian field, $g^{(3)}(\mathbf{r}, \mathbf{r}, \mathbf{r}) = 3! = 6$. By contrast, $g^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'') = 1$ at all points for a coherent state.

2.2. Quantum description of a system of N identical bosons

2.2.1. Second quantization formalism

In the framework of first quantization, the way to describe a system of N identical bosons consists of writing a completely symmetrical wave function $\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ in a 3N-D-space (we ignore the spin to keep notation simple). The observables of the system are described by symmetric Hermitian operators, for instance:

• the one-body density (or spatial density) $\hat{\rho}_{I}(\mathbf{r})$ in \mathbf{r} is written:

$$\hat{\rho}_{\mathrm{I}}(\boldsymbol{r}) = \sum_{i=1}^{N} \delta(\hat{\boldsymbol{r}}_{i} - \boldsymbol{r})$$
(14)

• the two-body density $\hat{\rho}_{II}(\boldsymbol{r}, \boldsymbol{r}')$ can be written:

$$\hat{\rho}_{\mathrm{II}}(\boldsymbol{r},\boldsymbol{r}') = \sum_{i=1}^{N} \sum_{j \neq i} \delta(\hat{\boldsymbol{r}}_{i} - \boldsymbol{r}) \delta(\hat{\boldsymbol{r}}_{j} - \boldsymbol{r}')$$
(15)

• the interaction energy is then:

$$\widehat{V}_{\text{int}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} V(\widehat{\boldsymbol{r}}_{i} - \widehat{\boldsymbol{r}}_{j}) = \frac{1}{2} \iint \mathrm{d}^{3} \boldsymbol{r} \,\mathrm{d}^{3} \boldsymbol{r}' \,V(\boldsymbol{r} - \boldsymbol{r}')\widehat{\rho}_{\text{II}}(\boldsymbol{r}, \boldsymbol{r}') \tag{16}$$

However, it is more convenient to describe such a N-boson system by using the *second quantization* formalism, similarly to the usual approach in quantum optics. Given an orthonormal basis of single particle states $\{|\psi_i\rangle\}$, one can build a Fock space with a basis $\{|n_1, n_2, \ldots, n_i, \ldots\rangle\}$ characterized by the number n_i of bosons in each individual state $|\psi_i\rangle$. The occupation numbers can take values from 0 to N and obey $\sum_i n_i = N$.

One can then define the annihilation and creation operators \hat{a}_i and \hat{a}_i^+ which respectively annihilate and create a boson in the state $|\psi_i\rangle$:

$$\hat{a}_i |n_i\rangle = \sqrt{n_i} |n_i - 1\rangle \tag{17}$$

$$\hat{a}_i^+|n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle \tag{18}$$

$$\hat{a}_i|0\rangle = 0 \tag{19}$$

where $|0\rangle$ is the vacuum state. Equation (18) implies that $|n_i\rangle = ((\hat{a}_i^+)^{n_i}/\sqrt{n_i!})|0\rangle$. The creation and annihilation operators follow the commutation relations $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$.

By analogy with quantum optics, one can also define field operators

$$\widehat{\psi}(\mathbf{r}) = \sum_{i} \psi_i(\mathbf{r}) \widehat{a}_i \tag{20}$$

$$\widehat{\psi}^{\dagger}(\boldsymbol{r}) = \sum_{i} \psi_{i}^{*}(\boldsymbol{r}) \widehat{a}_{i}^{+}$$
(21)

where $\psi_i(\mathbf{r}) = \langle \mathbf{r} | \psi_i \rangle$ is the projection of the single state particle onto $| \mathbf{r} \rangle$.

To understand better the physical meaning of the field operators, we can evaluate the effect of $\hat{\psi}^{\dagger}(r)$ on the vacuum:

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$$\widehat{\psi}^{\dagger}(\boldsymbol{r})|0\rangle = \sum_{i} \psi_{i}^{*}(\boldsymbol{r}) \hat{a}_{i}^{+}|0\rangle = \sum_{i} \langle \psi_{i}|\boldsymbol{r}\rangle|\psi_{i}\rangle = \sum_{i} |\psi_{i}\rangle\langle\psi_{i}|\boldsymbol{r}\rangle$$
$$\widehat{\psi}^{\dagger}(\boldsymbol{r})|0\rangle = |\boldsymbol{r}\rangle$$
(22)

We see that $\hat{\psi}^{\dagger}(r)$ or $\hat{\psi}(r)$ creates or annihilates a boson in r, respectively. In other words, $\hat{\psi}(r)$ and $\hat{\psi}^{\dagger}(\mathbf{r})$ are, for the basis $\{|\mathbf{r}\rangle\}$, the equivalent of \hat{a}_i and \hat{a}_i^+ for the basis $\{|\psi_i\rangle\}$. Similarly, they obey the commutation relations $[\widehat{\psi}(\mathbf{r}), \widehat{\psi}^{\dagger}(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}').$

In second quantization, all symmetric observables can be expressed as products of creation and annihilation operators:

• one-particle observables such as $\hat{F} = \sum_{i=1}^{N} \hat{f}_i$ become:

$$\widehat{F} = \sum_{\alpha} \sum_{\beta} \langle \psi_{\beta} | \widehat{f} | \psi_{\alpha} \rangle \widehat{a}_{\beta}^{+} \widehat{a}_{\alpha}$$
(23)

$$\widehat{F} = \iint \mathrm{d}^{3} \boldsymbol{r} \, \mathrm{d}^{3} \boldsymbol{r}' \langle \boldsymbol{r} | \widehat{f} | \boldsymbol{r}' \rangle \widehat{\psi}^{\dagger}(\boldsymbol{r}) \widehat{\psi}(\boldsymbol{r}')$$
(24)

• two-particle observables such as $\widehat{G} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \widehat{g}(i, j)$ become:

$$\widehat{G} = \frac{1}{2} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \sum_{\delta} \langle \psi_{\delta}(1)\psi_{\gamma}(2)|\hat{g}(1,2)|\psi_{\beta}(2)\psi_{\alpha}(1)\rangle \hat{a}^{+}_{\delta} \hat{a}^{+}_{\gamma} \hat{a}_{\beta} \hat{a}_{\alpha}$$

$$\widehat{G} = \frac{1}{2} \iiint d^{3}\boldsymbol{r} d^{3}\boldsymbol{r}' d^{3}\boldsymbol{r}'' d^{3}\boldsymbol{r}''' \langle 1\,\boldsymbol{r}''', 2\,\boldsymbol{r}''|\hat{g}(1,2)|2\,\boldsymbol{r}', 1\,\boldsymbol{r}\rangle \widehat{\psi}^{\dagger}(\boldsymbol{r}'') \widehat{\psi}(\boldsymbol{r}') \widehat{\psi}(\boldsymbol{r}') \widehat{\psi}(\boldsymbol{r})$$
(25)
$$(25)$$

For example, the one- and two-body density operators $\hat{\rho}_{I}(\mathbf{r})$ and $\hat{\rho}_{II}(\mathbf{r}, \mathbf{r}')$ are written, in the basis $\{|\mathbf{r}\rangle\}$:

$$\hat{\rho}_{\rm I}(\boldsymbol{r}) = \widehat{\psi}^{\dagger}(\boldsymbol{r})\widehat{\psi}(\boldsymbol{r}) \tag{27}$$

$$\hat{\rho}_{\rm II}(\boldsymbol{r},\boldsymbol{r}') = \widehat{\psi}^{\dagger}(\boldsymbol{r})\widehat{\psi}^{\dagger}(\boldsymbol{r}')\widehat{\psi}(\boldsymbol{r}')\widehat{\psi}(\boldsymbol{r})$$
(28)

The relation between $\hat{\rho}_{\rm I}(\boldsymbol{r})$ and $\hat{\rho}_{\rm II}(\boldsymbol{r},\boldsymbol{r}')$ can be obtained by using the commutation relations $[\hat{\psi}(\boldsymbol{r}),\hat{\psi}^{\dagger}(\boldsymbol{r}')] = \delta(\boldsymbol{r}-\boldsymbol{r}')$ and $[\hat{\psi}(\boldsymbol{r}),\hat{\psi}(\boldsymbol{r}')] = 0$:

$$\hat{\rho}_{\mathrm{II}}(\boldsymbol{r},\boldsymbol{r}') = \hat{\rho}_{\mathrm{I}}(\boldsymbol{r})\hat{\rho}_{\mathrm{I}}(\boldsymbol{r}') - \delta(\boldsymbol{r}-\boldsymbol{r}')\hat{\rho}_{\mathrm{I}}(\boldsymbol{r})$$
(29)

2.2.2. Correlation functions

The correlation functions of bosonic fields exhibit strong analogies with those of quantum optics. Correlation functions are the average value of products of bosonic field operators:

- $G^{(1)}(\mathbf{r},\mathbf{r}') = \langle \widehat{\psi}^{\dagger}(\mathbf{r}) \widehat{\psi}(\mathbf{r}') \rangle$ accounts for the spatial coherence between \mathbf{r} and \mathbf{r}' . In particular, $G^{(1)}(\boldsymbol{r},\boldsymbol{r}) = \langle \hat{\psi}^{\dagger}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) \rangle = \langle \hat{\rho}_{\mathrm{I}}(\boldsymbol{r}) \rangle \text{ is the probability of finding a boson in } \boldsymbol{r}.$ • $G^{(2)}(\boldsymbol{r},\boldsymbol{r}') = \langle \hat{\psi}^{\dagger}(\boldsymbol{r})\hat{\psi}^{\dagger}(\boldsymbol{r}')\hat{\psi}(\boldsymbol{r}') \rangle = \langle \hat{\rho}_{\mathrm{II}}(\boldsymbol{r},\boldsymbol{r}') \rangle$ represents the probability of finding a boson in \boldsymbol{r} .
- and another one in r'.
- All higher order correlation functions are deduced similarly. For instance, $\langle \hat{\rho}_{I}(\mathbf{r}',t')\hat{\rho}_{I}(\mathbf{r},t)\rangle$ describes the correlations between the densities of bosons in two different points at two different times.

We want to stress the importance of correlation functions: indeed, most experimental detection signals can be expressed in terms of correlation functions. For instance, the 3-body collision rate is related to $G^{(3)}(\mathbf{r},\mathbf{r},\mathbf{r})$, and the visibility of interference fringes is proportional to $G^{(1)}(\mathbf{r},\mathbf{r}')$. We will also see in Section 5.1 that the spectrum and the total intensity of the light scattered in a given direction is described by the dynamic and static structure factors which are derived from correlation functions.

Generally, correlation functions are connected with the linear response theory. The response of a system of N bosons to a *weak* probe can indeed be described by a *linear* susceptibility which is connected to the correlation function of the unperturbed system through the fluctuation-dissipation theorem [5].

2.2.3. Superselection rules

For a system of bosonic atoms, all physical observables contain an equal number of creation and annihilation operators (see, for example, equations (23) and (25)), which means that they do not change the total number N of bosons. Unlike photons, bosonic atoms cannot be created or destroyed in an interaction process.

As a consequence, if one starts with a density operator $\hat{\rho}$ diagonal in N ($\langle N | \hat{\rho} | N' \rangle = 0$ if $N \neq N'$), it remains diagonal at all subsequent times.

Contrary to photons, a linear superposition $|\psi\rangle = \sum_N c_N |N\rangle$ has no physical meaning. This is in particular the case of coherent states. However, we have seen in Section 2.1.1 that a statistical mixture of coherent states $|\alpha\rangle$ with the same $|\alpha|$ and a phase uniformly distributed over $[0, 2\pi[$ is also a statistical mixture of Fock states $|N\rangle$. Such a statistical mixture does have a physical meaning for bosonic atoms.

2.3. Calculation of correlation functions in a few simple cases

2.3.1. Perfect gas of bosons in a box

Let us consider a non-interacting gas of bosons of mass m in a box of size L. An orthonormal basis of single particle states is made of plane waves with periodic boundary conditions:

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{L^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} \quad \text{with} \quad k_x = \frac{2\pi}{L} n_x \quad (n_x \in \mathbb{Z})$$
(30)

The corresponding field operators are written:

$$\widehat{\psi}(\boldsymbol{r}) = \frac{1}{L^{3/2}} \sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \quad \text{and} \quad \widehat{\psi}^{\dagger}(\boldsymbol{r}) = \frac{1}{L^{3/2}} \sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$$
(31)

In the absence of interaction, the Hamiltonian is simply:

$$\widehat{H} = \sum_{k} \varepsilon_k \hat{a}^{\dagger}_{k} \hat{a}_{k} \quad \text{with} \quad \varepsilon_k = \frac{\hbar^2 k^2}{2m}$$
(32)

In the grand canonical ensemble [6], the equilibrium density operator is written $\hat{\rho}_{eq} = e^{-\beta(\hat{H}-\mu\hat{N})}/Z_{G}$ where $\beta = 1/(k_{B}T)$ is the Boltzmann factor, μ is the chemical potential ($\mu < 0$ for bosons) and Z_{G} is the grand canonical partition function. The translational invariance of $\hat{\rho}_{eq}$ implies that:

$$\left\langle \hat{a}_{\boldsymbol{k}'}^{+} \hat{a}_{\boldsymbol{k}} \right\rangle = \operatorname{Tr}\left(\hat{\rho}_{\mathrm{eq}} \hat{a}_{\boldsymbol{k}'}^{+} \hat{a}_{\boldsymbol{k}} \right) = \left\langle n_{\boldsymbol{k}} \right\rangle \delta_{\boldsymbol{k} \, \boldsymbol{k}'} \tag{33}$$

where $\langle n_{k} \rangle = \langle \hat{a}_{k}^{+} \hat{a}_{k} \rangle$ is the mean number of particles in the state k.

One can then calculate the first order correlation function $G^{(1)}(\mathbf{r}, \mathbf{r}')$ of the system:

$$G^{(1)}(\boldsymbol{r},\boldsymbol{r}') = \operatorname{Tr}\left[\hat{\rho}_{eq}\hat{\psi}^{\dagger}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}')\right] = \frac{1}{L^3}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})} \langle \hat{a}_{\boldsymbol{k}}^{\dagger}\hat{a}_{\boldsymbol{k}} \rangle$$
(34)

In equation (34), one can replace the sum over k by an integral, provided that one accounts separately for the contribution of k = 0 which would otherwise be omitted (the density of states vanishes for k = 0). Using

$$\langle \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}} \rangle = \frac{z \,\mathrm{e}^{-\beta \varepsilon_k}}{1 - z \,\mathrm{e}^{-\beta \varepsilon_k}} = \sum_{l=1}^{\infty} z^l \,\mathrm{e}^{-l\beta \varepsilon_k}$$

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where $z = e^{\beta \mu}$ is the fugacity, one gets

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{N_0}{L^3} + \frac{1}{(2\pi)^3} \int d^3 \boldsymbol{k} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \sum_{l=1}^{\infty} z^l e^{-l\beta\hbar^2 k^2/2m}$$
$$= \frac{N_0}{L^3} + \frac{1}{\lambda_{\rm T}^3} \sum_{l=1}^{\infty} \frac{z^l}{l^{3/2}} \exp\left(-\frac{\pi(\boldsymbol{r}-\boldsymbol{r}')^2}{l\lambda_{\rm T}^2}\right)$$
(35)

with $N_0 = \langle n_0 \rangle$ the population of the ground state and

$$\lambda_{\rm T} = \sqrt{\frac{2\pi\hbar^2}{mk_{\rm B}T}}$$

the thermal de Broglie wavelength. Note that $G^{(1)}(\mathbf{r}, \mathbf{r}')$ only depends on $(\mathbf{r} - \mathbf{r}')$, which is a consequence of the translational invariance of $\hat{\rho}_{eq}$. In particular:

$$G^{(1)}(\boldsymbol{r},\boldsymbol{r}) = \rho_{\mathrm{I}}(\boldsymbol{r}) = \frac{N_0}{L^3} + \frac{1}{\lambda_{\mathrm{T}}^3} \sum_{\substack{l=1\\ =g_{3/2}(z)}}^{\infty} \frac{z^l}{l^{3/2}} = \frac{N}{L^3}$$
(36)

does not depend on r.

