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EFFECTIVE HAMILTONIAN APPROACH TO $g-2$ RELATIVISTIC CALCULATION

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Contents

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1. Introduction

The previous course [1] contains a few simple physical discussions of the spin anomaly $g-2$.

In section 4, which is actually a reprint of reference [2], a dynamical equation is derived for a nonrelativistic electron after elimination of the radiation field variables. The analysis of such an equation [§5 of reference 2] shows that, at the lowest order in the fine structure constant α and in $1/c$, the main effect of the electron-field coupling is to increase the mass appearing in the kinetic energy term [and in the rest mass energy term], whereas the coupling of the spin magnetic moment with a static magnetic field $\boldsymbol{B}_0$ remains unchanged. In other words, the cyclotron frequency of the orbiting charge is slowed down, as a consequence of the mass increase, but the Larmor frequency of the spin is not modified. When re-expressed in terms of the corrected mass appearing in the kinetic energy (or in the corrected cyclotron frequency), the spin magnetic moment appears to have a g-factor larger than 2. Such a simple treatment therefore leads to the correct sign for $g-2$.

Corrections to the Larmor frequency only appear at the next order in $1/c$, as shown in reference [3] (see also section 5.3 of the previous course [1]). They are due to the vacuum fluctuations of the radiation field which, among other effects, produce a random angular vibration of the spin, and, consequently, a decrease of its effective magnetic moment. A similar vibration also exists for the charge and has been considered by Welton [4] in order to interpret the Lamb shift. [The mass correction considered above does not remove the degeneracy between the two states $2s_{1/2}$ and $2p_{1/2}$ of hydrogen which have the same average kinetic energy.] The vibration of the charge in the Coulomb potential of the proton produces a correction of the potential energy which is not the same for the 2s and 2p states, and one gets in this way the correct order of magnitude for the $2s_{1/2}$–$2p_{1/2}$ splitting. For $g-2$, the main effect seems therefore to be the slowing down of the charge motion due to the mass increase and not the vibration of the spin (in the non-relativistic domain, electric effects predominate over magnetic

ones). This explains the failure of a Welton's type picture for explaining the sign of $g-2$ (see also reference [5]).

The previous treatments start from nonrelativistic Hamiltonians and they consider only the contribution of the nonrelativistic modes of the radiation field ($\hbar\omega \ll mc^2$). On the other hand, it has been suggested that the positive sign of $g-2$ could be due to pure relativistic effects, related to the complex dynamics of the Dirac electron, and corresponding for example to a modification of the "Zitterbewegung" induced by high frequency vacuum fluctuations [a brief review of such suggestions is given in §4.3 of reference [3]). It seems therefore interesting to try to extend the effective Hamiltonian approach of [3], in order to include the effect of relativistic modes ($\hbar\omega \gtrsim mc^2$), and to see if they change drastically the main conclusions given above. This is the main purpose of this lecture.

More precisely, we would like to determine the explicit frequency dependence (for all values of ω and not only for $\hbar\omega \ll mc^2$ as in [3]) of the various correction terms which describe the modification of the motion of a slow electron in a weak static magnetic field. To answer such a question, the powerful covariant Q.E.D. formalism, which deals with the S-matrix (and not with the Hamiltonian) doesn't seem appropriate. Furthermore, the integration parameters which appear in the various integrals of such a formalism are not directly related to the frequency of the photon emitted in the virtual intermediate state. This is why we have preferred here to extend, in the simplest possible way, the effective Hamiltonian method of reference [3] to the relativistic domain, using the "old fashioned perturbation theory" and limiting the calculations to the lowest order in the fine structure constant α.

In section 2, we introduce our method of calculation. We start from the full Hamiltonian H of the interacting quantized Dirac and Maxwell fields, given in section 6.3 of the previous course [1], and we derive from H an effective Hamiltonian in the single-electron zero-photon subspace. We then calculate the various terms of the effective Hamiltonian (section 3) and discuss their physical meaning (section 4).

2. General method

2.1. Hamiltonian H – definition of H_0 and V

As in section 6.3 of the previous course [1], the full Q.E.D. Hamiltonian can be written

$$H = H_R + H_D + H_I \tag{1}$$

where H_R is the free Maxwell field Hamiltonian,

$$H_R = \sum_i \hbar\omega_i (a_i^+ a_i + \tfrac{1}{2}), \tag{2}$$

(a_i^+ and a_i being the creation and annihilation operators of a photon of mode i), H_D is the free Dirac field Hamiltonian,

$$H_D = \int d^3r\, \Psi^+(\boldsymbol{r})\{\beta mc^2 + c\boldsymbol{\alpha}\cdot[\boldsymbol{p} - q\boldsymbol{A}_0(\boldsymbol{r})]\}\Psi(\boldsymbol{r}), \tag{3}$$

Ψ^+ and Ψ being the quantized Dirac fields, $\boldsymbol{A}_0$ the static vector potential (we suppose that there is no static scalar potential ϕ_0), and

$$H_I = V_{Coul} + U_\perp \tag{4}$$

is the interaction Hamiltonian which, in Coulomb gauge, contains the electrostatic interaction

$$V_{Coul} = \frac{q^2}{8\pi\varepsilon_0}\int\int d^3r\, d^3r' \frac{\Psi^+(\boldsymbol{r})\Psi(\boldsymbol{r})\Psi^+(\boldsymbol{r}')\Psi(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{5}$$

and the interaction

$$\begin{aligned} U_\perp &= -\int d^3r\, \boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}) \\ &= -qc\int d^3r\, \Psi^+(\boldsymbol{r})\boldsymbol{\alpha}\Psi(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}) \end{aligned} \tag{6}$$

of the Dirac current $\boldsymbol{j}(\boldsymbol{r})$ with the transverse Maxwell field $\boldsymbol{A}(\boldsymbol{r})$.

The splitting of H into an unperturbed part H_0 and a coupling $V = H - H_0$ will be different from the one considered in section 6.4 of [1]. We take now

$$H_0 = H_R + H_D \tag{7}$$

and, consequently, for V,

$$V = H - H_0 = H_I, \tag{8}$$

H_0 describes states containing an arbitrary number of noninteracting electrons e^-, positrons e^+ (orbiting in the static field $\boldsymbol{A}_0$) and photons. V describes the coupling between these three types of particles.

