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Physical interpretations for radiative corrections in the non-relativistic limit

J Dupont-Roc, C Fabre and C Cohen-Tannoudji
Laboratoire de Spectroscopie Hertzienne† de l'ENS et Collège de France, 24 rue Lhomond, 75231 Paris Cedex 05, France

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Abstract. We present a detailed physical discussion of the contribution of non-relativistic modes of the radiation field to electron radiative corrections (Lamb shift, $g - 2$). We show that these corrections can be described by a simple effective Hamiltonian, derived from a single-particle theory, and that two main physical effects are involved: the vibration of the electron charge and spin moment due to vacuum fluctuations, and the radiation reaction of the charge. We find that the positive sign of $g - 2$ is entirely due to the radiation reaction which slows down the cyclotron motion, whereas the Lamb shift results from the averaging of the Coulomb potential by the vibrating electron (Welton's picture). We discuss briefly many-particle effects and the contributions of relativistic modes. They do not seem to alter these conclusions.

1. Introduction

When an electron interacts with an electromagnetic wave, its position and the direction of its spin vibrate at the frequency ω of the incident wave. If, in addition, this electron is submitted to weak static electric or magnetic fields, a slow motion due to the static fields is superimposed on the high-frequency vibration. These two motions are actually not independent: the high-frequency vibration modifies the dynamical response of the electron to the static fields and perturbs its slow motion.

In a previous publication (Avan *et al* 1976) we have investigated this 'renormalisation' of the electron properties induced by the interaction with an electromagnetic wave. The new dynamical properties of the electron are described by an effective Hamiltonian $\mathscr{H}_{\text{eff}}$ valid at the high-frequency limit (ω large compared to the frequencies characterising the slow motion) and including all relativistic corrections up to $1/c^2$. The incident wave is described quantum mechanically (for reasons which will be explained later). If one only keeps the terms of $\mathscr{H}_{\text{eff}}$ proportional to the number N of photons (stimulated terms), the quantum aspects of the electromagnetic field do not play any role. An identical expression for $\mathscr{H}_{\text{eff}}$ would be derived from a semiclassical approach. Therefore, the semiclassical pictures given above to describe the electron motion are quite appropriate.

In this paper we try to describe, by such simple physical pictures, the modifications of the electron dynamical properties due to its interaction with the vacuum

† Associé au CNRS.

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('spontaneous renormalisation'). The problem of the physical interpretation of radiative corrections has already received a lot of attention (Welton 1948, Koba 1949, Feynman 1962, Bourret 1973, Itzykson 1974, Senitzky 1973, Lai *et al* 1974, 1975, Weisskopf 1974, Grotch and Kazes 1975, 1977, Baier and Mil'Shtein 1976), but some points remain unclear (for instance, is there a simple explanation for the electron spin magnetic moment anomaly?)

We present here an approach to these problems based on a comparison between the N-dependent and N-independent terms of the effective Hamiltonian describing the effect of a given mode of the electromagnetic field. In more physical terms, we try to understand the spontaneous renormalisation of the electron properties by comparing it with the stimulated one, which can be described by simple physical pictures. This is the reason why we have previously used a quantum description of the electromagnetic field although it was not essential for discussing the stimulated terms: the same calculation gives simultaneously the stimulated and spontaneous corrections and allows a term to term comparison. As in the previous paper, we consider only a non-relativistic mode ($\hbar\omega \ll mc^2$), with a frequency much higher than the characteristic frequencies of the electron (high-frequency limit) and we keep all relativistic corrections up to order $1/c^2$.

Of course, it could be objected that this approach to the electron radiative corrections is too naive. The covariant QED calculations of these corrections are well established and it seems more appropriate to try to extract directly some physical pictures from this theory. Actually, one cannot progress very far in this direction (Feynman 1962). It seems that there are fundamental reasons for that: the boundary conditions of the Feynman propagator which simplify considerably the algebraic computations imply in counterpart that several distinct physical processes are described by the same diagram (emission and reabsorption of photons, virtual creation and annihilation of an electron–positron pair ...). Furthermore, the explicit covariance of the theory does not allow an easy separation in a given reference frame between electric and magnetic effects (a given vertex describes the interaction of the electron charge and that of the spin magnetic moment as well). It is therefore not surprising that these covariant expressions can be discussed only in very general terms like emission and absorption of photons and that the connection with our naive daily image of the electron is difficult.

In order to derive more elementary but more precise physical pictures, one is led to give up the explicit covariance of the theory and to analyse the radiative corrections in a given reference frame, generally chosen such that the electron is not relativistic†. Along these lines, one can mention the first evaluation of the Lamb shift by Bethe (1947) and its physical interpretation by Welton (1948), and the calculation and interpretation of the electron self-energy by Weisskopf (1939).

In this paper we adopt this point of view and we introduce one more restriction by considering only, in a first step, the effect of non-relativistic modes ($\hbar\omega \ll mc^2$). It is therefore clear that the interpretations which will be proposed are only valid in this limited domain and do not cover *a priori* the whole physics of radiative corrections. On the other hand, all the involved physical processes can be easily identified. In a second step, which will be considered in a subsequent paper, we will get rid of the restriction of non-relativistic modes and show that the contribution of the relativistic mode does not change the physical conclusions derived here drastically.

† This does not imply that in the virtual intermediate states the electron remains non-relativistic.

This paper is divided into four parts. After a brief outline of the effective Hamiltonian method (§2), the spontaneous terms of this Hamiltonian are displayed and their physical content analysed (§3). Taking advantage of these results, we investigate in §4 the physical interpretation of two well known radiative corrections: the Lamb shift and the electron anomalous g factor. We compare our conclusions with those of various authors interested in these problems. Finally, we analyse in §5 the limitations of the present approach, and the improvements to be effected in the effective Hamiltonian method in order to establish closer connections with QED covariant calculations.

2. The effective Hamiltonian method

We present here a general outline of the theory. We give just the principle of the calculation and its physical basis rather than detailed expressions which are quite lengthy. More details can be found in Avan *et al* (1976).

