

Atomic motion in laser light: connection between semiclassical and quantum descriptions

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Abstract. The quantum kinetic equation describing slow atomic motion in laser light is derived by an operatorial method which provides mathematical expressions with a transparent physical structure. We prove in a general way that the coefficients appearing in this equation, which is of a Fokker–Planck type, are simply related to the mean value and to the correlation functions of the Heisenberg radiative force of the semiclassical approach, where the atomic position is treated classically. We derive in particular a new theoretical expression for the damping force responsible for radiative cooling and we interpret it in terms of linear response theory. We also obtain a new crossed $\mathbf{r}-\mathbf{p}$ derivative term, which does not appear in semiclassical treatments, but which we find to be very small in most situations. Finally, all the theoretical expressions derived in this paper are valid for any J_g to J_e transition and are not restricted to two-level atoms.

1. Introduction

The subject of atomic motion in resonant laser light has been intensively studied recently and applications as varied as cooling of an atomic beam (Prodan *et al* 1982, Balykin *et al* 1984), isotope separation (Bernhardt *et al* 1976) or radiative atomic trapping (Ashkin 1978, Ashkin and Gordon 1979, Dalibard *et al* 1983) have been investigated, from both theoretical or experimental points of view. Considering the various theoretical descriptions of this atomic motion, one can first make a distinction between ‘short interaction time’ treatments, where spontaneous emission processes can be neglected during the atom–laser interaction and ‘long interaction time’ treatments where, on the contrary, many spontaneous processes can occur during the interaction time. In the first case, one can write a Schrödinger equation for the atomic wavefunction (see e.g. Letokhov and Minogin 1981), and extract from this equation all the characteristics of the motion. In the second case, where the interaction time is long compared with the lifetime of the atomic excited levels, one has in principle to take into account the coupling of the atom with all the modes of the electromagnetic field responsible for spontaneous emission processes. The random character of spontaneous emission then causes a stochastic spreading of the atomic momentum distribution. This ‘long-time’ situation, which occurs very frequently in experiments, and which is the one we are interested in in this paper, is therefore much more complicated than the short-time limit.

Up to now, there have been two main approaches to the description of atomic motion in laser light in the long interaction time limit. The first one is based on a

classical treatment of the atom's position, assuming a very small atomic wavepacket (Cook 1979, 1980a, Gordon and Ashkin 1980). It is therefore possible to calculate, via optical Bloch equations, the stationary internal atomic state and then to find, by application of Ehrenfest's theorem, the equation of motion of the centroid of the atomic wavepacket. Such an approach brings out the notion of 'average radiative force'. It is also possible in this treatment to describe the spreading of the atomic momentum due to the randomness of spontaneous emission, in a way similar to the one used in noise theory: one introduces a momentum diffusion constant which is expressed in terms of the two-time autocorrelation function of the radiative force (Cook 1980a, Gordon and Ashkin 1980).

The second approach to atomic motion in laser light for long interaction times is a fully quantum treatment of both internal and external atomic degrees of freedom, based on the use of the Wigner transform of the atomic density matrix. Under some conditions, it is possible to eliminate, from the master equation describing the atomic dynamics, all the internal atomic variables and to get a closed equation for the Wigner phase-space distribution function. This equation is of a Fokker-Planck type, containing terms which describe not only the mean force, but also the diffusion of atomic momentum. It can be applied to various types of situations, such as slow atoms in any laser light (Cook 1980b), fast atoms in a fluctuating or weak field (Kazantsev 1978, Javanainen and Stenholm 1980, Cook 1980b), or fast atoms in a running or standing wave (Baklanov and Dubetskii 1976, Minogin 1980, 1981a, b, Letokhov and Minogin 1981, Kazantsev *et al* 1981a, b, Stenholm 1983, 1984b, Tanguy *et al* 1984, Minogin and Rozhdestvensky 1984).

This second approach seems probably more rigorous than the first one (i.e. semiclassical treatment) but it has an important disadvantage which lies in the complexity of the calculations which are involved. As a consequence, for most of its applications, calculations have been restricted to the case of two-level atoms. Furthermore, it is only at the end of the calculation, working on the explicit expressions of the coefficients of the Fokker-Planck equation, that one can relate this treatment to the semiclassical one.

The motivation of this paper is to try to fill the gap between these two approaches. We would like to present a new derivation of the Fokker-Planck equation leading, for the coefficients of this equation, to expressions directly given in terms of one- or two-time averages of the Heisenberg radiative force of the semiclassical approach. Such a derivation, which, in addition, is not limited to two-level atoms (it applies to any J_g to J_e transition), has therefore a more transparent structure. We thus prove, in a general way, that the momentum diffusion coefficient appearing in the Fokker-Planck equation exactly coincides with the one deduced in the semiclassical theory from the autocorrelation function of the radiative force. Furthermore, we establish some interesting new results. For example, we get for the friction coefficient of the Fokker-Planck equation, which is related to the linear term in the expansion of the radiative force in powers of the atomic velocity, an explicit expression in terms of two time averages of the radiative force, and we interpret this result as a linear response of the atomic dipole to the perturbation associated with the motion of the atom in the laser wave.

The paper is organised as follows. In § 2, we present our notations and we briefly recall the definition and the equation of evolution of the Wigner transform of the atomic density matrix. In § 3, we first show how to expand this equation of evolution for slow atoms. We then indicate the general principle of the elimination of fast internal atomic variables in terms of the slow one (Wigner function). We then apply,

in § 4, the previous results to the calculation of the atomic density matrix in the Wigner representation, and we physically interpret its expression. Finally, in § 5, we derive the Fokker-Planck equation for the Wigner function, we establish the connection between this equation and the 'semiclassical' theory, and we discuss the new results which appear in our derivation.

2. Evolution of the atomic density matrix in the Wigner representation

2.1. Notations and assumptions

The Hamiltonian of the system 'atom + field' is the sum of four parts:

$$H = H_A + H_F + V_{A-L} + V_{A-F}. \quad (2.1)$$

H_A is the atomic Hamiltonian, H_F the quantised field Hamiltonian. The laser field is supposed to be in a coherent state, so that we can treat it as a c -number field, and then split the atom-field coupling into two parts (Mollow 1975), the first one (V_{A-L}) describing the atom-laser coupling, and the second one (V_{A-F}) the atom-quantised-field coupling, the quantised field being taken in its ground state.

The atomic Hamiltonian is the sum of the kinetic energy of the atom and of its internal energy:

$$H_A = \frac{\mathbf{P}^2}{2m} + \frac{\hbar\omega_0}{2} \left(\sum_e Q_{ee} - \sum_g Q_{gg} \right) \quad (2.2)$$

where we use the general notation:

$$Q_{ab} = |a\rangle\langle b| \quad (2.3)$$

$|a\rangle, |b\rangle$ being internal atomic states. In (2.2), the summations bear respectively on the $(2J_e + 1)$ and $(2J_g + 1)$ Zeeman sublevels of the excited and ground energy levels, ω_0 being the atomic frequency.

The quantised electromagnetic field is expanded on the complete set of plane wave modes with wavevector \mathbf{k} , frequency $\omega = c|\mathbf{k}|$ and polarisation $\boldsymbol{\varepsilon}$. The Hamiltonian H_F of the quantised field is thus:

$$H_F = \sum_{\mathbf{k}\boldsymbol{\varepsilon}} \hbar\omega a_{\mathbf{k}\boldsymbol{\varepsilon}}^\dagger a_{\mathbf{k}\boldsymbol{\varepsilon}} \quad (2.4)$$

where $a_{\mathbf{k}\boldsymbol{\varepsilon}}$ and $a_{\mathbf{k}\boldsymbol{\varepsilon}}^\dagger$ are the destruction and creation operators of a photon in the mode $\mathbf{k}, \boldsymbol{\varepsilon}$.

The atom-quantised-field coupling V_{A-F} can be written in the electric-dipole approximation as:

$$V_{A-F} = -\mathbf{D} \cdot \mathbf{E}(\mathbf{R}) \quad (2.5)$$

where \mathbf{D} is the atomic dipole operator and $\mathbf{E}(\mathbf{R})$ the quantised field taken for the atomic position operator \mathbf{R} :

$$\mathbf{E}(\mathbf{R}) = \sum_{\mathbf{k}\boldsymbol{\varepsilon}} i \left(\frac{\hbar\omega}{2\varepsilon_0 L^3} \right)^{1/2} \boldsymbol{\varepsilon} \exp(i\mathbf{k} \cdot \mathbf{R}) a_{\mathbf{k}\boldsymbol{\varepsilon}} + \text{H.C.} \quad (2.6)$$

L^3 is the quantisation volume.

The atom-laser coupling is also taken in the electric dipole approximation:

$$V_{A-L}(\mathbf{R}) = -\mathbf{D} \cdot \mathcal{E}_{\text{laser}}(\mathbf{R}, t) \quad (2.7)$$

where $\mathcal{E}_{\text{laser}}(\mathbf{R}, t)$ is obtained by replacing, in the classical function $\mathcal{E}_{\text{laser}}(\mathbf{r}, t)$ describing the laser field, \mathbf{r} by the atomic position operator \mathbf{R} . We assume that $\mathcal{E}_{\text{laser}}$ is perfectly monochromatic and we split it into its positive and negative frequency part:

$$\mathcal{E}_{\text{laser}}(\mathbf{r}, t) = \mathcal{E}^{(+)}(\mathbf{r}) \exp(-i\omega_L t) + \mathcal{E}^{(-)}(\mathbf{r}) \exp(i\omega_L t). \quad (2.8)$$

As usual, all the calculations will be done using the rotating-wave approximation, which consists in keeping only the resonant terms in the atom-laser coupling (2.7). To this end, we introduce the raising \mathbf{D}^+ and lowering \mathbf{D}^- parts of the atomic dipole and we denote the reduced matrix element of the dipole between the ground and excited level by d . Putting

$$\mathbf{D}^+ = d\mathbf{S}^+ \quad \mathbf{D}^- = d\mathbf{S}^- \quad (2.9)$$

so that ($\mathbf{S}^+, \mathbf{S}^-$ dimensionless):

$$\mathbf{D} = d(\mathbf{S}^+ + \mathbf{S}^-) \quad (2.10)$$

the atom-laser coupling can be written, in the rotating-wave approximation:

$$V_{A-L} = -d[\mathbf{S}^+ \cdot \mathcal{E}^{(+)}(\mathbf{R}) \exp(-i\omega_L t) + \mathbf{S}^- \cdot \mathcal{E}^{(-)}(\mathbf{R}) \exp(i\omega_L t)]. \quad (2.11)$$