Let us now study the evolution of $G^{(1)}(\mathbf{r} - \mathbf{r}')$ for increasing values of $\rho = N/L^3$ at fixed T.

• For a very dilute gas verifying $\rho \lambda_T^3 \ll 1$, we have $N_0 \ll N$ and $z \ll 1$. We can thus neglect N_0/L^3 and use $g_{3/2}(z) \simeq z$ (i.e. keep only the term l = 1 in the sum over l) in equation (36). This yields $N/L^3 \simeq z/\lambda_T^3$, i.e. $z = N\lambda_T^3/L^3$. Making the same approximations in the expression of $G^{(1)}$ (equation (35)), we get:

$$G^{(1)}(\boldsymbol{r} - \boldsymbol{r}') \simeq \frac{z}{\lambda_{\rm T}^3} e^{-\pi (\boldsymbol{r} - \boldsymbol{r}')^2 / \lambda_{\rm T}^2} \simeq \frac{N}{L^3} e^{-\pi (\boldsymbol{r} - \boldsymbol{r}')^2 / \lambda_{\rm T}^2}$$
(37)

We recover the coherence length $\lambda_{\rm T}/\sqrt{\pi}$ of a classical Maxwell–Boltzmann gas (see *figure 1*).

- For larger values of ρ still remaining below the critical value, N_0 can still be neglected but z is larger and one has to take into account the contribution of terms with l > 1 in the sum over l. $G^{(1)}(\boldsymbol{r} \boldsymbol{r}')$ is then a sum of gaussian curves with increasing widths $\lambda_T/\sqrt{\pi}$, $\lambda_T\sqrt{2}/\sqrt{\pi}$, $\lambda_T\sqrt{3}/\sqrt{\pi}$, ... The coherence length thus increases.
- When one reaches the critical regime, one has z = 1, which corresponds to $\rho_{\text{crit}}\lambda_{\text{T}}^3 = \sum_{l=1}^{\infty} 1/l^{3/2} = 2.612$. At this point, one can still neglect N_0 which has not yet started to increase. Inserting z = 1 in





Figure 1. First order correlation function for a very dilute gas $(\rho \lambda_T^3 \ll 1)$.

Figure 2. First order correlation function for a Bose condensed gas $(\rho \lambda_T^3 > 1)$.

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equation (35), one gets:

$$G_{\rm crit}^{(1)}(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{\lambda_{\rm T}^3} \sum_{l=1}^\infty \frac{1}{l^{3/2}} \exp\left(-\frac{\pi(\boldsymbol{r} - \boldsymbol{r}')^2}{l\lambda_{\rm T}^2}\right)$$
(38)

- $G_{\text{crit}}^{(1)}(\boldsymbol{r}-\boldsymbol{r}')$ only depends on λ_{T} and does not vary with a further increase of ρ at fixed T.
- For $\rho > \rho_{\text{crit}}$, N_0 is no longer negligible and z remains equal to 1. We thus have:

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{N_0}{L^3} + G^{(1)}_{\text{crit}}(\boldsymbol{r} - \boldsymbol{r}')$$
(39)

and $G^{(1)}(\boldsymbol{r}, \boldsymbol{r}) = N_0/L^3 + \rho_{\rm crit}$, with $\rho_{\rm crit} = (N - N_0)/L^3$.

Once the critical density is attained, any further increase of ρ increases only N_0 , $G_{\text{crit}}^{(1)}$ remaining unchanged. This corresponds to an increase of the flat background in *figure 2*, i.e. to an *infinite* coherence length. A long range spatial order thus develops for increasing densities at constant temperature.

Let us now study the asymptotic behaviour of $G^{(1)}(\mathbf{r} - \mathbf{r}')$ for $s = |\mathbf{r} - \mathbf{r}'| \gg \lambda_{\mathrm{T}}$. Equation (34) clearly shows that $G^{(1)}(s)$ is the Fourier transform of $\langle n_{\mathbf{k}} \rangle$. The behaviour of $G^{(1)}(s)$ at large s is thus determined by that of $\langle n_{\mathbf{k}} \rangle$ at small k:

$$\langle n_{\mathbf{k}} \rangle = \frac{z}{\mathrm{e}^{\beta \hbar^2 k^2 / 2m} - z} \underset{k \to 0}{\simeq} \frac{z}{1 - z + \beta \frac{\hbar^2 k^2}{2m}} \underset{k \to 0}{\simeq} \frac{z}{\beta \hbar^2 / 2m} \frac{1}{k^2 + k_c^2} \quad \text{with}$$

$$k_c^2 = \frac{1 - z}{\beta \hbar^2 / 2m} = \frac{4\pi (1 - z)}{\lambda_{\mathrm{T}}^2}.$$

$$(40)$$

This yields:

$$G^{(1)}(s) \propto_{(s \gg \lambda_{\rm T})} \frac{\exp\left(-s\frac{\sqrt{4\pi(1-z)}}{\lambda_{\rm T}}\right)}{s} \quad (\text{Yukawa shape) for } z < 1$$
$$G^{(1)}(s) \propto_{(s \gg \lambda_{\rm T})} 1/s \quad (\text{Coulomb shape) for } z = 1 \tag{41}$$

Normalized correlation functions are often convenient because they do not depend on N. For a dilute gas, we have $g^{(1)}(\mathbf{r} - \mathbf{r}') = \exp(-\pi(\mathbf{r} - \mathbf{r}')^2/\lambda_{\mathrm{T}}^2)$ (see *figure 3a*). By contrast, in the case of a very dense gas, the first order normalized correlation function decreases slowly with s towards the asymptotic value $1 - \rho_{\mathrm{crit}}/\rho$ (see *figure 3b*). This implies that when $\rho \to \infty$, $g^{(1)}(s) \to 1$.

In this paragraph we have made all the calculations in the grand canonical ensemble, where the mathematics is the simplest. However, in present experiments on Bose–Einstein condensates the total number N of bosons is fixed, the canonical ensemble would be more appropriate. Both ensembles lead



Figure 3. Normalized first order correlation function for a very dilute gas (a) and for a Bose condensed gas (b).

to similar results concerning $\langle n_k \rangle$, hence $G^{(1)}$, which justifies the preceding calculations. By contrast, the fluctuations of the occupation numbers $\langle n_k \rangle$ can be quite different in the canonical and the grand canonical ensembles for a degenerate Bose gas [7], so that the grand canonical calculation of $G^{(2)}$, $G^{(3)}$,... is not reliable outside the regime $N \ll N_{\rm crit}$. In the following, we will rather use a Fock state to describe the condensate and to calculate the correlation functions.

Let us now calculate $G^{(2)}$, $G^{(3)}$,... for $N \ll N_{crit}$, using the grand canonical description. We can use Wick's theorem, which applies to a gaussian distribution, hence to a perfect gas in grand canonical equilibrium [8]: all average values can be expressed as sums of products of average values involving only two operators (one creation and one annihilation operator). The results obtained for $g^{(2)}$, $g^{(3)}$,... are thus the same as those found in quantum optics for a gaussian field [3]:

$$g^{(2)}(\boldsymbol{r}, \boldsymbol{r}') = 1 + |g^{(1)}(\boldsymbol{r}, \boldsymbol{r}')|^{2}$$

$$g^{(2)}(\boldsymbol{r}, \boldsymbol{r}) = 1 + 1 = 2$$

$$g^{(3)}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}'') = 1 + |g^{(1)}(\boldsymbol{r}, \boldsymbol{r}')|^{2} + |g^{(1)}(\boldsymbol{r}', \boldsymbol{r}'')|^{2} + |g^{(1)}(\boldsymbol{r}'', \boldsymbol{r})|^{2}$$

$$+ 2\operatorname{Re}[g^{(1)}(\boldsymbol{r}, \boldsymbol{r}')g^{(1)}(\boldsymbol{r}', \boldsymbol{r}'')g^{(1)}(\boldsymbol{r}'', \boldsymbol{r})]$$

$$(43)$$

$$g^{(3)}(\boldsymbol{r}, \boldsymbol{r}, \boldsymbol{r}) = 3! = 6$$

2.3.2. N non-interacting bosons in the ground state of a trap

Let us consider a basis of individual states $\{\psi_i(\mathbf{r})\}\$ including the ground state $\psi_1(\mathbf{r})$ of the trap. The state of the system can be described by the Fock state $|N\rangle = |n_1 = N, n_i = 0$ if $i \neq 1\rangle$. The field operators are written:

$$\widehat{\psi}(\mathbf{r}) = \psi_1(\mathbf{r})\hat{a}_1 + \sum_{i \neq 1} \psi_i(\mathbf{r})\hat{a}_i$$
(44)

$$\widehat{\psi}^{\dagger}(\mathbf{r}) = \psi_1^*(\mathbf{r})\hat{a}_1^+ + \sum_{i \neq 1} \psi_i^*(\mathbf{r})\hat{a}_i^+$$
(45)

Because \hat{a}_i with $i \neq 1$ commutes with \hat{a}_1 and \hat{a}_1^+ and because $\hat{a}_i |0_i\rangle = 0$, we can use simplified expressions of $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ to calculate the average value of normally ordered field operators:

$$\widehat{\psi}(\boldsymbol{r}) = \psi_1(\boldsymbol{r})\hat{a}_1 \quad \text{and} \quad \widehat{\psi}^{\dagger}(\boldsymbol{r}) = \psi_1^*(\boldsymbol{r})\hat{a}_1^+$$
(46)

We can then write the field correlation functions:

• First order correlation function:

$$G^{(1)}(\boldsymbol{r}, \boldsymbol{r}') = \psi_1^*(\boldsymbol{r})\psi_1(\boldsymbol{r}')\langle N|\hat{a}_1^+\hat{a}_1|N\rangle = N\psi_1^*(\boldsymbol{r})\psi_1(\boldsymbol{r}')$$
(47)

• Second order correlation function:

$$G^{(2)}(\boldsymbol{r}, \boldsymbol{r}') = \psi_1^*(\boldsymbol{r})\psi_1^*(\boldsymbol{r}')\psi_1(\boldsymbol{r}')\underbrace{\langle N|\hat{a}_1^+\hat{a}_1^+\hat{a}_1\hat{a}_1|N\rangle}_{=N(N-1)}$$

$$G^{(2)}(\boldsymbol{r}, \boldsymbol{r}') = N(N-1)|\psi_1(\boldsymbol{r})|^2|\psi_1(\boldsymbol{r}')|^2$$
(48)

The normalized second order correlation function is written:

$$g^{(2)}(\boldsymbol{r}, \boldsymbol{r}') = \frac{N(N-1)}{N^2} = 1 - \frac{1}{N} \simeq 1 \quad \text{if } N \gg 1$$
(49)

• Third order correlation function:

$$G^{(3)}(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}'') = \psi_{1}^{*}(\boldsymbol{r})\psi_{1}^{*}(\boldsymbol{r}')\psi_{1}(\boldsymbol{r}'')\psi_{1}(\boldsymbol{r}')\psi_{1}(\boldsymbol{r})\underbrace{\langle N|\hat{a}_{1}^{+}\hat{a}_{1}^{+}\hat{a}_{1}\hat{a}_{1}\hat{a}_{1}\hat{a}_{1}|N\rangle}_{=N(N-1)(N-2)}$$

$$G^{(3)}(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}'') = N(N-1)(N-2)|\psi_{1}(\boldsymbol{r})|^{2}|\psi_{1}(\boldsymbol{r}')|^{2}|\psi_{1}(\boldsymbol{r}'')|^{2} \qquad (50)$$

This yields for the normalized third order correlation function:

$$g^{(3)}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}'') = \frac{N(N-1)(N-2)}{N^3} = \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right)$$
(51)

$$\simeq 1 \quad \text{if } N \gg 1$$
 (52)

Similarly to what we have calculated in equations (47), (48) and (50), all correlation functions G^(k) with k ≪ N of a single mode Fock state |n₁ = N, n_i = 0 if i ≠ 1⟩ are equal to a product of 2k functions, namely k functions √Nψ₁ and k functions √Nψ₁^{*}. We thus recover the same factorization as for a coherent state |α⟩ of the 'mode' ψ₁ with α = √N (see Section 2.1.2). Consequently, as shown in equations (49) and (51), all normalized correlation functions g^(k) with k ≪ N are equal to 1.

3. 'Macroscopic' matter waves

In this section, we first introduce the 'macroscopic' matter wave concept in the framework of the variational approximation. We then use it to study the coherence properties of a single Bose–Einstein condensate and the question of the relative phase between two separate condensates.

3.1. Variational approximation

3.1.1. The Gross-Pitaevskii equation

Let us consider N identical bosons in a trap described by a potential V_{ext} . For $T \ll T_{\text{C}}$ they all condense in the ground state of the Hamiltonian:

$$\widehat{H} = \sum_{i=1}^{N} \left[\frac{\widehat{p}_{i}^{2}}{2m} + V_{\text{ext}}(\widehat{r}_{i}) \right] + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} V(\widehat{r}_{ij})$$
(53)

where $V(\hat{r}_{ij}) = V(|\hat{r}_i - \hat{r}_j|)$ is the interaction potential between pairs of bosons. At very low temperatures, namely when the de Broglie wavelength λ_T becomes much larger than the range of $V(\hat{r}_{ij})$, only s-wave scattering between pairs of bosons remains significant, and we can approximate $V(\hat{r}_{ij})$ by:

$$V(\hat{r}_{ij}) = g\delta\left(|\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j|\right) \quad \text{with} \quad g = \frac{4\pi\hbar^2}{m}a \tag{54}$$

where a is the scattering length.