2.1. *Notation and the basic Hamiltonian*

We consider an electron (mass m, charge e), subjected to static electric and magnetic fields ($\boldsymbol{E}_0$ and $\boldsymbol{B}_0$) deriving from the potentials ϕ_0 and A_0 in the Coulomb gauge. The electron interacts with a mode of the electromagnetic field, quantised in the Coulomb gauge in a box of volume L^3. The mode is characterised by its wavevector $\boldsymbol{k}$, and its polarisation $\boldsymbol{\epsilon}$, supposed to be real. We shall also use $\omega = ck$ and $\boldsymbol{\kappa} = \boldsymbol{k}/k$. Annihilation and creation operators in the mode are a and $a^{\dagger}$. Since we want to include relativistic corrections, the Pauli Hamiltonian is not sufficient to describe the electron and its interaction with the static and radiation fields. Using the Foldy–Wouthuysen transformation along the line discussed by Bjorken and Drell (1964) (also Feynman 1961), it is possible to derive from the Dirac Hamiltonian a Hamiltonian which acts only on two components spinors and includes all relativistic corrections up to $1/c^2$. Since here the radiation field is quantised, some care must be taken when commuting the field operators. One finds that a constant term V^0 (given in appendix 1) appears as well as the usual Hamiltonian (Pauli Hamiltonian + Darwin, spin–orbit and relativistic mass corrections) so that the total Hamiltonian $\mathscr{H}$ is:

$$\mathscr{H} = \hbar\omega a^{\dagger}a + mc^2 + V^0 + \frac{\pi^2}{2m} + e\phi_0 - \frac{e\hbar}{2m}\boldsymbol{\sigma}.\boldsymbol{B}_{\mathrm{t}} - \frac{e\hbar^2}{8m^2c^2}\boldsymbol{\nabla}.\boldsymbol{E}_{\mathrm{t}}$$
$$- \frac{e\hbar}{8m^2c^2}\boldsymbol{\sigma}.(\boldsymbol{E}_{\mathrm{t}} \times \boldsymbol{\pi} - \boldsymbol{\pi} \times \boldsymbol{E}_{\mathrm{t}}) - \frac{1}{2mc^2}\left(\frac{\pi^2}{2m} - \frac{e\hbar}{2m}\boldsymbol{\sigma}.\boldsymbol{B}_{\mathrm{t}}\right)^2 \tag{2.1}$$

where

$$\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}_{\mathrm{t}}. \tag{2.2}$$

The subscript 't' refers to the total field (static fields + radiation).

2.2. *The effective Hamiltonian*

The total Hamiltonian $\mathscr{H}$ can be split into three parts: the radiation field Hamiltonian $\hbar\omega a^{\dagger}a$, the electronic Hamiltonian in the external static fields $\mathscr{H}_{\mathrm{e}}$, and the coupling $\mathscr{H}_I$ between the electron and the radiation

$$\mathscr{H} = \hbar\omega a^\dagger a + \mathscr{H}_e + \mathscr{H}_I. \tag{2.3}$$

If $\mathscr{H}_I$ is neglected the energy levels of the total system are bunched in well defined multiplicities $\mathscr{E}_N$ corresponding to a definite number of photons N. $\mathscr{H}_I$ has a diagonal part, operating inside each multiplicity $\mathscr{E}_N$, and an off-diagonal part, coupling multiplicities with a different number of photons. When $\mathscr{H}_I$ is taken into account, the eigenstates of the total Hamiltonian no longer correspond to a well defined value of N. In the evolution of the electronic variables, some frequencies close to ω and its multiples appear, corresponding to the classical picture of an electron vibrating at the field frequency. The energy levels in each multiplicity are also modified and this corresponds to a modification of the slow motion of the electron. This is precisely what we are interested in. It is certainly possible to determine the perturbed energy levels by ordinary second-order perturbation theory. However, it is more convenient to choose another method (Primas 1963) similar to the one used in solid-state physics to remove inter-band coupling or to describe multi-particle effects (Blount 1962, Nozieres and Pines 1958, see also Kittel 1963). The idea is to construct a unitary transformation which eliminates the off-diagonal parts of the total Hamiltonian, at least to a given order. The transformed Hamiltonian $\mathscr{H}'$ has the same eigenvalues as the initial Hamiltonian and has the following structure:

$$\mathscr{H}' = \hbar\omega a^\dagger a + \mathscr{H}_{\mathrm{eff}} \tag{2.4}$$

where $\mathscr{H}_{\mathrm{eff}}$, called the effective Hamiltonian, acts only within a given multiplicity, i.e. operates on electronic variables, and depends on field variables only through $a^\dagger a$ and $aa^\dagger$. If we restrict ourselves to a second-order calculation with respect to the radiation field, $\mathscr{H}_{\mathrm{eff}}$ takes the following form in the multiplicity $\mathscr{E}_N$:

$$\mathscr{H}_{\mathrm{eff}} = \mathscr{H}_e + (N+1)R + NS \tag{2.5}$$

where R and S are pure electronic operators, given in appendix 1. The last two terms of (2.5) describe the modification of the electron dynamical properties due to the coupling with the mode $(\boldsymbol{k}, \boldsymbol{\epsilon})$. They include, in particular, the effects of virtual transitions to upper and lower multiplicities.

3. Explicit form of the spontaneous terms: physical interpretation

3.1. General structure

In addition to the unperturbed Hamiltonian $\mathscr{H}_e$, the effective Hamiltonian (2.5) contains a term proportional to N:

$$N(R+S) \tag{3.1}$$

which represents the 'stimulated' corrections induced by the incident photons and discussed in detail in Avan *et al* (1976).

The same calculation yields a N-independent term, R, describing the effect of the coupling with the empty mode, i.e. the contribution of the $(\boldsymbol{k}, \boldsymbol{\epsilon})$ mode to the spontaneous corrections which we intend to study in this paper.

In order to make a connection with the physical pictures worked out for the stimulated corrections, it is appropriate to write the spontaneous corrections as:

$$R = \tfrac{1}{2}(R + S) + \tfrac{1}{2}(R - S). \tag{3.2}$$

The first term of (3.2), that we will call

$$\mathscr{H}_{\mathrm{fl}} = \tfrac{1}{2}(R + S), \tag{3.3}$$

has the same structure as (3.1), N being replaced by $\frac{1}{2}$. Its physical content is the same as the one of the stimulated terms (3.1), except that the energy $N\hbar\omega$ of the incident N photons is replaced by the zero point energy $\frac{1}{2}\hbar\omega$ of the mode. All the semiclassical pictures developed for the stimulated terms can therefore be transposed to $\mathscr{H}_{\mathrm{fl}}$ by just replacing the incident wave by the vacuum fluctuations†.

The second term

$$\mathscr{H}_{\mathrm{r}} = \tfrac{1}{2}(R - S) \tag{3.4}$$

is new and its physical interpretation will be given below. Finally, the effective Hamiltonian of the electron interacting with the empty $(\boldsymbol{k}, \boldsymbol{\epsilon})$ mode can be written as:

$$\mathscr{H}_{\mathrm{eff}} = \mathscr{H}_{\mathrm{e}} + \mathscr{H}_{\mathrm{fl}} + \mathscr{H}_{\mathrm{r}}. \tag{3.5}$$

3.2. *Characteristic energies*

The expressions giving R and S are quite complicated. In order to characterise the magnitude of the various terms appearing in these expressions, it is useful to introduce the characteristic energies of the problem which are: the rest energy of the electron mc^2, the photon energy $\hbar\omega$, an energy $\mathscr{E}_{\mathrm{b}}$ characterising the coupling of the electron to the static fields, and finally

$$\mathscr{E}_{\mathrm{v}}^{0} = \frac{e^2}{2m}\,\frac{\hbar}{2\epsilon_0\omega L^3} \tag{3.6}$$

representing the kinetic energy associated with the electron vibration in the vacuum fluctuations of the mode $(\boldsymbol{k}, \boldsymbol{\epsilon})$.