2.2. *Relevant manifolds of* H_0

Since we are interested in the dynamics of a single slow electron, with no incident (real) photon, we must first consider the manifold $\mathscr{E}_1$ of H_0 corresponding to one e^- and zero photon. Such a manifold is non-degenerate since H_D is the second quantized form of the single particle Hamiltonian

$$\mathscr{H}_D = \beta mc^2 + c\boldsymbol{\alpha} \cdot \boldsymbol{\pi}_0 , \tag{9a}$$

$$\pi_0 = \boldsymbol{p} - q\boldsymbol{A}_0(\boldsymbol{r}) , \tag{9b}$$

not restricted, as in section 6.4 of [1], to the βmc^2 term. The energy spectrum of $\mathscr{E}_1$ is therefore the (positive) energy spectrum of a single

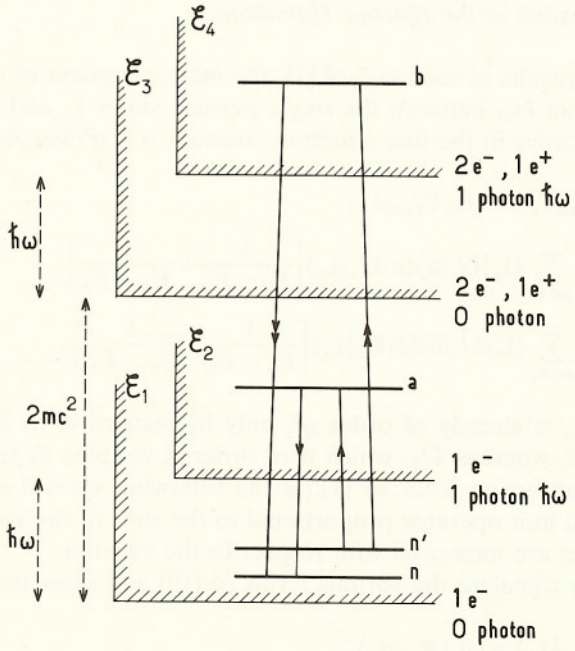

Fig. 1. Relevant manifolds of H_0.

electron orbiting in a static magnetic field. We will call u_n (v_n) the eigenspinors of $\mathscr{H}_D$ corresponding to a positive (negative) energy. We are interested here in the matrix element of the effective Hamiltonian H_{eff} between two states n and n' of the bottom of the $\mathscr{E}_1$ manifold (associated with the nonrelativistic spinors u_n and $u_{n'}$ of $\mathscr{H}_D$).

Since the electron can virtually emit a photon ω, we must also consider the manifold $\mathscr{E}_2$, corresponding to one e^- and one photon ω, and which starts at a distance $\hbar\omega$ above the bottom of $\mathscr{E}_1$ (fig. 1). Starting from n', the combined system can make a virtual transition to a state a of $\mathscr{E}_2$, and then come back to n (single arrow of fig. 1). Contrarily to n and n', there is no restriction on the state a which can be relativistic for the electron.

Finally, by studying the selection rules of the coupling V, it appears that one must also consider the manifold $\mathscr{E}_4$, corresponding to two e^-, one e^+, and one photon ω (creation of a pair e^-–e^+ and of a photon ω from $\mathscr{E}_1$), and which starts at a distance $2mc^2+\hbar\omega$ from the bottom of $\mathscr{E}_1$ (fig. 1). As for $\mathscr{E}_2$, the combined system can follow the path $n' \to b \to n$ where b is an arbitrary state of $\mathscr{E}_4$ (double arrows of fig. 1).

2.3. Expression of the effective Hamiltonian

From the results of section 5 of [1], the matrix element of the effective Hamiltonian H_{eff} between the single particle states 1_n and $1_{n'}$ of $\mathscr{E}_1$, is, to lowest order in the fine structure constant $\alpha = q^2/4\pi\varepsilon_0\hbar c$

$$\begin{aligned}\langle 1_n|H_{\text{eff}}|1_{n'}\rangle &= \langle 1_n|V_{\text{Coul}}|1_{n'}\rangle \\ &+ \tfrac{1}{2}\sum_{a\in\mathscr{E}_2}\langle 1_n|U_\perp|a\rangle\langle a|U_\perp|1_{n'}\rangle\left[\frac{1}{E_n-E_a}+\frac{1}{E_{n'}-E_a}\right] \\ &+ \tfrac{1}{2}\sum_{b\in\mathscr{E}_4}\langle 1_n|U_\perp|b\rangle\langle b|U_\perp|1_{n'}\rangle\left[\frac{1}{E_n-E_b}+\frac{1}{E_{n'}-E_b}\right]. \qquad (10)\end{aligned}$$

Since V_{Coul} is already of order q^2, only its restriction to $\mathscr{E}_1$ has to be considered, whereas $U_\perp$, which is of order q, couples $\mathscr{E}_1$ to $\mathscr{E}_2$ and $\mathscr{E}_4$.

As in subsection 6.4.3. of [1], in the following we will also subtract from H_{eff} a unit operator proportional to the shift of the vacuum, since all energies are measured with respect to the vacuum.

We now calculate the various terms of (10) and show that

$$\langle 1_n|H_{\text{eff}}|1_{n'}\rangle = \langle u_n|\mathscr{H}_{\text{eff}}|u_{n'}\rangle\,, \qquad (11)$$

where $\mathcal{H}_{\text{eff}}$ is a single-particle Dirac Hamiltonian. Since u_n and $u_{n'}$ are nonrelativistic spinors, we will then transform $\mathcal{H}_{\text{eff}}$ into an "even" form (only acting upon two-component spinors) more appropriate for physical discussions.