Generally, the ground state of \hat{H} cannot be determined exactly. In the absence of interactions however, it is a product state: all the bosons are in the ground state of the single particle Hamiltonian $\hat{h} = \hat{p}^2/2m + V_{\text{ext}}(\mathbf{r})$. In the presence of weak interactions, one can still approximate the ground state of \hat{H} by a product state:

$$|\psi_{\rm g}\rangle = |\psi(1)\rangle \cdots |\psi(i)\rangle \cdots |\psi(N)\rangle \tag{55}$$

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where all the bosons are in the same state $|\psi\rangle$. Obviously, $|\psi_g\rangle$ is completely symmetric. Contrary to the non-interacting case, $|\psi\rangle$ is no longer the ground state of \hat{h} but has to be determined as the one minimizing:

$$\frac{\langle \psi_{\rm g} | \hat{H} | \psi_{\rm g} \rangle}{\langle \psi_{\rm g} | \psi_{\rm g} \rangle}$$

Let us first calculate $\langle \psi_{g} | \hat{H} | \psi_{g} \rangle$:

$$\langle \psi_{\mathbf{g}} | \hat{H} | \psi_{\mathbf{g}} \rangle = N \int d^{3} \boldsymbol{r} \, \psi^{*}(\boldsymbol{r}) \left[-\frac{\hbar^{2}}{2m} \Delta + V_{\text{ext}} \right] \psi(\boldsymbol{r}) + \frac{N(N-1)}{2} \iint d^{3} \boldsymbol{r} \, d^{3} \boldsymbol{r}' \, \psi^{*}(\boldsymbol{r}) \psi^{*}(\boldsymbol{r}') V(|\boldsymbol{r} - \boldsymbol{r}'|) \psi(\boldsymbol{r}') \psi(\boldsymbol{r})$$
(56)

We now look for the minimum of $\langle \psi_g | \hat{H} | \psi_g \rangle$ with $\langle \psi_g | \psi_g \rangle = 1$ fixed. Because ψ is a priori a complex number, we can consider the variations $\delta \psi$ and $\delta \psi^*$ of ψ and ψ^* respectively as independent. Using the method of Lagrange multipliers, the approximate ground state $|\psi_g\rangle$ has to satisfy:

$$\delta \left[\langle \psi_{\rm g} | \hat{H} | \psi_{\rm g} \rangle \right] - \lambda \delta \langle \psi_{\rm g} | \psi_{\rm g} \rangle = 0 \tag{57}$$

where λ is the Lagrange multiplier associated with the constraint $\langle\psi_{\rm g}|\psi_{\rm g}\rangle=1.$

Inserting the expression (56) of $\langle \psi_g | \hat{H} | \psi_g \rangle$ in equation (57) and cancelling the coefficient of $\delta \psi^*$ yields:

$$\left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}}(\boldsymbol{r})\right]\psi(\boldsymbol{r}) + (N-1)\left[\int \mathrm{d}^3\boldsymbol{r}' \,V\big(|\boldsymbol{r}-\boldsymbol{r}'|\big)\big|\psi(\boldsymbol{r}')\big|^2\right]\psi(\boldsymbol{r}) = \lambda\psi(\boldsymbol{r}) \tag{58}$$

We recover the Gross–Pitaevskii equation [9,10], which has a straightforward interpretation: each boson evolves in the external potential V_{ext} and in the *mean-field* potential produced by the (N-1) other bosons.

Let us now try and find an interpretation for the parameter λ which was introduced formally as a Lagrange multiplier. After multiplying equation (58) by $\psi^*(\mathbf{r})$ and integrating over \mathbf{r} , one gets:

$$\lambda = \int d^{3}\boldsymbol{r} \,\psi^{*}(\boldsymbol{r}) \left[-\frac{\hbar^{2}}{2m} \Delta + V_{\text{ext}}(\boldsymbol{r}) \right] \psi(\boldsymbol{r}) + \underbrace{(N-1)}_{\simeq N} \iint d^{3}\boldsymbol{r} \,d^{3}\boldsymbol{r}' \,\psi^{*}(\boldsymbol{r}) \psi^{*}(\boldsymbol{r}') V(|\boldsymbol{r}-\boldsymbol{r}'|) \psi(\boldsymbol{r}') \psi(\boldsymbol{r})$$
(59)

$$\lambda \simeq \frac{\partial}{\partial N} \langle \psi_{\rm g} | \hat{H} | \psi_{\rm g} \rangle = \frac{\mathrm{d}}{\mathrm{d}N} \langle \psi_{\rm g} | \hat{H} | \psi_{\rm g} \rangle - \underbrace{\frac{\delta \langle \psi_{\rm g} | \hat{H} | \psi_{\rm g} \rangle}{\delta \psi}}_{=0} \frac{\partial \psi}{\partial N} \tag{60}$$

 $\lambda = \frac{\mathrm{d}}{\mathrm{d}N} \langle \psi_{\mathrm{g}} | \hat{H} | \psi_{\mathrm{g}} \rangle$ thus represents the chemical potential which was previously noted μ . We can simplify equation (58) by replacing $V(\boldsymbol{r} - \boldsymbol{r}')$ by $g\delta(\boldsymbol{r} - \boldsymbol{r}')$ (equation (54)). This gives

$$\left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}}(\boldsymbol{r})\right]\psi(\boldsymbol{r}) + (N-1)g|\psi(\boldsymbol{r})|^2\psi(\boldsymbol{r}) = \mu\psi(\boldsymbol{r})$$
(61)

This equation is analogous to the one found in non-linear optics for the light field. One can thus generalize to matter waves some effects well known in non-linear optics such as four-wave mixing [11], solitons [12], etc.

The ground state energy is generally the sum of three contributions:

- (i) the kinetic energy $E_{\rm kin} = N \int d^3 \boldsymbol{r} \, \psi^*(\boldsymbol{r}) (-\hbar^2/(2m)) \Delta \psi(\boldsymbol{r})$,
- (ii) the 'trapping energy' $E_{\text{ext}} = N \int d^3 \boldsymbol{r} |\psi(\boldsymbol{r})|^2 V_{\text{ext}}(\boldsymbol{r}),$
- (iii) the interaction energy $E_{\text{int}} = \frac{1}{2}N(N-1)g\int d^3r |\psi(\mathbf{r})|^4$.

For a homogeneous condensate in a box of size L, the wave function $\psi = 1/L^{3/2}$ is uniform and the only remaining contribution comes from the mean-field interaction energy $E_{\text{int}} \simeq \frac{1}{2}g\rho N$.

3.1.2. Macroscopic matter waves

We consider here a basis $\{|\psi_i\rangle\}$ of single particle states including the solution ψ_1 of the Gross–Pitaevskii equation. ψ_1 is generally real ($\psi_1^* = \psi_1$). In the variational description of the condensate at T = 0, the quantum state of the system is the Fock state $|N\rangle = |n_1 = N, n_i = 0$ if $i \neq 1\rangle$.

We have seen in Section 2.3.2 that for a Fock state, all the correlation functions $G^{(1)}, \ldots, G^{(k)}$ with $k \ll N$ are products of $2, 4, \ldots, 2k$ functions $\sqrt{N}\psi_1$. This is very similar to the situation of coherent optical fields (also called *quasi-classical* fields) for which the correlation functions are products of classical fields (see Section 2.1.2).

In the following, we associate to the condensate a macroscopic (or 'giant') matter wave $\sqrt{N}\psi_1(\mathbf{r})$. Note however that this macroscopic matter wave description fails when the variational approximation is no longer valid.

3.2. Coherence properties of a Bose-Einstein condensate

3.2.1. Coherence length of a trapped condensate

The coherence length of the condensate is the characteristic length over which the first order correlation function $G^{(1)}(\mathbf{r}, \mathbf{r}')$ decreases. In the macroscopic matter wave description, we have:

$$G^{(1)}(\mathbf{r}, \mathbf{r}') = N\psi_1(\mathbf{r})\psi_1(\mathbf{r}')$$
(62)

This shows that $G^{(1)}(\mathbf{r}, \mathbf{r}')$ vanishes when $|\mathbf{r} - \mathbf{r}'|$ becomes larger than the spatial extent σ_0 of $\psi_1(\mathbf{r})$. For a non-interacting condensate in a harmonic trap with frequency ω_0 , the coherence length of the condensate is equal to the width $\sigma_0 = \sqrt{\hbar/m\omega_0}$ of the ground state of the trap. In the presence of repulsive interactions, which correspond to a positive scattering length a, the size of the condensate increases.

To have a better insight, we now compare the different length scales of the problem, namely the de Broglie wavelength $\lambda_{\rm T} = \sqrt{2\pi\hbar^2/(mk_{\rm B}T)}$, the size of the thermal cloud $\Delta r \simeq \sqrt{k_{\rm B}T/(m\omega_0^2)}$ and the size of the trap ground state $\sigma_0 = \sqrt{\hbar/(m\omega_0)}$. In the situation considered here, we have $\hbar\omega_0 \ll k_{\rm B}T$, which yields:

$$\lambda_{\rm T} \ll \sigma_0 \ll \Delta r \tag{63}$$

This equation shows that there is a long range order associated with the condensate. It also reminds us that the spatial extension of the trap is finite, so that $G^{(1)}(\mathbf{r}, \mathbf{r}')$ depends on \mathbf{r} and \mathbf{r}' and not only on $\mathbf{r} - \mathbf{r}'$. We thus have to define the coherence length λ_c of the condensate more precisely than we have done with equation (62).

The global spatial coherence G(a) is given by the sum of all spatial coherences between pairs of points (r, r + a) separated by a:

$$G(\boldsymbol{a}) = \int d^3 \boldsymbol{r} \, G^{(1)}(\boldsymbol{r}, \boldsymbol{r} + \boldsymbol{a})$$
(64)

The coherence length is then defined as the characteristic decay length of G(a) when |a| increases from 0 to $+\infty$.

In fact, there is a simple relationship between the global spatial coherence G(a) and the momentum distribution $\mathcal{P}(p)$ of the gas:

$$G(\boldsymbol{a}) = N \int \mathrm{d}^{3}\boldsymbol{r} \langle \boldsymbol{r} | \psi_{1} \rangle \langle \psi_{1} | \boldsymbol{r} + \boldsymbol{a} \rangle = N \iiint \mathrm{d}^{3}\boldsymbol{r} \,\mathrm{d}^{3}\boldsymbol{p} \,\mathrm{d}^{3}\boldsymbol{p}' \langle \boldsymbol{r} | \boldsymbol{p} \rangle \langle \boldsymbol{p} | \psi_{1} \rangle \langle \psi_{1} | \boldsymbol{p}' \rangle \langle \boldsymbol{p}' | \boldsymbol{r} + \boldsymbol{a} \rangle$$
(65)

Since $\langle \boldsymbol{r} | \boldsymbol{p} \rangle = e^{i \boldsymbol{p} \cdot \boldsymbol{r} / \hbar} / (2\pi\hbar)^{3/2}$, integrating over \boldsymbol{r} yields:

$$\int d^3 \boldsymbol{r} \langle \boldsymbol{r} | \boldsymbol{p} \rangle \langle \boldsymbol{p}' | \boldsymbol{r} + \boldsymbol{a} \rangle = \frac{1}{(2\pi\hbar)^3} e^{-i\boldsymbol{p}' \cdot \boldsymbol{a}/\hbar} \int d^3 \boldsymbol{r} e^{i(\boldsymbol{p}-\boldsymbol{p}') \cdot \boldsymbol{r}/\hbar} = e^{-i\boldsymbol{p}' \cdot \boldsymbol{a}/\hbar} \delta(\boldsymbol{p}-\boldsymbol{p}')$$
(66)

Using equation (66), equation (65) transforms into:

$$G(\boldsymbol{a}) = \int \mathrm{d}^{3}\boldsymbol{p} \,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{a}/\hbar} \underbrace{N\langle \boldsymbol{p}|\psi_{1}\rangle\langle\psi_{1}|\boldsymbol{p}\rangle}_{=\mathcal{P}(\boldsymbol{p})}$$
(67)

G(a) is thus simply the Fourier transform of $\mathcal{P}(p)$.

3.2.2. A few experimental tests of spatial coherence

In this paragraph, we present a few recent experimental measurements of the coherence length of a condensate.

- Determination of P(p) by Bragg spectroscopy [13]. This experiment, performed at MIT, is based on a velocity-selective two-photon transition coupling two states with the same internal quantum numbers but with different momenta. From the spectrum obtained by sweeping the frequency difference between the two counter-propagating beams, one can derive the momentum distribution P(p) and in particular its width Δp. The value of Δp measured by J. Stenger et al. is consistent with the Heisenberg limit Δp ~ ħ/σ₀, where σ₀ is the spatial width of the solution ψ₁ of the Gross–Pitaevskii equation [13]. This shows that the coherence length of the condensate is equal to its spatial extent and that there is a single phase throughout the condensate.
- Direct measurement of G(a) [14]. The NIST experiment is a 'self-interference' experiment: a pair of short pulses of stationary wave creates two copies of the condensate separated by a. An absorption imaging technique allows one to measure the overlap integral of the two copies as they interfere. From the decay time of the interference fringes, E. Hagley et al. also find that the phase of the condensate is uniform [14], which amounts to saying that the coherence length of the condensate is equal to its spatial extent.
- Interference of two matter waves extracted from two different points of the condensate [15]. The method used by the Munich group to extract a matter wave from their condensate consists of a position-selective output-coupler using a continuous RF wave. Using two RF waves with different frequencies, one couples out two matter waves from two different points *r* and *r' = r + a* of the condensate. The visibility of the interference fringes between these matter waves is directly related to the first order correlation function *G*⁽¹⁾(*r*, *r'*). I. Bloch et al. have studied the evolution of *G*⁽¹⁾(*r*, *r'*) across the phase transition and have found clear evidence that the coherence length of the gas changes from λ_c = λ_T when *T > T_c* to λ_c ≃ σ₀ when *T < T_c*.

Higher order correlation functions have also been investigated experimentally.

• In particular, the measurement of the mean interaction energy $\langle \hat{V}_{int} \rangle$ of the condensate [16,17] yields some information about the *second order correlation function* $\mathbf{g}^{(2)}(\boldsymbol{r},\boldsymbol{r})$ [18]. Indeed, the mean

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interaction energy is equal to:

$$\left\langle \widehat{V}_{\text{int}} \right\rangle = \frac{1}{2} \iint \mathrm{d}^{3} \boldsymbol{r} \, \mathrm{d}^{3} \boldsymbol{r}' \, V \left(|\boldsymbol{r} - \boldsymbol{r}'| \right) G^{(2)}(\boldsymbol{r}, \boldsymbol{r}') \tag{68}$$

Replacing $V(|\mathbf{r} - \mathbf{r}'|)$ by its approximate form $4\pi\hbar^2 a\delta(\mathbf{r} - \mathbf{r}')/m$ (equation (54)) and using equation (12), one gets:

$$\left\langle \widehat{V}_{\text{int}} \right\rangle = \frac{2\pi\hbar^2}{m} a \int d^3 \boldsymbol{r} \left[\rho_{\text{I}}(\boldsymbol{r}) \right]^2 g^{(2)}(\boldsymbol{r}, \boldsymbol{r}) \tag{69}$$

We have already calculated $g^{(2)}(\mathbf{r}, \mathbf{r})$ in the cases of a thermal field and of a Fock state (see Section 2.1.2 and equation (49), respectively):

 $g^{(2)}(\boldsymbol{r},\boldsymbol{r}) = 2$ for a thermal cloud (70)

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r}) \simeq 1$$
 for a Fock state (71)

The measured values of $\langle \hat{V}_{int} \rangle$ are consistent with $g^{(2)}(\mathbf{r}, \mathbf{r}) = 1$ but exclude $g^{(2)}(\mathbf{r}, \mathbf{r}) = 2$. This means that the transition to Bose–Einstein condensation is associated with a strong suppression of density fluctuations.

• The three-body collision loss rate is proportional to the probability to have three bosons very close to each other and thus to the *third order correlation function* $G^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r}) = [\rho_{\rm I}(\boldsymbol{r})]^3 g^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})$. As seen in Section 2.1.2 and in equation (51), respectively, $g^{(3)}(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})$ is equal to 6 for a thermal cloud and to 1 for a Fock state. The measurement of the three-body loss rate was performed by the JILA group [19]. After an appropriate normalization to account for the variation of the one-body density from one experiment to the other, the three-body loss rate is found to be 7.4 (2.6) times smaller for a condensate than for an ultracold thermal cloud.

3.3. Relative phase between two Bose–Einstein condensates

In this section, we assume that we have two well separated condensates, i.e. two condensates in two different traps centered in r_1 and r_2 , respectively. The solutions of the Gross–Pitaevskii equation for each trap are denoted ψ_1 and ψ_2 , respectively, and for the sake of simplicity we assume $\langle \psi_1 | \psi_2 \rangle = 0$ as well as $\psi_1(r_2) = \psi_2(r_1) = 0$.