2.2. Evolution of the reduced atomic density matrix

The atomic system is coupled by V_{A-F} (2.5) to all the modes of the electromagnetic field, this coupling being the cause of spontaneous emission. The first step of our calculation is then to take into account this coupling and to derive a master equation for the reduced atomic density matrix $\rho_A = \text{Tr}_F(\rho)$, describing the effect of the atom-quantised-field interaction. The approximations used in this derivation are based on the smallness of the correlation time τ_c of the quantised electromagnetic field. It is then possible to consider only one interaction process between the atom and the field during the time τ_c , and also to neglect the free flight of the atom during τ_c . We will not derive here explicitly the master equation for ρ_A , since it is now a well known procedure (Cohen-Tannoudji 1977). We just indicate the final result for the contribution $(d\rho_A/dt)_{\text{vac}}$ of the 'atom-vacuum quantised field' coupling, to the evolution of the reduced atomic matrix ρ_A :

$$\begin{aligned} \left(\frac{d}{dt} \rho_A \right)_{\text{vac}} = & -\frac{\Gamma}{2} [(\mathbf{S}^+ \cdot \mathbf{S}^-) \rho_A + \rho_A (\mathbf{S}^+ \cdot \mathbf{S}^-)] \\ & + \Gamma \int \frac{d^2 \kappa}{8\pi/3} \sum_{\boldsymbol{\varepsilon} \perp \boldsymbol{\kappa}} (\mathbf{S}^- \cdot \boldsymbol{\varepsilon}^*) [\exp(-i\mathbf{k} \cdot \mathbf{R}) \rho_A \exp(i\mathbf{k} \cdot \mathbf{R})] (\mathbf{S}^+ \cdot \boldsymbol{\varepsilon}). \end{aligned} \quad (2.12)$$

In this expression $\boldsymbol{\kappa}$ is a unit vector and \mathbf{k} is defined by:

$$\mathbf{k} = \frac{\omega_0}{c} \boldsymbol{\kappa}. \quad (2.13)$$

Γ is the natural linewidth of the excited level:

$$\Gamma = \frac{d^2 \omega_0^3}{3\pi \varepsilon_0 \hbar c^3}. \quad (2.14)$$

The first line of equation (2.12) describes the de-excitation of the excited Zeeman sublevels. For example, using the normalisation for S^+ and S^- (P_e projector on excited level):

$$S^+ \cdot S^- = P_e \quad (2.15)$$

one gets the following evolution for the population of level $|e\rangle$:

$$\left(\frac{\partial}{\partial t} \langle e | \rho_A | e \rangle \right)_{\text{vac}} = -\Gamma \langle e | \rho_A | e \rangle. \quad (2.16)$$

This first line of (2.12) also describes the damping of 'Zeeman coherences' in the excited level and of 'optical coherences'.

The second line of (2.12) describes the 'feeding' by spontaneous emission of the ground-state Zeeman sublevels. Note the presence of $\exp(i\mathbf{k} \cdot \mathbf{R})$ and $\exp(-i\mathbf{k} \cdot \mathbf{R})$, so that a rate equation as simple as (2.16) cannot be obtained with this term. As we will see in § 2.4, these terms $\exp(\pm i\mathbf{k} \cdot \mathbf{R})$ describe the recoil of the atom in the spontaneous emission process: the population of the ground state, corresponding to a given momentum \mathbf{p} , can be fed by spontaneous emission of atoms in the excited state with a momentum $\mathbf{p} + \hbar\mathbf{k}$, where $\hbar\mathbf{k}$ is the spontaneous photon momentum.

2.3. Transformation to the rotating reference frame

In order to eliminate all time dependences in the coefficients of the equation of evolution of the atomic density matrix, we now put

$$\begin{aligned} \langle e | \hat{\rho}_A | g \rangle &= \langle e | \rho_A | g \rangle \exp(i\omega_L t) \\ \langle g | \hat{\rho}_A | e \rangle &= \langle g | \rho_A | e \rangle \exp(-i\omega_L t) \\ \langle e | \hat{\rho}_A | e' \rangle &= \langle e | \rho_A | e' \rangle \\ \langle g | \hat{\rho}_A | g' \rangle &= \langle g | \rho_A | g' \rangle. \end{aligned} \quad (2.17)$$

The evolution of ρ_A is given by

$$\begin{aligned} \frac{d}{dt} \hat{\rho}_A &= \frac{1}{i\hbar} [\hat{H}_A + \hat{V}_{A-L}, \hat{\rho}_A] - \frac{\Gamma}{2} [(S^+ \cdot S^-) \hat{\rho}_A + \hat{\rho}_A (S^+ \cdot S^-)] \\ &\quad + \Gamma \int \frac{d^2\kappa}{8\pi/3} \sum_{\epsilon \perp \kappa} (S^- \cdot \epsilon^*) [\exp(-i\mathbf{k} \cdot \mathbf{R}) \hat{\rho}_A \exp(i\mathbf{k} \cdot \mathbf{R})] (S^+ \cdot \epsilon) \end{aligned} \quad (2.18)$$

where \hat{H}_A and \hat{V}_{A-L} are the time-independent operators

$$\hat{H}_A = \frac{\mathbf{P}^2}{2m} - \frac{\hbar\delta}{2} \left(\sum_e Q_{ee} - \sum_g Q_{gg} \right) \quad (2.19a)$$

$$\hat{V}_{A-L} = -d(S^+ \cdot \mathcal{E}^{(+)}(\mathbf{R}) + S^- \cdot \mathcal{E}^{(-)}(\mathbf{R})) \quad (2.19b)$$

$$\delta = \omega_L - \omega_0. \quad (2.19c)$$

2.4. Wigner representation of the atomic density matrix

As explained in the introduction, the Wigner representation is very well adapted to the study of atomic motion in a light wave. In this representation, the density operator $\rho_A(t)$ is represented by the $[(2J_g + 1) + (2J_e + 1)]^2$ matrix $W(\mathbf{r}, \mathbf{p}, t)$ (Wigner 1932,

Takabayasi 1954, De Groot and Suttorp 1972):

$$W(\mathbf{r}, \mathbf{p}, t) = \frac{1}{h^3} \int d^3u \langle \mathbf{r} + \frac{1}{2}\mathbf{u} | \hat{\rho}_A(t) | \mathbf{r} - \frac{1}{2}\mathbf{u} \rangle \exp(-i\mathbf{p} \cdot \mathbf{u}/\hbar). \quad (2.20)$$

We also define the Wigner function $f(\mathbf{r}, \mathbf{p}, t)$ which is the trace of W :

$$f(\mathbf{r}, \mathbf{p}, t) = \text{Tr}(W(\mathbf{r}, \mathbf{p}, t)). \quad (2.21)$$

We can now write the equation of evolution for $W(\mathbf{r}, \mathbf{p}, t)$: starting from (2.18), we obtain after a straightforward calculation

$$\begin{aligned} \frac{\partial}{\partial t} W(\mathbf{r}, \mathbf{p}, t) = & -\frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} W(\mathbf{r}, \mathbf{p}, t) + \frac{i\delta}{2} \left(\sum_e Q_{ee} - \sum_g Q_{gg}, W(\mathbf{r}, \mathbf{p}, t) \right) \\ & + \frac{i}{\hbar} \int d^3k \exp(i\mathbf{k} \cdot \mathbf{r}) [W(\mathbf{r}, \mathbf{p} + \frac{1}{2}\hbar\mathbf{k}, t) \mathcal{V}(\mathbf{k}) - \mathcal{V}(\mathbf{k}) W(\mathbf{r}, \mathbf{p} - \frac{1}{2}\hbar\mathbf{k}, t)] \\ & - \frac{1}{2} \Gamma [(S^+ \cdot S^-) W(\mathbf{r}, \mathbf{p}, t) + W(\mathbf{r}, \mathbf{p}, t) (S^+ \cdot S^-)] \\ & + \Gamma \int \frac{d^2\kappa}{8\pi/3} \sum_e (S^- \cdot \boldsymbol{\varepsilon}^*) W(\mathbf{r}, \mathbf{p} + \hbar\mathbf{k}, t) (S^+ \cdot \boldsymbol{\varepsilon}) \end{aligned} \quad (2.22)$$

where we have introduced the Fourier transform $\mathcal{V}(\mathbf{k})$ of the operator $\hat{V}_{A-L}(\mathbf{r})$:

$$\hat{V}_{A-L}(\mathbf{r}) = \int d^3k \exp(i\mathbf{k} \cdot \mathbf{r}) \mathcal{V}(\mathbf{k}). \quad (2.23)$$

The first two terms in (2.22) come from H_A , via respectively the kinetic energy and the internal energy. The second line describes the atom-laser coupling and the third and fourth lines come from the atom-vacuum coupling. Note in the fourth line the presence of $W(\mathbf{r}, \mathbf{p} + \hbar\mathbf{k}, t)$ and not $W(\mathbf{r}, \mathbf{p}, t)$, which, as we already mentioned (§ 2.2), is a signature of the recoil of the atom in spontaneous emission.

2.5. Elimination of free flight

The last step of this section is to eliminate, in equation (2.22), the free flight term $-(\mathbf{p}/m) \partial W / \partial \mathbf{r}$. Such an elimination will indeed simplify the calculations of the next sections. This will be achieved simply by introducing the following change of function:

$$\tilde{W}(\mathbf{r}, \mathbf{p}, t) = W\left(\mathbf{r} + \frac{\mathbf{p}}{m}(t - t_0), \mathbf{p}, t\right) \quad (2.24a)$$

$$\tilde{f}(\mathbf{r}, \mathbf{p}, t) = f\left(\mathbf{r} + \frac{\mathbf{p}}{m}(t - t_0), \mathbf{p}, t\right) \quad (2.24b)$$

where t_0 is an arbitrary reference time. The equation of evolution of \tilde{W} can now be obtained in a straightforward way; we first write again (2.22) for $(\mathbf{r} + (\mathbf{p}/m)(t - t_0), \mathbf{p}, t)$ instead of $(\mathbf{r}, \mathbf{p}, t)$:

$$(i) \quad \left(\frac{\partial}{\partial t} W(\mathbf{r}, \mathbf{p}, t) \right)_{\mathbf{r}} \text{ becomes } \left(\frac{\partial}{\partial t} W\left(\mathbf{r} + \frac{\mathbf{p}}{m}(t - t_0), \mathbf{p}, t\right) \right)_{\mathbf{r} + (\mathbf{p}/m)(t - t_0)}$$

and we have

$$\begin{aligned}
 & \left(\frac{\partial}{\partial t} W \left(\mathbf{r} + \frac{\mathbf{p}}{m} (t - t_0), \mathbf{p}, t \right) \right)_{\mathbf{r} + (\mathbf{p}/m)(t-t_0)} \\
 &= \left(\frac{\partial}{\partial t} W \left(\mathbf{r} + \frac{\mathbf{p}}{m} (t - t_0), \mathbf{p}, t \right) \right)_r - \frac{\mathbf{p}}{m} \cdot \left(\frac{\partial}{\partial \mathbf{r}} W \left(\mathbf{r} + \frac{\mathbf{p}}{m} (t - t_0), \mathbf{p}, t \right) \right)_t \\
 &= \frac{\partial}{\partial t} \tilde{W}(\mathbf{r}, \mathbf{p}, t) - \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} \tilde{W}(\mathbf{r}, \mathbf{p}, t)
 \end{aligned} \tag{2.25}$$

$$(ii) \quad -\frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} W(\mathbf{r}, \mathbf{p}, t) \text{ becomes } -\frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} \tilde{W}(\mathbf{r}, \mathbf{p}, t)$$

$$\begin{aligned}
 (iii) \quad W(\mathbf{r}, \mathbf{p} + \tfrac{1}{2}\hbar\mathbf{k}) & \text{ becomes } W \left(\mathbf{r} + \frac{\mathbf{p}}{m} (t - t_0), \mathbf{p} + \tfrac{1}{2}\hbar\mathbf{k}, t \right) \\
 &= \tilde{W} \left(\mathbf{r} - \frac{\hbar\mathbf{k}}{2m} (t - t_0), \mathbf{p} + \tfrac{1}{2}\hbar\mathbf{k}, t \right).
 \end{aligned}$$