3. Calculation of the effective Hamiltonian

3.1. Contribution of the Coulomb interaction

3.1.1. Identification and calculation of the various terms

Introducing the Fourier transform of $1/|\boldsymbol{r} - \boldsymbol{r}'|$, we first write V_{Coul} as

$$V_{\text{Coul}} = \frac{q^2}{16\pi^3\varepsilon_0} \int\int\int \mathrm{d}^3r\, \mathrm{d}^3r'\, \mathrm{d}^3k\, \frac{\rho(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \rho(\boldsymbol{r}')\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}}{k^2}, \tag{12a}$$

$$\rho(\boldsymbol{r}) = \Psi^+(\boldsymbol{r})\Psi(\boldsymbol{r}). \tag{12b}$$

We then expand $\Psi(\boldsymbol{r})$ and $\Psi^+(\boldsymbol{r})$ in terms of the creation and annihilation operators for one electron (c^+, c) and one positron (b^+, b) (see section 6.3 of [1])

$$\Psi^+(\boldsymbol{r}) = \sum_q [c_q^+ u_q^+(\boldsymbol{r}) + b_q v_{\bar{q}}^+(\boldsymbol{r})],$$

$$\Psi(\boldsymbol{r}) = \sum_s [c_s u_s(\boldsymbol{r}) + b_s^+ v_{\bar{s}}(\boldsymbol{r})], \tag{13b}$$

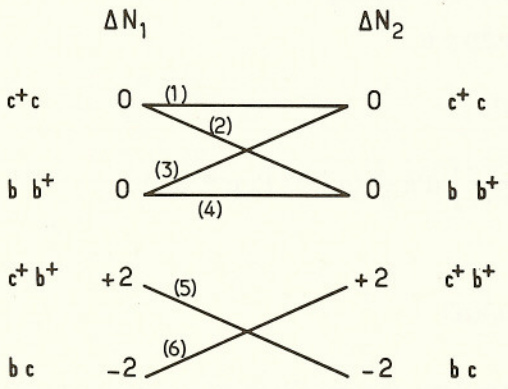

Fig. 2. Identification of the terms of V_{Coul} conserving the total number of particles.

$\Psi^+(\boldsymbol{r})\Psi(\boldsymbol{r})$ contains four terms varying the total number of particles by $\Delta N_1 = 0, \pm 2$ (left column of fig. 2). The same results hold for $\Psi^+(\boldsymbol{r}')\Psi(\boldsymbol{r}')$ and ΔN_2 (right column of fig. 2)

There are therefore six ways of combining a term of $\Psi^+(\boldsymbol{r})\Psi(\boldsymbol{r})$ and a term of $\Psi^+(\boldsymbol{r}')\Psi^+(\boldsymbol{r}')$ with $\Delta N_1 + \Delta N_2 = 0$ in order to get a nonzero matrix element of V_{Coul} between two states with the same number of particles as in (10). We now calculate the contribution of these six terms.

Contribution of term 1. This term can be written

$$(1) = \frac{q^2}{16\pi^3\varepsilon_0} \int \frac{\mathrm{d}^3k}{k^2} \sum_q \sum_r \sum_s \sum_t \left[\int \mathrm{d}^3r\, u_q^+(\boldsymbol{r})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, u_r(\boldsymbol{r}) \right]$$
$$\times \left[\int \mathrm{d}^3r'\, u_s^+(\boldsymbol{r}')\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'}\, u_t(\boldsymbol{r}') \right] [\langle 1_n | c_q^+ c_r c_s^+ c_t | 1_{n'} \rangle - \delta_{nn'} \langle 0 | c_q^+ c_r c_s^+ c_t | 0 \rangle] . \tag{14}$$

The $-\delta_{nn'}$ term of the bracket corresponds to the subtraction of the vacuum shift. The two integrals over $\boldsymbol{r}$ and $\boldsymbol{r}'$ give the product of two matrix elements

$$\langle u_q | \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} | u_r \rangle \langle u_s | \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} | u_t \rangle . \tag{15}$$

The two matrix elements of the last line of (14) are equal to

$$\langle 1_n | c_q^+ c_r c_s^+ c_t | 1_{n'} \rangle = \delta_{nq}\delta_{rs}\delta_{tn'} ,$$
$$\langle 0 | c_q^+ c_r c_s^+ c_t | 0 \rangle = 0 . \tag{16}$$

Finally, we get

$$(1) = \frac{q^2}{16\pi^3\varepsilon_0} \int \mathrm{d}^3k\, \frac{1}{k^2} \langle u_n | \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, P_+\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} | u_{n'} \rangle , \tag{17}$$

where

$$P_+ = \sum_s |u_s\rangle\langle u_s| \tag{18}$$

is the projector into the manifold of positive energy states of the single-particle Dirac Hamiltonian $\mathscr{H}_{\text{D}}$.

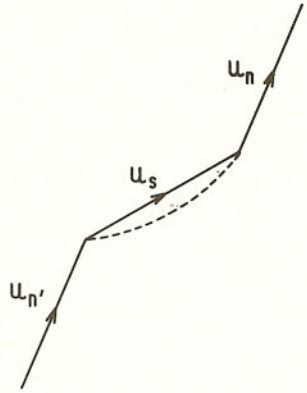

Fig. 3. Coulomb interaction: diagram for term (1).

We can also introduce a diagrammatic representation of the results, using full upwards (downwards) arrows for electron (positron) states, a dotted line for electrostatic interaction, a wavy line for transverse photons. The diagram associated with (17) is represented on fig. 3.

Contribution of terms 2 and 3. We have

$$(2) = \frac{q^2}{16\pi^3\varepsilon_0}\int \frac{\mathrm{d}^3k}{k^2}\sum_q\sum_r\sum_s\sum_t \langle u_q|\mathrm{e}^{\mathrm{i}k\cdot r}|u_r\rangle\langle v_{\bar{s}}|\mathrm{e}^{-\mathrm{i}k\cdot r}|v_{\bar{t}}\rangle \times [\langle 1_n|c_q^+c_rb_sb_t^+|1_{n'}\rangle - \delta_{nn'}\langle 0|c_q^+c_rb_sb_t^+|0\rangle] . \tag{19}$$

Using

$$\langle 1_n|c_q^+c_rb_sb_t^+|1_{n'}\rangle = \delta_{st}\delta_{rn'}\delta_{qn} ,$$
$$\langle 0|c_q^+c_rb_sb_t^+|0\rangle = 0 , \tag{20}$$

this expression becomes

$$(2) = \frac{q^2}{16\pi^3\varepsilon_0}\int \frac{\mathrm{d}^3k}{k^2}\langle u_n|\mathrm{e}^{\mathrm{i}k\cdot r}|u_{n'}\rangle\sum_s\langle v_{\bar{s}}|\mathrm{e}^{-\mathrm{i}k\cdot r}|v_{\bar{s}}\rangle . \tag{21}$$

In order to interpret such a result, we first note that

$$\sum\langle v_{\bar{s}}|\mathrm{e}^{-\mathrm{i}k\cdot r}|v_{\bar{s}}\rangle \tag{22}$$

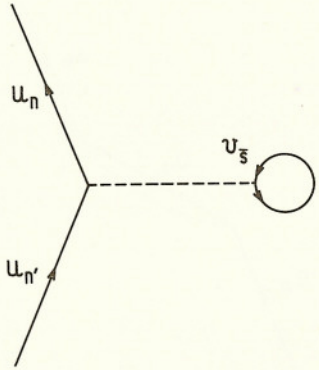

Fig. 4. Coulomb interaction: diagram for terms (2) and (3).

is the Fourier transform of the charge distribution in the vacuum. Expression (21) therefore represents the electrostatic interaction of the electron with the charge distribution in the vacuum [see also the diagrammatic representation of (21) on fig. 4].