The high-frequency, weak-coupling and non-relativistic approximations which have been made in the derivation of $\mathscr{H}_{\mathrm{eff}}$ are valid if

$$mc^2 \gg \hbar\omega \gg \mathscr{E}_{\mathrm{v}}^{0}, \mathscr{E}_{\mathrm{b}}. \tag{3.7}$$

In the various calculations, we keep:

(i) all terms in $1/c$ and $1/c^2$;

(ii) all terms linear in $\mathscr{E}_{\mathrm{v}}^{0}$ (second-order calculation with respect to the radiation field);

(iii) all terms linear in $\mathscr{E}_{\mathrm{b}}$, and the major quadratic terms such as $\mathscr{E}_{\mathrm{b}}^{2}/mc^2$, $\mathscr{E}_{\mathrm{v}}^{0}\mathscr{E}_{\mathrm{b}}^{2}/(\hbar\omega)^2$ (we neglect smaller terms of the order of $\mathscr{E}_{\mathrm{v}}^{0}\mathscr{E}_{\mathrm{b}}^{2}/\hbar\omega mc^2$).

† Working in the Heisenberg picture, Senitzky (1973) and Milonni *et al* (1973) have shown that there are some ambiguities when one tries to isolate vacuum fluctuation effects from those related to the radiation reaction. The effective Hamiltonian approach presented here seems to allow a clear separation: vacuum fluctuation terms are identified by comparison with the stimulated ones; the remaining terms will be shown to coincide with those describing radiation reaction effects in classical electrodynamics (see §3.4). The connection between these two different points of view will be considered elsewhere.

3.3. Terms analogous to stimulated terms: $\mathcal{H}_{fl}$

From the previous discussion, one can bypass the calculation of $\mathcal{H}_{\rm fl}$ and simply replace $\mathscr{E}_{\rm v}$ by $\mathscr{E}_{\rm v}^0$ in the expression of the stimulated corrections. One gets†:

$$\mathcal{H}_{\rm fl} = \mathscr{E}_{\rm v}^0 - \frac{\mathscr{E}_{\rm v}^0}{mc^2}\frac{\pi_0^2}{2m} + e\phi_0' - \frac{e}{2m}(A_0' \cdot \pi_0 + \pi_0 \cdot A_0') + W_{\rm d} + \frac{\mathscr{E}_{\rm v}^0}{mc^2}\frac{e\hbar}{2m}[2\boldsymbol{\sigma} \cdot \boldsymbol{B}_0 - (\boldsymbol{\kappa} \cdot \boldsymbol{\sigma})(\boldsymbol{\kappa} \cdot \boldsymbol{B}_0)] \tag{3.8}$$

where

$$\pi_0 = p - eA_0 \tag{3.9}$$

$$\phi_0' = \frac{\mathscr{E}_{\rm v}^0}{m\omega^2}(\boldsymbol{\epsilon} \cdot \nabla)^2 \phi_0 \tag{3.10a}$$

$$\nabla \times A_0' = \frac{\mathscr{E}_{\rm v}^0}{m\omega^2}(\boldsymbol{\epsilon} \cdot \nabla)^2 \boldsymbol{B}_0 \tag{3.10b}$$

$$W_{\rm d} = \frac{\mathscr{E}_{\rm v}^0 e^2}{m^2\omega^2}[\boldsymbol{B}_0^2 - (\boldsymbol{\epsilon} \cdot \boldsymbol{B}_0)^2]. \tag{3.11}$$

The physical interpretation of these various terms is straightforward (see §3 of Avan *et al* (1976)). The vibration kinetic energy in the vacuum fluctuations, $\mathscr{E}_{\rm v}^0$, adds to the electron rest mass mc^2 and corresponds to a mass increase:

$$\delta_1 m = \mathscr{E}_{\rm v}^0/c^2. \tag{3.12}$$

The second term of (3.8) is the correction of the kinetic energy associated with $\delta_1 m$. Due to its vibration, the electron averages the static electric and magnetic fields over a finite extension. This introduces the corrections ϕ_0' and A_0' to the static potentials ϕ_0 and A_0. $W_{\rm d}$ is the diamagnetic energy associated with the electron vibration. The last term represents a modification to the spin magnetic energy. It shows that the electron g factor is reduced by the vacuum fluctuations and becomes anisotropic. This results from the combined effect of four different processes. First there is an oscillation and consequently a spreading of the spin magnetic moment due to its coupling with the magnetic vacuum fluctuations. An additional reduction of the g factor results from the mass increase $\delta_1 m$ which also affects the Bohr magneton $e\hbar/2m$. Furthermore, as the electron vibrates in the electric field of the mode, it 'sees' a motional magnetic field which is found not to average to 0 in the presence of the static field $\boldsymbol{B}_0$; similarly, it explores the magnetic field of the mode over a finite extension (the dipole approximation is not made!), and this can give rise to a 'rectification' of the magnetic fluctuations which is also proportional to $\boldsymbol{B}_0$. These motional and rectified magnetic fields are coupled to the spin and contribute to the spin magnetic energy.

† Since we have supposed the polarisation $\boldsymbol{\epsilon}$ to be real, several terms derived in Avan *et al* (1976) vanish in (3.8). Let us recall that, when $\boldsymbol{\epsilon}$ is complex, the electron vibration is circular or elliptical, giving rise to a magnetic moment $\boldsymbol{\mu}$ which is responsible for additional magnetic couplings.

3.4. The new terms

Under the approximations given in §3.2, there are four new terms:

$$\mathscr{H}_{\mathrm{r}} = \mathscr{E}_{\mathrm{v}}^{0}\frac{\hbar\omega}{2mc^{2}} - \mathscr{E}_{\mathrm{v}}^{0}\frac{2(\boldsymbol{\epsilon}\cdot\boldsymbol{\pi}_{0})^{2}}{m\hbar\omega} - \mathscr{E}_{\mathrm{v}}^{0}\frac{\hbar\omega}{2mc^{2}} - \mathscr{E}_{\mathrm{v}}^{0}\frac{2(\boldsymbol{\epsilon}\cdot\boldsymbol{\pi}_{0})(\boldsymbol{\kappa}\cdot\boldsymbol{\pi}_{0})(\boldsymbol{\epsilon}\cdot\boldsymbol{\pi}_{0})}{\hbar\omega m^{2}c}. \tag{3.13}$$

The first term comes directly from the Foldy–Wouthuysen transformation used to derive the basic non-relativistic Hamiltonian (2.1). Let us recall that it originates from the non-commutation of the field operators. It is a constant and has no dynamical consequences. Furthermore, it represents a small correction to the rest mass energy of relative order $\mathscr{E}_{\mathrm{v}}^{0}\hbar\omega/(mc^{2})^{2}$, so that it will be neglected in the following.