We finally get for the equation of evolution of \tilde{W} :

$$\begin{aligned}
 \frac{\partial}{\partial t} \tilde{W}(\mathbf{r}, \mathbf{p}, t) &= \tfrac{1}{2}i\delta \left(\sum_e Q_{ee} - \sum_g Q_{gg}, \tilde{W}(\mathbf{r}, \mathbf{p}, t) \right) \\
 &+ \frac{i}{\hbar} \int d^3k \left\{ \exp \left[i\mathbf{k} \cdot \left(\mathbf{r} + \frac{\mathbf{p}}{m} (t - t_0) \right) \right] \right. \\
 &\times \left(\tilde{W} \left(\mathbf{r} - \frac{\hbar\mathbf{k}}{2m} (t - t_0), \mathbf{p} + \tfrac{1}{2}\hbar\mathbf{k}, t \right) \mathcal{V}(\mathbf{k}) \right. \\
 &\left. \left. - \mathcal{V}(\mathbf{k}) \tilde{W} \left(\mathbf{r} + \frac{\hbar\mathbf{k}}{2m} (t - t_0), \mathbf{p} - \tfrac{1}{2}\hbar\mathbf{k}, t \right) \right) \right\} \\
 &- \tfrac{1}{2}\Gamma[(\mathbf{S}^+ \cdot \mathbf{S}^-) \tilde{W}(\mathbf{r}, \mathbf{p}, t) + \tilde{W}(\mathbf{r}, \mathbf{p}, t)(\mathbf{S}^+ \cdot \mathbf{S}^-)] \\
 &+ \Gamma \int \frac{d^2\kappa}{8\pi/3} \sum_{\epsilon} (\mathbf{S}^- \cdot \boldsymbol{\epsilon}^*) \tilde{W} \left(\mathbf{r} - \frac{\hbar\mathbf{k}}{m} (t - t_0), \mathbf{p} + \hbar\mathbf{k}, t \right) (\mathbf{S}^+ \cdot \boldsymbol{\epsilon}).
 \end{aligned} \tag{2.26}$$

This equation, as (2.18) or (2.22), is an exact expression. The free flight term has been eliminated so that the equation of evolution of \tilde{W} does not contain any spatial derivative. The counterpart of this elimination is that the exponential $\exp(i\mathbf{k} \cdot \mathbf{r})$ in the atom-laser coupling term is now $\exp\{i\mathbf{k} \cdot [\mathbf{r} + (\mathbf{p}/m)(t - t_0)]\}$ (Doppler effect) and also that the equation is no longer local in \mathbf{r} since it involves \mathbf{r} and $\mathbf{r} \pm (\hbar\mathbf{k}/m)(t - t_0)$.

Remark. One can note that the transformation (2.24), from W to \tilde{W} , is the equivalent of the following unitary transformation on the density operator.

$$\tilde{\rho}_A(t) = \exp \left(i \frac{\mathbf{p}^2(t - t_0)}{2m\hbar} \right) \rho_A(t) \exp \left(-i \frac{\mathbf{p}^2(t - t_0)}{2m\hbar} \right) \tag{2.27}$$

which corresponds to an interaction representation with respect to the atomic kinetic energy.

3. Principle of the adiabatic elimination of fast internal atomic variables

Equation (2.26), obtained at the end of the previous section, describes all the dynamics for an atom in a monochromatic light wave. Unfortunately, this equation involves a great number of coupled matrix elements $(4(J_e + J_g + 1)^2)$, each of these being a function of $(\mathbf{r}, \mathbf{p}, t)$, and its general solution is impossible to obtain, even for the simplest case of a two-level atom ($J_e = J_g = 0$). The purpose of this section is then to show that under certain conditions concerning the atomic momentum distribution, it is possible to extract from (2.26) a closed equation for the trace \tilde{f} of \tilde{W} . More precisely, we want to express the time derivative $\partial \tilde{f}(\mathbf{r}, \mathbf{p}, t)/\partial t$ in terms of $\tilde{f}(\mathbf{r}, \mathbf{p}, t)$ and its \mathbf{r} and \mathbf{p} derivatives.

3.1. Validity conditions and principle of the procedure

In all this paper, we will first limit ourselves to situations where the momentum width Δp of \tilde{W} is large compared with the photon momentum $\hbar k$:

$$\varepsilon_1 = \hbar k / \Delta p \ll 1. \quad (3.1)$$

This means that a single-photon absorption or emission process changes only very slightly the atomic momentum distribution. (For sodium atoms, this corresponds to a velocity spread large compared with 3 cm s^{-1}). Note that such an assumption concerning the smallness of the elementary steps of a given process is very often the starting point of a Fokker-Planck treatment of this process (see e.g. Van Kampen 1981). Secondly, we will only consider in this paper slow atoms such as those found in laser cooling experiments. More precisely, we assume that these atoms travel over a small distance (compared with the optical wavelength λ) during the internal relaxation time Γ^{-1} :

$$v\Gamma^{-1} \ll \lambda \quad \text{or} \quad \varepsilon_2 = kv/\Gamma \ll 1 \quad (3.2)$$

where v is a typical atomic velocity (root mean square velocity). For sodium atoms, this gives $v \ll 6 \text{ m s}^{-1}$. This assumption has important consequences concerning the form of the solution of equation (2.26). Since the displacement of the atom during the internal relaxation time is very small, the internal variables are at every time 'nearly' in their steady state, following quasi-adiabatically the external motion. In other terms, provided condition (3.2) is fulfilled, internal atomic variables appear as fast components of \tilde{W} , while $\tilde{f} = \text{Tr}(\tilde{W})$, the variations of which describe the modification of motion due to the laser light, is the only slow component of \tilde{W} .

Remarks

(i) Conditions (3.1) and (3.2), which put a lower and an upper bound on the atomic velocity are compatible only if:

$$\frac{\hbar^2 k^2 / 2m}{\hbar \Gamma} \ll 1. \quad (3.3)$$

This condition, which is supposed to be fulfilled in the following, means that the recoil energy has to be very small compared with the natural width or, in other words, that the atom is still in resonance with the laser light after a single photon absorption or emission.

(ii) Since there are two expansion parameters, ε_1 and ε_2 , it is important to know their respective orders of magnitude, for a given situation, in order to expand at the correct order the initial equation (2.26). We are mostly interested here in the radiative cooling limit where one has (see e.g. Wineland and Itano 1979):

$$\Delta p^2/2m \sim \hbar\Gamma \quad (3.4)$$

so that

$$\varepsilon_1 \sim \varepsilon_2 \sim \left(\frac{\hbar^2 k^2/2m}{\hbar\Gamma} \right)^{1/2} \ll 1. \quad (3.5)$$

In such a case, ε_1 and ε_2 are small parameters with the same order of magnitude, and equation (2.26) has then to be expanded to the same order in ε_1 and ε_2 .

(iii) Because of the Heisenberg inequality

$$\Delta r \geq \hbar/\Delta p \quad (3.6)$$

condition (3.2) actually gives a lower bound for the spatial width Δr of the atomic distribution function:

$$\Delta r \gg \hbar k/m\Gamma. \quad (3.7)$$

Note that, because of (3.3), this lower bound is much smaller than the optical wavelength.

We can now outline the procedure which will be followed in this paper: we are looking for the time derivative $\partial \tilde{f}/\partial t$, which we will calculate for simplicity at time $t = t_0$ (t_0 can actually take any value so that this choice $t = t_0$ does not introduce any restriction). In order to get this time derivative at time t_0 , we will need the evolution of internal variables of \tilde{W} on a time interval prior to t_0 and of the order of the relaxation time Γ^{-1} of these variables. We will then expand equation (2.26) on a time interval:

$$|t - t_0| \leq \text{a few } \Gamma^{-1}. \quad (3.8)$$

This expansion gives for the variable p , using (3.1):

$$\begin{aligned} W(\mathbf{r}, \mathbf{p} + \hbar \mathbf{k}, t) &= W(\mathbf{r}, \mathbf{p}, t) + \hbar \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}} W(\mathbf{r}, \mathbf{p}, t) \\ &+ \sum_{ij} \frac{\hbar^2 k_i k_j}{2} \frac{\partial^2}{\partial p_i \partial p_j} W(\mathbf{r}, \mathbf{p}, t) + \dots \end{aligned} \quad (3.9)$$

For the variable \mathbf{r} , using (3.7) and (3.8), we get

$$\begin{aligned} \tilde{W}\left(\mathbf{r} + \frac{\hbar \mathbf{k}}{m}(t - t_0), \mathbf{p}, t\right) &= \tilde{W}(\mathbf{r}, \mathbf{p}, t) + \frac{\hbar \mathbf{k}}{m}(t - t_0) \frac{\partial}{\partial \mathbf{r}} \tilde{W}(\mathbf{r}, \mathbf{p}, t) \\ &+ \sum_{ij} \frac{\hbar^2 k_i k_j}{2m^2} (t - t_0)^2 \frac{\partial^2}{\partial r_i \partial r_j} \tilde{W}(\mathbf{r}, \mathbf{p}, t) + \dots \end{aligned} \quad (3.10)$$

We can also expand the exponential

$$\exp\left[i\mathbf{k} \cdot \left(\mathbf{r} + \frac{\mathbf{p}}{m}(t - t_0)\right)\right] = \exp(i\mathbf{k} \cdot \mathbf{r}) \left(1 + i \frac{(\mathbf{k} \cdot \mathbf{p})}{m}(t - t_0) + \dots\right). \quad (3.11)$$

We thus obtain in this way an expansion of $\partial \tilde{W}(\mathbf{r}, \mathbf{p}, t)/\partial t$ in terms of its \mathbf{r} and \mathbf{p} derivatives at the same point, same momentum and same time. In the next section

(§ 3.2), we study the zeroth order of this expansion and we show in particular that the trace \tilde{f} does not evolve at the lowest order. This will be the starting point of the adiabatic elimination (§§ 3.3–3.5) of the rapid internal variables of \tilde{W} to the benefit of the slow one \tilde{f} .

