COURSE 1

INTRODUCTION TO QUANTUM ELECTRODYNAMICS

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General introduction

This series of six lectures is intended to present a simple introduction to quantum electrodynamics (Q.E.D.) Q.E.D. is actually a very broad subject which, obviously, cannot be covered in six lectures, and it is necessary to make some choices. We will focus here on physical mechanisms, trying for example to give some simple physical pictures for radiative processes, but we will not enter into problems related to the formal structure of the theory, such as relativistic or gauge invariance.

In all lectures except the last one, we will restrict ourselves to systems containing a *fixed* number of *nonrelativistic* particles (having a velocity v much smaller than the speed of light c). We will therefore introduce a Q.E.D. formalism where particles are described by Schrödinger wave functions, and radiation by a quantized Maxwell field. Such an approximation applies to most situations occurring in atomic and molecular physics, where weakly bound electrons interact with microwaves or optical fields. This also explains why we use the Coulomb gauge throughout the whole course. The electrostatic interaction, which explicitly appears in such a gauge, is an excellent approximation for the electromagnetic interaction between nonrelativistic charged particles, and corrections to this approximation, such as retardation effects, or coupling with the transverse radiation field, are easily handled by perturbation methods.

We start in lecture 1 by a brief review of classical electrodynamics. By working in reciprocal space (i.e., by taking the spatial Fourier transform of the fields), we easily identify the relevant dynamical variables of the field and put the classical theory in a form suitable for the quantization which is then performed in lecture 2. We give in this lecture the basic commutation relations and the expression of the Hamiltonian describing the interaction between nonrelativistic particles and the quantized radiation field. Lecture 3 is devoted to a discussion of the physical content of such a theory. We review various types of observables and various types of states of the quantized

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radiation field exhibiting corpuscular aspects (photons) or wave aspects (interference phenomena) or both (wave-particle duality).

After this general presentation of Q.E.D., we consider in lecture 4 radiative processes such as spontaneous emission or radiative corrections, and we try to analyze the physical mechanisms responsible for such processes. Starting from the coupled Heisenberg equations of the electron-field system, we try to eliminate the field variables and to get a dynamical equation for the electron. We discuss in detail the respective contributions of the interaction of the electron with the quantized vacuum field (vacuum fluctuations) and with its own field (radiation reaction).

Another possible approach to radiative corrections is the so-called effective Hamiltonian method, discussed in lecture 5. One tries in such an approach to derive a Hamiltonian describing how the slow motion of a weakly bound electron is modified by virtual emissions and reabsorptions of photons. We review in lecture 5 some results obtained along these lines and which not only confirm those of lecture 4, but extend them to higher orders in 1/c.

Finally, in lecture 6, we try to give an idea of a more symmetric formulation of Q.E.D. where not only radiation but matter is described by a quantized relativistic field (in this case, the quantized Dirac field with elementary excitations corresponding to electrons and positrons). We also show that the single-particle nonrelativistic Hamiltonian used in the first part of this course for discussing the dynamics of a single electron coupled to the quantized radiation field, can be considered as an effective Hamiltonian in the one-particle subspace of the full relativistic Q.E.D. Hamiltonian. The general results of lecture 6 will be also useful for the discussion presented in the seminar [1] following this course, describing a theoretical work done in collaboration with Jacques Dupont-Roc and dealing with the contribution of relativistic modes of the radiation field ($\hbar \omega \ge mc^2$) to the electron spin anomaly g-2.

The presentation of the first three lectures closely follows the one used in a book on introductory Q.E.D. which is being written in collaboration with Jacques Dupont-Roc and Gilbert Grynberg [2]. Lecture 4 is actually a reprint of a paper written in collaboration with Jean Dalibard and Jacques Dupont-Roc and just published in Journal de Physique [3]. Lecture 5 reviews the results derived in references [4] and [5], and representing works done in collaboration with Jacques Dupont-Roc, Claude Fabre and Paul Avan. I am very grateful to all these colleagues for their close collaboration and for their contribution to this course. Finally, I would like to thank Mrs Catherine Marthouret for the typing of such a long manuscript.

1. Classical electrodynamics

1.1. Introduction

In this chapter, we first introduce (section 1.2) the basic equations of classical electrodynamics, the Maxwell–Lorentz equations, which describe the coupled evolution of the electromagnetic field and of a set of charged particles.