The second term is the most interesting one. Let us remark that since $\mathscr{E}_{\mathrm{v}}^{0}$ is proportional to $\hbar$ (see (3.6)), this term is actually independent of $\hbar$, so that it may have a classical interpretation. We claim that it represents the radiation reaction of the electron charge, more precisely, the contribution of the mode $(\boldsymbol{k}, \boldsymbol{\epsilon})$ to this effect. The current associated with the electron (velocity $\boldsymbol{\pi}_{0}/m$) acts as a source term for the mode $(\boldsymbol{k}, \boldsymbol{\epsilon})$; it gives rise to a vector potential, proportional to $\boldsymbol{\epsilon}(\boldsymbol{\epsilon}\cdot\boldsymbol{\pi}_{0})$. The correction to the total energy of the system, i.e. the energy of the generated field, plus its interaction energy with the electron, coincides exactly with the second term of (3.13). A complete classical derivation of this result is given in appendix 2. As a consequence of this effect, the electron inertia is increased along the $\boldsymbol{\epsilon}$ direction. There is a close connection between this extra inertia and the electromagnetic mass associated with the static Coulomb energy in the electron rest frame. We will come back to this point later.

The third term appears, in the calculation, in the following form:

$$-\frac{1}{\hbar\omega}\left[\left(\frac{\hbar}{2\epsilon_{0}\omega L}\right)^{1/2}\right]^{2}\left(\frac{e\hbar}{2m}\sigma(\boldsymbol{k}\times\boldsymbol{\epsilon})\right)^{2} \tag{3.14}$$

which suggests one can interpret it as the radiation reaction of the spin magnetic moment: this moment acts as a source term for the mode. When one takes into account the energy of the field so created, plus the interaction of this field with the spin moment, one finds (3.14) exactly (see appendix 2). This expression reduces to the third term of (3.13) if one remembers that for a one-half spin σ_{x}^{2} is unity. Actually, from now on we will neglect this third term, for the same reasons as for the first one.

The last term is a correction to the radiation reaction of the electron charge due to the Doppler effect: the mode frequency appears to be $\omega[1 - (\boldsymbol{\kappa}\cdot\boldsymbol{\pi}_{0}/mc)]$ when the electron moves with the velocity $\boldsymbol{\pi}_{0}/m$.

Remark. The ω dependence of the radiation reaction is not the same for the charge and for the spin. The charge is coupled to the potential vector, whereas the spin is coupled to the magnetic field which has one extra ω factor. Thus, the radiation reaction of the charge predominates at low frequencies, while that of the spin grows as ω increases. Furthermore, the spin radiation reaction is a relativistic effect in $1/c^{2}$ and vanishes in the non-relativistic limit.

4. Application to the interpretation of two radiative corrections: Lamb shift and $g - 2$

Up to now we have considered the contribution of a single mode to the radiative corrections. The next step would be an integration over all modes which involves an angular averaging and a summation over ω weighted by the mode density $8\pi k^2\,\mathrm{d}k(2\pi/L)^{-3}$. Since our calculations are not valid for relativistic frequencies, we will restrict ourselves to the angular averaging, which amounts to evaluating the mean contribution of a mode in the 'shell' of frequency ω. Thus one keeps only the isotropic parts of $\mathscr{H}_{\mathrm{eff}}$ and one gets in this way the electron effective Hamiltonian:

$$\mathscr{H}_{\mathrm{eff}} = mc^2 + \mathscr{E}_{\mathrm{v}}^0 + \frac{\pi_0^2}{2m}\left(1 - \frac{\mathscr{E}_{\mathrm{v}}^0}{mc^2} - \frac{4}{3}\frac{\mathscr{E}_{\mathrm{v}}^0}{\hbar\omega}\right) + e\phi_0 + e\frac{\mathscr{E}_{\mathrm{v}}^0}{3m\omega^2}\Delta\phi_0$$
$$- \frac{e}{2m}(A_0'\cdot\pi_0 + \pi_0\cdot A_0') - \frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\left(1 - \frac{5}{3}\frac{\mathscr{E}_{\mathrm{v}}^0}{mc^2}\right) + W_{\mathrm{d}} + \mathscr{H}_{\mathrm{fs}} \qquad (4.1)$$

where

$$\mathrm{rot}\,A_0' = \frac{\mathscr{E}_{\mathrm{v}}^0}{3m\omega^2}\Delta\boldsymbol{B}_0 \qquad (4.2)$$

$$W_{\mathrm{d}} = \frac{2\mathscr{E}_{\mathrm{v}}^0}{3m^2\omega^2}e^2B_0^2. \qquad (4.3)$$

$\mathscr{H}_{\mathrm{fs}}$ is the well known fine-structure Hamiltonian (independent of $\mathscr{E}_{\mathrm{v}}^0$):

$$\mathscr{H}_{\mathrm{fs}} = \frac{e\hbar^2}{8m^2c^2}\Delta\phi_0 - \frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\boldsymbol{E}_0\times\boldsymbol{\pi}_0) - \frac{1}{2mc^2}\left(\frac{\pi_0^2}{2m} - \frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\right)^2. \qquad (4.4)$$

The various corrections appearing in (4.1) are of two types:

(i) Modification of the electron mass, due either to the electron vibration in the vacuum fluctuations, or to the radiation reaction. The spin g factor is sensitive to the first effect (the kinetic and spin magnetic energies both contain corrections in $\mathscr{E}_{\mathrm{v}}^0/mc^2$), but not to the second one (corrections in $\mathscr{E}_{\mathrm{v}}^0/\hbar\omega$ only appear in the kinetic energy).

(ii) Electric and magnetic form factors due to the vibration of the charge and of the spin magnetic moment.

Let us now apply these general results to the interpretation of two important radiative corrections.