3.2. Zeroth-order expansion: optical Bloch equations

To zeroth order in ε_1 and ε_2 , equation (2.26) can be written:

$$\frac{\partial}{\partial t} \tilde{W}(\mathbf{r}, \mathbf{p}, t) = \mathcal{L}_{\text{Bloch}}(\mathbf{r}) \cdot \tilde{W}(\mathbf{r}, \mathbf{p}, t) \quad (3.12)$$

where $\mathcal{L}_{\text{Bloch}}$ is the following operator acting on \tilde{W} :

$$\begin{aligned} \mathcal{L}_{\text{Bloch}}(\mathbf{r}) \cdot \tilde{W}(\mathbf{r}, \mathbf{p}, t) = & \frac{1}{i\hbar} \left[\frac{\hbar\delta}{2} \left(\sum_g Q_{gg} - \sum_e Q_{ee} \right) + \hat{V}_{\text{A-L}}(\mathbf{r}), \tilde{W}(\mathbf{r}, \mathbf{p}, t) \right] \\ & - \frac{1}{2} \Gamma [(S^+ \cdot S^-) \tilde{W}(\mathbf{r}, \mathbf{p}, t) + \tilde{W}(\mathbf{r}, \mathbf{p}, t) (S^+ \cdot S^-)] \\ & + \Gamma \int \frac{d^2\kappa}{8\pi/3} \sum_{\varepsilon \perp \kappa} (S^- \cdot \varepsilon^*) \tilde{W}(\mathbf{r}, \mathbf{p}, t) (S^+ \cdot \varepsilon). \end{aligned} \quad (3.13)$$

$\mathcal{L}_{\text{Bloch}}$ is the so-called Bloch operator (Allen and Eberly 1975), giving the evolution of the reduced atomic density matrix ρ_A for an atom 'at rest in \mathbf{r} ' (the variable \mathbf{p} remains spectator):

$$\frac{d}{dt} \rho_A = \mathcal{L}_{\text{Bloch}}(\mathbf{r}) \cdot \rho_A. \quad (3.14)$$

The first line of (3.13) describes the atomic free evolution and the atom–laser coupling, the second and third lines describe the relaxation due to spontaneous emission. An important quantity associated with the Bloch operator (3.13) is the steady-state density matrix, $\sigma_s(\mathbf{r})$, satisfying:

$$\mathcal{L}_{\text{Bloch}}(\mathbf{r}) \cdot \sigma_s(\mathbf{r}) = 0 \quad (3.15a)$$

$$\text{Tr } \sigma_s(\mathbf{r}) = 1. \quad (3.15b)$$

Note that the characteristic time for reaching this steady state is Γ^{-1} (relaxation time of internal variables†).

Since $\mathcal{L}_{\text{Bloch}}$ is time independent, equation (3.12) can be formally integrated, between two times t_1 and t_2 ($t_2 \geq t_1$):

$$\tilde{W}(\mathbf{r}, \mathbf{p}, t_2) = \exp[\mathcal{L}_{\text{Bloch}}(\mathbf{r})(t_2 - t_1)] \tilde{W}(\mathbf{r}, \mathbf{p}, t_1). \quad (3.16)$$

As soon as $t_2 - t_1$ is larger than Γ^{-1} , (3.16) can be simplified since $\tilde{W}(\mathbf{r}, \mathbf{p}, t_2)$ is then proportional to the steady-state matrix $\sigma_s(\mathbf{r})$. The proportionality coefficient, equal to $\text{Tr}(\tilde{W}(\mathbf{r}, \mathbf{p}, t_2))$ according to (3.15b), is also equal to $\text{Tr}(\tilde{W}(\mathbf{r}, \mathbf{p}, t_1))$ since the trace of \tilde{W} remains unchanged during the time evolution (3.12), as a consequence of

$$\frac{d}{dt} (\text{Tr } \tilde{W}) = \text{Tr}(\mathcal{L}_{\text{Bloch}}(\mathbf{r}) \cdot \tilde{W}) = 0 \quad (3.17)$$

† We eliminate here the rare cases where there are several steady states, or where the single steady state is reached only after a time long compared with Γ^{-1} , due to a very weak laser excitation implying long pumping times.

which can be easily shown from (3.13) (conservation of the total population at point \mathbf{r}). Equation (3.16) can then be written in this case

$$t_2 - t_1 \geq \Gamma^{-1} \rightarrow \tilde{W}(\mathbf{r}, \mathbf{p}, t_2) = \tilde{f}(\mathbf{r}, \mathbf{p}, t_1) \sigma_s(\mathbf{r}). \quad (3.18)$$

3.3. The slow variable \tilde{f}

Since the trace \tilde{f} of \tilde{W} does not evolve at zeroth order (cf (3.17)), $d\tilde{f}/dt$ is at least of order one in ε_1 and ε_2 , contrary to the other components of \tilde{W} . In other words, \tilde{f} is a slowly varying function compared with the internal atomic variables. Starting from a formal expansion of (2.26) up to order two in ε_1 and ε_2 :

$$\frac{\partial}{\partial t} \tilde{W}(\mathbf{r}, \mathbf{p}, t) = (\mathcal{L}_{\text{Bloch}}(\mathbf{r}) + \mathcal{L}_1(\mathbf{r}, \mathbf{p}, t) + \mathcal{L}_2(\mathbf{r}, \mathbf{p}, t)) \cdot \tilde{W}(\mathbf{r}, \mathbf{p}, t) \quad (3.19)$$

we can write for the evolution of \tilde{f} :

$$\frac{\partial}{\partial t} \tilde{f}(\mathbf{r}, \mathbf{p}, t) = \text{Tr}[(\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t) + \mathcal{L}_2(\mathbf{r}, \mathbf{p}, t)) \cdot \tilde{W}(\mathbf{r}, \mathbf{p}, t)] \quad (3.20)$$

(\mathcal{L}_1 and \mathcal{L}_2 represent respectively the first- and second-order contributions).

The next step is now to calculate formally $\tilde{W}(\mathbf{r}, \mathbf{p}, t)$ up to order one, and to express it in terms of \tilde{f} and its \mathbf{r}, \mathbf{p} derivatives (§ 3.4). Using (3.20), we will then get the requested closed equation giving the evolution of \tilde{f} up to order two (§ 3.5). Such a procedure is called an adiabatic elimination of fast internal variables to the benefit of the slow variable \tilde{f} (see e.g. Gardiner 1983, Stenholm 1984a, b).

3.4. Calculation of the density matrix

We are looking here for a density operator \tilde{W} having the following form:

$$\tilde{W}(\mathbf{r}, \mathbf{p}, t) = \tilde{f}(\mathbf{r}, \mathbf{p}, t) \sigma_s(\mathbf{r}) + \tilde{\chi}(\mathbf{r}, \mathbf{p}, t). \quad (3.21)$$

In this expression $\tilde{f} \sigma_s$ is the part of \tilde{W} which follows adiabatically the external atomic motion, and $\tilde{\chi}$ the deviation of \tilde{W} from its adiabatic part, which we hope to be small, and which we want to calculate up to order one.

We first note, using (2.21) and (3.15b), that

$$\text{Tr}(\tilde{\chi}(\mathbf{r}, \mathbf{p}, t)) = 0. \quad (3.22)$$

We now insert (3.21) in the equation of evolution of \tilde{W} (3.19); using (3.15a) and (3.20), we obtain at order one:

$$\frac{\partial}{\partial t} \tilde{\chi} = \mathcal{L}_{\text{Bloch}} \cdot \tilde{\chi} + \mathcal{L}_1 \cdot (\tilde{f} \sigma_s + \tilde{\chi}) - \text{Tr}[\mathcal{L}_1 \cdot (\tilde{f} \sigma_s + \tilde{\chi})] \sigma_s. \quad (3.23)$$

The perturbative resolution of (3.23) is now straightforward; at zeroth order, one gets:

$$\frac{\partial}{\partial t} \tilde{\chi} = \mathcal{L}_{\text{Bloch}} \cdot \tilde{\chi} \quad (3.24)$$

or

$$\tilde{\chi}(\mathbf{r}, \mathbf{p}, t) = \exp[\mathcal{L}_{\text{Bloch}}(\mathbf{r})(t - t_i)] \tilde{\chi}(\mathbf{r}, \mathbf{p}, t_i) \quad (3.25)$$

for any time t_i prior to t . Provided the atom-laser coupling has been on for a period

larger than Γ^{-1} , one can choose $t_i < t - \Gamma^{-1}$ so that, using (3.19) and (3.22):

$$\tilde{\chi}(\mathbf{r}, \mathbf{p}, t) = 0 \text{ at order zero.} \quad (3.26)$$

Putting (3.26) in (3.23), we obtain at order one:

$$\frac{\partial}{\partial t} \tilde{\chi} = \mathcal{L}_{\text{Bloch}} \cdot \tilde{\chi} + \mathcal{L}_1 \cdot \tilde{f} \sigma_s - \text{Tr}(\mathcal{L}_1 \cdot \tilde{f} \sigma_s) \sigma_s, \quad (3.27)$$

which can be integrated:

$$\begin{aligned} \tilde{\chi}(\mathbf{r}, \mathbf{p}, t) = \int_{t_i}^t dt' \exp[\mathcal{L}_{\text{Bloch}}(\mathbf{r})(t-t')] & [\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t') \cdot \tilde{f}(\mathbf{r}, \mathbf{p}, t') \sigma_s(\mathbf{r}) \\ & - \text{Tr}(\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t') \cdot \tilde{f}(\mathbf{r}, \mathbf{p}, t') \sigma_s(\mathbf{r})) \sigma_s(\mathbf{r})]. \end{aligned} \quad (3.28)$$

Since the quantity inside the bracket has a zero trace, the only times which contribute to the integral verify $t - t' \leq \Gamma$ so that one can replace the lower bound t_i of the integral by $-\infty$.