In view of the quantization which will be performed in the subsequent chapters, we then show (section 1.3) that classical electrodynamics takes a simpler form when the basic equations are written in reciprocal space through a spatial Fourier transform of the fields. Such a transformation allows a simple splitting of the electromagnetic field into its longitudinal and transverse parts. It also appears that the longitudinal field, and its contribution to important physical observables, such as the total energy, can be reexpressed in terms of the dynamical variables of the particles.

A further important step (section 1.4) is the introduction of some simple linear combinations of the transverse electric and magnetic fields in reciprocal space which evolve independently from each other in absence of particles, and which therefore describe the normal modes of vibration of the free field. These new dynamical variables for the field, the so-called normal variables, play a central role in the theory since they will become, after quantization, the creation and annihilation operators for photons. We will end this chapter by giving the expression of all important physical observables in terms of these normal variables.

1.2. Basic equations in ordinary space

1.2.1. Maxwell-Lorentz equations [6, 7, 8]

There are two sets of basic equations. First, Maxwell equations, relating the electric field E(r, t) and the magnetic field B(r, t) to the charge density $\rho(r, t)$ and current density j(r, t)

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{\varepsilon_0} \rho(\boldsymbol{r},t), \qquad (1.1a)$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0, \qquad (1.1b)$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial}{\partial t} \boldsymbol{B}(\boldsymbol{r},t),$$
 (1.1c)

$$\nabla \times \boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c^2} \frac{\partial}{\partial t} \boldsymbol{E}(\boldsymbol{r},t) + \frac{1}{\varepsilon_0 c^2} \boldsymbol{j}(\boldsymbol{r},t) \,. \tag{1.1d}$$

Then, Newton-Lorentz equations describing the dynamics of each particle α , with mass m_{α} , charge q_{α} , positions $r_{\alpha}(t)$, velocity $v_{\alpha}(t)$, under the effect of the electric and magnetic forces exerted by the fields

$$m_{\alpha} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{r}_{\alpha}(t) = q_{\alpha} [\boldsymbol{E}(\boldsymbol{r}_{\alpha}, t) + \boldsymbol{v}_{\alpha}(t) \times \boldsymbol{B}(\boldsymbol{r}_{\alpha}, t)]$$
(1.2)

Equations (1.2) are only valid for slow, nonrelativistic particles.

From (1.1a) and (1.1d), it follows that

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0.$$
(1.3)

Such a continuity equation expresses the local conservation of the total electric charge

$$Q = \int \mathrm{d}^3 r \,\rho(\mathbf{r},t) \,. \tag{1.4}$$

It is important to express ρ and j in terms of the variables of the particles

$$\rho(\mathbf{r},t) = \sum q_{\alpha} \delta[\mathbf{r} - \mathbf{r}_{\alpha}(t)], \qquad (1.5a)$$

$$\mathbf{j}(\mathbf{r},t) = \sum_{\alpha} q_{\alpha} \mathbf{v}_{\alpha}(t) \delta[\mathbf{r} - \mathbf{r}_{\alpha}(t)] .$$
(1.5b)

One can check that equations (1.5) are consistent with (1.3).

Equations (1.1) and (1.2) form two sets of coupled equations. The evolution of the fields depends on the particles through ρ and j. The motion of the particles depends on the fields E and B. Equations (1.1)

are first-order partial differential equations, whereas eqs. (1.2) are second-order differential equations. It follows that the "state" of the total system field plus particles is determined, at a given time t_0 , by the knowledge of the field **E** and **B** in all points **r** of space, and by the knowledge of the position r_{α} and velocity v_{α} of each particle α

$$\{ E(r, t_0), B(r, t_0), r_{\alpha}(t_0), v_{\alpha}(t_0) \}.$$
(1.6)

It is important to keep in mind that, in Maxwell equations (1.1), r is not a dynamical variable (as r_{α}), but a continuous parameter labelling the field variables.

