J. Phys. B: At. Mol. Phys. 17 (1984) 4623-4641. Printed in Great Britain

Deflection of an atomic beam by a laser wave: transition between diffractive and diffusive regimes

C Tanguy†, S Reynaud and C Cohen-Tannoudji

Laboratoire de Spectroscopie Hertzienne de l'Ecole Normale Supérieure et Collège de France, 24 rue Lhomond, F 75231 Paris Cedex 05, France

Received 21 June 1984

Abstract. The exchange of momentum between atoms and photons in a deflection experiment is usually described by different formalisms depending on whether the interaction time T is short or long compared with the radiative lifetime τ_R. We present here a new approach to this problem leading to a single theoretical expression valid in both limits and therefore allowing the transition between them to be studied. We interpret in this way the resonant Kapitza–Dirac effect and the optical Stern and Gerlach effect appearing in the short-time limit ($T \ll \tau_R$) as well as the deflection profiles usually deduced from a Fokker–Planck equation in the long-time limit ($T \gg \tau_R$). The transition between these two regimes is interpreted in terms of momentum transfer due to absorption and stimulated emission of laser photons, convoluted by the distribution of recoil due to spontaneously emitted photons.

1. Introduction

We present in this paper a theoretical treatment of the deflection of an atomic beam by a laser wave, allowing the study of the transition between regimes corresponding to short and long interaction times (as compared with the atomic radiative lifetime).

More precisely, we consider a monoenergetic atomic beam propagating along the $0z$ axis, crossing at right angles a monochromatic laser wave propagating along $0x$, which can be a progressive or a standing wave. One measures the final distribution of the atomic momentum along $0x$. All atoms are supposed to have the same velocity $v_z = v_0$ along $0z$, and the time of flight $T = l/v_0$ through the width l of the interaction zone is the *interaction time*. All subsequent calculations will be done in the initial atomic rest frame moving with the velocity v_0 along $0z$. Another important time is the *radiative* lifetime $\tau_R = \Gamma^{-1}$ of the atomic excited state e (Γ is the natural width of e).

We shall call $v_\perp$ the root-mean-square velocity of the atoms along $0x$, due to an imperfect collimation of the atomic beam and to the transfer of momentum from the laser beam. We suppose in this paper that

$$kv_\perp \ll \Gamma \tag{1.1}$$

i.e. that the Doppler effect in the laser–atom interaction is negligible compared with

† Present address: Groupe de Physique des Solides de l'Ecole Normale Supérieure, 24 rue Lhomond, F 75231 Paris Cedex 05, France.

0022-3700/84/234623+19$02.25

the natural width of e. We also suppose that

$$kv_\perp \ll T^{-1}. \tag{1.2}$$

Condition (1.2), which can be also written $v_\perp T \ll \lambda$, where λ is the laser wavelength, means that the atom does not move appreciably along $0x$ during the interaction time, so that it will be possible to neglect the perpendicular displacement of the atoms during T. Since $v_\perp$ is at least equal to the recoil velocity $\hbar k/M$ associated with the absorption or emission of a photon of energy $\hbar\omega = \hbar ck$ (M is the atomic mass), conditions (1.1) and (1.2) imply that

$$E_{\text{rec}} \ll \hbar\Gamma, \hbar T^{-1} \tag{1.3}$$

where $E_{\text{rec}} = \hbar^2 k^2/2M$ is the recoil energy.

Conditions (1.1) and (1.2) are the basic assumptions considered in this paper. Note that we do not introduce any restriction on T/τ_R which can be small or large compared with one. The main motivation of this paper is actually to try to present a unified treatment of the various physical effects which can be observed in the domain $T \ll (kv_\perp)^{-1}$ and which are usually described by different formalisms depending on whether T is very small or very large compared with τ_R. In particular, we would like to study the transition between the regime $T \ll \tau_R$, which we will call the 'diffractive regime' (for reasons which will become clear later on), and which is usually described by a Schrödinger equation (Cook and Bernhardt 1978, Bernhardt and Shore 1981, Arimondo *et al* 1981, Compagno *et al* 1982) and the regime $T \gg \tau_R$, which we will call the 'diffusive regime' and which is usually described by a Fokker–Planck equation (Javanainen and Stenholm 1980, Cook 1980a, Letokhov and Minogin 1981, Minogin 1981, Kazantsev *et al* 1981).

The paper is organised in the following way. In § 2 we introduce the theoretical framework. We start from the 'generalised optical Bloch equations' in position representation, which describe the coupled evolution of both internal and translational atomic degrees of freedom during the interaction time. The possibility of neglecting the free flight along $0x$ (condition (1.2)) introduces great simplifications and allows us to derive a simple relation between the incoming and outgoing Wigner functions, which can be interpreted in terms of a 'quasi-probability' $G(x, q, T)$ for a momentum transfer q to an atom crossing the laser in x. The basic problem is then to understand how $G(x, q, T)$ changes when T/τ_R increases from very small to very great values. The short-time limit ($T \ll \tau_R$) is considered in § 3. We show that $G(x, q, T)$ reduces in this limit to a comb of δ functions of q, describing discrete exchanges of momentum between the atom and the laser wave and we interpret in this way two important physical effects, the resonant Kapitza–Dirac effect (Delone *et al* 1980) and the optical Stern–Gerlach effect (Kazantsev 1978, Cook 1978). The long-time limit ($T \gg \tau_R$) is then considered in § 4. We show that $G(x, q, T)$ can be, in such a case, approximated by a Gaussian function of q. The connection with the Fokker–Planck equation approach is made by noting that $G(x, q, T)$ is nothing but the Green's function of such an equation. We show how the $G(x, q, T)$ function can be used for interpreting the shape and the width of the deflection profile. Finally, we consider in § 6 the domain of intermediate times ($T \sim \tau_R$), and we analyse $G(x, q, T)$ in terms of momentum transfer due to the absorption of laser photons 'convoluted' by the distribution of recoil due to the spontaneously emitted photons. This will allow us to show how the approach

followed in this paper can be related to the problem of photon statistics in resonance fluorescence (Mandel 1979, Cook 1980b, Cook 1981, Stenholm 1983, Reynaud 1983 and references therein).

2. Theoretical framework

2.1. Representations of the atomic density operator

We consider in this paper two-level atoms. We denote: g the ground state, e the excited state, $\hbar\omega_0$ the energy separation between e and g, $\boldsymbol{D}$ the atomic dipole moment operator, $\boldsymbol{d} = \langle e|\boldsymbol{D}|g\rangle$ the matrix element of $\boldsymbol{D}$ between e and g, $\boldsymbol{R}$ and $\boldsymbol{P}$ the position and momentum of the centre of mass.

In the position representation (basis of eigenstates $|\boldsymbol{r}'\rangle$ of $\boldsymbol{R}$ with eigenvalues $\boldsymbol{r}'$), the density operator σ is represented by the matrix

$$\sigma_{ab}(\boldsymbol{r}', \boldsymbol{r}'') = \langle a, \boldsymbol{r}'|\sigma|b, \boldsymbol{r}''\rangle \tag{2.1}$$

with $a, b = e$ or g. It will be convenient, for the following, to introduce the following change of variable

$$\begin{aligned} \boldsymbol{r} &= \tfrac{1}{2}(\boldsymbol{r}' + \boldsymbol{r}'') \\ \boldsymbol{u} &= \boldsymbol{r}' - \boldsymbol{r}'' \end{aligned} \tag{2.2}$$

and to define the density matrix in the '$\{\boldsymbol{r}, \boldsymbol{u}\}$ representation' by

$$\sigma_{ab}(\boldsymbol{r}, \boldsymbol{u}) = \langle a, \boldsymbol{r} + \tfrac{1}{2}\boldsymbol{u}|\sigma|b, \boldsymbol{r} - \tfrac{1}{2}\boldsymbol{u}\rangle. \tag{2.3}$$

By Fourier transform with respect to $\boldsymbol{u}$, one gets the well known 'Wigner representation' of σ (Wigner 1932, Takabayasi 1954, De Groot and Suttorp 1972)

$$w_{ab}(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{h^3}\int \mathrm{d}^3u\, \sigma_{ab}(\boldsymbol{r}, \boldsymbol{u}) \exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{u}/\hbar). \tag{2.4}$$

The Wigner representation is more generally used than the $\{\boldsymbol{r}, \boldsymbol{u}\}$ one. However, for the situation considered in this paper, calculations are simpler in the $\{\boldsymbol{r}, \boldsymbol{u}\}$ representation. A similar representation is actually used by Stenholm (1983), and $\boldsymbol{u}$ (which is denoted $\boldsymbol{x}$) is considered as representing the 'amount of off-diagonality of the density matrix in the position representation' (this clearly appears in the second equation (2.2)). Note however that the $\boldsymbol{r}$ dependence of $\sigma_{ab}(\boldsymbol{r}, \boldsymbol{u})$ is not introduced by Stenholm (1983), whereas it will play an important role in the following calculations, when the laser amplitude is $\boldsymbol{r}$ dependent (for example, for a standing wave).