We take a basis $\{|\psi_i\rangle\}$ of individual states including $|\psi_1\rangle$ and $|\psi_2\rangle$. In our case, the only relevant basis states in the Fock space are $|n_1, n_2, n_i = 0$ for $i \neq 1, 2\rangle$. In the following, we simplify the notation by omitting n_i for $i \neq 1, 2$. We finally impose that the total number of bosons $N = n_1 + n_2$ is fixed.

3.3.1. Phase states

We now try to find a quantum state describing the two condensates with a well defined relative phase θ . Such a state must exhibit a spatial coherence between two points situated one in each condensate, for instance between r_1 and r_2 :

$$G^{(1)}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \left\langle \widehat{\psi}^{\dagger}(\boldsymbol{r}_1) \widehat{\psi}(\boldsymbol{r}_2) \right\rangle \neq 0$$
(72)

As all modes $i \neq 1, 2$ are empty, we can, when computing normally ordered products of field operators, replace $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^{\dagger}(\mathbf{r})$ by:

$$\psi(\mathbf{r}) = \psi_1(\mathbf{r})\hat{a}_1 + \psi_2(\mathbf{r})\hat{a}_2$$
 (73)

$$\widehat{\psi}^{\dagger}(\mathbf{r}) = \psi_1^*(\mathbf{r})\hat{a}_1^+ + \psi_2^*(\mathbf{r})\hat{a}_2^+ \tag{74}$$

Because $\psi_1(\boldsymbol{r}_2) = \psi_2(\boldsymbol{r}_1) = 0$, we have

$$\widehat{\psi}(\boldsymbol{r}_1) = \psi_1(\boldsymbol{r}_1) \hat{a}_1 \quad \text{and} \quad \widehat{\psi}(\boldsymbol{r}_2) = \psi_2(\boldsymbol{r}_2) \hat{a}_2$$
(75)

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The simplest idea would be to consider Fock states $|n_1, n_2\rangle$. Let us calculate the first order correlation function in (r_1, r_2) :

$$G^{(1)}(\boldsymbol{r}_{1},\boldsymbol{r}_{2}) = \langle n_{1}, n_{2} | \hat{\psi}^{\dagger}(\boldsymbol{r}_{1}) \hat{\psi}(\boldsymbol{r}_{2}) | n_{1}, n_{2} \rangle = \psi_{1}^{*}(\boldsymbol{r}_{1}) \psi_{2}(\boldsymbol{r}_{2}) \langle n_{1}, n_{2} | \hat{a}_{1}^{+} \hat{a}_{2} | n_{1}, n_{2} \rangle$$

$$= \psi_{1}^{*}(\boldsymbol{r}_{1}) \psi_{2}(\boldsymbol{r}_{2}) \sqrt{n_{2}(n_{1}+1)} \langle n_{1}, n_{2} | n_{1}+1, n_{2}-1 \rangle$$

$$G^{(1)}(\boldsymbol{r}_{1}, \boldsymbol{r}_{2}) = 0$$
(76)

The absence of any spatial coherence between two separated condensates in a state $|n_1, n_2\rangle$ shows that two condensates in Fock states do not possess a well defined relative phase.

Another idea consists of taking all N bosons in the same linear combination of ψ_1 and ψ_2 :

$$|\eta_1, \eta_2, \theta\rangle = \eta_1 |\psi_1\rangle + \eta_2 e^{-i\theta} |\psi_2\rangle \tag{77}$$

where η_1, η_2 are real and verify $\eta_1^2 + \eta_2^2 = 1$, and θ is the relative phase between the two condensates.

We now study the properties of such states, called phase states. In the following, we use the simplified expression $|N,\theta\rangle$ for the phase state $|N,\eta_1,\eta_2,\theta\rangle$ corresponding to N bosons in the state $|\eta_1,\eta_2,\theta\rangle$. To find the expression of $|N,\theta\rangle$, we introduce the creation operator $\hat{a}^+_{\eta_1\eta_2,\theta}$ of a boson in the state $|\eta_1,\eta_2,\theta\rangle$:

$$\hat{a}^{+}_{\eta_1 \eta_2 \theta} = \eta_1 \hat{a}^{+}_1 + \eta_2 e^{-i\theta} \hat{a}^{+}_2$$
(78)

We have:

$$|N,\theta\rangle = \frac{1}{\sqrt{N!}} \left[\hat{a}_{\eta_{1}\eta_{2}\theta}^{+} \right]^{N} |0\rangle = \frac{1}{\sqrt{N!}} \left[\eta_{1}\hat{a}_{1}^{+} + \eta_{2} e^{-i\theta}\hat{a}_{2}^{+} \right]^{N} |0\rangle$$
$$= \frac{1}{\sqrt{N!}} \sum_{\substack{n_{1}=0\\n_{2}=N-n_{1}}}^{N} \frac{N!}{n_{1}!n_{2}!} \eta_{1}^{n_{1}} \eta_{2}^{n_{2}} e^{-in_{2}\theta} \underbrace{(\hat{a}_{1}^{+})^{n_{1}}(\hat{a}_{2}^{+})^{n_{2}}|0\rangle}_{=\sqrt{n_{1}!}\sqrt{n_{2}!}|n_{1},n_{2}\rangle}$$
$$|N,\theta\rangle = \sum_{\substack{n_{1}=0\\n_{2}=N-n_{1}}}^{N} \sqrt{\frac{N!}{n_{1}!n_{2}!}} \eta_{1}^{n_{1}} \eta_{2}^{n_{2}} e^{-in_{2}\theta}|n_{1},n_{2}\rangle}$$
(79)

A phase state appears as a linear superposition of Fock states $|n_1, n_2\rangle$ with $n_1 + n_2 = N$ fixed. Equation (79) shows that a phase state is not a product of one state of 'mode' 1 by one of 'mode' 2, but exhibits some quantum correlations between the two condensates. Let us now study the distribution of n_1 and n_2 :

$$\mathcal{P}(n_1) = \frac{N!}{n_1! n_2!} \eta_1^{2n_1} \eta_2^{2n_2} \quad \text{with} \quad \begin{cases} \eta_2^2 = 1 - \eta_1^2 \\ n_2 = N - n_1 \end{cases}$$
(80)

The distribution of n_1 is thus a binomial distribution, with the standard results:

$$\begin{cases} \overline{n}_1 = N\eta_1^2 & \text{and} \quad \overline{n}_2 = N\eta_2^2 = N - \overline{n}_1 \\ (\Delta n_1)^2 = N\eta_1^2 (1 - \eta_1^2) = N\eta_1^2 \eta_2^2 = (\Delta n_2)^2 \end{cases}$$
(81)

To characterize the phase state $|N, \theta\rangle$, it is also interesting to study the distribution of $n = n_1 - n_2$, which can take values from -N to N by steps of 2. Its mean value is $\overline{n} = \overline{n}_1 - \overline{n}_2$, while its dispersion is $\Delta n = 2\Delta n_1 = 2\sqrt{N\eta_1\eta_2}$. The dispersion of n in a phase state is thus large in absolute value, but remains small compared to N.

The entanglement of the two condensates in a phase state appears in the density matrix $\hat{\rho} = |N, \theta\rangle \langle N, \theta|$ as off-diagonal elements coupling states with different values of $n = n_1 - n_2$. The extent of this off-

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diagonality in n is of the order of \sqrt{N} . By contrast, the reduced density matrix $\hat{\rho}^{(1)}$ of the condensate 1 contains only diagonal elements. Indeed:

$$\hat{\rho}^{(1)} = \operatorname{Tr}_{2} |N, \theta\rangle \langle N, \theta| = \sum_{n_{2}} \langle n_{2} | N, \theta\rangle \langle N, \theta | n_{2} \rangle = \sum_{n_{1}} \frac{N!}{n_{1}! (N - n_{1})!} \eta_{1}^{2n_{1}} \eta_{2}^{2(N - n_{1})} |n_{1}\rangle \langle n_{1}|$$

$$\hat{\rho}^{(1)} = \sum_{n_{1}} \mathcal{P}(n_{1}) |n_{1}\rangle \langle n_{1}| \quad \text{with} \quad \mathcal{P}(n_{1}) = \frac{N!}{n_{1}! (N - n_{1})!} \eta_{1}^{2n_{1}} \eta_{2}^{2(N - n_{1})}$$
(82)

Similarly, the condensate 2 considered individually is represented by a statistical mixture of Fock states $|n_2\rangle$ with a binomial distribution for n_2 .

What do the correlation functions of a phase state look like? Let us calculate for instance $G^{(2)}(\mathbf{r},\mathbf{r}') = \langle N,\theta | \hat{\psi}^{\dagger}(\mathbf{r}) \hat{\psi}^{\dagger}(\mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) | N, \theta \rangle$.

We first calculate:

$$\widehat{\psi}(\boldsymbol{r})|N,\theta\rangle = \frac{1}{\sqrt{N!}}\widehat{\psi}(\boldsymbol{r})\left(\widehat{a}_{\eta_{1}\eta_{2}\theta}^{+}\right)^{N}|0\rangle = \frac{1}{\sqrt{N!}}\left[\widehat{\psi}(\boldsymbol{r}),\left(\widehat{a}_{\eta_{1}\eta_{2}\theta}^{+}\right)^{N}\right]|0\rangle$$
(83)

where we have used $\hat{\psi}(\mathbf{r})|0\rangle = 0$. The commutator of equation (83) can be expressed in a more convenient form by using the properties of the creation and annihilation operators:

$$\begin{bmatrix} \widehat{\psi}(\mathbf{r}), \left(\hat{a}_{\eta_{1}\eta_{2}\theta}^{+}\right)^{N} \end{bmatrix} = \begin{bmatrix} \psi_{1}(\mathbf{r})\hat{a}_{1} + \psi_{2}(\mathbf{r})\hat{a}_{2}, \left(\eta_{1}\hat{a}_{1}^{+} + \eta_{2}e^{-i\theta}\hat{a}_{2}^{+}\right)^{N} \end{bmatrix}$$
$$= \left(\psi_{1}(\mathbf{r})\frac{\partial}{\partial\hat{a}_{1}^{+}} + \psi_{2}(\mathbf{r})\frac{\partial}{\partial\hat{a}_{2}^{+}}\right) \left(\eta_{1}\hat{a}_{1}^{+} + \eta_{2}e^{-i\theta}\hat{a}_{2}^{+}\right)^{N}$$
$$= N\left(\eta_{1}\psi_{1}(\mathbf{r}) + \eta_{2}e^{-i\theta}\psi_{2}(\mathbf{r})\right) \left(\eta_{1}\hat{a}_{1}^{+} + \eta_{2}e^{-i\theta}\hat{a}_{2}^{+}\right)^{N-1}$$
$$\left[\widehat{\psi}(\mathbf{r}), \left(\hat{a}_{\eta_{1}\eta_{2}\theta}^{+}\right)^{N}\right] = N\left(\eta_{1}\psi_{1}(\mathbf{r}) + \eta_{2}e^{-i\theta}\psi_{2}(\mathbf{r})\right) \left(\hat{a}_{\eta_{1}\eta_{2}\theta}^{+}\right)^{N-1}$$
(84)

Inserting this expression into equation (83) leads to:

$$\widehat{\psi}(\boldsymbol{r})|N,\theta\rangle = \frac{N}{\sqrt{N!}} \left(\eta_1 \psi_1(\boldsymbol{r}) + \eta_2 e^{-i\theta} \psi_2(\boldsymbol{r})\right) \left(\hat{a}^+_{\eta_1 \eta_2 \theta}\right)^{N-1} |0\rangle$$
$$= \sqrt{N} \left[\eta_1 \psi_1(\boldsymbol{r}) + \eta_2 e^{-i\theta} \psi_2(\boldsymbol{r})\right] |N-1,\theta\rangle$$
(85)

Similar calculations give:

$$\widehat{\psi}(\mathbf{r}')\widehat{\psi}(\mathbf{r})|N,\theta\rangle = \sqrt{N}\sqrt{N-1} \left[\eta_1\psi_1(\mathbf{r}') + \eta_2 e^{-i\theta}\psi_2(\mathbf{r}')\right] \left[\eta_1\psi_1(\mathbf{r}) + \eta_2 e^{-i\theta}\psi_2(\mathbf{r})\right]|N-2,\theta\rangle$$
(86)

Finally, using equation (86) with $\sqrt{N(N-1)} \simeq \sqrt{N^2} = N$, one gets:

$$G^{(2)}(\boldsymbol{r},\boldsymbol{r}') = \psi^*(\boldsymbol{r})\psi^*(\boldsymbol{r}')\psi(\boldsymbol{r}')\psi(\boldsymbol{r}) \quad \text{where}$$
(87)

$$\psi(\mathbf{r}) = \underbrace{\eta_1 \sqrt{N}}_{\sqrt{\overline{n}_1}} \psi_1(\mathbf{r}) + \underbrace{\eta_2 \sqrt{N}}_{\sqrt{\overline{n}_2}} e^{-i\theta} \psi_2(\mathbf{r})$$
(88)

The generalization to all correlation functions $G^{(k)}$ with $k \ll N$ is straightforward: they all factorize in products of 2k functions ψ and ψ^* . We can thus conclude that two interfering macroscopic matter waves $\sqrt{\overline{n}_1}\psi_1(\mathbf{r})$ and $\sqrt{\overline{n}_2}\psi_2(\mathbf{r})e^{-i\theta}$ can be associated with two condensates in a phase state.

Note that the phase states we have studied in this paragraph are a particular kind of quantum states with a well defined relative phase between the two condensates. One can think of other quantum states leading to similar results, for instance a statistical mixture of products of coherent states of the two

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modes $|\alpha_1 = \sqrt{\overline{n_1}} e^{i\theta_1} \rangle \otimes |\alpha_2 = \sqrt{\overline{n_2}} e^{i\theta_2} \rangle$ with $\theta_2 = \theta_1 - \theta$ and θ_1 uniformly distributed in $[0, 2\pi[$. The corresponding density operator is written [20]:

$$\hat{\rho} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta_1 \left| \sqrt{\overline{n}_1} \,\mathrm{e}^{\mathrm{i}\theta_1} \right\rangle \otimes \left| \sqrt{\overline{n}_2} \,\mathrm{e}^{\mathrm{i}(\theta_1 - \theta)} \right\rangle \left\langle \sqrt{\overline{n}_1} \,\mathrm{e}^{\mathrm{i}\theta_1} \right| \otimes \left\langle \sqrt{\overline{n}_2} \,\mathrm{e}^{\mathrm{i}(\theta_1 - \theta)} \right| \tag{89}$$

with \overline{n}_1 , \overline{n}_2 and θ fixed. One can show that $\hat{\rho}$ is diagonal in N and can be rewritten as a statistical mixture of phase states $|N, \theta\rangle$ with θ fixed and N distributed in an interval of size \sqrt{N} around \overline{N} .