1.2.2. Some important constants of motion

From the basic equations (1.1) and (1.2), and from the expression (1.5) of ρ and j, one can show that the following functions of E, B, r_{α} , v_{α} :

$$\boldsymbol{H} = \sum_{\boldsymbol{\alpha}} \frac{1}{2} m_{\boldsymbol{\alpha}} \boldsymbol{v}_{\boldsymbol{\alpha}}^{2}(t) + \frac{\varepsilon_{0}}{2} \int \mathrm{d}^{3} \boldsymbol{r} \left[\boldsymbol{E}^{2}(\boldsymbol{r}, t) + c^{2} \boldsymbol{B}^{2}(\boldsymbol{r}, t) \right], \qquad (1.7)$$

$$\boldsymbol{P} = \sum_{\alpha} m_{\alpha} \boldsymbol{v}_{\alpha}(t) + \varepsilon_0 \int d^3 \boldsymbol{r} \, \boldsymbol{E}(\boldsymbol{r}, t) \times \boldsymbol{B}(\boldsymbol{r}, t) \,, \tag{1.8}$$

$$\boldsymbol{J} = \sum_{\alpha} \boldsymbol{r}_{\alpha}(t) \times \boldsymbol{m}_{\alpha} \boldsymbol{v}_{\alpha}(t) + \varepsilon_0 \int d^3 \boldsymbol{r} \, \boldsymbol{r} \times \left[\boldsymbol{E}(\boldsymbol{r}, t) \times \boldsymbol{B}(\boldsymbol{r}, t) \right], \tag{1.9}$$

are constants of motion, i.e., independent of t. See for example [8].

H is the total energy of the combined field plus particles system, P the total momentum, J the total angular momentum.

1.2.3. Potentials – Gauge invariance

From (1.1b) and (1.1c) one can show that E and B can always be written

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t), \qquad (1.10a)$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r},t) - \nabla \boldsymbol{U}(\boldsymbol{r},t), \qquad (1.10b)$$

where A is a vector field, called the vector potential, U a scalar field, called the scalar potential. A first advantage of introducing A and U is that two Maxwell equations are automatically satisfied. Other ad-

vantages appear in the Lagrangian and Hamiltonian formulation of electrodynamics.

Inserting (1.10) in the last two Maxwell equations gives the equations of motion of A and U

$$\Delta U(\mathbf{r},t) = -\frac{1}{\varepsilon_0} \rho(\mathbf{r},t) - \nabla \cdot \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r},t), \qquad (1.11a)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) \mathbf{A}(\mathbf{r}, t) = \frac{1}{\varepsilon_0 c^2} \mathbf{j}(\mathbf{r}, t) - \nabla \left[\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{1}{c^2}\frac{\partial}{\partial t} U(\mathbf{r}, t)\right].$$
(1.11b)

We get second-order partial differential equations, instead of firstorder as in (1.1). Actually (1.11a) is not an equation of motion for U, but rather relates U to $\partial A/\partial t$. The state of the field is now determined by the knowledge of $A(r, t_0)$ and $(\partial/\partial t)A(r, t_0)$ for all r [provided that relation (1.11a) remains valid for all t].

From (1.10), it follows that E and B remain invariant in the following "gauge transformation"

$$A(\mathbf{r}, t) \to A'(\mathbf{r}, t) = A(\mathbf{r}, t) + \nabla F(\mathbf{r}, t), \qquad (1.12a)$$

$$U(\mathbf{r},t) \to U'(\mathbf{r},t) = U(\mathbf{r},t) - \frac{\partial}{\partial t}F(\mathbf{r},t).$$
(1.12b)

There is therefore a certain redundancy in the potentials since the same physical fields E and B can be described by several different potentials A and U. Such a redundancy can be reduced by choosing a gauge condition which fixes $\nabla \cdot A [\nabla \times A]$ is already determined by (1.10a)].

The two most frequently used gauges are the Lorentz gauge and the Coulomb gauge.

Lorentz gauge is defined by

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) + \frac{1}{c^2} \frac{\partial}{\partial t} U(\mathbf{r},t) = 0.$$
(1.13)

One can check that it is always possible to choose in (1.12) a function F such that condition (1.13) is fulfilled. In Lorentz gauge, equations (1.11) take a more symmetric form

$$\Box U(\mathbf{r}, t) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}, t)$$
(1.14a)

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$$\Box \mathbf{A}(\mathbf{r},t) = \frac{1}{\varepsilon_0 c^2} \mathbf{j}(\mathbf{r},t)$$
(1.14b)

where $\Box = \frac{\partial^2}{c^2 \partial t^2} - \Delta$ is the Dalembertian. This is due to the fact that the Lorentz condition is relativistically invariant (it keeps the same form in two different Lorentz frames). Using covariant notations, equations (1.13) and (1.14) can be written

$$\partial_{\mu}A^{\mu} = 0,$$

$$\partial_{\mu} = \left\{ \frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right\}, \qquad A^{\mu} = \left\{ \frac{U}{c}, A \right\},$$

$$\partial_{\nu}\partial^{\nu}A^{\mu} = \frac{1}{\varepsilon_{0}c^{2}}j^{\mu},$$

$$j^{\mu} = \left\{ c\rho, j \right\},$$
(1.16)

where A^{μ} and j^{μ} are 4-vectors respectively associated with the potentials and the current.