Actually, in the presence of a uniform magnetic field $\boldsymbol{B}_0$, which is the situation considered here, the charge distribution in vacuum is uniform (it must be, as the static field $\boldsymbol{B}_0$, invariant under any translation). It doesn't give rise to any electric field. It follows that the contribution of (21) is equal to zero.

It must be emphasized however that, for other situations, for example in the presence of a static scalar potential ϕ_0, the contribution of (21) would not be zero, but would be associated with vacuum polarization effects (and also charge renormalization).

A similar calculation shows that the contribution of term 3 is equal to zero.

Contribution of term 4. One gets

$$(4) = \frac{q^2}{16\pi^3\varepsilon_0}\int \frac{\mathrm{d}^3k}{k^2}\sum_q\sum_r\sum_s\sum_t \langle v_{\bar{q}}|\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}|v_{\bar{r}}\rangle\langle v_{\bar{s}}|\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}|v_{\bar{t}}\rangle$$

$$\times\left[\langle 1_n|b_q b_r^+ b_s b_t^+|1_{n'}\rangle - \delta_{nn'}\langle 0|b_q b_r^+ b_s b_t^+|0\rangle\right]. \qquad (23)$$

The two matrix elements of the bracket are equal to $\delta_{nn'}\delta_{qr}\delta_{st}$ and therefore cancel each other.

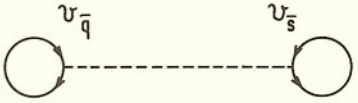

Fig. 5. Coulomb interaction: diagram for term (4).

The diagrammatic representation of term 4 is shown on fig. 5. Such a term actually represents the electrostatic interaction in vacuum, which is the same in presence or in absence of the electron, and which therefore disappears when one subtracts the vacuum shift.

Contribution of term 5. Such a term is proportional to

$$\langle 1_n|c_q^+ b_r^+ b_s c_t|1_{n'}\rangle - \delta_{nn'}\langle 0|c_q^+ b_r^+ b_s c_t|0\rangle\,, \tag{24}$$

and is therefore equal to zero.

Contribution of term 6. One gets

$$(6) = \frac{q^2}{16\pi^3\varepsilon_0}\int \frac{\mathrm{d}^3k}{k^2}\sum_q\sum_r\sum_s\sum_t \langle v_{\bar{q}}|\mathrm{e}^{\mathrm{i}k\cdot r}|u_r\rangle\langle u_s|\mathrm{e}^{-\mathrm{i}k\cdot r}|v_{\bar{t}}\rangle$$
$$\times\left[\langle 1_n|b_q c_r c_s^+ b_t^+|1_{n'}\rangle - \delta_{nn'}\langle 0|b_q c_r c_s^+ b_t^+|0\rangle\right]. \tag{25}$$

The second term of the bracket is equal to $\delta_{nn'}\delta_{rs}\delta_{qt}$. Writing

$$c_r c_s^+ = \delta_{rs} - c_s^+ c_r \tag{26}$$

one transforms the first matrix element of the bracket into

$$\delta_{rs}\delta_{nn'}\delta_{qt} - \delta_{sn}\delta_{qt}\delta_{rn'}\,, \tag{27}$$

so that expression (25) becomes (after changing $\boldsymbol{k}$ into $-\boldsymbol{k}$ in the integral)

$$(6) = -\frac{q^2}{16\pi^3\varepsilon_0}\int \frac{\mathrm{d}^3k}{k^2}\langle u_n|\mathrm{e}^{\mathrm{i}k\cdot r}\,P_-\,\mathrm{e}^{-\mathrm{i}k\cdot r}|u_{n'}\rangle\,, \tag{28}$$

where

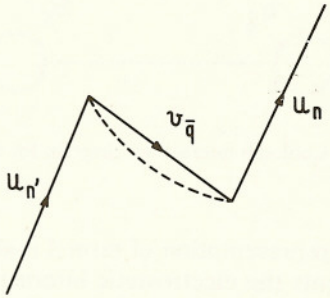

Fig. 6. Coulomb interaction: diagram for term (6).

$$P_- = \sum_q |v_{\bar{q}}\rangle\langle v_{\bar{q}}| \tag{29}$$

is the projector into the manifold of negative energy states of $\mathscr{H}_D$.

In the diagrammatic representation of (28) (see fig. 6), we have now in the intermediate state a positron line (instead of an electron line as in fig. 3).

Combining all the previous results, we finally get

$$\langle 1_n|V_{\text{Coul}}|1_{n'}\rangle = \frac{q^2}{16\pi^3\varepsilon_0}\int \frac{d^3k}{k^2}\langle u_n|e^{ik\cdot r}(P_+ - P_-)\,e^{-ik\cdot r}\,|u_{n'}\rangle\,. \tag{30}$$

Remark. If we come back to section 6.4 of the previous course [1], we can note that the calculation of the contribution of the Coulomb term to the effective Hamiltonian considered in the previous course is exactly the same as here, with the only difference that the single-particle Hamiltonian $\mathscr{H}_D$ given in (9) has to be replaced by $\mathscr{H}_0 = \beta mc^2$. Expression (30) remains therefore valid provided we replace u_n and $u_{n'}$ by u_{0p} and $u_{0p'}$, $P_\pm$ by $P_{0\pm}$, where u_{0p} are the positive energy eigenspinors of $\mathscr{H}_0$, $P_{0\pm}$ the projectors into the positive and negative energy manifolds of $\mathscr{H}_0$. Since $e^{\pm ik\cdot r}$ commute with $\mathscr{H}_0$, they also commute with $P_{0\pm}$ and we get

$$\begin{aligned}\langle 1_p|V_{\text{Coul}}|1_{p'}\rangle &= \frac{q^2}{16\pi^3\varepsilon_0}\int \frac{d^3k}{k^2}\langle u_{0p}|P_{0+} - P_{0-}|u_{0p'}\rangle \\ &= \frac{q^2}{16\pi^3\varepsilon_0}\,\delta_{pp'}\int \frac{d^3k}{k^2} = \frac{q^2 k_M}{4\pi^2\varepsilon_0}\,\delta_{pp'}\,,\end{aligned} \tag{31}$$

where k_M is the cut off in the k integral. This demonstrates the result used in section 6.4 of [1].