4.1. Electron in a Coulomb static field: Lamb shift

Omitting the constant terms $(mc^2 + \mathscr{E}_{\mathrm{v}}^0)$, using the fact that $A_0 = \boldsymbol{B}_0 = 0$, one gets for $\mathscr{H}_{\mathrm{eff}}$:

$$\mathscr{H}_{\mathrm{eff}} = \frac{p^2}{2m}\left(1 - \frac{\mathscr{E}_{\mathrm{v}}^0}{mc^2} - \frac{4}{3}\frac{\mathscr{E}_{\mathrm{v}}^0}{\hbar\omega}\right) + e\phi_0 + e\frac{\mathscr{E}_{\mathrm{v}}^0}{3m\omega^2}\Delta\phi_0 + \mathscr{H}_{\mathrm{fs}}. \qquad (4.5)$$

The correction to the kinetic energy term is the contribution of the modes ω to the mass renormalisation. It does not change the relative position of the energy levels and, in particular, does not remove the degeneracy between the $2S_{1/2}$ and $2P_{1/2}$ states. The only remaining radiative correction is the third term, which we have interpreted as due to the averaging of the electrostatic potential by the electron vibrating in the vacuum fluctuations. We arrive in this way at the well known interpretation of the Lamb shift (Welton 1948). Let us remark that the high-frequency condition (3.7) implies here that $\hbar\omega$ is larger than one rydberg.

4.2. *Electron in a static magnetic field: $g - 2$ anomaly*

Keeping only the significant terms, we get for the effective Hamiltonian of an electron in a uniform static magnetic field $\boldsymbol{B}_0$:

$$\mathscr{H}_{\text{eff}} = \frac{\pi_0^2}{2m}\left(1 - \frac{\mathscr{E}_\text{v}^0}{mc^2} - \frac{4}{3}\frac{\mathscr{E}_\text{v}^0}{\hbar\omega}\right) - \frac{e\hbar}{2m}\boldsymbol{\sigma}.\boldsymbol{B}_0\left(1 - \frac{5}{3}\frac{\mathscr{E}_\text{v}^0}{mc^2}\right). \tag{4.6}$$

The first term describes the cyclotron motion of the electric charge. Radiative corrections reduce the cyclotron frequency by a factor

$$\left(1 - \frac{\mathscr{E}_\text{v}^0}{2mc^2} - \frac{4}{3}\frac{\mathscr{E}_\text{v}^0}{\hbar\omega}\right).$$

The second term describes the spin Larmor precession, the frequency of which is also reduced by a factor

$$\left(1 - \frac{5}{3}\frac{\mathscr{E}_\text{v}^0}{mc^2}\right).$$

Going through the calculations, one easily identifies the $\mathscr{E}_\text{v}^0/mc^2$ terms as due to vacuum fluctuations, and the $\mathscr{E}_\text{v}^0/\hbar\omega$ terms as due to the radiation reaction of the charge (see equations (3.8) and (3.13)).

It is clear that neglecting the radiation reaction would lead to a Larmor frequency less than the cyclotron frequency ($\frac{5}{3}$ is larger than 1) and consequently to $g - 2 < 0$. Therefore, any attempt to understand the $g - 2$ anomaly as resulting from the vibration of the electric charge and of the spin moment in the vacuum fluctuations is doomed to failure.

The important point to realise is that the radiation reaction further reduces the cyclotron frequency and does not affect the Larmor frequency. In addition, in the non-relativistic limit where $\hbar\omega$ is much smaller than mc^2, this effect is by far the most important. Thus, when the radiation reaction is included, the net effect of non-relativistic modes is to reduce the cyclotron frequency more than the Larmor one, and consequently their contribution to $g - 2$ is positive.

To summarise, the positive sign of $g - 2$ appears in the non-relativistic limit as due to the fact that a charge is a source more efficiently coupled to the radiation field than a magnetic moment; this explains why the radiation reaction slows down the cyclotron motion more than the spin precession.

4.3. *Discussion about some previous interpretations of $g - 2$*

Numerous attempts have been made to give a simple explanation of $g - 2$, at least of its sign. Without pretending to give an account of all of them, we discuss here some of the physical ideas which have been put forward.

After the success of the Welton's picture for the Lamb shift, models only taking into account the magnetic coupling of the spin moment with the radiation field have been considered (Welton 1948, Senitzky 1973). As mentioned above, one finds in this case a negative $g - 2$, unless unrealistic frequency spectra are introduced for the magnetic vacuum fluctuations (Bourret 1973, Itzykson 1974).

This failure led to the idea that the positive sign of $g - 2$ has something to do with the complex dynamics of the Dirac electron, requiring the introduction of negative energy or multi-particle states. This point of view, suggested by Welton (1948), has been investigated by Koba (1949), and later on by Weisskopf (1974), Lai *et al* (1974), Grotch and Kazes (1976). The physical idea developed by Koba is that the electron spin magnetic moment is related to the Zitterbewegung (see also Huang 1952), which can be visualised as a small ring current. Under the effect of vacuum fluctuations, this ring current not only rolls and pitches, which reduces its magnetic moment, but also vibrates in its own plane, increasing its surface, and consequently its magnetic moment. The second effect is expected to predominate, which explains the positive sign of $g - 2$. Actually, Koba does not calculate precisely the perturbed magnetic moment, but something like the 'delocalisation' of the electron, which is, of course, increased. In our opinion, this delocalisation does not imply necessarily an enhancement of the magnetic moment: if, for instance, the fluctuations of the electron position and velocity are in quadrature, the mean magnetic moment remains unchanged. Furthermore, this model seems questionable for frequencies smaller than mc^2: in this case, the electromagnetic field appears as constant both in space and time on the Zitterbewegung scale and one would rather expect the ring current to be displaced as a whole, without change of its internal structure†.

To summarise, we do not think that it has been really demonstrated that the positive sign of $g - 2$ is related to a modification of the Zitterbewegung by the vacuum fluctuations. This picture is certainly wrong for non-relativistic modes and its validity at higher frequencies is not obvious.

In fact, these various treatments overlook an important point. It is not sufficient to consider the modification of the spin magnetic moment. The perturbed Larmor frequency has to be compared to the perturbed cyclotron frequency, as was done in Avan *et al* (1976) and in this paper. In other words, the mass renormalisation has to be performed as has been emphasised by Grotch and Kazes (1975). With such a point of view, we have shown in this paper that the positive sign of $g - 2$ arises quite naturally, even for non-relativistic modes. Using a slightly different method, Grotch and Kazes (1977) arrive at the same conclusion.

5. Some critical remarks

As mentioned in the introduction, the approach followed in this paper has its own imperfections and limitations. In this section, we analyse them and discuss some possible improvements.

† In this matter we are actually following Weisskopf (1974) who considers that only high-frequency modes ($\hbar\omega > mc^2$) could change the Zitterbewegung.

Another argument for the non-enhancement of the spin magnetic moment can be derived from the calculations of Avan *et al* (1976), which are in the non-relativistic limit strictly equivalent to the ones in the Dirac representation, and which do not predict any effect of this type.