Coulomb gauge is defined by

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r},t) = 0. \tag{1.17}$$

Equations (1.11) then become

$$\Delta U(\mathbf{r},t) = -\frac{1}{\varepsilon_0} \rho(\mathbf{r},t), \qquad (1.18a)$$

$$\Box \mathbf{A}(\mathbf{r},t) = \frac{1}{\varepsilon_0 c^2} \mathbf{j}(\mathbf{r},t) - \frac{1}{c^2} \nabla \frac{\partial}{\partial t} U(\mathbf{r},t) .$$
(1.18b)

Equation (1.18a) is the Poisson equation for U. Manifest covariance is lost, but other advantages of the Coulomb gauge will appear in the subsequent chapters.

1.3. Electrodynamics in reciprocal space

1.3.1. Spatial Fourier transform – notations

We note $\mathscr{C}(k, t)$ the spatial Fourier transform of E(r, t). E and \mathscr{C} are linked by the following relations:

$$\mathscr{E}(\mathbf{k},t) = \frac{1}{(2\pi)^{3/2}} \int d^3 \mathbf{r} \, \mathbf{E}(\mathbf{r},t) \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \,, \tag{1.19a}$$

$$E(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, \mathscr{E}(\mathbf{k},t) \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \,. \tag{1.19b}$$

Table 1 gives the notations chosen for the Fourier transforms of a few other physical quantities

Table 1

$$\begin{split} E(\mathbf{r},t) &\longleftrightarrow \mathscr{E}(k,t) \\ B(\mathbf{r},t) &\longleftrightarrow \mathscr{B}(k,t) \\ A(\mathbf{r},t) &\longleftrightarrow \mathscr{A}(k,t) \\ U(\mathbf{r},t) &\longleftrightarrow \mathscr{A}(k,t) \\ \rho(\mathbf{r},t) &\longleftrightarrow \mathscr{P}(k,t) \\ j(\mathbf{r},t) &\longleftrightarrow \mathbf{j}(k,t) \end{split}$$

From the reality of E(r, t), we get

$$\mathscr{E}^*(\mathbf{k},t) = \mathscr{E}(-\mathbf{k},t) \,. \tag{1.20}$$

In the following, we will frequently use the Parseval-Plancherel identity

$$\int d^3 r F^*(\mathbf{r}) G(\mathbf{r}) = \int d^3 k \, \mathscr{F}^*(\mathbf{k}) \, \mathscr{G}(\mathbf{k}) \tag{1.21}$$

(where \mathcal{F} and \mathcal{G} are the Fourier transforms of F and G), and the fact that the Fourier transform of a product is proportional to the convolution product of the Fourier transforms

$$\mathscr{F}(\boldsymbol{k})\mathscr{G}(\boldsymbol{k}) \longleftrightarrow \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 \boldsymbol{r}' \, F(\boldsymbol{r}') G(\boldsymbol{r}-\boldsymbol{r}') \,. \tag{1.22}$$

Finally, in order to simplify the notations, we will often write \dot{r}_{α} instead of $(d/dt)r_{\alpha}(t)$, \dot{E} instead of $(\partial/\partial t)E(r, t)$, $\ddot{\mathcal{E}}$ instead of $(\partial^2/\partial t^2)\mathcal{E}(\mathbf{k}, t)$... when there is no risk of confusion.