3.1.2. Operatorial form for the contribution of the Coulomb term
It is useful now to use the identity

$$P_+ - P_- = \mathcal{H}_D/\sqrt{\mathcal{H}_D^2}\,, \tag{32}$$

which can be easily checked by applying the two members of (32) to any eigenstate $|u_n\rangle$ or $|v_n\rangle$ of $\mathcal{H}_D$. The denominator of (32) is easily transformed. From the anticommutation relations of the Dirac matrices, we get

$$\begin{aligned}\mathcal{H}_D^2 &= m^2c^4 + c^2(\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_0)^2 \\ &= m^2c^4 + c^2\boldsymbol{\pi}_0^2 - q\hbar c^2\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\,.\end{aligned} \tag{33}$$

On the other hand, since $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is a translation operator for $\boldsymbol{p}$ (and also for $\boldsymbol{\pi}_0 = \boldsymbol{p} - q\boldsymbol{A}_0$), we have

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,\boldsymbol{\pi}_0\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{\pi}_0 - \hbar\boldsymbol{k}\,, \tag{34}$$

so that

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,\frac{\mathcal{H}_D}{\sqrt{\mathcal{H}_D^2}}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = \frac{\beta mc^2 + c\boldsymbol{\alpha}\cdot(\boldsymbol{\pi}_0 - \hbar\boldsymbol{k})}{[m^2c^4 + c^2(\boldsymbol{\pi}_0 - \hbar\boldsymbol{k})^2 - q\hbar c^2\boldsymbol{\sigma}\cdot\boldsymbol{B}_0]^{1/2}}\,. \tag{35}$$

We now use the fact that the electron is not relativistic in the initial and final states u_n and $u_{n'}$. If, in the denominator of (35), we neglect the terms in $\boldsymbol{\pi}_0$ and $\boldsymbol{\sigma}\cdot\boldsymbol{B}_0$, we get

$$\frac{1}{(m^2c^4 + \hbar^2c^2k^2)^{1/2}} = \frac{1}{mc^2}\,\frac{1}{(1+x^2)^{1/2}}\,, \tag{36}$$

where

$$x = \hbar\omega/mc^2\,. \tag{37}$$

We can then make an expansion of the denominator of (35) in powers of $\boldsymbol{\pi}_0$ and $\boldsymbol{\sigma}\cdot\boldsymbol{B}_0$. Keeping terms in $\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_0$, $\boldsymbol{\pi}_0^2$, $\boldsymbol{\sigma}\cdot\boldsymbol{B}_0$, we thus get from

(35), after the angular integration,

$$\langle 1_n|V_{\text{Coul}}|1_{n'}\rangle = \langle u_n|\mathcal{V}_{\text{Coul}}|u_{n'}\rangle\,, \tag{38}$$

where

$$\mathcal{V}_{\text{Coul}} = \frac{\alpha}{\pi}\int_0^{x_M} dx\left\{\beta mc^2\frac{1}{(1+x^2)^{1/2}} + c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_0\frac{1}{(1+x^2)^{1/2}}\left[1-\frac{x^2}{3(1+x^2)}\right]\right.$$

$$\left. -\beta\frac{\boldsymbol{\pi}_0^2}{2m}\frac{1}{(1+x^2)^{5/2}} + \beta\frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\frac{1}{(1+x^2)^{3/2}}\right\}, \tag{39}$$

with

$$\alpha = \frac{q^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137}\,, \qquad x_M = \frac{\hbar\omega_M}{mc^2}\,. \tag{40}$$

It must be noted that there is now no restriction on the cut off x_M which can be larger than 1. We have a complete relativistic calculation to order 1 in α. It also appears on (39) that the divergence is only logarithmic in x_M, and not linear as in the nonrelativistic theory (see expression 31).

3.2. Contribution of transverse photons

The general scheme of the calculation is the same as for V_{Coul}. The two operators $U_\perp$, given in (6), and appearing in (10), contain the same four operators c^+c, bb^+, c^+b^+ and bc as the two operators $\Psi^+\Psi$ appearing in V_{Coul}, so that we can still use fig. 2.

The only differences are, first, that we must take the matrix elements of $\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}\exp(\pm i\boldsymbol{k}\cdot\boldsymbol{r})$ instead of $\exp(\pm i\boldsymbol{k}\cdot\boldsymbol{r})$ (where $\boldsymbol{\varepsilon}$ is the polarization of the transverse photon), secondly, that we have now energy denominators and sum over intermediate states instead of $\int d^3k/k^2$.

So, we will just give here the diagrams associated with the six terms of fig. 2 and the final result.

Term 1

In the intermediate state, we have one electron and one transverse photon.

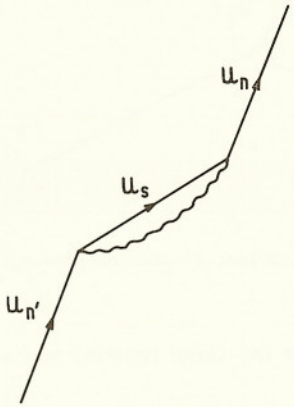

Fig. 7. Contribution of transverse photons: diagram (1).

Terms 2 and 3

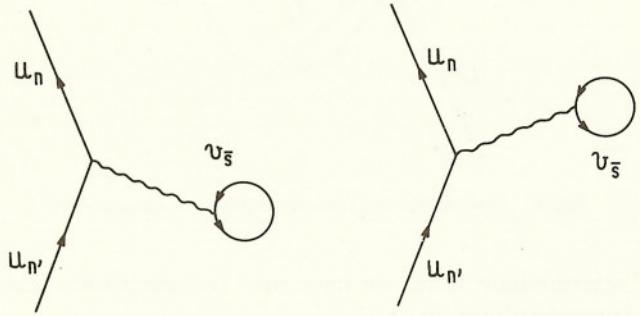

Fig. 8. Contribution of transverse photons: diagrams (2) and (3).