We now want to address the question of the conjugate variable of the relative phase θ . We consider a statistical mixture of phase states $|N, \theta\rangle$ with N fixed and θ distributed according to a distribution $W(\theta)$:

$$\hat{\rho} = \int_0^{2\pi} \mathrm{d}\theta \, W(\theta) |N, \theta\rangle \langle N, \theta| \tag{90}$$

Developing $\hat{\rho}$ over the basis of the Fock states $|n_1, n_2\rangle$ (see equation (79)) yields:

$$\hat{\rho} = \sum_{n_1=0}^{N} \sum_{n_1'=0}^{N} \sqrt{\frac{N!}{n_1! n_2!}} \sqrt{\frac{N!}{n_1'! n_2'!}} \eta_1^{n_1+n_1'} \eta_2^{n_2+n_2'} \left[\int_0^{2\pi} \mathrm{d}\theta \, W(\theta) \,\mathrm{e}^{\mathrm{i}(n_2'-n_2)\theta} \right] |n_1, n_2\rangle \langle n_1', n_2'| \tag{91}$$

If one notices that $N = n_1 + n_2 = n'_1 + n'_2$ implies $n'_2 - n_2 = n_1 - n'_1$ and thus $n - n' = n_1 - n_2 - (n'_1 - n'_2) = 2(n'_2 - n_2)$, the integral in equation (91) can be rewritten:

$$\mathcal{I} = \int_0^{2\pi} \mathrm{d}\theta \, W(\theta) \,\mathrm{e}^{\mathrm{i}(n_2' - n_2)\theta} = \int_0^{2\pi} \,\mathrm{d}\theta \, W(\theta) \,\mathrm{e}^{\mathrm{i}(n - n')\theta/2} \tag{92}$$

Equation (92) shows that the integral \mathcal{I} multiplying the off-diagonal elements of $\hat{\rho}$ between two states with different values of $n = n_1 - n_2$ is all the smaller as the phase distribution $W(\theta)$ is broad. In particular, if $W(\theta) = 1/2\pi$ is flat, the integral over θ gives a delta function $\delta_{nn'}$ and $\hat{\rho}$ is diagonal, not only in N, but also in $n = n_1 - n_2$ which thus appears as the conjugate variable of the relative phase θ .

3.3.2. Emergence of a relative phase as a result of detection processes

Generally, two independent condensates do not find themselves in a state with a well-defined relative phase, but rather in a product of Fock states $|N_1, N_2\rangle = |N_1\rangle \otimes |N_2\rangle$. Such a state corresponds to a zero dispersion on $n = n_1 - n_2$, and thus to a totally undefined relative phase.

However, the detection processes induce a dispersion on n, so that the relative phase of the two condensates becomes better and better known. Indeed, the first detected boson can come either from mode 1 or from mode 2. After this detection, the state vector becomes:

$$|\psi\rangle = \alpha |N_1 - 1, N_2\rangle + \beta |N_1, N_2 - 1\rangle \tag{93}$$

where α and β are coefficients depending on the position of the first detected boson.

Similarly, the second detection process changes the state vector into:

$$|\psi\rangle = \lambda |N_1 - 2, N_2\rangle + \mu |N_1 - 1, N_2 - 1\rangle + \nu |N_1, N_2 - 2\rangle$$
(94)

The off-diagonality of $\hat{\rho} = |\psi\rangle\langle\psi|$ in $n = n_1 - n_2$ increases with the number of detected bosons and a relative phase θ builds up. Note that θ is an unpredictable random variable and takes different values from one experimental realization to the other.

More precisely, the emergence of a relative phase between two condensates has been studied analytically by Y. Castin and J. Dalibard [21] and numerically by J. Javanainen and S.M. Yoo [22] as well as by the groups of P. Zoller [20] and of D.F. Walls [23].

4. Beyond the variational approximation

In Section 3, we have described the condensate within the variational approximation. This approach gives an approximate expression only for the ground state, but does not yield any information about the excited states or the elementary excitations. Besides, one may wonder if the approximation of the ground state by a product state is sufficient, and what the first corrections to this treatment are. Indeed, the interpretation of some physical effects requires to go beyond the product state description, as we will see in Section 5 for the total intensity of the light scattered by a condensate.

In this paragraph, we briefly review the Bogolubov treatment [24] for a homogeneous condensate, which gives analytical results (see also [6], Chapter 19). This approach can be extended to a gas of bosons in a trap [25].

The Bogolubov approach gives accurate results when $T \ll T_c$ and $\rho a^3 \ll 1$, with ρ the spatial density and a the scattering length. Let us first write the second quantized Hamiltonian \hat{H} , using the field operators $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ for a gas of bosons in a box of size L (see equation (31)):

$$\widehat{H} = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}} + \widehat{V}_{\text{int}} \quad \text{with}$$

$$\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m} \quad \text{and} \quad \widehat{V}_{\text{int}} = \frac{1}{2} \iint d^3 \boldsymbol{r} \, d^3 \boldsymbol{r}' \, \widehat{\psi}^{\dagger}(\boldsymbol{r}) \widehat{\psi}^{\dagger}(\boldsymbol{r}') V\big(|\boldsymbol{r} - \boldsymbol{r}'|\big) \widehat{\psi}(\boldsymbol{r}') \widehat{\psi}(\boldsymbol{r})$$
(95)

If one uses the approximate expression of the interaction potential $V(|\mathbf{r} - \mathbf{r}'|) = g\delta(\mathbf{r} - \mathbf{r}')$ with $g = 4\pi\hbar^2 a/m$, one can write:

$$\widehat{V}_{\text{int}} = \frac{g}{2L^3} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}} \hat{a}^+_{\mathbf{k}_1} \hat{a}^+_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_2 + \mathbf{k}} \hat{a}_{\mathbf{k}_1 - \mathbf{k}}$$
(96)

We now use the fact that for $T \ll T_c$ the bosons are essentially condensed in $\mathbf{k} = \mathbf{0}$, i.e. $\langle \hat{a}_{\mathbf{0}}^+ \hat{a}_{\mathbf{0}} \rangle \gg \sum_{\mathbf{k}\neq\mathbf{0}} \langle \hat{a}_{\mathbf{k}}^+ \hat{a}_{\mathbf{k}} \rangle$. One can transform the Hamiltonian \hat{H} into an approximate Hamiltonian \hat{H}_{eff} by neglecting all terms of \hat{H} that contain more than two operators $\hat{a}_{\mathbf{k}}$ or $\hat{a}_{\mathbf{k}}^+$ with $\mathbf{k}\neq\mathbf{0}$ and by replacing $\hat{a}_{\mathbf{0}}$ and $\hat{a}_{\mathbf{0}}^+$ by \sqrt{N} .

A special treatment has to be applied to the term $\hat{a}_{0}^{+}\hat{a}_{0}^{+}\hat{a}_{0}\hat{a}_{0}$ in order to take into account first order terms of the form $n_{k} = \hat{a}_{k}^{+}\hat{a}_{k}$ (with $k \neq 0$):

$$\hat{a}_{0}^{+} \hat{a}_{0}^{+} \hat{a}_{0} \hat{a}_{0} = \hat{a}_{0}^{+} (\hat{a}_{0} \hat{a}_{0}^{+} - 1) \hat{a}_{0} = (\hat{a}_{0}^{+} \hat{a}_{0})^{2} - \hat{a}_{0}^{+} \hat{a}_{0} = \left(N - \sum_{k \neq 0} \hat{a}_{k}^{+} \hat{a}_{k}\right)^{2} - \left(N - \sum_{k \neq 0} \hat{a}_{k}^{+} \hat{a}_{k}\right)$$
$$\simeq N(N-1) - (2N-1) \sum_{k \neq 0} \hat{a}_{k}^{+} \hat{a}_{k}$$
$$\hat{a}_{0}^{+} \hat{a}_{0}^{+} \hat{a}_{0} \hat{a}_{0} \simeq N^{2} - 2N \sum_{k \neq 0} \hat{a}_{k}^{+} \hat{a}_{k}$$
(97)

Finally, using $N/L^3 = \rho$, one gets:

$$\widehat{H}_{\text{eff}} = \frac{1}{2}g\rho N + \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}} + \frac{1}{2}g\rho \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \left[\hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{\boldsymbol{k}} + \hat{a}_{-\boldsymbol{k}}^{\dagger} \hat{a}_{-\boldsymbol{k}} + \hat{a}_{\boldsymbol{k}}^{\dagger} \hat{a}_{-\boldsymbol{k}}^{\dagger} + \hat{a}_{\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}} \right]$$
(98)

A rapid inspection of \hat{H}_{eff} shows that this Hamiltonian conserves the total momentum. It is a quadratic function of \hat{a}_{k}^{+} and \hat{a}_{k} and can therefore be diagonalized by introducing normal modes.

To find these modes, let us write the Heisenberg equation for \hat{a}_{k} :

$$i\hbar\dot{\hat{a}}_{\boldsymbol{k}} = \left[\hat{a}_{\boldsymbol{k}}, \widehat{H}_{\text{eff}}\right] = \frac{\partial\widehat{H}_{\text{eff}}}{\partial\hat{a}_{\boldsymbol{k}}^{+}}, \qquad i\hbar\dot{\hat{a}}_{\boldsymbol{k}} = (\varepsilon_{k} + g\rho)\hat{a}_{\boldsymbol{k}} + g\rho\hat{a}_{-\boldsymbol{k}}^{+}$$
(99)

As the evolution of \hat{a}_{k} is coupled to \hat{a}^{+}_{-k} , we also calculate \dot{a}^{+}_{-k} :

$$i\hbar \dot{\hat{a}}_{-\boldsymbol{k}}^{+} = \left[\hat{a}_{-\boldsymbol{k}}^{+}, \widehat{H}_{\text{eff}}\right] = -\frac{\partial H_{\text{eff}}}{\partial \hat{a}_{-\boldsymbol{k}}}, \qquad i\hbar \dot{\hat{a}}_{-\boldsymbol{k}}^{+} = -(\varepsilon_{k} + g\rho)\hat{a}_{-\boldsymbol{k}}^{+} - g\rho\hat{a}_{\boldsymbol{k}}$$
(100)

Equations (99) and (100) define a closed linear system of equations for \hat{a}_k and \hat{a}^+_{-k} . As a consequence, there are two linear combinations of \hat{a}_k and \hat{a}^+_{-k} which evolve independently from each other at frequencies $\omega(k)$ given by the eigenvalues $\pm \omega(k)$ of the matrix:

$$\frac{1}{\hbar} \begin{pmatrix} \varepsilon_k + g\rho & g\rho \\ -g\rho & -(\varepsilon_k + g\rho) \end{pmatrix}$$
(101)

One finds:

$$\hbar\omega(k) = \sqrt{\varepsilon_k(\varepsilon_k + 2g\rho)} = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2g\rho\right)}$$
(102)

Let us now write explicitly the eigenvectors \hat{b}_{k} and \hat{b}_{-k}^{+} of the linear system of equations (99) and (100):

$$\hat{b}_{\boldsymbol{k}} = u_k \hat{a}_{\boldsymbol{k}} + v_k \hat{a}^+_{-\boldsymbol{k}} \quad \text{and} \quad \hat{b}^+_{-\boldsymbol{k}} = v_k \hat{a}_{\boldsymbol{k}} + u_k \hat{a}^+_{-\boldsymbol{k}}$$
(103)

where u_k and v_k (which are real and depend only on $k = |\mathbf{k}|$) are normalized to ensure $[\hat{b}_k, \hat{b}_k^+] = 1$. This condition yields $u_k^2 - v_k^2 = 1$, which allows one to write:

$$u_k = \cosh \theta_k \quad \text{and} \quad v_k = \sinh \theta_k \tag{104}$$

Diagonalizing the matrix (101), one gets:

$$\tanh 2\theta_k = \frac{\rho g}{\varepsilon_k + \rho g} \tag{105}$$

The system of equation (103) can be inverted, which gives:

$$\hat{a}_{k} = \cosh\theta_{k}\hat{b}_{k} - \sinh\theta_{k}\hat{b}_{-k}^{+} \tag{106}$$

$$\hat{a}_{-\boldsymbol{k}}^{+} = -\sinh\theta_{k}\hat{b}_{\boldsymbol{k}} + \cosh\theta_{k}\hat{b}_{-\boldsymbol{k}}^{+} \tag{107}$$

We now write and study the effective Hamiltonian \hat{H}_{eff} in terms of \hat{b}_k and \hat{b}_k^+ . Inserting equations (106) and (107) into (98) and using (105), one gets:

$$\widehat{H}_{\text{eff}} = E_0 + \sum_{\boldsymbol{k}} \hbar \omega(k) \widehat{b}_{\boldsymbol{k}}^+ \widehat{b}_{\boldsymbol{k}}$$
(108)

• The operators \hat{b}_{k}^{+} and \hat{b}_{k} respectively create and annihilate an elementary excitation of the system with an energy $\hbar\omega(k)$. The dispersion relation (see equation (102)):

$$\hbar\omega(k) = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2g\rho\right)}$$

can be simplified by introducing a wave vector k_0 defined by

$$g\rho = \frac{4\pi\hbar^2}{m}a\rho = \frac{\hbar^2 k_0^2}{2m}$$

The wave vector $k_0 = \sqrt{8\pi a\rho}$ is associated with a length ξ_0 called *healing length*:

$$\xi_0 = \frac{1}{k_0} = \frac{1}{\sqrt{8\pi a\rho}}$$
(109)

Equation (102) can then be written:

$$\omega(k) = \frac{\hbar}{2m} \sqrt{k^2 (k^2 + 2k_0^2)} \tag{110}$$

- For $k \ll k_0$, equation (110) gives a phonon type spectrum $\omega(k) = ck$ where:

$$c = \frac{1}{\sqrt{2}} \frac{\hbar k_0}{m} = \sqrt{\frac{g\rho}{m}} = \sqrt{\frac{\mu}{m}}$$
(111)

and $\mu = g\rho$ is the chemical potential.

- For $k \gg k_0$, the spectrum resembles a free particle spectrum but for a constant offset:

$$\omega(k) = \frac{\hbar k^2}{2m} + \frac{\hbar k_0^2}{2m} = \frac{\hbar k^2}{2m} + \frac{g\rho}{\hbar}$$

We show in *figure 4* the dispersion relation for the elementary excitations in the Bogolubov theory.

• The ground state energy E_0 of the system can be calculated as:

$$E_0 = \frac{1}{2}g\rho N + \sum_{\boldsymbol{k}} \varepsilon_k v_k^2 + \sum_{\boldsymbol{k}\neq\boldsymbol{0}} g\rho \left(v_k^2 - u_k v_k\right)$$
(112)





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However, the approximation of the interaction potential \hat{V}_{int} by a δ -function leads to a divergence of the sum over k in equation (112). A regularized expression of the approximate interaction potential has to be used to overcome this difficulty (see [6], Chapter 13). One then gets ([6], Section 19.4):

$$\frac{E_0}{N} = \frac{1}{2}g\rho \left[1 + \frac{128}{15\sqrt{\pi}} (\rho a^3)^{1/2} \right]$$
(113)

It is also interesting to have a physical insight into the ground state |φ₀⟩ of the system in the Bogolubov theory. |φ₀⟩ must verify b̂_k|φ₀⟩ = 0 for all k ≠ 0. The expression for b̂_k is given by equation (103). However, one should pay attention to the fact that b̂_k⁺ and b̂_k respectively create and annihilate an elementary excitation, but should conserve the total number N of particles. By replacing â₀⁺ and â₀ by √N, we have lost the explicit conservation of N. To overcome this difficulty, we use an improved expression for b̂_k⁺ and b̂_k, namely:

$$\hat{b}_{\boldsymbol{k}}^{+} = \cosh\theta_{k}\,\hat{a}_{\boldsymbol{k}}^{+}\frac{\hat{a}_{\boldsymbol{0}}}{\sqrt{N}} + \sinh\theta_{k}\,\frac{\hat{a}_{\boldsymbol{0}}^{+}}{\sqrt{N}}\hat{a}_{-\boldsymbol{k}} \tag{114}$$

$$\hat{b}_{\boldsymbol{k}} = \cosh\theta_k \frac{\hat{a}_{\boldsymbol{0}}^+}{\sqrt{N}} \hat{a}_{\boldsymbol{k}} + \sinh\theta_k \, \hat{a}_{-\boldsymbol{k}}^+ \frac{\hat{a}_{\boldsymbol{0}}}{\sqrt{N}} \tag{115}$$

Using the expression (115) for \hat{b}_{k} , the ground state $|\phi_{0}\rangle$ has to verify:

$$\hat{b}_{\boldsymbol{k}}|\phi_0\rangle = \frac{1}{\sqrt{N}} \left[\cosh\theta_k \,\hat{a}^{\dagger}_{\boldsymbol{0}} \hat{a}_{\boldsymbol{k}} + \sinh\theta_k \,\hat{a}^{\dagger}_{-\boldsymbol{k}} \hat{a}_{\boldsymbol{0}}\right] |\phi_0\rangle = 0 \quad \forall \, \boldsymbol{k} \neq \boldsymbol{0}$$
(116)

To go further, we assume a ground state $|\phi_0\rangle$ of the form:

$$|\phi_0\rangle = \alpha |n_0 = N, 0, 0, \ldots\rangle + \sum_{k \neq 0} \beta_k |n_0 = N - 2, n_k = 1, n_{-k} = 1, 0, \ldots\rangle + \cdots$$
 (117)

In this expression, the terms $|n_0 = N - 2, n_k = 1, n_{-k} = 1, 0, ...\rangle$ correspond to the virtual excitation of pairs of particles +k, -k when two particles with k = 0 interact, with conservation of the total momentum. Note that the structure of the ground state written in equation (117) can be deduced from a general expression, using the condition (116) (see [6], Section 19.4).