Remark. The previous calculations could be also presented in terms of projection operators (see e.g. Agarwal 1974, Stenholm 1984a, b). Let $P(\mathbf{r})$ and $Q(\mathbf{r})$ be the operators in Liouville space defined by their action on any vector \tilde{W} of this space:

$$\begin{aligned} P(\mathbf{r}) \cdot \tilde{W} &= \sigma_s(\mathbf{r}) \text{Tr}(\tilde{W}) \\ Q(\mathbf{r}) &= \mathbb{1} - P(\mathbf{r}). \end{aligned} \quad (3.29)$$

One easily checks that P and Q project respectively on slow and fast subspaces of equation (3.19). Using this general formulation, we get for $\tilde{\chi}$:

$$\tilde{\chi}(\mathbf{r}, \mathbf{p}, t) = Q(\mathbf{r}) \tilde{W}(\mathbf{r}, \mathbf{p}, t) \quad (3.30)$$

and $\mathcal{L}_{\text{Bloch}}(\mathbf{r})$ can be shown to obey the following equalities

$$P(\mathbf{r}) \cdot \mathcal{L}_{\text{Bloch}}(\mathbf{r}) = \mathcal{L}_{\text{Bloch}}(\mathbf{r}) \cdot P(\mathbf{r}) = 0 \quad (3.31a)$$

$$\mathcal{L}_{\text{Bloch}}(\mathbf{r}) = Q(\mathbf{r}) \mathcal{L}_{\text{Bloch}}(\mathbf{r}) Q(\mathbf{r}). \quad (3.31b)$$

The expression (3.28) for $\tilde{\chi}(\mathbf{r}, \mathbf{p}, t)$ can then be written in the following way:

$$\tilde{\chi}(\mathbf{r}, \mathbf{p}, t) = \int_{-\infty}^t dt' \exp[\mathcal{L}_{\text{Bloch}}(\mathbf{r})(t-t')] Q(\mathbf{r}) \mathcal{L}_1(\mathbf{r}, \mathbf{p}, t') P(\mathbf{r}) \cdot \tilde{W}(\mathbf{r}, \mathbf{p}, t'). \quad (3.32)$$

3.5. Evolution of the Wigner function

We now come to the last step of this section, i.e. we insert the expression of \tilde{W} , obtained in (3.21) and (3.28), in the equation of evolution (3.20) of \tilde{f} . This equation (3.20) is 'a priori' valid for any time t close to t_0 . Actually, we write it for $t = t_0$ since this choice will simplify notably the calculations of §§ 4 and 5; we then get:

$$\begin{aligned} \left. \frac{\partial}{\partial t} \tilde{f} \right|_{t=t_0} &= \text{Tr}(\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t_0) \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r})) \\ &+ \text{Tr}(\mathcal{L}_2(\mathbf{r}, \mathbf{p}, t_0) \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r})) \\ &+ \text{Tr} \left(\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t_0) \int_{-\infty}^{t_0} dt' \exp[\mathcal{L}_{\text{Bloch}}(\mathbf{r})(t_0-t')] \right. \end{aligned}$$

$$\begin{aligned}
& \times \mathcal{L}_1(\mathbf{r}, \mathbf{p}, t') \cdot \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r}) \Big) \\
& - \text{Tr} \left(\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t_0) \int_{-\infty}^{t_0} dt' \exp[\mathcal{L}_{\text{Bloch}}(\mathbf{r})(t_0 - t')] \right. \\
& \left. \times \text{Tr}(\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t') \cdot \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r})) \sigma_s(\mathbf{r}) \right). \quad (3.33)
\end{aligned}$$

Note that we have replaced in the two time integrals $\tilde{f}(\mathbf{r}, \mathbf{p}, t')$ by $\tilde{f}(\mathbf{r}, \mathbf{p}, t_0)$. This is possible since only times t' close to t_0 contribute to the integral, and for such times, the difference $\tilde{f}(\mathbf{r}, \mathbf{p}, t') - \tilde{f}(\mathbf{r}, \mathbf{p}, t_0)$ is of order one in ε_1 and ε_2 and would then give a contribution to $\partial \tilde{f} / \partial t|_{t=t_0}$ of order three at least. To summarise, we have derived in this section two main results. First, we have obtained an expression of the atomic density matrix (3.21)–(3.28). In § 4, we will discuss the physical content of this expression. Secondly, we have obtained a closed equation of evolution for the Wigner function (3.33). In § 5, we will show, using the expressions of \mathcal{L}_1 and \mathcal{L}_2 extracted from (2.26), that this equation is of a Fokker–Planck type and we will calculate its coefficients.

Note finally that equation (3.33) deals with the function \tilde{f} defined in (2.24b) and not with the real Wigner function f . However, since we are now considering a time t equal to the reference time t_0 , we have from (2.24b):

$$\tilde{f}(\mathbf{r}, \mathbf{p}, t_0) = f(\mathbf{r}, \mathbf{p}, t_0) \quad (3.34a)$$

$$\left. \frac{\partial}{\partial t} \tilde{f} \right|_{t=t_0} = \left. \frac{\partial}{\partial t} f \right|_{t=t_0} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} f \Big|_{t=t_0}. \quad (3.34b)$$

4. Perturbative calculation of the atomic density operator

In this section, we first determine, using (2.26), the operator $\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t)$ (§ 4.1). We then use this expression to calculate, using (3.21) and (3.28), the atomic density operator (§ 4.2). Finally, we interpret physically our results (§ 4.3).

4.1. Rate of variation at order one

The expansion of (2.26), up to order one in ε_1 and ε_2 , can be written, using (3.9), (3.10) and (3.11):

$$\frac{\partial}{\partial t} \tilde{W}(\mathbf{r}, \mathbf{p}, t) = (\mathcal{L}_{\text{Bloch}}(\mathbf{r}) + \mathcal{L}_1(\mathbf{r}, \mathbf{p}, t)) \cdot \tilde{W}(\mathbf{r}, \mathbf{p}, t) \quad (4.1)$$

with

$$\begin{aligned}
\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t) \cdot \tilde{W}(\mathbf{r}, \mathbf{p}, t) &= \frac{i(t - t_0)}{m\hbar} [\mathbf{p} \cdot \mathbf{F}(\mathbf{r}), \tilde{W}(\mathbf{r}, \mathbf{p}, t)] \\
&- \frac{1}{2} \left\{ \left(\frac{\partial}{\partial \mathbf{p}} - \frac{t - t_0}{m} \frac{\partial}{\partial \mathbf{r}} \right) \tilde{W}(\mathbf{r}, \mathbf{p}, t), \mathbf{F}(\mathbf{r}) \right\}_+. \quad (4.2)
\end{aligned}$$

We have used in (4.2) the notation:

$$\{A, B\}_+ = AB + BA \quad (4.3)$$

and we have put

$$\mathbf{F}(\mathbf{r}) = - \int d^3k \, i\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{r}) \mathcal{V}(\mathbf{k}) = -\nabla \hat{V}_{A-L}(\mathbf{r}). \quad (4.4)$$

We will call $\mathbf{F}(\mathbf{r})$ the radiative force operator at point \mathbf{r} , since it is the quantum equivalent of a classical force, usually defined as the opposite of the potential gradient[†].

4.2. Calculation of the atomic density operator

We can now insert the expression of \mathcal{L}_1 in (3.28), which gives $\tilde{\chi}$, and then get \tilde{W} up to order one. Since there are actually three terms in \mathcal{L}_{12} , one commutator of \tilde{W} with $\mathbf{p} \cdot \mathbf{F}(\mathbf{r})$ and two anticommutators between $\partial \tilde{W} / \partial \mathbf{p}$, $\partial \tilde{W} / \partial \mathbf{r}$ and \mathbf{F} , there are three corresponding contributions from $\tilde{\chi}$ to \tilde{W} ; taking (3.28) for $t = t_0$ and putting $\tau = t - t'$ in the integral, we immediately get[‡]

$$\tilde{W}(\mathbf{r}, \mathbf{p}, t_0) = \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r}, \mathbf{p}, t_0) + \tilde{\chi}(\mathbf{r}, \mathbf{p}, t_0) \quad (4.5a)$$

with

$$\begin{aligned} \tilde{\chi}(\mathbf{r}, \mathbf{p}, t_0) = & -\tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \frac{i}{m\hbar} \int_0^\infty d\tau \exp(\mathcal{L}_{\text{Bloch}}(\mathbf{r})\tau) [\mathbf{p} \cdot \mathbf{F}(\mathbf{r}), \sigma_s(\mathbf{r})] \\ & - \frac{\partial}{\partial \mathbf{p}} \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \int_0^\infty d\tau \exp(\mathcal{L}_{\text{Bloch}}(\mathbf{r})\tau) \\ & \times [\tfrac{1}{2}(\mathbf{F}(\mathbf{r})\sigma_s(\mathbf{r}) + \sigma_s(\mathbf{r})\mathbf{F}(\mathbf{r})) - \boldsymbol{\phi}(\mathbf{r})\sigma_s(\mathbf{r})] \\ & - \frac{\partial}{\partial \mathbf{r}} \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \int_0^\infty d\tau \frac{\tau}{m} \exp(\mathcal{L}_{\text{Bloch}}(\mathbf{r})\tau) \\ & \times [\tfrac{1}{2}(\mathbf{F}(\mathbf{r})\sigma_s(\mathbf{r}) + \sigma_s(\mathbf{r})\mathbf{F}(\mathbf{r})) - \boldsymbol{\phi}(\mathbf{r})\sigma_s(\mathbf{r})]. \end{aligned} \quad (4.5b)$$

The three terms of (4.5b) respectively come from the three contributions of \mathcal{L}_1 to $\tilde{\chi}$. We have put in (4.5b):

$$\boldsymbol{\phi}(\mathbf{r}) = \text{Tr}(\mathbf{F}(\mathbf{r})\sigma_s(\mathbf{r})). \quad (4.6)$$

$\boldsymbol{\phi}$ is the average, in the stationary state, of the radiative force. It is equal to the stationary force calculated in the semiclassical approach (cf introduction).

4.3. Physical interpretation

In addition to the adiabatic term $\tilde{f}\sigma_s$, there are, via $\tilde{\chi}$, three contributions to \tilde{W} that we want now to discuss.

Consider the first term of (4.5b):

$$-\tilde{f} \frac{i}{m\hbar} \int_0^\infty d\tau \exp(\mathcal{L}_{\text{Bloch}}\tau) [\mathbf{p} \cdot \mathbf{F}, \sigma_s].$$

[†] Note that, because of the isotropy of the quantised field in the vacuum state, the relaxation terms in (2.26) do not contribute to \mathcal{L}_1 .