They describe the interaction of the electron with the transverse field of the vacuum current. In a uniform magnetic field $\boldsymbol{B}_0$, the vacuum current is equal to zero (because of rotational and translational invariances) so that the contribution of these two terms vanishes.

Term 4

Such a term describes the transverse interactions in the vacuum which are the same in presence or in absence of the electron and which therefore do not contribute to H_{eff}.

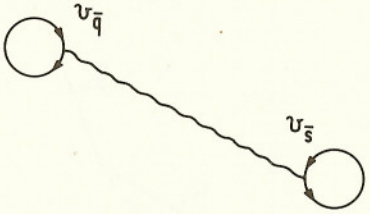

Fig. 9. Contribution of transverse photons: diagram (4).

Term 5

It is equal to zero for the same reasons as for V_{Coul}.

Term 6

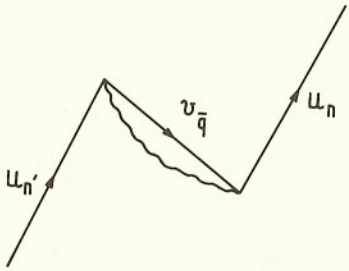

Fig. 10. Contribution of transverse photons: diagram (6).

In the intermediate state, we have now two electrons, one positron and one transverse photon.

Finally, only terms 1 and 6 contribute and give

$$\langle 1_n|H_{\text{eff}}^{\perp}|1_{n'}\rangle = \langle u_n|\mathscr{H}_{\text{eff}}^{\perp}|u_{n'}\rangle$$

$$= \frac{q^2c^2}{2}\int \mathrm{d}^3k \sum_{\varepsilon} \frac{\hbar}{2\varepsilon_0\omega(2\pi)^3}$$

$$\times \langle u_n|(\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\left[\frac{P_+}{E_n-\hbar\omega-\mathscr{H}_{\mathrm{D}}}+\frac{P_-}{E_n+\hbar\omega-\mathscr{H}_{\mathrm{D}}}\right](\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}|u_{n'}\rangle$$

$$+ \text{same term with } E_n \to E_{n'}\,. \tag{41}$$

Now, we use

$$\frac{P_+}{E_n - \hbar\omega - \mathscr{H}_D} + \frac{P_-}{E_n + \hbar\omega - \mathscr{H}_D} = \frac{1}{E_n - \mathscr{H}_D - (P_+ - P_-)\hbar\omega} \tag{42}$$

and (32), which gives

$$\frac{1}{E_n - \mathscr{H}_D - \hbar\omega\mathscr{H}_D/\sqrt{\mathscr{H}_D^2}} = \frac{E_n + \mathscr{H}_D + \hbar\omega\mathscr{H}_D/\sqrt{\mathscr{H}_D^2}}{E_n^2 - [\mathscr{H}_D^2 + \hbar^2\omega^2 + 2\hbar\omega\sqrt{\mathscr{H}_D^2}]}. \tag{43}$$

The effect of the two $\exp(\pm i\boldsymbol{k}\cdot\boldsymbol{r})$ operators in (41) is to replace everywhere $\boldsymbol{\pi}_0$ by $\boldsymbol{\pi}_0 - \hbar\boldsymbol{k}$. As for V_{Coul}, we expand the fraction in (43) in a power series of $\boldsymbol{\pi}_0$, $\boldsymbol{\sigma}\cdot\boldsymbol{B}_0$ and $E_n^2 - m^2c^4$. We then sum over $\boldsymbol{\varepsilon}$ perpendicular to $\boldsymbol{k}$, integrate over solid angle, and get in this way a single-particle Hamiltonian $\mathscr{H}_{\text{eff}}^{(\text{trans})}$ analogous to (39), and given by

$$\begin{aligned}\mathscr{H}_{\text{eff}}^{(\text{trans})} = \frac{\alpha}{\pi}\int_0^{x_M} \mathrm{d}x \Bigg\{ & \beta mc^2\left[x - \frac{x^2}{(1+x^2)^{1/2}}\right] \\ & + c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_0\left[\frac{2x}{3} - \frac{2x^2}{3(1+x^2)^{1/2}} - \frac{3x^2+2}{3(1+x^2)^{3/2}}\right] \\ & - \beta\frac{\boldsymbol{\pi}_0^2}{2m}\left[\frac{4x}{3} - \frac{4x^2}{3(1+x^2)^{1/2}} - \frac{x^2(2x^2+5)}{3(1+x^2)^{5/2}}\right] \\ & + \beta\frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\left[2x - \frac{2x^2}{(1+x^2)^{1/2}} - \frac{3x^2+4}{3(1+x^2)^{3/2}}\right]\Bigg\}. \end{aligned} \tag{44}$$

3.3. Transformation to an "even" effective Hamiltonian; final results

The sum of (39) and (44) is a single-particle Hamiltonian which still contains an "odd" operator, the term proportional to $\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_0$, which connects the two manifolds $\mathscr{E}_+^0$ and $\mathscr{E}_-^0$ of βmc^2. We can, by a unitary transformation, transform the sum of (39), (44) (and of the unperturbed Hamiltonian) into an "even" Hamiltonian (operating only within $\mathscr{E}_+^0$ and $\mathscr{E}_-^0$, i.e., acting only upon two-components spinors). From the results of section 6.4 of the previous course [1], it follows that any Dirac Hamiltonian of the form

$$\beta mc^2(1+\varepsilon) + c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_0(1+\varepsilon') + \beta\mathscr{E} \tag{45}$$

where ε, $\varepsilon' \ll 1$ and $\mathscr{E}$ is an "even" operator ($\mathscr{E} \ll mc^2$) can be transformed into

$$\beta\left[mc^2(1+\varepsilon)+\frac{c^2(\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_0)^2(1+\varepsilon')^2}{2mc^2(1+\varepsilon)}+\mathscr{E}\right]$$
$$=\beta\left[mc^2(1+\varepsilon)+\mathscr{E}+\left(\frac{\boldsymbol{\pi}_0^2}{2m}-\frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\right)(1+2\varepsilon'-\varepsilon)\right]. \tag{46}$$