1.3.2. Field equations in reciprocal space

Since the gradient operator ∇ in *r*-space becomes the multiplication by ik in k-space, Maxwell equations (1.1) become in k-space

$$\mathbf{i}\boldsymbol{k}\cdot\boldsymbol{\mathscr{C}} = \frac{1}{\varepsilon_0}\,\rho\,,\tag{1.23a}$$

$$\mathbf{i}\mathbf{k}\cdot\boldsymbol{\mathscr{B}}=0\,,\tag{1.23b}$$

$$\mathbf{i}\mathbf{k}\times\mathscr{C}=-\dot{\mathscr{B}}\,,\tag{1.23c}$$

$$\mathbf{i}\mathbf{k} \times \mathcal{B} = \frac{1}{c^2} \dot{\mathcal{E}} + \frac{1}{\varepsilon_0 c^2} \mathbf{j}.$$
 (1.23d)

It clearly appears on (1.23) that $\dot{\mathscr{C}}(k)$ and $\dot{\mathscr{B}}(k)$ only depend on the values of $\mathscr{C}(k)$, $\mathscr{B}(k)$, $\rho(k)$, j(k) at the same k. Maxwell equations, which are partial differential equations in *r*-space, become strictly local equations in *k*-space, which is a great simplification.

The continuity equation (1.3) now reads

$$\mathbf{i}\mathbf{k}\cdot\mathbf{j}+\dot{\boldsymbol{\rho}}=0. \tag{1.24}$$

The relations (1.10) between the fields and the potentials are now

$$\mathcal{B} = i\mathbf{k} \times \mathcal{A}, \qquad (1.25a)$$

$$\mathscr{E} = -\dot{\mathscr{A}} - ik\mathscr{U}, \qquad (1.25b)$$

the gauge transformation (1.12),

$$\mathcal{A} \to \mathcal{A}' = \mathcal{A} + \mathrm{i}k\mathcal{F},\tag{1.26a}$$

$$\mathcal{U} \to \mathcal{U}' = \mathcal{U} - \dot{\mathcal{F}},$$
 (1.26b)

and the potential equations (1.11),

$$k^2 \mathcal{U} = \frac{1}{\epsilon_0} \rho + \mathrm{i} \mathbf{k} \cdot \dot{\mathbf{A}}, \qquad (1.27a)$$

$$\frac{1}{c^2}\ddot{\mathcal{A}} + k^2\mathcal{A} = \frac{1}{\varepsilon_0 c^2}\,\boldsymbol{j} - \mathrm{i}\boldsymbol{k}\left(\mathrm{i}\boldsymbol{k}\cdot\mathcal{A} + \frac{1}{c^2}\,\dot{\mathcal{U}}\right). \tag{1.27b}$$

1.3.3. Longitudinal and transverse vector fields

By definition, a longitudinal vector field $V_{\parallel}(r)$ is a vector field such that

$$\nabla \times V_{\parallel}(\mathbf{r}) = \mathbf{0} \,. \tag{1.28a}$$

Such a relation becomes in k-space

$$\mathbf{i}\mathbf{k} \times \mathcal{V}_{\parallel}(\mathbf{k}) = \mathbf{0} \ . \tag{1.28b}$$

On the other hand, a transverse vector field $V_{\perp}(r)$ satisfies

$$\nabla \cdot \boldsymbol{V}_{\perp}(\boldsymbol{r}) = 0, \qquad (1.29a)$$

$$i\boldsymbol{k} \cdot \boldsymbol{\mathcal{V}}_{\perp}(\boldsymbol{k}) = 0. \qquad (1.29b)$$

Comparing (1.28a) and (1.28b), or (1.29a) and (1.29b) shows that the denomination longitudinal or transverse has a proper geometrical meaning in k-space. For a longitudinal vector field, $\mathcal{V}_{\parallel}(k)$ is parallel to k for all k, for a transverse vector field, $\mathcal{V}_{\perp}(k)$ is perpendicular to k for all k.

It must be emphasized that a vector field is longitudinal (or transverse), only if (1.28) [or (1.29)] is satisfied for all \mathbf{r} , [or all \mathbf{k}]. For example, in the presence of a point charged particle in \mathbf{r}_{α} , $\nabla \cdot \mathbf{E}$ is according to (1.1a) equal to zero nearly everywhere, except at the position \mathbf{r}_{α} of the particle. \mathbf{E} cannot thus be considered as a transverse field. This appears more clearly in \mathbf{k} -space since $\mathbf{k} \cdot \mathbf{\mathcal{E}}$ is then proportional to $\exp(-i\mathbf{k} \cdot \mathbf{r}_{\alpha})$ which is obviously nonzero everywhere.