Inserting equation (117) into equation (116) and writing $\sqrt{(N-1)/N} \simeq 1$, one gets the relation between the coefficients α and β_k :

$$\beta_{k} = -\alpha \tanh \theta_{k} \tag{118}$$

After normalization, the ground state is totally determined by equation (118). Compared with the approximate ground state considered in the variational approximation, the Bogolubov approach accounts for the fact that pairs of atoms are transferred from the state with k = 0 to states k, -k with $k \neq 0$ under the effect of atomic interactions.

If one calculates the mean number of particles $N - N_0$ out of the k = 0 state, one finds the quantum depletion $(N - N_0)/N$ of the ground state:

$$\frac{N - N_0}{N} = \frac{8}{3\sqrt{\pi}} \left(\rho a^3\right)^{1/2} \tag{119}$$

We finally want to stress that equations (113) and (119) explicitly give the lowest order term in the small dimensionless parameter ρa^3 .

5. Probing a condensate

In this section, we study detection signals obtained from the scattering of a probe particle by the N-boson system. Generally, scattering experiments (neutron scattering for instance) are widely used to investigate the properties of N-body systems. Indeed, observing how a probe particle is scattered by a given system yields some information about its spatial structure, its elementary excitation spectrum, etc.

In this paper we discuss only the *linear regime*, where the excitation due to the probe can be treated perturbatively. This means that the system's response is entirely determined by its properties in the absence of the probe. As usual in the weak perturbation regime, we neglect multiple scattering and we consider each scattering process as independent from the other ones. In this regime the linear response theory applies, so that we can define a dynamic structure factor $S(q, \omega)$ and a static structure factor S(q) [26] to describe the response of the system.

We first derive the dynamic and static structure factors within the Born approximation, similarly to what was done in the context of superfluid helium [26]. We then discuss the origin of interference in detection signals.

5.1. Probing the wave function of the condensate

Let us consider a probe particle with position R interacting with each boson *i* through an interaction potential $U(R - r_i)$. We first write the interaction Hamiltonian \hat{H}_{int} in the first quantization form:

$$\widehat{H}_{int} = \sum_{i=1}^{N} U(\boldsymbol{R} - \hat{\boldsymbol{r}}_{i}) = \int d^{3}\boldsymbol{r} U(\boldsymbol{R} - \boldsymbol{r}) \sum_{i=1}^{N} \delta(\boldsymbol{r} - \hat{\boldsymbol{r}}_{i})$$
$$\widehat{H}_{int} = \int d^{3}\boldsymbol{r} U(\boldsymbol{R} - \boldsymbol{r}) \hat{\rho}_{I}(\boldsymbol{r})$$
(120)

where $\hat{\rho}_{I}(\mathbf{r}) = \sum_{i=1}^{N} \delta(\mathbf{r} - \hat{\mathbf{r}}_{i})$ is the one-body spatial density. In second quantization, we can also write $\hat{\rho}_{I}(\mathbf{r}) = \hat{\psi}^{\dagger}(\mathbf{r})\hat{\psi}(\mathbf{r})$. Using the basis $\{|\mathbf{k}\rangle\}$ of single particle plane waves $e^{i\mathbf{k}\cdot\mathbf{r}}/L^{3/2}$ in a box of size L with periodic boundary conditions, the one-particle density $\hat{\rho}_{I}(\mathbf{r})$ can be rewritten as:

$$\hat{\rho}_{\mathrm{I}}(\boldsymbol{r}) = \frac{1}{L^3} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} \hat{a}_{\boldsymbol{k}'}^{\dagger} \hat{a}_{\boldsymbol{k}} = \frac{1}{L^3} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{K}} \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{r}} \hat{a}_{\boldsymbol{k}+\boldsymbol{K}}^{\dagger} \hat{a}_{\boldsymbol{k}}$$
(121)

In this expression, $\sum_{k} \hat{a}^{+}_{k+K} \hat{a}_{k}$ appears as the Fourier transform $\hat{\rho}(K) = \int d^{3}r e^{iK \cdot r} \hat{\rho}_{I}(r)$ of $\hat{\rho}_{I}(r)$, which is written as $\hat{\rho}(K) = \sum_{i=1}^{N} e^{iK \cdot \hat{r}_{i}}$ in first quantization. In second quantization, using equation (23) with $\hat{f} = e^{iK \cdot \hat{r}}$ gives:

$$\hat{\rho}(\boldsymbol{K}) = \sum_{i=1}^{N} e^{i\boldsymbol{K}\cdot\hat{\boldsymbol{r}}_{i}} = \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} \frac{1}{L^{3}} \underbrace{\left[\int d^{3}\boldsymbol{r} \, e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}} \, e^{i\boldsymbol{K}\cdot\boldsymbol{r}} \, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \right]}_{L^{3}\delta(\boldsymbol{k}'-\boldsymbol{K}-\boldsymbol{k})} \hat{a}_{\boldsymbol{k}'}^{+} \hat{a}_{\boldsymbol{k}}$$

$$\hat{\rho}(\boldsymbol{K}) = \sum_{i=1}^{N} e^{i\boldsymbol{K}\cdot\hat{\boldsymbol{r}}_{i}} = \sum_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}+\boldsymbol{K}}^{+} \hat{a}_{\boldsymbol{k}}$$
(122)

Inserting equation (121) into (120) and using equation (122) leads to:

$$\widehat{H}_{\text{int}} = \sum_{\boldsymbol{k}} \sum_{\boldsymbol{K}} e^{-i\boldsymbol{K}\cdot\boldsymbol{R}} \hat{a}^{+}_{\boldsymbol{k}+\boldsymbol{K}} \hat{a}_{\boldsymbol{k}+\boldsymbol{K}} \hat{a}_{\boldsymbol{k}} \frac{1}{L^{3}} \underbrace{\int d^{3}\boldsymbol{r} \, U(\boldsymbol{R}-\boldsymbol{r}) \, e^{i\boldsymbol{K}\cdot(\boldsymbol{R}-\boldsymbol{r})}}_{=\mathcal{U}(\boldsymbol{K})} = \frac{1}{L^{3}} \sum_{\boldsymbol{K}} \mathcal{U}(\boldsymbol{K}) \, e^{-i\boldsymbol{K}\cdot\boldsymbol{R}} \sum_{\boldsymbol{k}} \hat{a}^{+}_{\boldsymbol{k}+\boldsymbol{K}} \hat{a}_{\boldsymbol{k}}$$

$$\widehat{H}_{\text{int}} = \frac{1}{L^3} \sum_{\mathbf{K}} \mathcal{U}(\mathbf{K}) \,\mathrm{e}^{-\mathrm{i}\mathbf{K}\cdot\mathbf{R}} \widehat{\rho}(\mathbf{K}) \tag{123}$$

In this expression, $\mathcal{U}(\mathbf{K})$ is the Fourier transform of $U(\mathbf{r})$.

5.1.1. Scattering of a probe particle by the system of N bosons

Let us consider a scattering process where the probe particle is scattered from an initial state with momentum $p_i = \hbar k_i$ and energy ε_i to a final state with momentum $p_f = \hbar k_f$ and energy ε_f . Meanwhile the N-boson system changes from an initial state ϕ_i having an energy E_i to a final state ϕ_f with an energy E_f .

In order to evaluate the scattering amplitude associated with this process, we first calculate the matrix element of \hat{H}_{int} between the initial and final states $|\mathbf{k}_i, \phi_i\rangle$ and $|\mathbf{k}_f, \phi_f\rangle$ of the global system:

$$\langle \boldsymbol{k}_{f}, \phi_{f} | \widehat{H}_{\text{int}} | \boldsymbol{k}_{i}, \phi_{i} \rangle = \langle \boldsymbol{k}_{f}, \phi_{f} | \frac{1}{L^{3}} \sum_{\boldsymbol{K}} \mathcal{U}(\boldsymbol{K}) e^{-i\boldsymbol{K}\cdot\boldsymbol{R}} \hat{\rho}(\boldsymbol{K}) | \boldsymbol{k}_{i}, \phi_{i} \rangle$$
$$= \sum_{\boldsymbol{K}} \mathcal{U}(\boldsymbol{K}) \langle \phi_{f} | \hat{\rho}(\boldsymbol{K}) | \phi_{i} \rangle \delta(\boldsymbol{k}_{i} - \boldsymbol{k}_{f} - \boldsymbol{K})$$
$$\langle \boldsymbol{k}_{f}, \phi_{f} | \widehat{H}_{\text{int}} | \boldsymbol{k}_{i}, \phi_{i} \rangle = \mathcal{U}(\boldsymbol{q}) \langle \phi_{f} | \hat{\rho}(\boldsymbol{q}) | \phi_{i} \rangle \quad \text{with} \quad \boldsymbol{q} = \boldsymbol{k}_{i} - \boldsymbol{k}_{f}$$
(124)

Let us now analyze the structure of equation (124), where the δ -function $\delta(\mathbf{k}_i - \mathbf{k}_f - \mathbf{K})$ accounts for the momentum conservation. The term $\mathcal{U}(\mathbf{q})$ describes the diffusion of the probe particle by a single boson, independently from the quantum state of the N-boson system. By contrast, the last term $\langle \phi_f | \hat{\rho}(\mathbf{q}) | \phi_i \rangle$, which does not depend on the interaction potential, corresponds to the interference effects between the contributions of the different bosons of the target.

Within the Born approximation, the transition rate r_{if} between the initial and the final state is given by the Fermi golden rule:

$$r_{if} = \frac{2\pi}{\hbar} \left| \mathcal{U}(\boldsymbol{q}) \right|^2 \left| \langle \phi_f | \hat{\rho}(\boldsymbol{q}) | \phi_i \rangle \right|^2 \delta(E_f - E_i - \hbar\omega)$$
(125)

where $\hbar \omega = \varepsilon_i - \varepsilon_f$ is the energy transfer from the probe to the N-boson system.

The formalism that we have developed here to describe the scattering of a probe particle can be extended to Bragg scattering, where the absorption of a photon of energy $\hbar\omega_1$ is followed by the stimulated emission of a photon of energy $\hbar\omega_2$. In this process, the energy and momentum transfers write $\hbar\omega = \hbar(\omega_1 - \omega_2)$ and $\boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_2$, respectively. One can scan ω by keeping for instance ω_1 fixed and scanning ω_2 . Strictly speaking, one also modifies \boldsymbol{q} . However, in the usual case where the energy transfer is small ($\omega \ll \omega_1, \omega_2$), the momentum transfer is essentially determined by the angle between \boldsymbol{k}_1 and \boldsymbol{k}_2 , and depends only weakly on the change of $|\boldsymbol{k}_2|$. In the following, we will therefore neglect the dependence of \boldsymbol{q} upon ω .

5.1.2. Dynamic and static structure factors

The dynamic structure factor $S(q, \omega)$ corresponds to the terms in equation (125) that depend only on the N-boson system. More precisely, we have:

$$S(\boldsymbol{q},\omega) = \sum_{f} \left| \langle \phi_f | \hat{\rho}(\boldsymbol{q}) | \phi_i \rangle \right|^2 \delta(E_i - E_f + \hbar\omega)$$
(126)

In this expression, we sum over the final states of the bosonic system to account for the fact that we measure only the final state of the scattered particle. Equation (126) can be easily generalized to the case when the initial state is a statistical mixture of states ϕ_i with weights π_i :

$$S(\boldsymbol{q},\omega) = \sum_{i} \pi_{i} \sum_{f} \left| \langle \phi_{f} | \hat{\rho}(\boldsymbol{q}) | \phi_{i} \rangle \right|^{2} \delta(E_{i} - E_{f} + \hbar\omega)$$
(127)

The variations of $S(q, \omega)$ with ω are directly related to the dependence of the scattering cross-section on ω in the direction q. This gives access to the energy spectrum of the elementary excitations.

The static structure factor $S(q) = \int d(\hbar\omega) S(q,\omega)$ represents the total scattered intensity for a given momentum transfer q. S(q) can be rewritten:

$$S(\boldsymbol{q}) = \int \mathrm{d}(\hbar\omega) \, S(\boldsymbol{q},\omega) = \sum_{f} \left| \langle \phi_{f} | \hat{\rho}(\boldsymbol{q}) | \phi_{i} \rangle \right|^{2}, \qquad S(\boldsymbol{q}) = \langle \phi_{i} | \hat{\rho}^{\dagger}(\boldsymbol{q}) \hat{\rho}(\boldsymbol{q}) | \phi_{i} \rangle \tag{128}$$

This shows that S(q) depends only on ϕ_i and q. The static structure factor thus yields some information on ϕ_i .

We now express the structure factors $S(q, \omega)$ and S(q) in terms of correlation functions. Indeed, one can write the δ -function of equation (126) as $\delta(E_i - E_f + \hbar\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp[-i(E_i - E_f - \hbar\omega)t/\hbar] dt$, hence:

$$S(\boldsymbol{q},\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t \,\mathrm{e}^{-\mathrm{i}\omega t} \sum_{f} \langle \phi_i | \hat{\rho}^{\dagger}(\boldsymbol{q}) | \phi_f \rangle \underbrace{\langle \phi_f | \hat{\rho}(\boldsymbol{q}) | \phi_i \rangle \,\mathrm{e}^{\mathrm{i}(E_f - E_i)t/\hbar}}_{\langle \phi_f | \,\mathrm{e}^{\mathrm{i}\widehat{H}t/\hbar}\hat{\rho}(\boldsymbol{q}) \,\mathrm{e}^{-\mathrm{i}\widehat{H}t/\hbar} | \phi_i \rangle}$$
$$S(\boldsymbol{q},\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t \,\mathrm{e}^{-\mathrm{i}\omega t} \langle \phi_i | \hat{\rho}^{\dagger}(\boldsymbol{q},t=0) \hat{\rho}(\boldsymbol{q},t) | \phi_i \rangle$$
(129)

In this expression, $S(\boldsymbol{q},\omega)$ appears as the time Fourier transform of the average value in ϕ_i of the product of two Heisenberg operators $\hat{\rho}(\boldsymbol{q})$ taken at two different times. Noting that the product of Fourier transforms $\hat{\rho}^{\dagger}(\boldsymbol{q},t=0)\hat{\rho}(\boldsymbol{q},t)$ can be expressed as the Fourier transform of the corresponding convolution product, one can also consider $S(\boldsymbol{q},\omega)$ as the spatio-temporal Fourier transform of $\int d^3 \boldsymbol{r}' \langle \phi_i | \hat{\rho}_{\rm I}(\boldsymbol{r}',t=0) \times \hat{\rho}_{\rm I}(\boldsymbol{r}'+\boldsymbol{r},t) | \phi_i \rangle$.