Applying such a result to the sum of (39) and (44) finally gives the even form of the total effective Hamiltonian

$$\mathscr{H}_{\text{eff}}=mc^2\left[1+\frac{\alpha}{\pi}\int_0^{x_M}f_1(x)\,dx\right]$$
$$+\frac{\boldsymbol{\pi}_0^2}{2m}\left[1-\frac{\alpha}{m}\int_0^{x_M}f_2(x)\,dx\right]-\frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\left[1-\frac{\alpha}{\pi}\int_0^{x_M}f_3(x)\,dx\right], \tag{47}$$

where

$$f_1(x)=\frac{1}{(1+x^2)^{1/2}}+x-\frac{x^2}{(1+x^2)^{1/2}},$$
$$f_2(x)=x\left[1-\frac{x}{(1+x^2)^{1/2}}+\frac{4}{3x(1+x^2)^{1/2}}-\frac{x}{3(1+x^2)^{3/2}}-\frac{2x}{(1+x^2)^{5/2}}\right],$$
$$f_3(x)=x\left[\frac{5}{3}\left(1-\frac{x}{(1+x^2)^{1/2}}\right)+\frac{2}{3}\frac{x}{(1+x^2)^{3/2}}\right]. \tag{48}$$

The graphs of the three functions f_1, f_2, f_3 are represented on fig. 11.

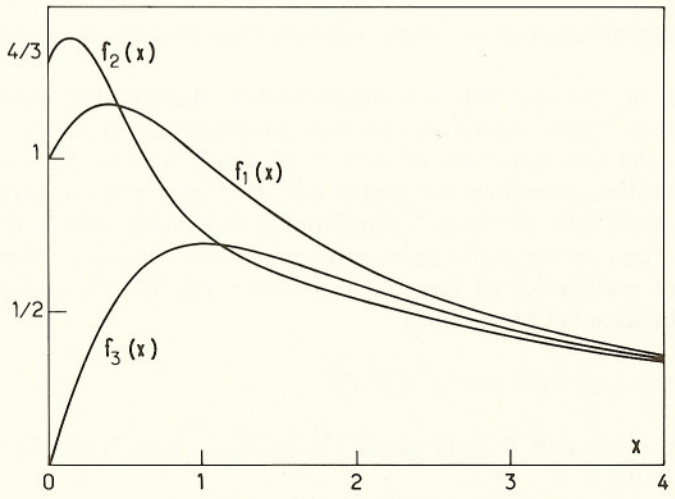

Fig. 11. Graphs of the three functions f_1, f_2, f_3.

Remark

One can check that the same result is obtained if one works in the Lorentz gauge and if one computes the contribution of longitudinal and temporal photons.

4. Physical discussion

4.1. Comparison with nonrelativistic calculations

From (47) and (48), it is possible first to calculate the contribution of nonrelativistic modes by expanding $f_1(x)$, $f_2(x)$, $f_3(x)$ around $x = 0$.

To lowest order, we have

$$\begin{aligned} f_1 &= 1 + x + \dots \\ f_2(x) &= \tfrac{4}{3} + \dots \\ f_3(x) &= 0 + \dots . \end{aligned} \tag{49}$$

[Since f_1 is multiplied by mc^2 in (47), we must expand it one order further than f_2 and f_3]. Inserting (49) into the integrals appearing in (47), and taking a cut-off $x_M \ll 1$, we get

$$\mathcal{H}_{\text{eff}} = (m + \delta m_1 + \delta m_2)c^2 + \frac{\boldsymbol{\pi}_0^2}{2m}\left(1 - \frac{4}{3}\frac{\delta m_1}{m}\right) - \frac{q\hbar}{2m}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\,, \tag{50}$$

with

$$\frac{\delta m_1}{m} = \frac{\alpha}{\pi}\,x_M, \qquad \frac{\delta m_2}{m} = \frac{\alpha}{2\pi}\,x_M^2\,. \tag{51}$$

This is in complete agreement with the results of [2]. δm_1 is the mass correction associated with self-reaction, δm_2 the mass correction due to vacuum fluctuations. At this order, there are no modifications of the spin magnetic energy.

By pushing the expansions of f_1, f_2, f_3 one step further in $1/c$,

$$\begin{aligned} f_1(x) &= 1 + x - \tfrac{3}{2}x^2 + \dots , \\ f_2(x) &= \tfrac{4}{3} + x + \dots , \\ f_3(x) &= \tfrac{5}{3}x + \dots , \end{aligned} \tag{52}$$

we can now make the connection with the results of [3]. Using (47) and (52), we get now

$$\mathscr{H}_{\text{eff}} = (m + \delta m_1 + \delta m_2 - \delta m_3)c^2 + \frac{\boldsymbol{\pi}_0^2}{2m}\left(1 - \frac{4}{3}\frac{\delta m_1}{m} - \frac{\delta m_2}{m}\right) - \frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\left(1 - \frac{5}{3}\frac{\delta m_2}{m}\right), \tag{53}$$

with

$$\frac{\delta m_3}{m} = \frac{\alpha}{2\pi}x_{\text{M}}^3 . \tag{54}$$

The new terms appearing in (53) have a simple physical meaning. The term $-\delta m_3 c^2$ represents a decrease of the electron self-energy due to many-particle effects. We note that, if the nonrelativistic calculations of [3] were done with an Hamiltonian derived from the single-particle Dirac equation by a Foldy–Wouthuysen transformation, we would get here a correction with the opposite sign. This shows that the relativistic corrections to be added to the Hamiltonian must be derived from the second quantized Dirac–Maxwell Hamiltonian (as in section 6.4 of [1]), and *not* from the single-particle Dirac Hamiltonian.

The term $-(\delta m_2/m)(\boldsymbol{\pi}_0^2/2m)$ represents a correction to the kinetic energy due to the mass correction associated with vacuum fluctuations. It appears at a higher order in $1/c$ than the contribution of self-reaction (term in $\delta m_1/m$).

Finally, the last term of (54) represents a correction associated with a decrease of the spin magnetic moment. As shown in [3] and [6], such a correction is due to the angular vibration of the spin in vacuum fluctuations, plus some "crossed" relativistic effects between the vibration of the charge and the motion of the spin. It is actually possible to give a complete semi-classical interpretation of the factor 5/3 appearing in the last term of (54). One can show also [3] that, at this order, self-reaction does not introduce any dynamical effect on the spin. It gives rise to a term proportional to $\sigma^2 = 1$, which shifts the two spin states by the same amount, so that the motion of the spin is not changed.