2.2. Description of the laser field

We denote

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) = \boldsymbol{\varepsilon}\mathcal{E}_0(\boldsymbol{r}) \cos(\omega t + \phi(\boldsymbol{r})) \tag{2.5}$$

the monochromatic laser field with frequency ω, amplitude $\mathcal{E}_0(\boldsymbol{r})$, phase $\phi(\boldsymbol{r})$ and polarisation $\boldsymbol{\varepsilon}$ ($\mathcal{E}_0$ and ϕ are real functions of $\boldsymbol{r}$, $\boldsymbol{\varepsilon}$ is supposed to be independent of $\boldsymbol{r}$ and linear). A plane progressive wave corresponds to

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) = \boldsymbol{\varepsilon}\mathcal{E}_0 \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) \tag{2.6}$$

i.e., to a uniform amplitude $\mathscr{E}_0$ and to a phase $\phi(\boldsymbol{r}) = -\boldsymbol{k}\cdot\boldsymbol{r}$, whereas a plane standing wave

$$\boldsymbol{\mathscr{E}}(\boldsymbol{r}, t) = \boldsymbol{\varepsilon}\mathscr{E}_0 \cos \boldsymbol{k}\cdot\boldsymbol{r} \cos \omega t \tag{2.7}$$

has a zero uniform phase and a sinusoidal amplitude $\mathscr{E}_0(\boldsymbol{r}) = \mathscr{E}_0 \cos \boldsymbol{k}\cdot\boldsymbol{r}$.

It is convenient to introduce the positive ($\mathscr{E}^+$) and negative ($\mathscr{E}^-$) frequency components of $\boldsymbol{\mathscr{E}}$ defined by

$$\boldsymbol{\mathscr{E}}(\boldsymbol{r}, t) = \boldsymbol{\varepsilon}\mathscr{E}^+(\boldsymbol{r}) \exp(-\mathrm{i}\omega t) + \boldsymbol{\varepsilon}\mathscr{E}^-(\boldsymbol{r}) \exp(\mathrm{i}\omega t). \tag{2.8}$$

From (2.5) and (2.8), it follows that

$$\mathscr{E}^{\pm}(\boldsymbol{r}) = \tfrac{1}{2}\mathscr{E}_0(\boldsymbol{r}) \exp(\mp \mathrm{i}\phi(\boldsymbol{r})). \tag{2.9}$$

The laser–atom coupling is characterised by the $\boldsymbol{r}$-dependent Rabi frequency

$$\omega_1(\boldsymbol{r}) = -(\boldsymbol{\varepsilon}\cdot\boldsymbol{d})\mathscr{E}_0(\boldsymbol{r})/\hbar. \tag{2.10}$$

We will also use the following coupling parameter

$$\kappa(\boldsymbol{r}) = -(\boldsymbol{\varepsilon}\cdot\boldsymbol{d})\mathscr{E}^+(\boldsymbol{r})/\hbar \tag{2.11}$$

related to $\omega_1(\boldsymbol{r})$ by

$$\kappa(\boldsymbol{r}) = \tfrac{1}{2}\omega_1(\boldsymbol{r}) \exp(-\mathrm{i}\phi(\boldsymbol{r})). \tag{2.12}$$

2.3. *Generalised optical Bloch equations*

The equations of motion of the $\sigma_{ab}(\boldsymbol{r}, \boldsymbol{u})$ will be called the 'generalised optical Bloch equations' (GOBE), since they generalise the well known optical Bloch equations by including both internal (a, b) and external ($\boldsymbol{r}$, $\boldsymbol{u}$) quantum numbers. They can be written

$$\left(\frac{\partial}{\partial t} - \frac{\mathrm{i}\hbar}{M}\frac{\partial^2}{\partial r\,\partial u}\right)\sigma_{ee}(\boldsymbol{r}, \boldsymbol{u})$$

$$= -\Gamma\sigma_{ee}(\boldsymbol{r}, \boldsymbol{u}) - \mathrm{i}[\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})\sigma_{ge}(\boldsymbol{r}, \boldsymbol{u}) - \kappa^*(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})\sigma_{eg}(\boldsymbol{r}, \boldsymbol{u})] \tag{2.13a}$$

$$\left(\frac{\partial}{\partial t} - \frac{\mathrm{i}\hbar}{M}\frac{\partial^2}{\partial r\,\partial u}\right)\sigma_{gg}(\boldsymbol{r}, \boldsymbol{u})$$

$$= +\Gamma\chi(\boldsymbol{u})\sigma_{ee}(\boldsymbol{r}, \boldsymbol{u}) - \mathrm{i}[\kappa^*(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})\sigma_{eg}(\boldsymbol{r}, \boldsymbol{u}) - \kappa(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})\sigma_{ge}(\boldsymbol{r}, \boldsymbol{u})] \tag{2.13b}$$

$$\left(\frac{\partial}{\partial t} - \frac{\mathrm{i}\hbar}{M}\frac{\partial^2}{\partial r\,\partial u}\right)\sigma_{eg}(\boldsymbol{r}, \boldsymbol{u})$$

$$= [\mathrm{i}(\omega-\omega_0) - \tfrac{1}{2}\Gamma]\sigma_{eg}(\boldsymbol{r}, \boldsymbol{u}) - \mathrm{i}[\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})\sigma_{gg}(\boldsymbol{r}, \boldsymbol{u}) - \kappa(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})\sigma_{ee}(\boldsymbol{r}, \boldsymbol{u})] \tag{2.13c}$$

$$\left(\frac{\partial}{\partial t} - \frac{\mathrm{i}\hbar}{M}\frac{\partial^2}{\partial r\,\partial u}\right)\sigma_{ge}(\boldsymbol{r}, \boldsymbol{u})$$

$$= [-\mathrm{i}(\omega-\omega_0) - \tfrac{1}{2}\Gamma]\sigma_{ge}(\boldsymbol{r}, \boldsymbol{u}) - \mathrm{i}[\kappa^*(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})\sigma_{ee}(\boldsymbol{r}, \boldsymbol{u}) - \kappa^*(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})\sigma_{gg}(\boldsymbol{r}, \boldsymbol{u})]. \tag{2.13d}$$

In these equations, the terms in $(-\mathrm{i}\hbar/M)\,\partial^2/\partial r\,\partial u$ describe the effect of free flight (they come from the commutator of σ with the kinetic energy operator $\boldsymbol{P}^2/2M$). The terms proportional to the natural width Γ of e describe the relaxation due to spontaneous

emission. The population σ_{ee} of e and the 'optical coherences' σ_{eg} (or σ_{ge}) are damped with rates respectively equal to Γ and $\frac{1}{2}\Gamma$. The term in $\Gamma\chi(\boldsymbol{u})$ in (2.13b) describes the transfer of atoms from e to g by spontaneous emission. $\chi(\boldsymbol{u})$ is equal to

$$\chi(\boldsymbol{u}) = \int \mathrm{d}^2 n\, \phi(\boldsymbol{n}) \exp(-\mathrm{i}\omega_0 \boldsymbol{n}\cdot\boldsymbol{u}/c) \tag{2.14}$$

where $\phi(\boldsymbol{n})$ is the normalised angular distribution of spontaneous emission in the direction $\boldsymbol{n} = \boldsymbol{k}/k$. From the normalisation condition $\int \mathrm{d}^2 n\, \phi(\boldsymbol{n}) = 1$, it follows that

$$\chi(\boldsymbol{0}) = 1. \tag{2.15}$$

Finally, the terms in κ and κ^* describe the interaction with the laser field. They come from the commutator of σ with the interaction Hamiltonian $-\boldsymbol{D}\cdot\boldsymbol{\mathscr{E}}(\boldsymbol{R}, t)$ (in the rotating-wave approximation). Actually, equations (2.13) are written in a 'rotating frame' representation, which eliminates any explicit time dependence in $\exp(\pm\mathrm{i}\omega t)$.