[‡] The r derivative of \tilde{W} actually gives rise to two terms $(\partial \tilde{f} / \partial \mathbf{r}) \sigma_s$ and $\tilde{f} d\sigma_s / d\mathbf{r}$. However the characteristic scale of variation for σ_s is larger than λ so that $|d\sigma_s / d\mathbf{r}|$ is of the order or smaller than $k\sigma_s$. The contribution of $\tilde{f} d\sigma_s / d\mathbf{r}$ can then be shown to be at least of order two, and is therefore neglected in (4.5b). The other term $(d\tilde{f} / d\mathbf{r}) \sigma_s$ cannot be neglected since spatial distributions narrower than λ can be considered (the only restriction on the spatial spread is (3.7)).

Adding this term to the adiabatic one $\tilde{f}\sigma_s$, we can write the result as $\tilde{f}\sigma$ with:

$$\sigma = \sigma_s - \frac{i}{m\hbar} \int_0^\infty d\tau \tau \exp(\mathcal{L}_{\text{Bloch}} \tau) [\mathbf{p} \cdot \mathbf{F}, \sigma_s]. \quad (4.7)$$

Such a density matrix σ has actually a clear physical meaning. Note first that the integral term in (4.7) is just the linear atomic response at time t_0 , to the time-dependent perturbation $(\mathbf{p} \cdot \mathbf{E}/m\hbar)(t-t_0)$. σ is then the (first-order) solution at time t_0 , of the equation:

$$\frac{d}{dt} \sigma = \mathcal{L}_{\text{Bloch}}(\mathbf{r}) \sigma - \frac{1}{i\hbar} \left[\frac{\mathbf{p} \cdot \mathbf{F}(\mathbf{r})}{m} (t-t_0), \sigma \right]. \quad (4.8)$$

Now, coming back to the definition of $\mathbf{F}(\mathbf{r})$, we get:

$$-\frac{\mathbf{p} \cdot \mathbf{F}}{m} (t-t_0) = \hat{V}_{\text{A-L}} \left(\mathbf{r} + \frac{\mathbf{p}}{m} (t-t_0) \right) - \hat{V}_{\text{A-L}}(\mathbf{r}) \quad (4.9)$$

so that σ is actually the solution of:

$$\frac{d}{dt} \sigma = \mathcal{L}_{\text{Bloch}} \left(\mathbf{r} + \frac{\mathbf{p}}{m} (t-t_0) \right) \sigma. \quad (4.10)$$

The matrix σ then appears to be the atomic density operator including the first-order correction due to the atomic motion. It contains, in particular, the modification of the internal state due to the Doppler effect in a plane wave. We will then call the first line of (4.5b) the ‘Doppler term’.

Consider now the second and third contributions in (4.5b):

$$\begin{aligned} & -\frac{\partial \tilde{f}}{\partial \mathbf{p}} \int_0^\infty d\tau \exp(\mathcal{L}_{\text{Bloch}} \tau) \left[\frac{1}{2} (\mathbf{F}\sigma_s + \sigma_s \mathbf{F}) - \phi \sigma_s \right] \\ & -\frac{\partial \tilde{f}}{\partial \mathbf{r}} \int_0^\infty d\tau \frac{\tau}{m} \exp(\mathcal{L}_{\text{Bloch}} \tau) \left[\frac{1}{2} (\mathbf{F}\sigma_s + \sigma_s \mathbf{F}) - \phi \sigma_s \right]. \end{aligned}$$

These two terms have a structure which is different from the ‘Doppler term’. They are indeed not proportional to \tilde{f} , but to its derivatives with respect to \mathbf{r} and \mathbf{p} . For this reason, they cannot be taken into account by simply replacing the stationary state σ_s , by another one, σ , as we have done for the Doppler term. Actually, these terms describe a correlation between internal and external variables. More precisely, the position and momentum atomic distributions are split into several parts, each of these corresponding to a given atomic internal state. This can easily be seen on the simple example of a two-level atom moving in a strong resonant standing wave (no gradient of phase, zero detuning and Rabi frequency larger than Γ). We introduce the dressed levels:

$$\begin{aligned} |1\rangle &= \frac{1}{\sqrt{2}} (|e\rangle + |g\rangle) \\ |2\rangle &= \frac{1}{\sqrt{2}} (|e\rangle - |g\rangle) \end{aligned} \quad (4.11)$$

where $|g\rangle$ and $|e\rangle$ are the ground and excited levels. These dressed states are eigenstates of the dressed-atom Hamiltonian (‘atom + laser’ Hamiltonian in the ‘rotating’ frame)

with energies respectively equal to $\frac{1}{2}\hbar\omega_1$ and $-\frac{1}{2}\hbar\omega_1$ (Reynaud and Cohen-Tannoudji 1982, Reynaud 1983). In this dressed set, σ_s and \tilde{F} have the following expressions, at the lowest order in Γ/ω_1 :

$$\sigma_s = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \mathbf{F} = -\frac{\hbar\nabla\omega_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.12)$$

and \tilde{W} can be easily calculated:

$$\tilde{W}(\mathbf{r}, \mathbf{p}, t_0) = \frac{1}{2}\tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\partial\tilde{f}}{\partial\mathbf{p}} \frac{\hbar\nabla\omega_1}{2\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\partial\tilde{f}}{\partial\mathbf{r}} \frac{\hbar\nabla\omega_1}{m\Gamma^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.13)$$

which can also be written:

$$\tilde{W}(\mathbf{r}, \mathbf{p}, t_0) = \frac{1}{2} \begin{pmatrix} \tilde{f}\left(\mathbf{r} + 2\frac{\hbar\nabla\omega_1}{m\Gamma^2}, \mathbf{p} + \frac{\hbar\nabla\omega_1}{\Gamma}, t_0\right) & 0 \\ 0 & \tilde{f}\left(\mathbf{r} - 2\frac{\hbar\nabla\omega_1}{m\Gamma^2}, \mathbf{p} - \frac{\hbar\nabla\omega_1}{\Gamma}, t_0\right) \end{pmatrix}. \quad (4.14)$$

We then find a diagonal density matrix with equal global weights on the two dressed levels. However, we see also that there is a slight splitting between the position or momentum distributions corresponding to levels 1 and 2. For example, if the total position distribution is centred in \mathbf{r}_0 , then the position distribution 'for level 1' is centred in $\mathbf{r}_0 - 2\hbar\nabla\omega_1/m\Gamma^2$, and 'for level 2' in $\mathbf{r}_0 + 2\hbar\nabla\omega_1/m\Gamma^2$. Such a splitting has a straightforward interpretation: an atom in level 1 (level 2) experiences a force $\mathbf{F} = -\frac{1}{2}\hbar\nabla\omega_1$ ($+\frac{1}{2}\hbar\nabla\omega_1$, respectively); this force acts during a random time, of the order of $T = 2/\Gamma$, which is the radiative lifetime of both levels 1 and 2 (Reynaud 1983). It follows that this force produces a momentum splitting $\Delta\mathbf{p} \sim \mathbf{F} \cdot T \sim \pm\hbar\nabla\omega_1/\Gamma$ and a position splitting $\Delta\mathbf{r} \sim m^{-1}\mathbf{F} \cdot T^2 \sim \pm\hbar\nabla\omega_1/m\Gamma^2$ which are precisely those found above.

5. Fokker-Planck equation for the atomic Wigner function

In this last section, we derive the equation of motion for the Wigner function \tilde{f} . This is done first at order one (§ 5.1), and then at order two (§ 5.2). We get in this way a Fokker-Planck equation and we show that the coefficients of this equation are related to correlation functions of the radiative force operator (§ 5.3). Finally, we discuss the physical content of our results (§ 5.4).

5.1. Rate of variation of the Wigner function at order one: mean radiative force

At order one, the evolution of the Wigner function \tilde{f} is given by (cf (3.20)):

$$\left. \frac{\partial\tilde{f}}{\partial t} \right|_{t=t_0} = \text{Tr}(\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t_0)\tilde{f}(\mathbf{r}, \mathbf{p}, t_0)\sigma_s(\mathbf{r})). \quad (5.1)$$

For $t = t_0$, the only non-vanishing part of the expression (4.2) of \mathcal{L}_1 is:

$$\mathcal{L}_1(\mathbf{r}, \mathbf{p}, t_0)\tilde{W}(\mathbf{r}, \mathbf{p}, t_0) = -\frac{1}{2} \left\{ \frac{\partial}{\partial\mathbf{p}} \tilde{W}(\mathbf{r}, \mathbf{p}, t_0), \mathbf{F}(\mathbf{r}) \right\}_+ \quad (5.2)$$

so that

$$\left. \frac{\partial \tilde{f}}{\partial t} \right|_{t=t_0} = -\text{Tr} \left(\frac{\partial}{\partial \mathbf{p}} (\tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r})) \cdot \mathbf{F}(\mathbf{r}) \right) = -\frac{\partial \tilde{f}}{\partial \mathbf{p}} \cdot \boldsymbol{\phi}(\mathbf{r}) \quad (5.3)$$

using the definition (4.6) of $\boldsymbol{\phi}$. Coming back to f , via equation (3.34), we finally have at order one:

$$\left. \frac{\partial}{\partial t} f(\mathbf{r}, \mathbf{p}, t) \right|_{t=t_0} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} f(\mathbf{r}, \mathbf{p}, t_0) = -\frac{\partial}{\partial \mathbf{p}} f(\mathbf{r}, \mathbf{p}, t_0) \cdot \boldsymbol{\phi}(\mathbf{r}). \quad (5.4)$$

This equation is the same as the one describing the evolution of a classical phase-space distribution $f(\mathbf{r}, \mathbf{p}, t)$ for a particle with mass m , submitted to a force equal to the steady-state radiative force $\boldsymbol{\phi}$.