Similarly, S(q) appears as the spatial Fourier transform of $\int d^3 \mathbf{r}' \langle \phi_i | \hat{\rho}_{\mathrm{I}}(\mathbf{r}') \hat{\rho}_{\mathrm{I}}(\mathbf{r}'+\mathbf{r}) | \phi_i \rangle$. Let us now write the one-body density operators in terms of field operators (27):

$$\hat{\rho}_{\mathrm{I}}(\mathbf{r}')\hat{\rho}_{\mathrm{I}}(\mathbf{r}'+\mathbf{r}) = \hat{\psi}^{\dagger}(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}^{\dagger}(\mathbf{r}+\mathbf{r}')\hat{\psi}(\mathbf{r}+\mathbf{r}') = \hat{\psi}^{\dagger}(\mathbf{r}')\left[\hat{\psi}^{\dagger}(\mathbf{r}+\mathbf{r}')\hat{\psi}(\mathbf{r}')+\delta(\mathbf{r})\right]\hat{\psi}(\mathbf{r}+\mathbf{r}')$$

$$\hat{\rho}_{\mathrm{I}}(\mathbf{r}')\hat{\rho}_{\mathrm{I}}(\mathbf{r}'+\mathbf{r}) = \delta(\mathbf{r})\hat{\psi}^{\dagger}(\mathbf{r}')\hat{\psi}(\mathbf{r}') + \hat{\psi}^{\dagger}(\mathbf{r}')\hat{\psi}^{\dagger}(\mathbf{r}+\mathbf{r}')\hat{\psi}(\mathbf{r}+\mathbf{r}')\hat{\psi}(\mathbf{r}')$$
(130)

Finally, one gets:

$$\int \mathrm{d}^{3}\boldsymbol{r}' \langle \phi_{i} | \hat{\rho}_{\mathrm{I}}(\boldsymbol{r}') \hat{\rho}_{\mathrm{I}}(\boldsymbol{r}'+\boldsymbol{r}) | \phi_{i} \rangle = N \delta(\boldsymbol{r}) + \int \mathrm{d}^{3}\boldsymbol{r}' G^{(2)}(\boldsymbol{r}',\boldsymbol{r}'+\boldsymbol{r})$$
(131)

where the last term represents the total probability to find in the state $|\phi_i\rangle$ two particles separated by r.

5.1.3. Static structure factor of a homogeneous condensate

We now calculate the static structure factor S(q) in the simple case of a homogeneous condensate, both in the variational approximation and using the Bogolubov theory. We then compare these calculations to experimental results.

Within the variational approximation, the ground state of a homogeneous condensate is the solution of the Gross–Pitaevskii equation (61). Because of the translational invariance of the system, the ground state is the state k = 0, so that the initial quantum state of the condensate is |φ_i⟩ = |n₀ = N, n_i = 0 for i ≠ 0⟩. In order to evaluate S(q) = ⟨φ_i|ρ[†](q)ρ̂(q)|φ_i⟩, we calculate:

$$\hat{\rho}(\boldsymbol{q})|\phi_i\rangle = \sum_{\boldsymbol{k}} \hat{a}^+_{\boldsymbol{k}+\boldsymbol{q}} \underbrace{\hat{a}_{\boldsymbol{k}}|\phi_i\rangle}_{\text{if } \substack{=0\\ \boldsymbol{k}\neq\boldsymbol{0}}} = \hat{a}^+_{\boldsymbol{q}} \hat{a}_{\boldsymbol{0}}|\phi_i\rangle = \sqrt{N}|n_0 = N - 1, n_{\boldsymbol{q}} = 1\rangle$$
(132)

This yields immediately:

$$S(\boldsymbol{q}) = N \tag{133}$$

• We now calculate the static structure factor in the framework of the *Bogolubov theory*, which is valid for $T \ll T_c$ and $\rho a^3 \ll 1$. Looking for an approximate expression for $\hat{\rho}(\boldsymbol{q})$, we can neglect in $\hat{\rho}^{\dagger}(\boldsymbol{q})\hat{\rho}(\boldsymbol{q})$ all terms containing less than two operators \hat{a}_0 or \hat{a}_0^+ , similarly to what we have done in Section 4. We can also replace \hat{a}_0 and \hat{a}_0^+ by \sqrt{N} . Such a treatment amounts to neglecting in $\hat{\rho}(\boldsymbol{q})$ all terms containing less than one operator \hat{a}_0 or \hat{a}_0^+ :

$$\hat{\rho}(\boldsymbol{q}) = \sum_{\boldsymbol{k}} \hat{a}^{+}_{\boldsymbol{k}+\boldsymbol{q}} \hat{a}_{\boldsymbol{k}} \simeq \hat{a}^{+}_{\boldsymbol{q}} \hat{a}_{0} + \hat{a}^{+}_{0} \hat{a}_{-\boldsymbol{q}} \simeq \sqrt{N} (\hat{a}^{+}_{\boldsymbol{q}} + \hat{a}_{-\boldsymbol{q}})$$
(134)

We now express $\hat{\rho}(q)$ in terms of the operators \hat{b}_{k}^{+} and \hat{b}_{k} creating and annihilating an elementary excitation with momentum k. Using equations (106) and (107), one gets from equation (134):

$$\hat{\rho}(\boldsymbol{q}) \simeq \sqrt{N} \left[(\cosh \theta_q - \sinh \theta_q) \hat{b}^+_{\boldsymbol{q}} + (\cosh \theta_q - \sinh \theta_q) \hat{b}_{-\boldsymbol{q}} \right]$$
(135)

We have already seen that the calculation of S(q) involves that of $\hat{\rho}(q)|\phi_i\rangle$. If we assume the initial state $|\phi_i\rangle$ of the condensate to be the Bogolubov ground state $|\phi_0\rangle$, which verifies $\hat{b}_k |\phi_0\rangle = 0 \forall k$, then we get:

$$\hat{\rho}(\boldsymbol{q})|\phi_0\rangle = \sqrt{N}(\cosh\theta_q - \sinh\theta_q)\hat{b}^+_{\boldsymbol{q}}|\phi_0\rangle \tag{136}$$

Hence:

$$S(\boldsymbol{q}) = N(\cosh\theta_q - \sinh\theta_q)^2 \langle \phi_0 | \hat{b}_{\boldsymbol{q}} \hat{b}_{\boldsymbol{q}}^+ | \phi_0 \rangle = N(\cosh\theta_q - \sinh\theta_q)^2 \langle \phi_0 | 1 - \hat{b}_{\boldsymbol{q}}^+ \hat{b}_{\boldsymbol{q}} | \phi_0 \rangle$$

$$S(\boldsymbol{q}) = N(\cosh\theta_q - \sinh\theta_q)^2$$
(137)

Using equation (105), we can write the explicit dependence of S(q) on q:

$$S(\boldsymbol{q}) = N(\cosh 2\theta_q - \sinh 2\theta_q) = \frac{N}{\sqrt{1 - \tanh^2 2\theta_q}} (1 - \tanh 2\theta_q) = N\sqrt{\frac{1 - \tanh 2\theta_q}{1 + \tanh 2\theta_q}}$$
$$S(\boldsymbol{q}) = N\sqrt{\frac{\varepsilon_q}{\varepsilon_q + 2\rho g}} \quad \text{with} \quad \varepsilon_q = \frac{\hbar^2 q^2}{2m}$$
(138)

In the limit $q \to \infty$, the condensate is in the free-particle regime where $\varepsilon_q \gg \rho g = \mu$ (μ being the chemical potential). In this situation one finds $S(q) \to N$. We recover here the same result as with the variational approach.

By contrast, the limit $q \to 0$ corresponds to the phonon-like regime (see Section 4). This corresponds to $\varepsilon_q \ll \rho g$ and therefore leads to:

$$S(\boldsymbol{q}) \simeq_{\boldsymbol{q} \to 0} N \sqrt{\frac{\varepsilon_{\boldsymbol{q}}}{2\rho g}} \propto q$$
 (139)

According to the Bogolubov approach, S(q) vanishes linearly with q when $q \rightarrow 0$. The discrepancy between equations (133) and (139) gives evidence for the fact that the approximation of the condensate wave-function by a product state is not sufficient to account properly for all physical effects.

• These theoretical predictions can be compared to the *experimental results* obtained by W. Ketterle and coworkers [27]. The authors have used small angle stimulated Bragg scattering to probe excitations of a condensate, and they explored both the free-particle and phonon regimes by changing the atomic density for a fixed momentum transfer. The results show indeed a significant weakening of the line strength of the Bragg resonance when one increases the density to enter the phonon regime.

Relative phase and interference

Finally, one can give a simple interpretation for the behaviour of the static structure factor of a condensate. We have seen in Section 4, equation (117), that the ground state wave function $|\phi_0\rangle$ of a homogeneous condensate in the Bogolubov theory is a linear superposition of $|n_0 = N, 0, 0, \ldots\rangle$ and of Fock states of the kind $|n_0 = N - 2, n_k = 1, n_{-k} = 1, 0, 0, \ldots\rangle$. Therefore, there are two different ways of imparting a momentum q to the condensate: one can either start from $|n_0 = N, 0, 0, \ldots\rangle$ and promote a zero-momentum particle to momentum q, or start with $|n_0 = N - 2, n_q = 1, n_{-q} = 1, 0, 0, \ldots\rangle$ and transfer a particle from momentum -q to zero-momentum. The resulting state for both paths is $|n_0 = N - 1, n_q = 1, 0, 0, \ldots\rangle$, so that one has to sum the corresponding transition *amplitudes*. For high momentum transfers q however, the state $|n_0 = N - 2, n_q = 1, n_{-q} = 1, 0, 0, \ldots\rangle$ has a negligible weight in $|\phi_0\rangle$ and the static structure factor is simply equal to N, as for a perfect gas. By contrast, for $q \to 0$ a destructive interference between the two paths leads to a vanishing structure factor.

5.2. About interferences in detection signals

Following the discussion about the vanishing of the static structure factor of a condensate, we now wish to clarify the general concept of quantum interference. Indeed, thinking about interference brings to the mind the picture of interfering waves, although interference effects can as well be observed in cases where a description in terms of classical light or matter waves cannot be implemented. What are the 'objects' that interfere then?

For a *N*-boson system, when the ground state of the system is not a macroscopic matter wave, the correlation functions cannot be written as products of matter waves. One can gain some physical insight into the interference phenomenon by writing the correlation functions in a particular way. We consider here for example the second order correlation function $G^{(2)}(\mathbf{r} t, \mathbf{r}' t')$, which describes a double-detection signal. If the initial state of the system is a statistical mixture of states $|\chi_i\rangle$ with weights π_i , one has, $\{|u_f\rangle\}$ being an orthonormal basis of states:

$$G^{(2)}(\boldsymbol{r}\,t,\boldsymbol{r}'\,t') = \langle \hat{\psi}^{\dagger}(\boldsymbol{r},t)\hat{\psi}^{\dagger}(\boldsymbol{r}',t')\hat{\psi}(\boldsymbol{r}',t')\hat{\psi}(\boldsymbol{r},t)\rangle = \sum_{i}\pi_{i}\langle\chi_{i}|\hat{\psi}^{\dagger}(\boldsymbol{r},t)\hat{\psi}^{\dagger}(\boldsymbol{r}',t')\hat{\psi}(\boldsymbol{r},t)|\chi_{i}\rangle$$

$$= \sum_{i}\pi_{i}\langle\chi_{i}|\hat{\psi}^{\dagger}(\boldsymbol{r},t)\hat{\psi}^{\dagger}(\boldsymbol{r}',t')\underbrace{\left[\sum_{f}|u_{f}\rangle\langle u_{f}|\right]}_{=1}\hat{\psi}(\boldsymbol{r}',t')\hat{\psi}(\boldsymbol{r},t)|\chi_{i}\rangle$$

$$G^{(2)}(\boldsymbol{r}\,t,\boldsymbol{r}'\,t') = \sum_{i}\pi_{i}\sum_{f}\left|\langle u_{f}|\hat{\psi}(\boldsymbol{r}',t')\hat{\psi}(\boldsymbol{r},t)|\chi_{i}\rangle\right|^{2}$$
(140)

The detection signal is thus obtained by averaging over the initial states $|\chi_i\rangle$ and summing over the final states $|u_f\rangle$ the transition probabilities $|A_{fi}|^2 = |\langle u_f | \hat{\psi}(\mathbf{r}', t') \hat{\psi}(\mathbf{r}, t) | \chi_i \rangle|^2$. Let us now consider a single transition amplitude A_{fi} . Because $\hat{\psi}(\mathbf{r}, t)$ is a linear superposition of operators corresponding to the different 'modes' $\psi_i(\mathbf{r})$, A_{fi} is a sum of amplitudes associated to detections in different modes.

For a two-mode problem, for instance, $\widehat{\psi}(\mathbf{r},t) = \hat{a}_1(t)\psi_1(\mathbf{r}) + \hat{a}_2(t)\psi_2(\mathbf{r})$, so that A_{fi} is the sum of 4 amplitudes, as shown in *figure 5*.

The transition probability $|A_{fi}|^2$ contains crossed terms, i.e. terms of the kind $\psi_2^*(\mathbf{r})\psi_1(\mathbf{r})$ which correspond to detections in different modes. Therefore, one can say that the 'objects' that interfere are *transition amplitudes*, giving rise to fringes in the detection signal, provided that there exists at least two possible paths between the initial state and a given final state.

In the simple case of a two-boson state $|\chi_i\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle$ with one boson having momentum \mathbf{k}_1 and another one having a momentum \mathbf{k}_2 , if the two detections are simultaneous (t = t'), then only two interfering paths exist:

(1) the process where \mathbf{k}_1 is detected in \mathbf{r} and \mathbf{k}_2 in \mathbf{r}' has an amplitude $\exp(i\mathbf{k}_1 \cdot \mathbf{r}) \exp(i\mathbf{k}_2 \cdot \mathbf{r}')$;

(2) the one with k_1 detected in r' and k_2 in r has an amplitude $\exp(ik_1 \cdot r') \exp(ik_2 \cdot r)$.



Both paths end in the vacuum state, and the total transition amplitude A is the sum of the two contributions, namely $A \propto \exp(i\mathbf{k}_1 \cdot \mathbf{r}) \exp(i\mathbf{k}_2 \cdot \mathbf{r}') + \exp(i\mathbf{k}_1 \cdot \mathbf{r}') \exp(i\mathbf{k}_2 \cdot \mathbf{r})$. The transition probability then exhibits interference fringes:

$$|A|^{2} \propto 1 + 1 + 2 \operatorname{Re}\left[e^{i(\boldsymbol{k}_{1} - \boldsymbol{k}_{2}) \cdot (\boldsymbol{r} - \boldsymbol{r}')}\right] = 2\left[1 + \cos(\boldsymbol{k}_{1} - \boldsymbol{k}_{2}) \cdot (\boldsymbol{r} - \boldsymbol{r}')\right]$$
(141)

One can easily see that the probability of detecting one boson in r does not depend on r, since it is proportional to $|\exp(i\mathbf{k}_1 \cdot \mathbf{r})|^2 + |\exp(i\mathbf{k}_2 \cdot \mathbf{r})|^2 = 2$. Once a boson is detected in r, however, the probability to detect the second one in r' is a sinusoidal function of r - r', as shown by equation (141).