By Fourier transform with respect to $\boldsymbol{u}$, equations (2.13) become the GOBE in the Wigner representation (Vorobev *et al* 1969, Baklanov and Dubetskii 1976, Javanainen and Stenholm 1980, Cook 1980a, Letokhov and Minogin 1981). Since the operator $-\mathrm{i}\hbar\partial/\partial\boldsymbol{u}$ is changed into $\boldsymbol{p}$ in such a transformation, the left-hand side of equations (2.13) becomes the 'hydrodynamic derivative' $\partial/\partial t + (\boldsymbol{p}/M)\cdot\partial/\partial\boldsymbol{r}$. The ordinary products of functions of $\boldsymbol{u}$ in the right-hand side become convolution products of functions of $\boldsymbol{k}$ expressing the momentum conservation in photon–atom interactions.

Suppose finally that we put $\boldsymbol{u} = \boldsymbol{0}$ in the right hand side of equations (2.13). Then, only $\kappa(\boldsymbol{r})$ and $\kappa^*(\boldsymbol{r})$ appear in the equations, and $\chi(\boldsymbol{u})$ is, according to (2.15), replaced by one, so that one gets the ordinary optical Bloch equations (Allen and Eberly 1975) (dealing only with internal variables, the atom being considered at rest in $\boldsymbol{r}$).

2.4. Simplifications appearing when free flight is neglected

Condition (1.2) means that one can neglect the spatial displacement of the atom along $0x$ during the interaction time T, even if it gets some momentum by absorbing and emitting photons. The same argument holds for the displacements along $0x$ and $0z$ (we recall that we are in the initial rest frame moving with velocity v_0 along $0z$). It is therefore possible to neglect the free-flight terms of equations (2.13) which describe the effect on σ of the spatial displacement of the atom. Such an approximation introduces great simplifications in the calculations (Tanguy 1983). Equations (2.13) become then *strictly local* in $\boldsymbol{r}$ and $\boldsymbol{u}$, i.e. they can be solved for each set $\{\boldsymbol{r}, \boldsymbol{u}\}$.

If we suppose the detection signal insensitive to the atomic internal state, it is convenient to introduce the trace of the density matrix over internal variables

$$F(\boldsymbol{r}, \boldsymbol{u}) = \sigma_{gg}(\boldsymbol{r}, \boldsymbol{u}) + \sigma_{ee}(\boldsymbol{r}, \boldsymbol{u}). \tag{2.16}$$

From the linearity and locality (in $\boldsymbol{r}$ and $\boldsymbol{u}$) of equations (2.13) (without free flight), it follows that the outgoing F function, $F_{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{u})$, depends linearly on the incoming one, $F_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{u})$, for each set $\boldsymbol{r}, \boldsymbol{u}$

$$F_{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{u}) = L(\boldsymbol{r}, \boldsymbol{u}, T) F_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{u}) \tag{2.17}$$

The 'linear filter' amplitude $L(\boldsymbol{r}, \boldsymbol{u}, T)$ depends of course on the interaction time T.

The explicit expression of L can be obtained by taking the Laplace transform of equations (2.13) (without free flight), which are transformed into algebraic equations. If

$$\tilde{L}(\boldsymbol{r}, \boldsymbol{u}, s) = \int_0^\infty \mathrm{d}t \exp(-st) L(\boldsymbol{r}, \boldsymbol{u}, t) \tag{2.18}$$

is the Laplace transform of $L(\boldsymbol{r}, \boldsymbol{u}, t)$, one finds that $\tilde{L}(\boldsymbol{r}, \boldsymbol{u}, s)$ can be written

$$\tilde{L}(\boldsymbol{r}, \boldsymbol{u}, s) = P_3(s)/P_4(s) \tag{2.19}$$

where P_3 and P_4 are polynomials of degree three and four in s with coefficients depending on $\boldsymbol{r}$ and $\boldsymbol{u}$:

$$\begin{aligned} P_3(s) = {} & 2(s+\tfrac{1}{2}\Gamma)\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})\kappa^*(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u}) \\ & +(s+\Gamma)[(s+\tfrac{1}{2}\Gamma)^2+(\omega-\omega_0)^2]+(s+\tfrac{1}{2}\Gamma)[|\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})|^2+|\kappa(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})|^2] \\ & -\mathrm{i}(\omega-\omega_0)[|\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})|^2-|\kappa(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})|^2] \end{aligned} \tag{2.20}$$

$$\begin{aligned} P_4(s) = {} & s(s+\Gamma)[(s+\tfrac{1}{2}\Gamma)^2+(\omega-\omega_0)^2]+[|\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})|^2-|\kappa(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})|^2]^2 \\ & -2\Gamma(s+\tfrac{1}{2}\Gamma)\chi(\boldsymbol{u})\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})\kappa^*(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u}) \\ & +2(s+\tfrac{1}{2}\Gamma)^2[|\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})|^2+|\kappa(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})|^2] \\ & +\mathrm{i}\Gamma(\omega-\omega_0)[|\kappa(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{u})|^2-|\kappa(\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{u})|^2]. \end{aligned} \tag{2.21}$$

(We have supposed that the initial internal atomic state is the ground state.)

A few important particular cases will be considered later on. We just point out here that the $\boldsymbol{u}$ dependence of $L(\boldsymbol{r}, \boldsymbol{u}, s)$ has two physical origins. First, the spatial dependence of the laser field, through the functions $\kappa(\boldsymbol{r}\pm\frac{1}{2}\boldsymbol{u})$ and $\kappa^*(\boldsymbol{r}\pm\frac{1}{2}\boldsymbol{u})$. Secondly, the angular properties of spontaneous emission through the function $\chi(\boldsymbol{u})$, which actually only appears in $P_4(s)$.

Finally, since equations (2.13), without free flight and with $\boldsymbol{u}=\boldsymbol{0}$, reduce to the ordinary Bloch equations for an atom at rest in $\boldsymbol{r}$, and since the trace of σ is a constant of motion for these equations, it follows that $F_{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{0}) = F_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{0})$, and, consequently, according to (2.17)

$$L(\boldsymbol{r}, \boldsymbol{0}, T) = 1. \tag{2.22}$$

Actually, it can be directly checked in (2.20) and (2.21) that $\tilde{L}(\boldsymbol{r}, \boldsymbol{0}, s) = 1/s$, which is the Laplace transform of 1.

2.5. *Propagator $G(\boldsymbol{r}, \boldsymbol{q}, T)$ of the Wigner function*

The Fourier transforms of $F_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{u})$ and $F_{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{u})$ with respect to $\boldsymbol{u}$ are the Wigner functions $w_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{p})$ and $w_{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{p})$ describing the incoming and outgoing external states. From equation (2.17), it follows that

$$w_{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{p}) = \int \mathrm{d}^3 q\, G(\boldsymbol{r}, \boldsymbol{q}, T) w_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{p}-\boldsymbol{q}) \tag{2.23}$$

where

$$G(\boldsymbol{r}, \boldsymbol{q}, T) = \frac{1}{h^3} \int \mathrm{d}^3 u\, L(\boldsymbol{r}, \boldsymbol{u}, T) \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{u}/\hbar). \tag{2.24}$$

$G(\boldsymbol{r}, \boldsymbol{q}, T)$ therefore appears as the propagator, or the Green's function, of the equation of motion of the Wigner function.

From the hermiticity of σ and from (2.16) and (2.3), one can show that $F(\boldsymbol{r}, \boldsymbol{u}) = F^*(\boldsymbol{r}, -\boldsymbol{u})$, and consequently, from (2.17), that $L(\boldsymbol{r}, \boldsymbol{u}, T) = L^*(\boldsymbol{r}, -\boldsymbol{u}, T)$. It then follows from (2.24) that $G(\boldsymbol{r}, \boldsymbol{q}, T)$ is a real function, which is, in addition, normalised in q

$$\int \mathrm{d}^3 q \, G(\boldsymbol{r}, \boldsymbol{q}, T) = 1 \tag{2.25}$$

as a consequence of (2.22). This suggests interpreting equation (2.23) by considering that 'an atom in $\boldsymbol{r}$ has a probability $G(\boldsymbol{r}, \boldsymbol{q}, T)$ to receive a momentum $\boldsymbol{q}$ during T' from the laser beam, and to have its momentum changed from $\boldsymbol{p} - \boldsymbol{q}$ to $\boldsymbol{p}$. Actually, G is not a true probability, since it can take negative values, but rather a 'quasi-probability'. It may also appear surprising to consider a momentum transfer in a given point, since such a picture seems to violate Heisenberg relations. Actually, $G(\boldsymbol{r}, \boldsymbol{q}, T)$ is a propagator and not a representation of a physical state, so that Heisenberg relations do not apply in principle to such a function. The physical initial and final states are described by $w_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{p})$ and $w_{\mathrm{out}}(\boldsymbol{r}, \boldsymbol{p})$ and one can show that the reduced distributions in $\boldsymbol{r}$ and $\boldsymbol{p}$

$$\mathscr{R}_{\substack{\mathrm{in} \\ \mathrm{out}}}(\boldsymbol{r}) = \int \mathrm{d}^3 p \, w_{\substack{\mathrm{in} \\ \mathrm{out}}}(\boldsymbol{r}, \boldsymbol{p}) \tag{2.26a}$$

$$\mathscr{P}_{\substack{\mathrm{in} \\ \mathrm{out}}}(\boldsymbol{p}) = \int \mathrm{d}^3 r \, w_{\substack{\mathrm{in} \\ \mathrm{out}}}(\boldsymbol{r}, \boldsymbol{p}) \tag{2.26b}$$

satisfy of course $\Delta r \Delta p > \hbar/2$.