5.1. *Lack of covariance*

The first anomaly to be noted is the absence in (4.1) of a correction to the rest mass energy of the order of $mc^2\mathscr{E}_v^0/\hbar\omega$. Since the radiation reaction increases the electron inertia by a $4m\mathscr{E}_v^0/3\hbar\omega$ term, a similar correction should also be present for the rest mass energy. It is easy to understand that such a correction does not appear in the theory because we have discarded from the beginning the electron Coulomb self-energy. Radiation reaction and Coulomb self-energy are in fact closely related: radiation reaction only expresses that, for accelerating the electron, one must also push the Coulomb field associated with it, and this results in an extra inertia. However, in the Coulomb gauge, the Coulomb self-energy is infinite and cannot be incorporated in our point of view where the contribution of each mode is isolated. The solution to this difficulty would be to leave the Coulomb gauge and to introduce longitudinal and temporal modes to describe the total field associated with the electron. The mean correction to the electron energy due to the ω longitudinal and temporal modes can be worked out without great difficulty. This additional correction results in a single term:

$$mc^2 \frac{\mathscr{E}_v^0}{\hbar\omega} \tag{5.1}$$

which must be added to the effective Hamiltonian (4.1). Thus we get a term with the correct order of magnitude, but not with the $\frac{4}{3}$ factor expected. Actually, one must not be surprised by this lack of covariance of the effective Hamiltonian. We have only considered the effect of a shell of modes with frequency ω, and this is obviously not a Lorentz invariant object. The covariance can be restored only when the contribution of all modes is taken into account (i.e. when the integration over ω is performed), and this cannot be done here as a consequence of our non-relativistic approximations.

5.2. *Many-particle effects*

A single-particle theory has been used throughout this paper. It seems generally accepted that for smoothly varying external fields (i.e. when the electron energy spectrum can be split in two well separated positive and negative energy multiplicities), the single-particle Dirac theory gives sensible results. However, when the interaction with the quantised radiation field is considered, as we do here, negative energy states cannot be so easily discarded. They have to be occupied in order to avoid radiative decays from positive to negative energy states. It is also well known that many-particle effects are essential to get an electron self-energy which diverges only logarithmically (Weisskopf 1939). Since we have included, in our single-particle theory, dynamical relativistic corrections up to $1/c^2$, it is not obvious that many-particle effects do not introduce in the contribution of low-frequency modes additional corrections of the same order of magnitude as those already considered. Thus, starting from a single-particle Dirac equation for dealing with electron radiative corrections has to be justified.

This can be done in the following way. Working in the second quantisation picture, one concentrates on states where all negative energy levels are occupied and where one extra electron has a positive non-relativistic energy. Such states are coupled by the radiation field to other ones in which electron–positron pairs and photons have been created. In the non-relativistic limit used here, all these states have an energy quite different from the initial one. Thus one can take into account these non-resonant couplings by introducing an effective Hamiltonian for the original electron. In this way, the one-particle states are decoupled from many-particle ones, the effect of virtual transitions to these states being described by an effective Hamiltonian which is an ordinary one-particle non-relativistic Hamiltonian.

We have done such a calculation and we have found that the effective Hamiltonian describing the dynamics of the extra positive-energy electron is actually almost identical to the one used in this paper (2.1) (when the same non-relativistic approximations are made). This result is not too surprising: when mc^2 is much larger than any other energy involved in the problem, the coupling with negative-energy states simulates quite well the coupling with many-particle states. There are, however, two differences. First, the correct theory contains the photon renormalisation and vacuum polarisation effects† which are absent here. Secondly, the V^0 term in (2.1), which arises from the quantum Foldy–Wouthuysen transformation, appears in the many-particle theory with a negative sign. This last point is consistent with the fact that many-particle effects reduce the electron self-energy. Let us recall that the V^0 term has finally been neglected (see §3.4).

In conclusion, the results of the calculations presented here appear to coincide with those derived from a consistent QED approach and hence are quite reliable. This would not have been the case for terms of higher order in $\mathscr{E}_{\mathrm{v}}^0$ or $1/c^2$.

5.3. *Effect of relativistic modes*

We have repeatedly emphasised that our conclusions only concern the effect of modes with a frequency much smaller than mc^2. If it happened that the main contribution to $g - 2$ comes from relativistic or ultra-relativistic modes, then the physical pictures developed here would be of little interest, since they are not relevant for them. Hence it seems very desirable to extend the previous calculations, particularly the effective Hamiltonian method, to all frequencies, within a many-particle theory. It is clear that such calculations would be less elegant than those of covariant QED. One can hope, however, that they would provide more physical insight into electron radiative corrections. We will consider such a generalisation in subsequent publications.

Preliminary results concerning $g - 2$ have already been obtained. The contribution of each shell of modes to $g - 2$ has been calculated. It appears that the main contribution comes from frequencies smaller than mc^2. The integration over ω gives the correct result $\alpha/2\pi$. Thus the physical pictures given here are not invalidated by the contribution of relativistic modes.

6. Summary

(i) The contribution to radiative corrections of the non-relativistic modes of the radiation field has been determined. The corresponding modifications of the electron dynamics are detailed explicitly in the form of a simple effective Hamiltonian derived from a single-particle theory in the non-relativistic limit.

† Let us recall that vacuum polarisation effects represent only 3% of the Lamb shift and do not contribute to $g - 2$.

(ii) A term to term comparison between the effective Hamiltonians describing stimulated and spontaneous corrections provides a clue to the physical interpretation of radiative corrections. Two main physical effects are involved: the vibration of the electron charge and spin moment due to vacuum fluctuations and the radiation reaction of the charge.

(iii) It is shown that non-relativistic modes of the radiation field contribute with a positive sign to $g - 2$. The radiation reaction is found to play an essential role in the explanation of $g - 2$ whereas the Lamb shift is simply interpreted by the averaging of the Coulomb potential by the vibrating charge (Welton's picture).

(iv) Many-particle effects and contributions of relativistic modes do not alter these conclusions significantly.

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Appendix 1

We give here some intermediate steps in the calculations which lead from the basic Hamiltonian (2.1) to the effective Hamiltonian (3.8) and (3.12). Further details may be found in Avan *et al* (1976).