Note that similar effects exist for photons [28] (see also [2], Comp. A_{III}). Concerning two-photon states, one clearly identifies the two interfering paths by introducing emitters E_1 and E_2 for the two photons in modes 1 and 2, respectively, as well as two detectors D and D' (see *figure 6*).

6. Matter waves versus light waves: effect of atom-atom interactions

The approach we have adopted in this article makes it clear that bosonic quantum gases exhibit many analogies with quantum optical fields. However, there are also some major differences, in particular those related with the interactions between atoms. We study in this section the effect of atom–atom interactions first on the relative phase of two condensates in different traps, and then on the behaviour of two condensates in the same trap.

6.1. Relative phase of two condensates in different traps

We use here notations similar to the ones of Section 3.3: the condensates 1 and 2 situated in r_1 and r_2 are associated with macroscopic matter waves $\psi_1(r)$ and $\psi_2(r)$, respectively. We now study the time evolution of the spatial coherence between the two condensates under the effect of interactions within each condensate (we neglect the interactions between the two condensates). Using the Heisenberg point of view, one has:

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$$\left\langle \hat{\psi}^{\dagger}(\boldsymbol{r}_{1},t)\hat{\psi}(\boldsymbol{r}_{2},t)\right\rangle = \psi_{1}^{*}(\boldsymbol{r}_{1})\psi_{2}(\boldsymbol{r}_{2})\left\langle \hat{a}_{1}^{+}(t)\hat{a}_{2}(t)\right\rangle \quad \text{with}$$
(142)

$$\hat{a}_{1}^{+}(t) = e^{i\hat{H}_{1}t/\hbar}\hat{a}_{1}^{+}e^{-i\hat{H}_{1}t/\hbar} \quad \text{and} \quad \hat{a}_{2}(t) = e^{i\hat{H}_{2}t/\hbar}\hat{a}_{2}e^{-i\hat{H}_{2}t/\hbar}$$
(143)

In equation (143), \hat{H}_i is the Hamiltonian of condensate *i*, including the interaction potential. Inserting a closure relation in equations (143) yields:

$$\hat{a}_{1}^{+}(t) = \sum_{n_{1}} \sqrt{n_{1} + 1} |n_{1} + 1\rangle \langle n_{1}| e^{i[E(n_{1}+1) - E(n_{1})]t/\hbar}$$
(144)

$$\hat{a}_2(t) = \sum_{n_2} \sqrt{n_2} |n_2 - 1\rangle \langle n_2| \,\mathrm{e}^{\mathrm{i}[E(n_2 - 1) - E(n_2)]t/\hbar} \tag{145}$$

We get:

$$\left\langle \hat{\psi}^{\dagger}(\boldsymbol{r}_{1},t)\hat{\psi}(\boldsymbol{r}_{2},t)\right\rangle = \sum_{n_{1},n_{2}}\psi_{1}^{*}(\boldsymbol{r}_{1})\psi_{2}(\boldsymbol{r}_{2})\sqrt{n_{2}}\sqrt{n_{1}+1}\langle n_{1},n_{2}|\hat{\rho}|n_{1}+1,n_{2}-1\rangle$$

$$\times e^{i[E(n_{1}+1,n_{2}-1)-E(n_{1},n_{2})]t/\hbar}$$
(146)

The spatial coherence between the two condensates appears as a sum of terms proportional to the coherence between $|n_1, n_2\rangle$ and $|n_1 + 1, n_2 - 1\rangle$ and oscillating at the corresponding Bohr frequencies.

Let us consider an initial state of the system with a well-defined spatial coherence, for instance a phase state $|N,\theta\rangle$ (see Section 3.3.1). In the following, we label the states $|n_1,n_2\rangle$ by $n = n_1 - n_2$, $n_1 + n_2 = N$ being fixed. One can write $n_{1,2} = \frac{1}{2}(N \pm n)$ and n can take values from -N to N in steps of 2. The spatial coherence $\langle \hat{\psi}^{\dagger}(\mathbf{r}_1,t)\hat{\psi}(\mathbf{r}_2,t)\rangle$ between the condensates is then proportional to $\sum_n \sqrt{(N-n)(N+n+1)}\langle n|\hat{\rho}|n+2\rangle \exp(\mathrm{i}[E(n+2)-E(n)]t/\hbar).$

To go further in the discussion while keeping the calculations as simple as possible, we consider the case of two homogeneous condensates in boxes of size L, with the same scattering length a; in this case the Hamiltonian is reduced to the interaction Hamiltonian (see Section 3.1.1), namely $\hat{H}_1 + \hat{H}_2 = g(\hat{N}_1^2 + \hat{N}_2^2)/(2L^3)$, with $g = 4\pi\hbar^2 a/m$ (54). This yields:

$$E(n) = \frac{g}{8L^3} \left[(N+n)^2 + (N-n)^2 \right] = \frac{g}{4L^3} \left(N^2 + n^2 \right)$$
(147)

As shown in *figure 7*, E(n) varies quadratically with n, so that the relevant quantity $E(n+2) - E(n) = g(n+1)/L^3$ is linear.

As a consequence of the atom-atom interactions within each condensate, the evolution frequencies of the coherences $\langle n|\hat{\rho}|n+2\rangle$ form a comb of equally spaced frequencies $g(n+1)/(\hbar L^3)$, n varying from -N to N. We have seen in Section 3.3.1 that the dispersion in n in a phase state is on the order of \sqrt{N} . Then the frequency spreading $\Delta \omega$ of the spatial coherence $\langle \hat{\psi}^{\dagger}(\mathbf{r}_1,t)\hat{\psi}(\mathbf{r}_2,t)\rangle$ is $\Delta \omega \simeq \sqrt{Ng}/(\hbar L^3)$, hence a





coherence time:

$$T_{\rm coh} \simeq \frac{\hbar L^3}{g\sqrt{N}} = \frac{\hbar\sqrt{N}}{\mu} \tag{148}$$

where $\mu = gN/L^3$ is the chemical potential.

Moreover, since the frequencies $g(n+1)/(\hbar L^3)$ are equally spaced by $g/\hbar L^3$, one expects a revival of the spatial coherence after a time:

$$T_{\rm rev} \simeq \frac{\hbar L^3}{g} = \frac{\hbar N}{\mu} \gg T_{\rm coh}$$
 (149)

6.2. Two condensates in the same trap

We now consider the situation where two ensembles of bosonic atoms 1 and 2, prepared in two different hyperfine or Zeeman sublevels, are both present in the same trap. Such a configuration was obtained experimentally with Rubidium atoms at JILA [29] and with Sodium atoms at MIT [30]. To describe the interactions with the same approximation as in the rest of the course, three coupling constants $g_{11} = g_1$, $g_{22} = g_2$, $g_{21} = g_{12}$ and three scattering lengths a_1 , a_2 , $a_{21} = a_{12}$ are needed.

We first try to elucidate whether the two condensates are spatially separated or mixed. If $g_{12} \gg g_1, g_2$, one expects that the two condensates would rather separate in order to minimize the interactions 1–2. To go further, we consider the simplified situation of a homogeneous system in a box of volume V, with N atoms in the state $|1\rangle$ and N atoms in $|2\rangle$. This system can find itself in two states: either the two components are mixed and occupy both the whole volume V, or they are separated, the component 1 or 2 occupying a volume xV or (1-x)V, respectively. We denote the total energy of the system by E_{I} in the former case, and by E_{II} in the latter. The calculation of E_{I} is straightforward:

$$E_{\rm I} = \frac{1}{2}g_1 \frac{N^2}{V} + \frac{1}{2}g_2 \frac{N^2}{V} + g_{12} \frac{N^2}{V} = \frac{N^2}{2V}(g_1 + g_2 + 2g_{12})$$
(150)

To calculate E_{II} , we can neglect g_{12} since the two condensates are separated. We get:

$$E_{\rm II} = \frac{1}{2}g_1 \frac{N^2}{xV} + \frac{1}{2}g_2 \frac{N^2}{(1-x)V} = \frac{N^2}{2V} \underbrace{\left[\frac{g_1}{x} + \frac{g_2}{1-x}\right]}_{G(x)}$$
(151)

The value of x can be determined by minimizing G(x), which yields $x = \sqrt{g_1}/(\sqrt{g_1} + \sqrt{g_2})$, hence:

$$G(x) = \sqrt{g_1} \left(\sqrt{g_1} + \sqrt{g_2} \right) + \sqrt{g_2} \left(\sqrt{g_1} + \sqrt{g_2} \right) = \left(\sqrt{g_1} + \sqrt{g_2} \right)^2$$
(152)

Finally, one gets:

$$E_{\rm II} = \frac{N^2}{2V} \left[g_1 + g_2 + 2\sqrt{g_1 g_2} \right] \tag{153}$$

Comparing equations (150) and (153), one can conclude that: (1) if $g_{12} > \sqrt{g_1g_2}$, $E_I > E_{II}$ and the two condensates separate; (2) if $g_{12} < \sqrt{g_1g_2}$, $E_I < E_{II}$ so that both components are mixed.

We now study the role played by the mutual interactions between the two condensates concerning the coherence time $T_{\rm coh}$ calculated in Section 6.1. We have indeed seen that the scrambling of the relative phase θ between the two condensates is due to the fact that the Bohr frequencies $\omega(n_1, n_2) = \frac{1}{\hbar}[E(n_1 + 1, n_2 - 1) - E(n_1, n_2)]$ appearing in the evolution of $\langle \hat{a}_1^+(t)\hat{a}_2(t) \rangle$ (see equation (146)) are

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spread over an interval $\delta \omega \sim \sqrt{Ng}/(\hbar L^3)$ when n_1 and n_2 vary over an interval $\Delta n \sim \sqrt{N}$ around \overline{n}_1 and \overline{n}_2 . How are these results modified when one takes g_{12} into account? We discuss here the homogeneous case, which is the simplest. Note that a more general theoretical study of a binary mixture of condensates has been performed by A. Sinatra and Y. Castin [31].

The relevant Bohr frequency $\omega(n_1, n_2)$ can be written:

$$\omega(n_1, n_2) = \frac{1}{\hbar} \left[E(n_1 + 1, n_2 - 1) - E(n_1, n_2) \right] = \frac{1}{\hbar} \left(\frac{\partial}{\partial n_1} - \frac{\partial}{\partial n_2} \right) E(n_1, n_2)$$

$$\omega(n_1, n_2) = \frac{1}{\hbar} \left[\mu_1(n_1, n_2) - \mu_2(n_1, n_2) \right]$$
(154)

When n_1 and n_2 vary over an interval Δn around \overline{n}_1 and \overline{n}_2 (with $\overline{n}_1 + \overline{n}_2 = n_1 + n_2 = N$ imposed by the super-selection rule), the Bohr frequency spreading $\Delta \omega$ is written:

$$\Delta\omega \simeq \omega(\overline{n}_1 + \Delta n, \overline{n}_2 - \Delta n) - \omega(\overline{n}_1, \overline{n}_2) \simeq \Delta n \left(\frac{\partial}{\partial n_1} - \frac{\partial}{\partial n_2} \right) \omega(n_1, n_2) \Big|_{n_1 = \overline{n}_1, n_2 = \overline{n}_2}$$
(155)

In the present case of a homogeneous mixture of two condensates in a box of volume $V = L^3$, the energy $E(n_1, n_2)$ is written:

$$E(n_1, n_2) = \frac{1}{2}g_1\frac{n_1^2}{V} + \frac{1}{2}g_2\frac{n_2^2}{V} + g_{12}\frac{n_1n_2}{V}$$
(156)

This gives, using equation (154):

$$\mu_{1,2}(n_1, n_2) = \frac{\partial}{\partial n_{1,2}} E(n_1, n_2) = g_{1,2} \frac{n_{1,2}}{V} + g_{12} \frac{n_{2,1}}{V}$$
(157)

$$\omega(n_1, n_2) = \frac{1}{\hbar} \left(g_1 \frac{n_1}{V} - g_2 \frac{n_2}{V} + g_{12} \frac{n_2 - n_1}{V} \right)$$
(158)

Finally, one gets:

$$\Delta\omega \simeq \frac{\Delta n}{\hbar V} (g_1 + g_2 - 2g_{12}) \tag{159}$$

Since $T_{\rm coh} \simeq 1/\Delta \omega$, one easily recovers the result of equation (148) by taking $g_{12} = 0$ and $\Delta n \simeq \sqrt{N}$. By contrast, if $g_{12} \neq 0$ the Bohr frequency spread is decreased and the coherence time increases. In particular, when $g_{12} \simeq (g_1 + g_2)/2$ one has $\Delta \omega = 0$ and thus $T_{\rm coh} \rightarrow \infty$.

Although this behaviour seems surprising, one can find a simple physical explanation for it. For a given Δn , the relevant quantity is the variation of $\mu_1 - \mu_2$ when n_1 increases by 1 and n_2 decreases by 1.

- If $g_{12} = 0$, μ_1 increases when n_1 increases because there are then more atoms interacting in condensate 1. Similarly, μ_2 decreases because n_2 decreases. As a result, $\mu_1 - \mu_2$ increases.
- If one takes into account the mutual interactions between the two condensates, then μ_1 increases with n_1 as above. However, at the same time the interaction of the atoms of condensate 1 with the ones of condensate 2 decreases because n_2 decreases (see equation (157)). Globally, the variation of μ_1 can be smaller than when $g_{12} = 0$, and it can even vanish. A symmetric argument applies for the variation of μ_2 when n_2 decreases.

Experimentally, the long coherence times observed at JILA with ⁸⁷Rb [32] possibly originate from the phenomenon we have discussed here, since the interaction parameters are such that $2g_{12} \simeq g_1 + g_2$.

7. Conclusion

In this article, we have shown that Bose–Einstein condensates can be described by quantum states analogous to those used in quantum optics, with the advantage that they do not violate any super-selection rule (see Section 2.2.3). Using a variational approximation, we have shown that it is possible to associate to the condensate a 3D matter wave allowing one to study many coherence properties. In particular, we have seen that a relative phase between two condensates can exist only if the state of the two condensates is a linear superposition of states $|n_1, n_2\rangle$ with different values of $n_1 - n_2$, n_1 and n_2 being the number of bosons in each condensate and $n_1 + n_2$ being fixed and equal to N.

Experimentally, the detection of a condensate is often achieved by particle or light scattering; we have introduced dynamical and static structure factors describing this scattering in the framework of the linear response theory. We have shown that some physical effects cannot be accounted for within the variational approximation but require a more elaborate approach, such as the Bogolubov theory.

Finally, we have studied the effects of atom–atom interactions on the dynamics of the condensate, and in particular on the scrambling of the relative phase of two condensates, either in two different traps or in the same trap.

Concerning future studies, a great challenge would consist of preparing a system of two condensates in states such as $c_1|n_1 = N, n_2 = 0\rangle + c_2|n_1 = 0, n_2 = N\rangle$. Such states could indeed be considered as linear superpositions of two different macroscopic states (sometimes called 'Schrödinger cats').

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