The propagator $G(\boldsymbol{r}, \boldsymbol{q}, T)$ will be the basic tool used in this paper. We shall determine in the following sections the structure of G in the limit of short ($T \ll \tau_{\mathrm{R}}$) and long ($T \gg \tau_{\mathrm{R}}$) interaction times, and we try to understand the evolution of G between these two regimes. We first relate the experimental signal measured by the detector to $G(\boldsymbol{r}, \boldsymbol{q}, T)$ for two extreme types of initial states.

2.6. Expression of the detection signal for two extreme types of initial states

Suppose first that the incoming atomic wavepacket has a width Δx along x much larger than the laser wavelength, and also a width Δp_x in p_x much smaller than the photon momentum $\hbar k$

$$\Delta x \gg \lambda \qquad \Delta p_x \ll \hbar k \tag{2.27}$$

(see however the remark at the end of this section). From now on, we will write only the components x and $p = p_x$ of $\boldsymbol{r}$ and $\boldsymbol{p}$ in w_{in} and w_{out}, since these components are those which are relevant for the deflection experiment. Condition (2.27) means that the width of $w_{\mathrm{in}}(x, p)$ in p around $p = 0$ is much smaller than the characteristic width of the q dependence of $G(x, q, T)$, which is of the order of $\hbar k$. It follows that, for an initial state satisfying (2.27), (2.23) can be approximated by

$$\begin{aligned} w_{\mathrm{out}}(x, p) &\simeq G(x, p, T) \int \mathrm{d}q \, w_{\mathrm{in}}(x, p - q) \\ &= G(x, p, T) X_{\mathrm{in}}(x) \end{aligned} \tag{2.28}$$

where $X_{in}(x)$ is the initial distribution in x (see 2.26a). The detector measures the final momentum distribution $\mathscr{P}_{out}(p)$, which is obtained by integrating $w_{out}(x, p)$ over x (see 2.26b). We thus get from (2.28)

$$\mathscr{P}_{out}(p) = \int dx\, G(x, p, T) X_{in}(x). \tag{2.29}$$

The second type of initial state which we will consider in the following corresponds to a width Δx much smaller than λ and, also, to a width Δp_x much larger than $\hbar k$

$$\Delta x \ll \lambda \qquad \Delta p_x \gg \hbar k. \tag{2.30}$$

These conditions correspond to a small incoming wavepacket crossing the laser wave in a well defined abscissa x_0. One can then show that

$$\mathscr{P}_{out}(p) = \int dq\, G(x_0, q, T) \mathscr{P}_{in}(p - q). \tag{2.31}$$

Since $F_{in}(x, u)$ is the Fourier transform of $w_{in}(x, p)$ with respect to p, $F_{in}(x, u)$ has a width in u around $u = 0$ much smaller than $1/k$. The presence of $F_{in}(x, u)$ in the right hand side of (2.17) suppresses in this case the contributions of the values of u which do not satisfy $ku \ll 1$. For the second type of initial state (2.30), it will therefore be possible to use an approximate value of $L(x, u, T)$ corresponding to the limit $ku \ll 1$.

Remark. We could also consider atomic beams for which $\Delta x\, \Delta p_x \gg \hbar$. Suppose for example that the incoming atomic state is a statistical mixture of small wavepackets of the type (2.30), with the centres of the wavepackets distributed along $0x$. The deflection profile of such a beam is just the statistical average of the deflection profiles corresponding to all the individual wavepackets. We could equivalently describe the atomic state as a statistical mixture of large wavepackets of the type (2.27), having different values of $\langle p_x \rangle$, i.e. crossing the laser beam with different angles.

3. Short-time limit

3.1. Structure of the propagator in the limit $T \ll \tau_R$

We suppose in this section that the interaction time T is very short compared with the radiative lifetime τ_R, so that spontaneous emission can be neglected during the time of flight of atoms through the laser beam. It follows that we can neglect the terms proportional to Γ in the polynomials $P_3(s)$ and $P_4(s)$ appearing in the expression of $\tilde{L}(x, u, s)$ (see expressions (2.19) to (2.21)). In particular, $\chi(u)$, which is multiplied by Γ (see (2.21)), vanishes. This means that the u dependence of $\tilde{L}$ is only due to the spatial dependence of the laser field, through the functions κ and κ^* appearing in (2.20) and (2.21). In order to interpret the physical content of $\tilde{L}(x, u, s)$ and then of the propagator $G(x, q, T)$, in the limit $T \ll \tau_R$, we consider now two important particular cases.

For a resonant ($\omega = \omega_0$) laser progressive wave, propagating along $0x$, we have, according to (2.12), (2.10) and (2.6):

$$\kappa(x) = \tfrac{1}{2}\omega_1 \exp(ikx) \tag{3.1}$$

where

$$\omega_1 = -(\boldsymbol{\varepsilon}\cdot\boldsymbol{d})\mathscr{E}_0/\hbar \tag{3.2}$$

is independent of x. Inserting (3.1) into (2.20) and (2.21), neglecting the terms in Γ and using $\omega = \omega_0$, one gets

$$\tilde{L}(x, u, s) = \frac{s^2 + \frac{1}{2}\omega_1^2[1+\exp(iku)]}{s(s^2+\omega_1^2)} = \frac{1+\exp(iku)}{2s} + \frac{1-\exp(iku)}{4(s+i\omega_1)} + \frac{1-\exp(iku)}{4(s-i\omega_1)}. \tag{3.3}$$

The inverse Laplace transform of (3.3) is

$$L(x, u, T) = \tfrac{1}{2}[1+\exp(iku)] + \tfrac{1}{2}[1-\exp(iku)]\cos\omega_1 T \tag{3.4}$$

so that the propagator $G(x, q, T)$, which is the Fourier transform of $L(x, u, T)$ with respect to u, appears to be equal to

$$G(x, q, T) = \cos^2\tfrac{1}{2}\omega_1 T\,\delta(q) + \sin^2\tfrac{1}{2}\omega_1 T\,\delta(q-\hbar k). \tag{3.5}$$

The physical meaning of (3.5) is very clear. The atom initially in the ground state g and crossing the laser beam has a probability $\cos^2\frac{1}{2}\omega_1 T$ of staying in the same state without absorbing a laser photon, and a probability $\sin^2\frac{1}{2}\omega_1 T$ of absorbing a laser photon and to get in this way a momentum $\hbar k$ along $0x$ (since spontaneous emission is neglected during T, only induced emission processes can take place after the atom has been excited, bringing back the laser–atom system in its initial state). Equation (3.5) describes a resonant Rabi precession between e and g, including momentum exchange (Luzgin 1980). For a non-resonant excitation ($\omega \neq \omega_0$), the structure of (3.5) remains the same, the coefficients of the two delta functions $\delta(q)$ and $\delta(q-\hbar k)$ corresponding to a non-resonant Rabi precession.