In the basic Hamiltonian (2.1), one isolates the unperturbed electronic Hamiltonian

$$\mathscr{H}_{\mathrm{e}} = mc^2 + \frac{\pi_0^2}{2m} + e\phi_0 - \frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0 + \frac{e\hbar^2}{8m^2c^2}\Delta\phi_0 + \frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\nabla\phi_0 \times \boldsymbol{\pi}_0)$$

$$- \frac{1}{2mc^2}\left(\frac{\pi_0^2}{2m} - \frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\right)^2 \qquad \text{(A.1.1)}$$

where

$$\boldsymbol{\pi}_0 = \boldsymbol{p} - e\boldsymbol{A}_0 \qquad \text{(A.1.2)}$$

and the coupling $\mathscr{H}_{\mathrm{I}}$ between the electron and the radiation field is written in the following form:

$$\mathscr{H}_{\mathrm{I}} = V^0 + V^- a + V^+ a^\dagger + V^{-+} aa^\dagger + V^{+-} a^\dagger a + \dots \qquad \text{(A.1.3)}$$

Terms in a^2, $a^{\dagger 2}$ and of higher order in a and $a^\dagger$ are neglected. One finds:

$$V^0 = \mathscr{E}_{\mathrm{v}}^0 \frac{\hbar\omega}{2mc^2} \qquad \text{(A.1.4)}$$

$$V^- = (V^+)^\dagger = \left(\frac{2\mathscr{E}_v^0}{m}\right)^{1/2}\Bigg\{-\epsilon\,.\,\pi_0\exp(\mathrm{i}k\,.\,r) - \frac{\mathrm{i}\hbar}{2}\sigma\,.\,(k\times\epsilon)\exp(\mathrm{i}k\,.\,r)$$

$$+\frac{e\hbar}{4mc^2}\sigma\,.\,(E_0\times\epsilon)\exp(\mathrm{i}k\,.\,r) - \frac{\mathrm{i}\hbar\omega}{4mc^2}\sigma\,.\,(\epsilon\times\pi_0)\exp(\mathrm{i}k\,.\,r)$$

$$+\frac{1}{4mc^2}\Bigg[[2\epsilon\,.\,\pi_0\exp(\mathrm{i}k\,.\,r) + \mathrm{i}\hbar\sigma\,.\,(k\times\epsilon)\exp(\mathrm{i}k\,.\,r)]$$

$$\times\left(\frac{\pi_0^2}{2m} - \frac{e\hbar}{2m}\sigma\,.\,B_0\right) + \mathrm{sym}\Bigg]\Bigg\} \qquad \text{(A.1.5)}$$

$$V^{-+} = (V^{+-})^\dagger = \mathscr{E}_v^0 - \frac{\mathscr{E}_v^0}{mc^2}\left(\frac{\pi_0^2}{2m} - \frac{e\hbar}{2m}\sigma\,.\,B_0\right) - \frac{\mathscr{E}_v^0}{mc^2}\left(\frac{(\epsilon\,.\,\pi_0)^2}{m} + \frac{\hbar^2\omega^2}{4mc^2}\right) + \ldots \qquad \text{(A.1.6)}$$

$\mathscr{E}_v^0$ is defined by formula (3.6). When the 'off-diagonal' elements of $\mathscr{H}_1$ are removed by a unitary transformation, one gets (2.4) and (2.5), where

$$R = V^0 + V^{-+} - \frac{1}{\hbar\omega}V^-V^+ - \frac{1}{2\hbar^2\omega^2}([\mathscr{H}_e, V^-]V^+ - V^-[\mathscr{H}_e, V^+])$$

$$-\frac{1}{2\hbar^3\omega^3}(\mathscr{H}_e[\mathscr{H}_e, V^-]V^+ - V^-[\mathscr{H}_e, V^+]\mathscr{H}_e + V^-\mathscr{H}_e[\mathscr{H}_e, V^+]$$

$$-[\mathscr{H}_e, V^-]\mathscr{H}_eV^+) + \ldots. \qquad \text{(A.1.7)}$$

S is identical to R with the exchange of + and − superscripts, and the change of the sign of ω and V^0.

Appendix 2

In this appendix we derive the radiation reaction of a particle with charge e and magnetic moment μ interacting with the mode (k, ϵ) of the electromagnetic field. The particle, as well as the radiation field, are treated *classically*. In order to make the parallel with the quantum calculation, we use the Hamiltonian formalism and we develop the electromagnetic field on the same plane-wave basis as in the quantum theory. However, the operators a and $a^\dagger$ are here classical variables α and α^*. Despite the presence of $\hbar$ in the intermediate calculations, due to the particular choice of the basis functions, it should not be overlooked that our point of view here is entirely classical.

The development of the electromagnetic field on the plane waves is the following:

$$A(\rho, t) = \sum_{k,\epsilon}\alpha_{k,\epsilon}(t)\mathscr{A}_{k,\epsilon}(\rho) + \mathrm{cc} \qquad \text{(A.2.1}a\text{)}$$

$$E(\rho, t) = \sum_{k,\epsilon}\alpha_{k,\epsilon}(t)\mathscr{E}_{k,\epsilon}(\rho) + \mathrm{cc} \qquad \text{(A.2.1}b\text{)}$$

$$B(\rho, t) = \sum_{k,\epsilon}\alpha_{k,\epsilon}(t)\mathscr{B}_{k,\epsilon}(\rho) + \mathrm{cc} \qquad \text{(A.2.1}c\text{)}$$

where

$$\mathscr{A}_{k,\epsilon} = \left(\frac{\hbar}{2\epsilon_0 \omega L^3}\right)^{1/2} \epsilon \exp(\mathrm{i}k \cdot \rho)$$

$$\mathscr{B}_{k,\epsilon} = \mathrm{i}k \times \mathscr{A}_{k,\epsilon} \tag{A.2.2}$$

$$\mathscr{E}_{k,\epsilon} = \mathrm{i}\omega \mathscr{A}_{k,\epsilon}.$$

The current density associated with the electron is the source of the field. It contains two terms, $J^{(1)}$ associated with the charge and $J^{(2)}$ associated with the magnetic moment: r being the electron position and π_0/m its velocity in the external fields, $J^{(1)}$ and $J^{(2)}$ are given by (see for instance Jackson 1971):

$$J^{(1)}(\rho) = e\frac{\pi_0}{m}\delta(r - \rho) \tag{A.2.3}$$

$$J^{(2)}(\rho) = \nabla_\rho \times [\mu\, \delta(r - \rho)]. \tag{A.2.4}$$

We need essentially the projections of these currents onto the mode (k, ϵ), defined by

$$\mathscr{J}_{k,\epsilon} = \int \mathrm{d}^3\rho\, \mathscr{A}_{k,\epsilon}(\rho) \cdot J(\rho). \tag{A.2.5}$$

We find

$$\mathscr{J}^{(1)}_{k,\epsilon} = e\frac{\pi_0}{m} \cdot \mathscr{A}^*_{k,\epsilon}(r) \tag{A.2.6}$$

$$\mathscr{J}^{(2)}_{k,\epsilon} = \mu \cdot \mathscr{B}^*_{k,\epsilon}(r). \tag{A.2.7}$$

With these notations, the Maxwell equations reduce to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_{k,\epsilon} = -\mathrm{i}\omega\, \alpha_{k,\epsilon} + \frac{\mathrm{i}}{\hbar}\mathscr{J}_{k,\epsilon}. \tag{A.2.8}$$

Since we assume that the frequencies associated with the motion of the electron in the static fields are low compared to ω, an approximate solution of (A.2.8) is

$$\alpha_{k,\epsilon} \simeq \frac{1}{\hbar\omega}\mathscr{J}_{k,\epsilon} = \frac{1}{\hbar\omega}\left(e\frac{\pi_0}{m} \cdot \mathscr{A}^*_{k,\epsilon} + \mu \cdot \mathscr{B}^*_{k,\epsilon}\right). \tag{A.2.9}$$

From this equation, using (A.2.1a, b, c), one gets the radiation field associated with the electron.