5.2. Variation of the Wigner function at order two: Fokker-Planck equation

In order to apply the formula (3.33), giving $\partial \tilde{f} / \partial t|_{t=t_0}$, we still have to evaluate the quantity:

$$\text{Tr}(\mathcal{L}_2(\mathbf{r}, \mathbf{p}, t_0) \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r})). \quad (5.5)$$

For that, we expand at order two the equation of evolution for \tilde{W} (2.26), keeping in mind that we only need \mathcal{L}_2 for $t = t_0$ and, furthermore, that we take the trace of $\mathcal{L}_2 \cdot W$. As a consequence, the atom-laser coupling is not going to contribute to (5.5), since the only non-zero part of this coupling at order two and for $t = t_0$ is a sum of commutators:

$$\sum_{ij} -\frac{i\hbar}{8} \left[\frac{\partial^2 \hat{W}}{\partial p_i \partial p_j}, \frac{\partial^2 \hat{V}_{A-L}}{\partial r_i \partial r_j} \right] \quad (5.6)$$

which have a zero trace. The relaxation term on the contrary has a non-vanishing contribution. For $t = t_0$, we find:

$$(\mathcal{L}_2 \tilde{W})_{\text{relaxation}} = \frac{\hbar^2 k^2 \Gamma}{2} \sum_{ij} \int \frac{d^2 \kappa}{8\pi/3} \kappa_i \kappa_j \sum_{\epsilon} (\mathbf{S}^+ \cdot \boldsymbol{\epsilon}^*) \frac{\partial^2 \tilde{W}}{\partial p_i \partial p_j} (\mathbf{S}^+ \cdot \boldsymbol{\epsilon}) \quad (5.7)$$

so that (5.5) becomes:

$$\text{Tr}(\mathcal{L}_2(\mathbf{r}, \mathbf{p}, t_0) \tilde{f}(\mathbf{r}, \mathbf{p}, t_0) \sigma_s(\mathbf{r})) = \frac{\hbar^2 k^2 \Gamma}{2} \sum_{ij} \frac{\partial^2 \tilde{f}}{\partial p_i \partial p_j} E_{ij} \quad (5.8a)$$

with

$$E_{ij} = \int \frac{d^2 \kappa}{8\pi/3} \kappa_i \kappa_j \sum_{\epsilon} \langle (\mathbf{S}^+ \cdot \boldsymbol{\epsilon}) (\mathbf{S}^- \cdot \boldsymbol{\epsilon}^*) \rangle_{\sigma_s}. \quad (5.8b)$$

The coefficients E_{ij} are related to the dipole radiation diagram for an atom in the stationary state σ_s . More precisely, E_{ij} is proportional to the average value of $(\hbar \mathbf{k}_i) \cdot (\hbar \mathbf{k}_j)$ where $\hbar \mathbf{k}$ is the momentum of a fluorescence photon. For a two-level atom, E_{ij} is also proportional to the stationary population of the excited level. In the

general case, we have the relation (using (2.15)):

$$\begin{aligned}\sum_i E_{ii} &= \int \frac{d^2\kappa}{8\pi/3} \sum_{\mathbf{e}} \langle (\mathbf{S}^+ \cdot \mathbf{e})(\mathbf{S}^- \cdot \mathbf{e}^*) \rangle_{\sigma_s} \\ &= \langle \mathbf{S}^+ \cdot \mathbf{S}^- \rangle_{\sigma_s} = \sum_{\mathbf{e}} \langle e | \sigma_s | e \rangle.\end{aligned}\quad (5.9)$$

$\sum_i E_{ii}$ is then the total population of the excited atomic level.

We are now able to derive from (3.33) the explicit expression for $\partial f / \partial t|_{t=t_0}$; in this expression, there will be a first-order term, already calculated in § 5.1, and two types of second-order terms. The first type is a term involving $\mathcal{L}_2(t_0)$ which we have just calculated in (5.8). The second type of second-order terms are those formed by products of $\mathcal{L}_1(t_0)$ and $\mathcal{L}_1(t)$; $\mathcal{L}_1(t_0)$ consists just in a \mathbf{p} derivative of \tilde{W} (see (5.2)), while $\mathcal{L}_1(t)$, for $t \neq t_0$, involves three terms (see (4.2)): a \mathbf{p} derivative, an \mathbf{r} derivative and a commutator of \tilde{W} with $\mathbf{p} \cdot \mathbf{F}(\mathbf{r})$. As a consequence, there will be three terms of this second type in $\partial f / \partial t|_{t=t_0}$: a double \mathbf{p}, \mathbf{p} derivative ('diffusion term'), a crossed \mathbf{p}, \mathbf{r} derivative ('crossed term') and a single \mathbf{p} derivative combined with a multiplication by $\mathbf{p} \cdot \mathbf{F}$ ('friction term'). We then get the following Fokker-Planck equation, coming back to f via (3.34):

$$\begin{aligned}\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{r}} &= -\frac{\partial f}{\partial \mathbf{p}} \cdot \boldsymbol{\Phi}(\mathbf{r}) + \frac{\hbar^2 k^2 \Gamma}{2} \sum_{ij} E_{ij}(\mathbf{r}) \frac{\partial^2 f}{\partial p_i \partial p_j} \\ &+ \sum_{ij} D_{ij}(\mathbf{r}) \frac{\partial^2 f}{\partial p_i \partial p_j} + \sum_{ij} \eta_{ij}(\mathbf{r}) \frac{\partial^2 f}{\partial p_i \partial r_j} + \sum_{ij} \gamma_{ij}(\mathbf{r}) \frac{\partial}{\partial p_i} (p_j f).\end{aligned}\quad (5.10)$$

On the first line of (5.10), we find the first-order term and the second-order term related to $\mathcal{L}_2(t_0)$. On the second line, we find the three terms coming from the product $\mathcal{L}_1(t_0) \cdot \mathcal{L}_1(t)$. Using (3.33), (4.2) and (5.2), one finds for the three coefficients D_{ij} , η_{ij} and γ_{ij} :

$$D_{ij}(\mathbf{r}) = \text{Tr} \left(F_i(\mathbf{r}) \int_0^\infty d\tau \exp(\mathcal{L}_{\text{Bloch}}(\mathbf{r})\tau) \left[\frac{1}{2} (F_j(\mathbf{r})\sigma_s(\mathbf{r}) + \sigma_s(\mathbf{r})F_j(\mathbf{r})) - \phi_j(\mathbf{r})\sigma_s(\mathbf{r}) \right] \right) \quad (5.11a)$$

$$\eta_{ij}(\mathbf{r}) = \text{Tr} \left(F_i(\mathbf{r}) \int_0^\infty d\tau \frac{\tau}{m} \exp(\mathcal{L}_{\text{Bloch}}(\mathbf{r})\tau) \left[\frac{1}{2} (F_j(\mathbf{r})\sigma_s(\mathbf{r}) + \sigma_s(\mathbf{r})F_j(\mathbf{r})) - \phi_j(\mathbf{r})\sigma_s(\mathbf{r}) \right] \right) \quad (5.11b)$$

$$\gamma_{ij}(\mathbf{r}) = \text{Tr} \left(F_i(\mathbf{r}) \int_0^\infty d\tau \frac{i\tau}{m\hbar} \exp(\mathcal{L}_{\text{Bloch}}(\mathbf{r})\tau) [F_j(\mathbf{r}), \sigma_s(\mathbf{r})] \right). \quad (5.11c)$$

5.3. Coefficients of the Fokker-Planck equation and correlation functions of the force operator

Before giving a physical interpretation for equation (5.10), it is useful to relate first the coefficients D_{ij} , η_{ij} and γ_{ij} to the correlation functions of the Heisenberg force $\mathbf{F}_H(\mathbf{r}, t)$ introduced in the semiclassical theory. More precisely, we want to establish here the two equalities:

$$\text{Tr}[A \exp(\mathcal{L}_{\text{Bloch}}\tau)(B\sigma_s)]^* = \langle B_H(0)A_H(\tau) \rangle_{\sigma_s} \quad (5.12a)$$

$$\text{Tr}[A \exp(\mathcal{L}_{\text{Bloch}}\tau)(\sigma_s B)]^* = \langle A_H(\tau)B_H(0) \rangle_{\sigma_s}. \quad (5.12b)$$

Putting $A = F_j$ and $B = F_j$ in (5.12), we will then be able to simplify the three expressions (5.11).

It is important to note that, in the left-hand side of equations (5.12), the various operators A, B, σ_s are pure atomic operators, actually internal atomic operators[†] for an atom at rest in \mathbf{r} , whereas, on the right-hand side, $A_H(\tau)$ and $B_H(0)$ are Heisenberg operators in the total space including the variables of the quantised field. More precisely, the evolution of $A_H(\tau)$, in the right-hand side of (5.12), is Hamiltonian, governed by the total Hamiltonian (2.1) where \mathbf{R} is treated as a fixed c number \mathbf{r} (see e.g. Gordon and Ashkin 1980), whereas $\mathcal{L}_{\text{Bloch}}$ describes a non-Hamiltonian evolution, including relaxation terms, for the reduced atomic variables.

Let us first transform the left-hand side of (5.12a). Expanding A on a set of eigenvectors, V_μ , of $\mathcal{L}_{\text{Bloch}}$ (in Liouville space), with eigenvalues λ_μ :

$$A = \sum_{\mu} \alpha_{\mu} V_{\mu} \quad (5.13)$$

we get

$$\text{Tr}[A \exp(\mathcal{L}_{\text{Bloch}} \tau) (B \sigma_s)]^* = \left\langle B \left(\sum_{\mu} \alpha_{\mu} \exp(\lambda_{\mu} \tau) V_{\mu} \right) \right\rangle_{\sigma_s}. \quad (5.14)$$

Before turning to the right-hand side of (5.12a), consider now one-time averages, $\langle A_H(\tau) \rangle$, of the Heisenberg operator $A_H(\tau)$ (for an arbitrary atomic state). Going back from the Heisenberg picture to the Schrödinger picture, and using the equation of motion (3.14) of the reduced atomic density matrix[‡], one can show that, for $\tau > 0$:

$$\langle A_H(\tau) \rangle = \left\langle \sum_{\mu} \alpha_{\mu} \exp(\lambda_{\mu} \tau) V_{\mu} \right\rangle. \quad (5.15)$$

In order to calculate the two-time average of the right-hand side of (5.12a), we can then use the quantum regression theorem (Lax 1968, Louisell 1973), which applies when the correlation time of the reservoir (in our case, the correlation time of vacuum fluctuations) is very short compared with the relaxation time (in our case, the radiative lifetime Γ^{-1}). Such a theorem states that, for $\tau > 0$, the τ dependence of the two-time average $\langle B_H(0) A_H(\tau) \rangle$ is given by the same equation as the τ dependence of the one-time average $\langle A_H(\tau) \rangle$. More precisely, if $\langle A_H(\tau) \rangle$ is given by (4.15), then:

$$\langle B_H(0) A_H(\tau) \rangle = \left\langle B_H(0) \left(\sum_{\mu} \alpha_{\mu} \exp(\lambda_{\mu} \tau) V_{\mu} \right) \right\rangle. \quad (5.16)$$

Using $B_H(0) = B$, and comparing (5.14) and (5.16), one deduces the equality (5.12a). We could prove (5.12b) in the same way.