For a resonant laser standing-wave propagating along $0x$, we have, from (2.12), (2.10) and (2.7)

$$\kappa(x) = \tfrac{1}{2}\Omega\cos kx \tag{3.6}$$

where Ω is still given by (3.2) and independent of x. The same calculations as above lead to the following expression of $\tilde{L}$

$$\tilde{L}(x, u, s) = \frac{s}{s^2+\Omega^2\sin^2 kx\sin^2\frac{1}{2}ku}$$
$$= \frac{1}{2}\left(\frac{1}{s+i\Omega\sin kx\sin\frac{1}{2}ku} + \frac{1}{s-i\Omega\sin kx\sin\frac{1}{2}ku}\right) \tag{3.7}$$

from which one deduces

$$L(x, u, T) = \cos(\Omega T\sin kx\sin\tfrac{1}{2}ku) \tag{3.8}$$

and

$$G(x, q, T) = \sum_{m=-\infty}^{+\infty} J_{2m}(\Omega T\sin kx)\,\delta(q-m\hbar k) \tag{3.9}$$

where J_{2m} are the Bessel functions of order $2m$. In the derivation of (3.9) from (3.8), we have used

$$\exp(i\alpha\sin\theta) = \sum_{n=-\infty}^{+\infty} J_n(\alpha)\exp(in\theta) \tag{3.10}$$

and

$$J_n(\alpha) = (-1)^n J_n(-\alpha) = (-1)^n J_{-n}(\alpha). \tag{3.11}$$

For a resonant standing wave, the propagator $G(x, q, T)$ is therefore a comb of delta functions with a spacing $\hbar k$. Mathematically, this comes from the fact that the laser amplitude is a sinusoidal function of x, so that $\tilde{L}(x, u, s)$ and $L(x, u, T)$, which depend on $\kappa(x \pm \frac{1}{2}u)$ and $\kappa^*(x \pm \frac{1}{2}u)$ are periodic functions of u, which can be expanded in a Fourier series of u, and which become by Fourier transform a comb of delta functions. Physically, such a structure is associated with the redistribution of photons between the two counterpropagating waves forming the standing wave. An atom initially in the ground state can absorb a photon from the wave propagating along the positive (or negative) direction of $0x$ and get in this way a momentum $+\hbar k$ (or $-\hbar k$) along $0x$. Then, by a stimulated emission process induced by the counterpropagating wave, it can emit a photon in the opposite direction and return to the ground state with a momentum $+2\hbar k$ (or $-2\hbar k$) along $0x$. One understands in this way how all integer multiples of $\hbar k$, $\pm n\hbar k$, can be found in the transfer of momentum from the laser beam to the atom. Such a result can be also understood from a wave point of view, as being due to a 'Bragg scattering' of the incoming atomic de Broglie wave by a 'grating of light' associated with the laser standing wave, as in the Kapitza–Dirac effect (see § 3.2 below).

Note finally that $G(x, q, T)$ given in (3.9) can take negative values (the Bessel functions J_{2m} are real but not always positive). This clearly shows that $G(x, q, T)$ is a quasi-probability of momentum transfer and not a probability. We use now the expression (3.9) for $G(x, q, T)$ for calculating the physical signal $\mathscr{P}_{\text{out}}(p)$ corresponding to the two extreme types of initial states considered in § 2.6. This will show how the approach used in this paper can be applied to the discussion of two important physical effects observable on the deflection profile of a monoenergetic atomic beam crossing at right angles a resonant laser standing wave (in the limit $T \ll \tau_R$).

3.2. *Resonant Kapitza–Dirac effect*

We suppose first that the incoming atomic wavepacket has a width Δx along $0x$ much larger than the laser wavelength λ (condition (2.27)). The final momentum distribution is then given by (2.29). Actually, the incoming atomic spatial distribution $X_{\text{in}}(x)$ along $0x$ varies very slowly with x, and $G(x, q, T)$ is a periodic function of x, so that (2.29) can be rewritten

$$\mathscr{P}_{\text{out}}(p) = \frac{1}{\lambda} \int_{-\lambda/2}^{+\lambda/2} \mathrm{d}x\, G(x, p, T). \tag{3.12}$$

Inserting the expression (3.9) of G into (3.12), one finally gets

$$\mathscr{P}_{\text{out}}(p) = \sum_{m=-\infty}^{+\infty} J_m^2(\tfrac{1}{2}\Omega T)\delta(p - m\hbar k) \tag{3.13}$$

which exactly coincides with the result derived by other methods (Bernhardt and Shore 1981, Arimondo *et al* 1981) for the deflection profile. Note that J_m^2 is always positive, so that $\mathscr{P}_{\text{out}}(p)$ is, as expected, a true probability. The structure of $\mathscr{P}_{\text{out}}(p)$, which appears as a series of equally spaced discrete peaks, is similar to the structure of the deflection profile of a monoenergetic electron beam crossing at right angles a standing wave (Kapitza and Dirac 1933). In the electron case, such a structure comes from

stimulated Compton scattering processes induced by the two counterpropagating waves forming the standing wave whereas, in the atomic case, the physical processes are, as we have seen above, resonant absorption and stimulated emission processes. This is why the effect described by (3.13) is called the *resonant* Kapitza–Dirac effect. It has been recently experimentally observed on sodium atoms (Moskowitz *et al* 1983).

Remark. If the detection zone is far from the interaction one, it is necessary to take into account the recoil due to spontaneous emission for those atoms which leave the interaction zone in the excited state e. The odd teeth of the comb, which correspond to such a situation, are therefore broadened and reduced.

3.3. Optical Stern and Gerlach effect

We consider now the opposite limit ($\Delta x \ll \lambda$) for the incoming wavepacket (condition (2.30)), so that we have now to use the expression (2.31) of $\mathscr{P}_{\text{out}}(p)$, where x_0 is the abscissa of the point at which the small incoming wavepacket crosses the laser standing wave.

As explained above (see end of § 2.6), it is possible, when condition (2.30) is fulfilled, to use $ku \ll 1$ in the expression of $L(x_0, u, T)$, i.e. replace $\sin \frac{1}{2}ku$ by $\frac{1}{2}ku$ in (3.8). This gives

$$L(x_0, u, T) = \tfrac{1}{2}\exp(\mathrm{i}u\delta k) + \tfrac{1}{2}\exp(-\mathrm{i}u\delta k) \tag{3.14}$$

with

$$\delta k = \tfrac{1}{2}\Omega k T \sin kx_0 \tag{3.15}$$

and consequently

$$G(x_0, q, T) = \tfrac{1}{2}[\delta(q - \hbar\delta k) + \delta(q + \hbar\delta k)]. \tag{3.16}$$

Inserting (3.16) in the expression (2.31) of $\mathscr{P}_{\text{out}}(p)$, we finally get

$$\mathscr{P}_{\text{out}}(p) = \tfrac{1}{2}\mathscr{P}_{\text{in}}(p - \hbar\delta k) + \tfrac{1}{2}\mathscr{P}_{\text{in}}(p + \hbar\delta k). \tag{3.17}$$

We therefore predict that the incoming wavepacket is split in two parts respectively translated by $+\hbar\delta k$ and $-\hbar\delta k$. The amount $\hbar\delta k$ of the translation is, according to (3.15), proportional to the interaction time T, and to the gradient in x_0 of the coupling parameter $\kappa(x)$ defined in (3.6). The effect described by (3.17) is the optical Stern and Gerlach effect and does not seem to have yet been observed.

Remark. It might be interesting to discuss the shape of $\mathscr{P}_{\text{out}}(p)$ for an incoming atomic state which is a statistical mixture of wavepackets of the type (2.30) (see remark at the end of § 2). Each wavepacket (2.30) gives rise to two wavepackets with a splitting in p, $2\hbar\delta k$, depending on the abscissa x_0 at which the wavepacket crosses the standing wave (see equation (3.15)). We have to calculate $\mathscr{P}_{\text{out}}(p)$ for each incoming wavepacket x_0 and then to average over x_0. We get in this way a smooth curve, symmetric with respect to $p = 0$, and with two maxima at the extremal deviations $\pm\hbar\delta k_{\text{M}} = \pm\hbar\Omega kT/2$, occurring for the values of x_0 such that $\sin kx_0 = \pm 1$. Such a curve is somewhat similar to the 'quasi-classical' continuous curve represented in figure 2 of Arimondo *et al* (1981). We could also reproduce such a curve by taking a statistical mixture of wavepackets (2.27). Each individual wavepacket gives a comb of δ functions centred around $\langle p_x \rangle$. Since the dispersion of the different values of $\langle p_x \rangle$ is much larger than $\hbar k$, the average of the various displaced combs gives rise again to a smooth curve.

4. Long-time limit ($T \gg \tau_R$)

We now show that the Green function $G(x, q, T)$ tends to an asymptotic Gaussian limit when the interaction time T is greater than the radiative lifetime τ_R. We then connect our approach with the Fokker–Planck equation often used in this situation. We finally obtain the shape of the deflection profile and particularly the variation of its width plotted against the interaction time T.