The energy of the total system is given by

$$\mathscr{H} = \frac{1}{2m}[\pi_0 - eA(r)]^2 - \mu \cdot B_0 - \mu \cdot B(r) + \sum_{k,\epsilon} \hbar\omega\, \alpha^*_{k,\epsilon}\alpha_{k,\epsilon} \tag{A.2.10}$$

where $A(r)$, $B(r)$ and $\hbar\omega\, \alpha^*_{k,\epsilon}\alpha_{k,\epsilon}$ are calculated from (A.2.9), (A.2.7), (A.2.6), (A.2.1a, b, c) and (A.2.2).

The energy correction $\delta\mathscr{H}$ due to the radiation reaction is defined by

$$\delta\mathscr{H} = \mathscr{H} - \left(\frac{\pi_0^2}{2m} - \frac{e\hbar}{2m}\sigma \cdot B_0\right). \tag{A.2.11}$$

Keeping only terms of second order with respect to the coupling between the electron and the radiation field, one gets for $\delta\mathscr{H}$:

$$\delta\mathscr{H} = -\frac{e}{m}\pi_0 \cdot A(r) - \mu \cdot B(r) + \sum_{k,\epsilon} \hbar\omega\,\alpha_{k,\epsilon}^{*}\alpha_{k,\epsilon} \tag{A.2.12}$$

$$= \sum_{k,\epsilon} - \left[\left(\frac{e}{m}\pi_0 \cdot \mathscr{A}_{k,\epsilon} + \mu\mathscr{B}_{k,\epsilon}\right)\alpha_{k,\epsilon} + \mathrm{cc}\right] + \hbar\omega\alpha_{k,\epsilon}^{*}\alpha_{k,\epsilon}. \tag{A.2.13}$$

$\delta\mathscr{H}$ is simply interpreted as the interaction energy of the charge and magnetic moment with the field which they have created, plus the energy of this field. From (A.2.9), the coefficient of $\alpha_{k,\epsilon}$ in the bracket is simply $\hbar\omega\alpha_{k,\epsilon}^{*}$, so that $\delta\mathscr{H}$ appears to be negative and equal to

$$\delta\mathscr{H} = -\sum_{k,\epsilon} \hbar\omega\,\alpha_{k,\epsilon}^{*}\alpha_{k,\epsilon}. \tag{A.2.14}$$

Replacing $\alpha_{k,\epsilon}$ by its expression (A.2.9) as a function of the electronic variables, one gets

$$\delta\mathscr{H} = \sum_{k,\epsilon} - \left(\frac{\hbar}{2\epsilon_0\omega L^3}\right)\frac{1}{\hbar\omega}\left[e^2\left(\frac{\epsilon\pi_0}{m}\right)^2 + (k \times \epsilon \cdot \mu)^2\right]. \tag{A.2.15}$$

The first term is the correction to the kinetic energy due to the radiation reaction of the charge. Introducing $\mathscr{E}_{\mathrm{v}}^{0}$ (see formula (3.6)), one finds the corresponding term of (2.7) exactly:

$$-\frac{\mathscr{E}_{\mathrm{v}}^{0}}{\hbar\omega}\,\frac{2(\epsilon\pi_0)^2}{m}. \tag{A.2.16}$$

The second term of (A.2.15) represents the radiation reaction of the magnetic moment. For the electron, we have

$$\mu_x^2 = \left(\frac{e\hbar}{2m}\right)^2 \tag{A.2.17}$$

so that we find again the third term of (2.7):

$$-\mathscr{E}_{\mathrm{v}}^{0}\hbar\omega/(2mc^2). \tag{A.2.18}$$

It is worth noting that there is no cross term between the charge radiation reaction and the spin one.

References

Avan P, Cohen-Tannoudji C, Dupont-Roc J and Fabre C 1976 *J. Physique* **37** 993
Baier V N and Mil'Shtein A I 1976 *Sov. Phys.–Dokl.* **21** 83
Bethe H A 1947 *Phys. Rev.* **72** 339
Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill) p 48
Blount E I 1962 *Solid State Physics* vol 13, ed F Seitz and D Turnbull (New York: Academic Press)
Bourret R 1973 *Lett. Nuovo Cim.* **7** 801
Feynman R P 1961 *Frontiers in Physics, Quantum Electrodynamics* (New York: Benjamin) p 50
——1962 *Théorie Quantique des Champs* (New York: Interscience)
Grotch H and Kazes E 1975 *Phys. Rev. Lett.* **35** 124
——1976 *Phys. Rev.* D **13** 2851, erratum D **15** 1184
——1977 *Am. J. Phys.* **45** 618

Huang K 1952 *Am. J. Phys.* **20** 479
Itzykson C 1974 *Commun. Math. Phys.* **36** 19
Jackson J D 1971 *Classical Electrodynamics* 2nd ed (New York: Wiley) pp 188, 233, 672
Kittel C 1963 *Quantum Theory of Solids* (New York: Wiley) problem 6, p 148
Koba Z 1949 *Prog. Theor. Phys.* **4** 319
Lai S B, Knight P L and Eberly J H 1974 *Phys. Rev. Lett.* **32** 494
——1975 *Phys. Rev. Lett.* **35** 126
Milonni P W, Ackerhalt J R and Smith W A 1973 *Phys. Rev. Lett.* **31** 958
Nozieres P and Pines D 1958 *Nuovo Cim.* **9** 470
Primas H 1963 *Rev. Mod. Phys.* **35** 710
Senitzky I R 1973 *Phys. Rev. Lett.* **31** 955
Weisskopf V 1939 *Phys. Rev.* **56** 72
——1974 *Lepton and Hadron Structures, Subnuclear Physics Series* vol 12 ed A Zichichi (New York: Academic Press)
Welton T A 1948 *Phys. Rev.* **74** 1157