4.2. Contribution of high-frequency modes – mass renormalization

For $x \gg 1$, we have

$$f_1(x) \simeq f_2(x) \simeq f_3(x) \simeq \frac{3}{2x} . \tag{55}$$

We get therefore the same logarithmic divergence, which will be "reabsorbed" in the mass renormalization.

Taking $x_M \gg 1$, we can write

$$\mathcal{H}_{\text{eff}} = mc^2\left[1+\frac{\alpha}{\pi}\left(\tfrac{3}{2}\log x_M + C_1\right)\right]$$
$$+\frac{\boldsymbol{\pi}_0^2}{2m}\left[1-\frac{\alpha}{\pi}\left(\tfrac{3}{2}\log x_M + C_2\right)\right]$$
$$-\frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\left[1-\frac{\alpha}{\pi}\left(\tfrac{3}{2}\log x_M + C_3\right)\right], \tag{56}$$

where C_1, C_2, C_3 are three constants,

$$C_1 = \tfrac{3}{2}\log 2 - \tfrac{1}{4},$$
$$C_2 = \tfrac{3}{2}\log 2 - \tfrac{7}{12},$$
$$C_3 = \tfrac{3}{2}\log 2 - \tfrac{13}{12}. \tag{57}$$

Since $\alpha = 1/137$, $\alpha \log x_M = 1$ when $x_M = e^{137}$ which is a huge number. We can therefore choose

$$1 \ll x_M \ll e^{137} \tag{58}$$

so that we describe correctly the effect of a wide spectrum of relativistic modes (since $x_M \gg 1$), while keeping

$$(\alpha/\pi)\log x_M \ll 1\,. \tag{59}$$

Using (59), we can write the second term of (56), to order 1 in α, as

$$\frac{\boldsymbol{\pi}_0^2}{2m}\left[1+\frac{\alpha}{\pi}\int_0^{x_M} f_2(x)\,dx\right]^{-1} = \frac{\boldsymbol{\pi}_0^2}{2\bar{m}}, \tag{60}$$

with

$$\bar{m} = m\left[1+\frac{\alpha}{\pi}\int_0^{x_M} f_2(x)\,dx\right]. \tag{61}$$

Such a mass $\bar{m}$, which appears in the kinetic energy term, is the experimental mass, which is measured in deflection experiments.

4.3. Spin anomaly $(g-2)/2$

We can now re-express the last term of $\mathcal{H}_{\text{eff}}$ in terms of the renormalized Bohr magneton $-q\hbar/2\bar{m}$,

$$-\frac{q\hbar}{2\bar{m}}\frac{\bar{m}}{m}\left[1-\frac{\alpha}{\pi}\int_0^{x_M} f_3(x)\,dx\right]\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\,. \tag{62}$$

Expression (62) can also be written

$$-g\frac{q}{2\bar{m}}\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\boldsymbol{B}_0\,, \tag{63}$$

where g is the g-factor given by

$$g=2\frac{\bar{m}}{m}\left[1-\frac{\alpha}{\pi}\int_0^{x_M} f_3(x)\,dx\right]. \tag{64}$$

Using (61), we finally get for the electron spin anomaly

$$a_e=\frac{g-2}{2}=\frac{\alpha}{\pi}\int_0^{x_M}[f_2(x)-f_3(x)]\,dx\,. \tag{65}$$

The contribution of f_2 in (65) is the contribution of the mass renormalization to the spin anomaly.

The graph of f_2-f_3 is represented on fig. 12. The integral of f_2-f_3 from 0 to ∞ is *convergent* and equal to 1/2. We therefore get for a_e a result independent of the cut-off x_M, as it should be

$$a_e=\alpha/2\pi\,. \tag{66}$$

The curve of fig. 12 shows how the various modes of the electromagnetic field contribute to $g-2$. It clearly appears on fig. 12 that the main contribution comes from the domain $x<1$ and that it is not necessary to invoke ultra-high relativistic modes for explaining the sign of $g-2$ (actually, the contribution of the domain $x>1$ to the integral is negative!). The quantitative results derived in this lecture therefore show that the main physical conclusions derived from nonrelativistic calculations are not drastically changed by including the effect of relativistic modes.

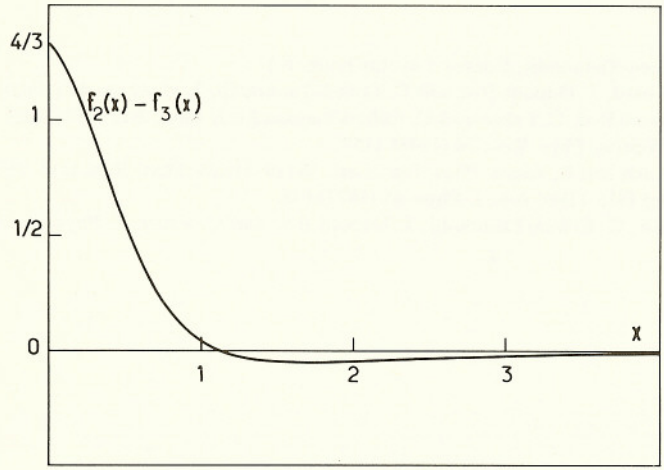

Fig. 12. Graph of the function $f_2 - f_3$.

Remark. If we come back to the expressions (57) of C_1, C_2, C_3, we see that

$$C_2 - C_3 = \tfrac{1}{2}, \tag{67}$$

which is the result used in the evaluation of the integral (65). But

$$C_1 \neq C_2, \tag{68}$$

which means that the constant term (after the logarithmic one) is not the same in the rest mass energy and in the kinetic energy. Such a surprising result is actually due to the noncovariant character of the cut off (see also the discussion of §2.1.4 of [2]). By using a covariant procedure for the cut off, we have checked that one obtains new values C_1', C_2', C_3' for C_1, C_2, C_3 such that

$$C_1' = C_2',$$

$$C_2' - C_3' = C_2 - C_3. \tag{69}$$

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[3] J. Dupont-Roc, C. Fabre and C. Cohen-Tannoudji, J. Phys. B11 (1978) 563.
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