4.1. Gaussian limit of the Green's function

The expression (2.19) of the Laplace transform $\tilde{L}(x, u, s)$ of the linear filter amplitude $L(x, u, T)$ has a rational form and can be split up in elementary fractions of the variable s

$$\tilde{L}(x, u, s) = \sum_{i=1}^{4} \frac{a_i(x, u)}{s - s_i(x, u)}. \tag{4.1}$$

The four roots s_i of the polynomial P_4 have been supposed distinct for the sake of simplicity. It follows that

$$L(x, u, T) = \sum_{i=1}^{4} a_i(x, u) \exp(T s_i(x, u)). \tag{4.2}$$

The real parts of the roots s_i are all negative. If s_1 is the root associated with the smallest damping, one gets in the long-time limit

$$T \gg \tau_R \qquad L(x, u, T) \simeq a_1(x, u) \exp(T s_1(x, u)). \tag{4.3}$$

Now, the Green's function $G(x, q, T)$ is the Fourier transform of $L(x, u, T)$. In other words, $L(x, u, T)$ is the first characteristic function of $G(x, q, T)$ considered as a distribution (more properly a quasi-distribution) of the variable q. We will rather consider the second characteristic function, $\ln L(x, u, T)$, which is a linear function of T in the long time limit as a consequence of equation (4.3)

$$T \gg \tau_R \qquad \ln L(x, u, T) \simeq T s_1(x, u). \tag{4.4}$$

It follows that the cumulants $\kappa_n(x, T)$ associated with the distribution G

$$\kappa_n(x, T) = \left[\left(\frac{\hbar}{i} \frac{\partial}{\partial u} \right)^n \ln L(x, u, T) \right]_{u=0} \tag{4.5}$$

are linear functions of T. Such a result is easy to understand: since the correlation time of the radiative forces is of the order of τ_R, the amounts of momentum transferred during different time intervals larger than τ_R can be considered as independent random variables. It is therefore not surprising to find the cumulants $\kappa_n(x, T)$ increasing linearly with T (cumulants are additive in the superposition of independent random variables).

We can now go further by applying the central limit theorem: in the limit $T \gg \tau_R$, the momentum transferred during T is the sum of many independent variables and becomes a Gaussian ('normal') variable characterised by two non-zero cumulants κ_1 and κ_2. Mathematically, such a theorem means that the cumulants κ_n of order greater than two (and which are proportional to T) can be neglected when scaled to the dispersion of the distribution $(\kappa_2)^{1/2}$

$$\kappa_n / (\kappa_2)^{n/2} \sim (\tau_R / T)^{(n/2)-1} \ll 1 \qquad \text{for } n \geq 3. \tag{4.6}$$

The Green's function can therefore be approximated by a Gaussian function of q

$$G(x, q, T) \sim \frac{1}{(2\pi\kappa_2)^{1/2}} \exp\left(-\frac{(q-\kappa_1)^2}{2\kappa_2}\right) \tag{4.7}$$

where κ_1 and κ_2, proportional to T, are the mean value and the variance of the momentum transfer for an atom located in x (as G is a quasiprobability, it might be more appropriate to call κ_1 and κ_2 a quasi mean value and a quasivariance).

4.2. Connection with a Fokker–Planck equation

Let us introduce the following notations

$$\begin{aligned} \kappa_1(x, T) &= F(x)T \\ \kappa_2(x, T) &= 2D(x)T. \end{aligned} \tag{4.8}$$

Expression (4.7) thus appears as the Green's function of the following Fokker–Planck equation

$$\left(\frac{\partial}{\partial t} + F(x)\frac{\partial}{\partial p} - D(x)\frac{\partial^2}{\partial p^2}\right) w(x, p, t) = 0 \tag{4.9}$$

(the free-flight term is omitted in (4.9); see § 1). Our approach, based on the solution of the generalised optical Bloch equations (see § 2.3), is therefore equivalent in the long-time limit to the description by a Fokker–Planck equation. At this stage, we want to emphasise that the expressions obtained in our approach for κ_1 and κ_2 are in complete agreement with the expressions obtained by Gordon and Ashkin (1980) for the mean force $F(x)$ and the momentum diffusion coefficient $D(x)$ (Tanguy 1983).

The connection between the generalised optical Bloch equations and the Fokker–Planck one can be derived in a more formal manner. As a matter of fact, one deduces from equation (2.17)

$$\frac{\partial}{\partial T} F_{\text{out}}(x, u, T) = \left(\frac{\partial}{\partial T} \ln L(x, u, T)\right) F_{\text{out}}(x, u, T) \tag{4.10}$$

and from (4.5)

$$\frac{\partial}{\partial T} F_{\text{out}}(x, u, T) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\partial \kappa_n}{\partial T}\right) \left(\frac{\mathrm{i}u}{\hbar}\right)^n F_{\text{out}}(x, u, T). \tag{4.11}$$

A Fourier transform with respect to u gives the evolution of the Wigner distribution

$$\frac{\partial}{\partial T} w(x, q, T) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\partial \kappa_n}{\partial T}\right) \left(-\frac{\partial}{\partial q}\right)^n w(x, q, T). \tag{4.12}$$

This equation is correct for short as well as long interaction times (in so far as free flight can be ignored). For long interaction times, simplifications can be introduced as a consequence of the central limit theorem (see discussion above). First, the coefficients $(\partial\kappa_n/\partial T)$ can be considered as constant (κ_n is proportional to T). Second, the equation can be truncated at the second order in $(\partial/\partial q)$ (the cumulants of order greater than two can be ignored).

4.3. Application to deflection profiles for $\Delta x \gg \lambda$

Applying the preceding results to deflection profiles, we shall limit ourselves to the case of a monoenergetic atomic beam (i.e. $\Delta x \gg \lambda$) crossing a laser beam, the latter being a running or a standing plane wave. $G(x, q, T)$ does not depend on x in the first case, and has a period of $\lambda/2$ (due to the $\omega_1^2(x)$ terms) in the second one. We can write

$$\mathscr{P}_{\text{out}}(p) = \frac{2}{\lambda} \int_{-\lambda/4}^{\lambda/4} \mathrm{d}x\, G(x, p, T). \tag{4.13}$$

It is now clear that the deflection profile can be obtained by a superposition of Gaussian curves corresponding to equally spaced values of x each of which is centred on $\bar{p}(x)$ $(= TF(x))$ and has a width $(2TD(x))^{1/2}$ (see also Kazantsev *et al* 1981).

In the case of a running plane wave, $F(x)$ does not depend on x, so that all curves are centred on the same value. We thus expect a bell-shaped deflection profile.

The case of a standing plane wave is more interesting as it gives very different results according as the detuning $\omega - \omega_0$ is zero or not. The expression of $F(x)$ is

$$F(x) = \hbar k(\omega - \omega_0) \frac{\Omega^2 \sin 2kx}{\Gamma^2 + 4(\omega - \omega_0)^2 + 2\Omega^2 \cos^2 kx} \tag{4.14}$$

with

$$\Omega = -\mathrm{d}\mathscr{E}_0/\hbar. \tag{4.15}$$

It appears that $F(x) = 0$ for $\omega = \omega_0$. The deflection profile will thus be, in the same way as above, a bell-shaped curve centred on $p = 0$. For $\omega \neq \omega_0$ however, $F(x)$ varies with x and we must add the contributions of Gaussian distributions centred on *different* points.

We have sketched in figure 1 $\bar{p}(x)$ $(= TF(x))$ as a function of x. The distribution of the $\bar{p}(x_i)$ corresponding to equally spaced values x_i of x is obviously denser in the neighbourhood of $\pm p_M$, corresponding to the extrema of $\bar{p}(x)$. Consequently, in the

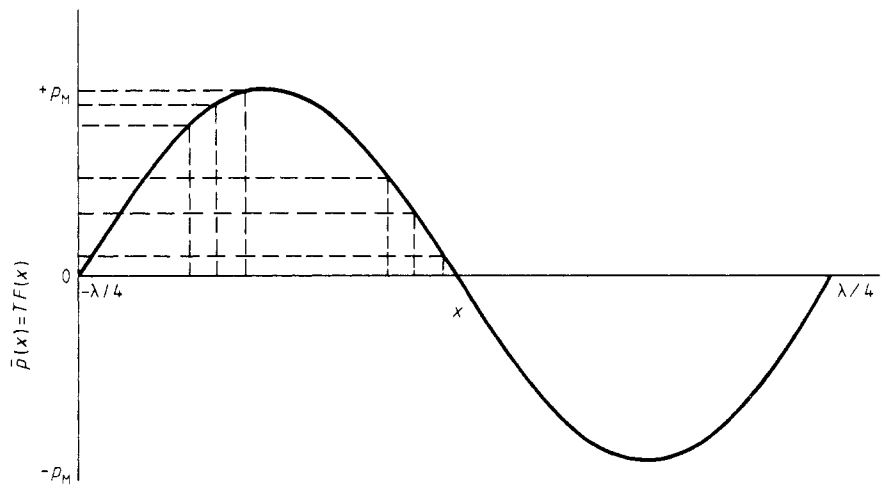

Figure 1. The distribution of the $\overline{p(x_i)}$ corresponding to equally spaced values of x_i is denser near the values $\pm p_M$ corresponding to the extrema of $\bar{p}(x)$.

construction of the deflection profile, there will be more Gaussian curves centred on $\pm p_M$ than on any other value of p. We thus predict that for $\omega \neq \omega_0$, $\mathscr{P}_{out}(p)$ should exhibit a structure with two peaks near $p = \pm p_M$ (which we call a 'rainbow structure'), provided that p_M is greater than the widths of the Gaussian distributions we are summing. Such structures are studied in more details by Tanguy *et al* (1983).