Finally, we can now re-express the coefficients D_{ij} , η_{ij} and γ_{ij} in the following way:

$$D_{ij}(\mathbf{r}) = \int_0^{\infty} d\tau \left[\left\langle \frac{1}{2} (F_{iH}(\mathbf{r}, \tau) F_{jH}(\mathbf{r}, 0) + F_{jH}(\mathbf{r}, 0) F_{iH}(\mathbf{r}, \tau)) \right\rangle_{\sigma_s} - \phi_i(\mathbf{r}) \phi_j(\mathbf{r}) \right] \quad (5.17)$$

$$\eta_{ij}(\mathbf{r}) = \frac{1}{m} \int_0^{\infty} d\tau \tau \left[\left\langle \frac{1}{2} (F_{iH}(\mathbf{r}, \tau) F_{jH}(\mathbf{r}, 0) + F_{jH}(\mathbf{r}, 0) F_{iH}(\mathbf{r}, \tau)) \right\rangle_{\sigma_s} - \phi_i(\mathbf{r}) \phi_j(\mathbf{r}) \right] \quad (5.18)$$

$$\gamma_{ij}(\mathbf{r}) = \frac{i}{m\hbar} \int_0^{\infty} d\tau \tau \left[\langle [F_{iH}(\mathbf{r}, \tau), F_{jH}(\mathbf{r}, 0)] \rangle_{\sigma_s} \right]. \quad (5.19)$$

[†] They can be also considered as vectors in the internal atomic Liouville space, $\mathcal{L}_{\text{Bloch}}$ being an operator in this Liouville space.

[‡] We need only this reduced matrix since A is an atomic operator.

5.4. Physical discussion

The Fokker-Planck equation (5.10) describes the motion of a particle with mass m , submitted to a force ($\partial f / \partial p_i$ terms), with momentum diffusion ($\partial^2 f / \partial p_i \partial p_j$ terms) and 'crossed effects' ($\partial^2 f / \partial p_i \partial r_j$ terms). We now discuss separately these three contributions.

5.4.1. Mean force. The total force appearing in (5.10) can be written:

$$\mathcal{F}_i(\mathbf{r}, \mathbf{p}) = \phi_i(\mathbf{r}) - \sum_j \gamma_{ij}(\mathbf{r}) p_j. \quad (5.20)$$

It is the sum of the stationary force ϕ and of a damping (or heating) force. We have obtained in (5.19) an operatorial expression for the damping tensor γ_{ij} , which, to our knowledge, is a new one. We have checked that for a two-level system, (5.20) gives back the usual friction force calculated for example, semiclassically by an explicit expansion of the atomic density matrix in terms of $\hbar v / \Gamma$ (Gordon and Ashkin 1980). However let us insist on the fact that (5.19) and (5.20) are valid for any $J_g - J_e$ transition.

We can find the damping force again by a semiclassical argument using a linear response theory, as we have done for the 'Doppler term' of the density matrix in § 4.3 (actually, the damping force and the 'Doppler term' come both from the component of $\mathcal{L}_1(t): [\mathbf{p} \cdot \mathbf{F}, W]$). In a semiclassical treatment, the atomic motion can be taken into account by replacing the atom-laser coupling $\hat{V}_{A-L}(\mathbf{r})$ by the time-dependent one $\hat{V}_{A-L}(\mathbf{r} + (\mathbf{p}/m)(t - t_0))$ which, at first order, can be written (cf (4.9)):

$$\hat{V}_{A-L}\left(\mathbf{r} + \frac{\mathbf{p}}{m}(t - t_0)\right) = \hat{V}_{A-L}(\mathbf{r}) + U(\mathbf{r}, \mathbf{p}, t) \quad (5.21)$$

where U is the operator:

$$U(\mathbf{r}, \mathbf{p}, t) = -\frac{t - t_0}{m} \mathbf{p} \cdot \mathbf{F}(\mathbf{r}). \quad (5.22)$$

If we look for the total average force acting on the moving atom (MA) at time t_0 , $\langle F_H(\mathbf{r}, t_0) \rangle_{MA}$, we find first the stationary force $\phi(\mathbf{r})$, resulting from the atom-laser coupling $V_{A-L}(\mathbf{r})$, and secondly a small correction, due to the perturbation $U(\mathbf{r}, \mathbf{p}, t)$, and that we can write, using linear response theory (Martin 1968):

$$\langle F_{iH}(\mathbf{r}, t_0) \rangle_{MA} = \phi_i(\mathbf{r}) - \frac{i}{\hbar} \int_{-\infty}^{t_0} dt \langle [F_{iH}(\mathbf{r}, t_0), U_H(\mathbf{r}, \mathbf{p}, t)] \rangle_{\sigma_s}. \quad (5.23)$$

Using the expression of the damping tensor γ_{ij} (5.19), one then checks immediately that (5.23) is identical to (5.20).

5.4.2. Diffusion terms. The total diffusion coefficient of equation (5.10) is

$$\mathcal{D}_{ij}(\mathbf{r}) = D_{ij}(\mathbf{r}) + \frac{1}{2} \hbar^2 k^2 \Gamma E_{ij}(\mathbf{r}). \quad (5.24)$$

As it is now well known (see e.g. Stenholm 1983), we find two contributions to the momentum diffusion coefficient. The term $\frac{1}{2} \hbar^2 k^2 \Gamma E_{ij}(\mathbf{r})$ is due to the fluctuations of the momentum carried away by fluorescence photons, while $D_{ij}(\mathbf{r})$ corresponds to fluctuations in momentum exchanges between the atom and the laser (fluctuations of the number of photons absorbed in a plane running wave, fluctuations of the dipole force in a focused wave).

An important result of this paper is the expression (5.17) for the coefficient D_{ij} . Such an expression had already been used in a semiclassical treatment of the atomic motion in a light wave (Cook 1980a, Gordon and Ashkin 1980) but it has never appeared, to our knowledge, in a fully quantum treatment leading to a Fokker-Planck equation. Note that expression (5.17) is the quantum analogue of the classical diffusion coefficients found in Brownian motion theory:

$$D_{ij} = \overline{\mathcal{F}_i p_j} - \overline{\mathcal{F}_i} \overline{p_j} \quad (5.25)$$

where \mathbf{p} is the momentum of the Brownian particle, and \mathcal{F} and the fluctuating force acting on it. Using:

$$\frac{d\mathbf{p}}{dt} = \mathcal{F} \quad (5.26)$$

one gets

$$D_{ij} = \int_{-\infty}^0 dt (\overline{\mathcal{F}_i(0) \mathcal{F}_j(t)} - \overline{\mathcal{F}_i(0)} \overline{\mathcal{F}_j(t)}) \quad (5.27)$$

which, for a stationary process, is the analogue of equation (5.17).

5.4.3. Crossed effects. These effects come from the crossed derivative of (5.10):

$$\eta_{ij} \frac{\partial^2 f}{\partial p_i \partial r_j}.$$

This seems to be a new term for free atoms, but one can note that a contribution of this type has already been found in the study of the motion of trapped ions (Javanainen 1981). In our problem, the origin of this term lies in the spatial splitting of the various components of the internal atomic density operator found in § 4.3. The action of the radiative force operator on these components causes a rotation in phase space of the total Wigner function, and this rotation is described by the $\partial^2 f / \partial p_i \partial r_j$ term.

Note that, although we could not eliminate this term from the beginning since it appears at order two, as damping and diffusion, its contribution seems to be very small in most interesting physical situations. For example, for a two-level system moving in a strong running wave, η_{ij} is found to be of the order of the recoil energy $\hbar^2 k^2 / 2m$. If the spatial extension of f is of the order or larger than the optical wavelength λ , we then get:

$$\eta_{ij} \frac{\partial^2 f}{\partial p_i \partial r_j} \leq \frac{\hbar^2 k^2 / m}{\hbar \Gamma} \hbar k \Gamma \frac{\partial f}{\partial p_i} \approx \varepsilon_1 \varepsilon_2 \boldsymbol{\phi} \cdot \frac{\partial f}{\partial \mathbf{p}} \quad (5.28)$$

where we have used the expression of the average force $\boldsymbol{\phi}$ in a strong running wave:

$$\boldsymbol{\phi} = \frac{1}{2} \hbar \mathbf{k} \Gamma. \quad (5.29)$$

Since $\boldsymbol{\phi} \cdot \partial f / \partial \mathbf{p}$ is a term of order one in ε_1 and ε_2 , $\eta_{ij} \partial^2 f / \partial p_i \partial r_j$ is in this case at least of order three and can therefore be neglected. More generally, it appears that this term contributes only in situations where, after a large number of spontaneous emissions (which is necessary to derive the Fokker-Planck equation), the spatial extension remains small compared with λ . Besides this very particular case, we can omit the

crossed term, so that we are left finally with the following Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{r}} = - \frac{\partial}{\partial \mathbf{p}} (\mathcal{F}(\mathbf{r}, \mathbf{p})f) + \sum_{ij} \mathcal{D}_{ij}(\mathbf{r}) \frac{\partial^2 f}{\partial p_i \partial p_j} \quad (5.30)$$

where \mathcal{F} , given by equation (5.20), includes the stationary radiative force and the damping force, and where \mathcal{D}_{ij} , given in (5.24), describes the diffusion due to the fluctuations of the radiative force and of the momentum carried away by spontaneous photons.

6. Conclusion

We have presented in this paper a full quantum treatment of atomic motion in laser light. Starting from the equation of motion of the reduced atomic density matrix, in the case of slow atoms (Doppler effect smaller than the natural width), we have eliminated the fast internal variables to the benefit of the slow external one, and we have derived a quantum kinetic equation for the Wigner function, which is of a Fokker-Planck type.

The originality of the approach presented in this paper consists in its operatorial character. Rather than calculating explicitly the coefficients of the Fokker-Planck equation, we have tried to keep mathematical expressions containing quantities with a clear physical meaning, such as the Bloch operator in \mathbf{r} , $\mathcal{L}_{\text{Bloch}}(\mathbf{r})$, or the radiative force operator in \mathbf{r} , $\mathbf{F}(\mathbf{r})$. This allowed us to compare our results with the semiclassical ones, where similar quantities are introduced 'by hand'.

We have thus shown that our momentum diffusion coefficient exactly coincides with the semiclassical one. We have also derived a new operatorial expression for the damping force responsible for radiative cooling, and we have interpreted it using a semiclassical linear response theory. Finally, a third term appears in our treatment, consisting in a crossed \mathbf{r}, \mathbf{p} derivative. Such a term, which is very small in most situations, has not been anticipated in semiclassical approaches.

The similarity between our results and the semiclassical ones calls for a last comment. The semiclassical treatments are restricted to situations where the atomic wavepacket is very small, and the force and the diffusion coefficient are calculated at the centre of this wavepacket. In the quantum kinetic equation, we find a force and a diffusion coefficient which are functions of \mathbf{r} and which, for each value of \mathbf{r} , coincide with the semiclassical values associated with a very small wavepacket in \mathbf{r} . It must be kept in mind however that the quantum kinetic equation can be applied to atomic wavepackets which are not restricted in size. This equation has therefore a wider range of applicability and allows the study of situations which could not be easily handled by semiclassical treatments.

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