4.4. Width of the profile versus the interaction time

The variance $\Delta p^2(T)$ of the momentum p in the final momentum distribution $\mathscr{P}_{out}(p)$ can be evaluated from (4.13), (4.7) and (4.8). One finds

$$\Delta p^2(T) = 2\bar{D}T + \Delta F^2 T^2. \tag{4.16}$$

The first term $2\bar{D}T$ is the average value of the variances $\kappa_2 = 2D(x)T$ of the Gaussian contributions associated with each value of x

$$\bar{D} = \frac{2}{\lambda}\int_{-\lambda/4}^{+\lambda/4} dx\, D(x). \tag{4.17}$$

The second term $\Delta F^2 T^2$ is the variance of the mean values $\kappa_1 = F(x)T$ of the same Gaussian contributions

$$\Delta F^2 = \left(\frac{2}{\lambda}\int_{-\lambda/4}^{+\lambda/4} dx\, F^2(x)\right) - \left(\frac{2}{\lambda}\int_{-\lambda/4}^{+\lambda/4} dx\, F(x)\right)^2. \tag{4.18}$$

When $F(x)$ is independent of x, i.e. for a running wave or a resonant standing one, this second term vanishes and the dispersion Δp of the deflection profile varies as $\sqrt{T}$. On the contrary, for a non-resonant standing wave, ΔF^2 is non-zero (and equal to the first term of (4.18)) and Δp varies as T.

Remark. Equation (4.16) could also be used for discussing the laser power dependence of the dispersion Δp of the deflection profile, which has been actually experimentally measured on a sodium beam (Arimondo *et al* 1979, Viala 1982). For a resonant excitation, F and ΔF^2 are equal to zero, and according to (4.16), Δp should vary as $(2\bar{D}T)^{1/2}$, i.e. as the square root of the laser power P_L, since $\bar{D}$ is proportional to P_L at high intensities. Such a result seems to be in good agreement with experimental observations (see also Minogin 1981 and Kazantsev *et al* 1981).

5. Intermediate times ($T \sim \tau_R$)

We finally come to the domain of intermediate times ($T \sim \tau_R$) where the Green function $G(x, q, T)$ can no longer be considered as a comb of δ functions or as a Gaussian function. We want to show that it has nevertheless a simple interpretation in terms of momentum conservation in absorption and emission processes.

5.1. Structure of the propagator for a laser running wave

The expression (2.19) of $\tilde{L}(x, u, s)$ can be written in the case of a laser running wave as

$$\tilde{L}(x, u, s) = \frac{a + a' \exp(iku)}{b[1 - c\chi(u)\exp(iku)]} \tag{5.1}$$

where a, a', b and c are functions of s only

$$a = (s+\Gamma)[(s+\tfrac{1}{2}\Gamma)^2+(\omega-\omega_0)^2]+(s+\tfrac{1}{2}\Gamma)\tfrac{1}{2}\omega_1^2$$
$$a' = (s+\tfrac{1}{2}\Gamma)\tfrac{1}{2}\omega_1^2$$
$$b = s(s+\Gamma)[(s+\tfrac{1}{2}\Gamma)^2+(\omega-\omega_0)^2]+(s+\tfrac{1}{2}\Gamma)^2\omega_1^2$$
$$bc = \Gamma(s+\tfrac{1}{2}\Gamma)\tfrac{1}{2}\omega_1^2. \tag{5.2}$$

Equation (5.1) can then be expanded into

$$\tilde{L}(x, u, s) = \frac{a+a'\exp(iku)}{b}\sum_{m=0}^{\infty} c^m(\chi(u))^m \exp(imku). \tag{5.3}$$

A Fourier transform with respect to u gives the corresponding expansion of $\tilde{G}(x, q, s)$

$$\tilde{G}(x, q, s) = \sum_{m=0}^{\infty}\frac{ac^m}{b}\delta(q-m\hbar k)\otimes\phi^{(m)}(q)$$
$$+\sum_{m=0}^{\infty}\frac{a'c^m}{b}\delta[q-(m+1)\hbar k]\otimes\phi^{(m)}(q) \tag{5.4}$$

where the symbol $\otimes$ represents the convolution product of two functions of q and $\phi^{(m)}(q)$ the Fourier transform of $(\chi(u))^m$. As the Fourier transform of $\chi(u)$ is just the normalised distribution $\phi(q)$ of the momentum transfer during a spontaneous emission process (see equation (2.14)), $\phi^{(m)}(q)$ is the autoconvolution product of $\phi(q)$ of order m

$$\phi^{(m)}(q) = \underbrace{\phi(q)\otimes\phi(q)\ldots\otimes\phi(q)}_{m \text{ times}}. \tag{5.5}$$

In other terms, $\phi^{(m)}(q)$ is the distribution of the recoil momentum given by m spontaneous emission processes.

The expression (5.4) of the propagator has thus a very clear interpretation if (ac^m/b) is associated with the probability for the atom starting from g to end in g after m spontaneous emissions (distribution of recoil $\phi^{(m)}(q)$), the number of absorbed laser photons being also m (momentum transfer $m\hbar k$). In a similar way, $(a'c^m/b)$ has to be associated with the probability for the atom to end in e after m spontaneous emissions (distribution of recoil $\phi^{(m)}(q)$), the number of absorbed laser photons being $(m+1)$ (momentum transfer $(m+1)\hbar k$). Using resonance fluorescence theory (Smirnov and Troshin 1981, Reynaud 1983 and references therein), one can actually calculate the statistics of the number of emitted photons; the results thus obtained entirely agree with this interpretation.

As the convolution product by a δ function is only a translation, equation (5.4) can be written in a simpler manner

$$\tilde{G}(x, q, s) = \sum_{m=0}^{\infty}\left(\frac{ac^m}{b}\phi^{(m)}(q-m\hbar k)+\frac{a'c^m}{b}\phi^{(m)}[q-(m+1)\hbar k]\right). \tag{5.6}$$

Remarks. (i) In the case of a laser running wave, the propagator $G(x, q, T)$ is actually independent of x, as it appears on equations (5.4) and (5.2). This is why it can be interpreted as a true probability distribution.

(ii) It is very simple to modify the expressions obtained in this section in order to take into account the emission of a last fluorescence photon by the atom during its

free flight from the interaction zone to the detector. The momentum transfer due to this spontaneous emission can actually be described by one more function $\phi(q)$ when the atom leaves the interaction zone in the excited state. More precisely, equation (5.6) becomes

$$\tilde{G}(x, q, s) = \sum_{m=0}^{\infty} \left(\frac{ac^m}{b} \phi^{(m)}(q - m\hbar k) + \frac{a'c^m}{b} \phi^{(m+1)}[q - (m+1)\hbar k] \right). \tag{5.7}$$

5.2. Structure of the propagator for a laser standing wave

In the case of a laser standing wave, the expression (2.19) of $\tilde{L}(x, u, s)$ can be written

$$\tilde{L}(x, u, s) = \frac{a}{b(1 - c\chi(u))} = \sum_{m=0}^{\infty} \frac{ac^m}{b} (\chi(u))^m \tag{5.8}$$

where a, b and c are functions of x, u and s

$$\begin{aligned} a &= P_3(s) = (s+\Gamma)[(s+\tfrac{1}{2}\Gamma)^2 + (\omega - \omega_0)^2] \\ &\quad + \tfrac{1}{4}\Omega^2[(s+\tfrac{1}{2}\Gamma)(1+\cos 2kx)(1+\cos ku) + \mathrm{i}(\omega - \omega_0) \sin 2kx \sin ku] \\ b &= s(s+\Gamma)[(s+\tfrac{1}{2}\Gamma)^2 + (\omega - \omega_0)^2] + \tfrac{1}{16}\Omega^4 \sin^2 2kx \sin^2 ku \\ &\quad + \tfrac{1}{4}\Omega^2[2(s+\tfrac{1}{2}\Gamma)^2(1+\cos 2kx \cos ku) - \mathrm{i}\Gamma(\omega - \omega_0) \sin 2kx \sin ku] \\ bc &= \Gamma(s+\tfrac{1}{2}\Gamma)\tfrac{1}{4}\Omega^2(\cos 2kx + \cos ku). \end{aligned} \tag{5.9}$$

It is worth noting that a, b and c are periodic functions of u, which is not the case for $\chi(u)$. The Fourier transform of (ac^m/b) with respect to u is therefore a comb of δ functions and the expansion of $\tilde{G}(x, q, s)$ corresponding to (5.8) can be written

$$\tilde{G}(x, q, s) = \sum_{m=0}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} \tilde{R}_m^n(x, s) \delta(q - n\hbar k) \right) \otimes \phi^{(m)}(q). \tag{5.10}$$

As in the preceding section, the function $\phi^{(m)}(q)$ (Fourier transform of $(\chi(u))^m$) is the distribution of the recoil momentum given by m spontaneous processes. Now, the parenthesis (Fourier transform of ac^m/b) represents the momentum transfer associated with the absorption or the stimulated emission of laser photons. The quantity $R_m^n(x, T)$ (inverse Laplace transform of $\tilde{R}_m^n(x, s)$) thus appears as the probability for an atom at point x to emit m fluorescence photons during the interaction time T and to redistribute photons between the two waves $+k$ and $-k$ in such a way that the momentum transferred in this redistribution is $n\hbar k$. More properly, $R_m^n(x, T)$ has to be considered as a quasi-probability since it may be negative. When the incoming atomic wavepacket can be considered as a plane wave (condition (2.27)), the final momentum distribution $\mathcal{P}_{\text{out}}(q)$ can be written (from 2.29 and 5.10)

$$\mathcal{P}_{\text{out}}(q) = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} \Pi_m^n(T) \phi^{(m)}(q - n\hbar k) \tag{5.11}$$

with

$$\Pi_m^n(T) = \frac{1}{\lambda} \int_{-\lambda/2}^{+\lambda/2} \mathrm{d}x \, R_m^n(x, T). \tag{5.12}$$

$\Pi_m^n(T)$ is a true probability (as $\mathcal{P}_{\text{out}}(q)$) and can be interpreted in a dressed-atom approach. The energy diagram of the atom dressed by the two types of laser photons

$\hbar k$ and $-\hbar k$ is sketched in figure 2. The initial state of the dressed atom is $|g, n_1, n_2\rangle$ (atom in g in presence of n_1 photons $\hbar k$ and n_2 photons $-\hbar k$). The states located on the same horizontal line can be populated through redistribution of photons between the two waves (the number of atomic plus laser excitations being conserved). Spontaneous emission allows states located lower than the initial state in the energy diagram to be populated (through the emission of fluorescence photons represented by wavy arrows in figure 2). Each state of the energy diagram is labelled by two quantum numbers m and n (m for the horizontal lines, n for the vertical columns). The quantity $\Pi_m^n(T)$ is simply the probability for the atom starting from $m=0$, $n=0$ (labels of $|g, n_1, n_2\rangle$) to be in the state m, n after an interaction time T. The equation (5.11) has thus a very clear interpretation since it expresses the conservation of the total momentum during the evolution of the dressed atom.

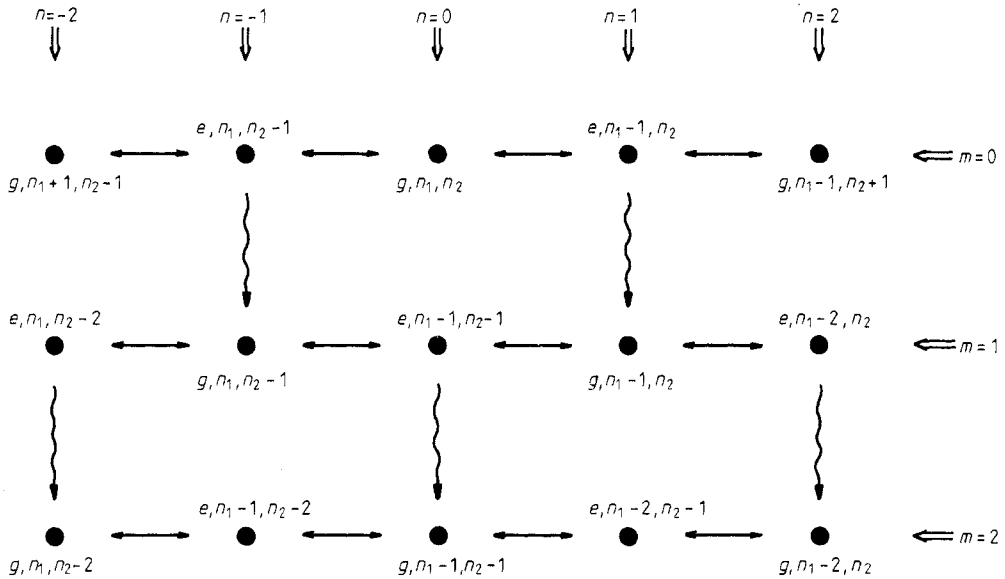

Figure 2. Energy diagram of the dressed atom. The states are labelled by three quantum numbers, g or e for the atomic state, n_1 and n_2 for the numbers of photons in the two laser modes, or equivalently by the two quantum numbers n and m describing their position in the diagram. The horizontal arrows are associated with the coherent redistribution of photons between the two waves (variation of n) whereas the vertical wavy arrows are associated with spontaneous emission (variation of m).

Remark. The final state of the atom is e when $n+m$ is an odd number. One can therefore take into account the free flight of the atom from the interaction zone to the detector by adding one function $\phi(q)$ in the terms of (5.10) or (5.11) for which $n+m$ is odd.

References

Allen L and Eberly J H 1975 *Optical Resonance and Two Level Atoms* (New York: Wiley)
Arimondo E, Bambini A and Stenholm S 1981 *Phys. Rev.* A **24** 898–909

Arimondo E, Lew H and Oka T 1979 *Phys. Rev. Lett.* **43** 753–7
Baklanov E V and Dubetskii B Ya 1976 *Opt. Spectrosc.* **41** 1–4
Bernhardt A F and Shore B W 1981 *Phys. Rev.* A **23** 1290–301
Compagno G, Peng J S and Persico F 1982 *Phys. Lett.* **88A** 285–8
Cook R J 1978 *Phys. Rev. Lett.* **41** 1788–91
—— 1980a *Phys. Rev.* A **22** 1078–98
—— 1980b *Opt. Commun.* **35** 347–50
—— 1981 *Phys. Rev.* A **23** 1243–50
Cook R J and Bernhardt A F 1978 *Phys. Rev.* A **18** 2533–7
De Groot S R and Suttorp L G 1972 *Foundations of Electrodynamics* (Amsterdam: North-Holland)
Delone G A, Grinchuk V A, Kuzmichev S D, Nagaeva M L, Kazantsev A P and Surdutovich G I 1980 *Opt. Commun.* **33** 149–52
Gordon J P and Ashkin A 1980 *Phys. Rev.* A **21** 1606–17
Javanainen J and Stenholm S 1980 *Appl. Phys.* **21** 35–45, 163–7
Kapitza P L and Dirac P A M 1933 *Proc. Camb. Phil. Soc.* **29** 297
Kazantsev A P 1978 *Sov. Phys.–Usp.* **21** 58–76
Kazantsev A P, Surdutovich G I and Yakovlev V P 1981 *J. Physique* **42** 1231–7
Letokhov V S and Minogin V G 1981 *Phys. Rep.* **73** 1–65
Luzgin S N 1980 *Theor. Math. Phys.* **43** 372–3
Mandel L 1979 *J. Opt.* **10** 51
Minogin V G 1981 *Opt. Commun.* **37** 442–6
Moskowitz P E, Gould P L, Atlas S R and Pritchard D E 1983 *Phys. Rev. Lett.* **51** 370
Reynaud S 1983 *Ann. Phys., Paris* **8** 315–70
Smirnov D F and Troshin A S 1981 *Sov. Phys.–JETP* **54** 848–51
Stenholm S 1983 *Phys. Rev.* A **27** 2513–22
Takabayasi T 1954 *Prog. Theor. Phys.* **11** 341–73
Tanguy C 1983 *Thèse de 3ème cycle* Paris VI
Tanguy C, Reynaud S, Matsuoka M and Cohen-Tannoudji C 1983 *Opt. Commun.* **44** 249–53
Viala F 1982 *Thèse de 3ème cycle* Paris XI
Vorobev F A, Rautian S G and Sokolovskii R I 1969 *Opt. Spectrosc.* **27** 398–401
Wigner E P 1932 *Phys. Rev.* **40** 749–59