Working in k-space also allows a very simple decomposition of any vector field $\mathcal{V}(k)$ into its longitudinal and tranverse parts:

$$\mathcal{V}(\boldsymbol{k}) = \mathcal{V}_{\parallel}(\boldsymbol{k}) + \mathcal{V}_{\perp}(\boldsymbol{k}) \,. \tag{1.30}$$

For each k, $\mathcal{V}_{\parallel}(k)$ is obtained by projecting $\mathcal{V}(k)$ along the unit vector κ in the k direction

$$\boldsymbol{\kappa} = \boldsymbol{k}/\boldsymbol{k} \,. \tag{1.31}$$

We have therefore

$$\mathcal{V}_{\parallel}(k) = \kappa[\kappa \cdot \mathcal{V}(k)], \qquad (1.32a)$$

$$\mathscr{V}_{\perp}(\boldsymbol{k}) = \mathscr{V}(\boldsymbol{k}) - \mathscr{V}_{\parallel}(\boldsymbol{k}), \qquad (1.32b)$$

 $V_{\parallel}(r)$ and $V_{\perp}(r)$ are then obtained from (1.32) by a spatial Fourier transform.

Remarks

(i) In k-space, the relation which exists between a vector field $\mathcal{V}(k)$ and its longitudinal or transverse part is a local one. For example, from (1.32), we have

$$\mathcal{V}_{\perp i}(\boldsymbol{k}) = \sum_{j} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \mathcal{V}_j(\boldsymbol{k}) , \qquad (1.33)$$

where i, j = x, y, z. Each component of $\mathcal{V}_{\perp}(k)$ in point k only depends on the components of $\mathcal{V}(k)$ for the same point k. If we take now the Fourier transform of (1.33), we get, according to (1.22),

$$V_{\perp i}(\boldsymbol{r}) = \sum_{j} \int d^{3}\boldsymbol{r}' \, \delta_{ij}^{\perp}(\boldsymbol{r} - \boldsymbol{r}') V_{j}(\boldsymbol{r}') \,, \qquad (1.34)$$

where

$$\delta_{ij}^{\perp}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3k \ e^{i\mathbf{k}\cdot\mathbf{r}} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$$
$$= \delta_{ij}\delta(\mathbf{r}) + \frac{\partial^2}{\partial r_i \partial r_j} \frac{1}{(2\pi)^3} \int d^3k \ e^{i\mathbf{k}\cdot\mathbf{r}} \frac{1}{k^2}$$
$$= \delta_{ij}\delta(\mathbf{r}) + \frac{1}{4\pi} \frac{\partial^2}{\partial r_i \partial r_i} \frac{1}{\mathbf{r}}$$
(1.35)

 $[\delta_{ij}^{\perp}(\mathbf{r})$ is also called the transverse δ -function]. Because of the last term of (1.35), the relation between $V_{\perp}(\mathbf{r})$ and $V(\mathbf{r})$ is *not* a local one. $V_{\perp}(\mathbf{r})$ depends on the values $V(\mathbf{r}')$ of V in all other points \mathbf{r}' .

(ii) The decomposition of a vector field (part of a 4-vector or of an antisymmetric 4-tensor) into its longitudinal and transverse parts is not relativistically invariant. A vector field which is transverse in a given Lorentz frame is in general no longer transverse in another Lorentz frame.

(iii) The fact that the decomposition (1.30) introduces nonlocal effects in *r*-space and is not relativistically invariant seems to indicate that such a decomposition is not interesting for the electromagnetic field. Actually this is not true. Considered as "dynamical" equations, i.e., as equations giving the rate of variation of the fields, Maxwell equations lead quite naturally to such a decomposition. As shown in the next sections, two of the four Maxwell equations just fix the longitudinal parts of the fields in terms of the charge distribution, whereas the other two equations give the rate of variation of the transverse fields. Such a point of view then allows the introduction of a convenient set of normal variables for the field and leads to a quantization scheme (in Coulomb gauge) where the main part of the interaction between nonrelativistic particles (electrostatic interaction) is singled out from the beginning. It is of course possible to follow a different path and to keep manifest covariance at every stage. This is important in Q.E.D. for an unambiguous elimination of the infinities (renormalization). On the other hand, this introduces some complications in the quantization scheme and, for an introduction to Q.E.D., we have preferred to take here a simpler approach.

1.3.4. Longitudinal electric and magnetic fields

Coming back to Maxwell equations, we see now that the two first equations (1.23a) and (1.23b) actually give the longitudinal part of \mathscr{E} and \mathscr{B} . The second equation (1.23b) expresses that the magnetic field is purely transverse

$$\mathscr{B}_{\parallel} = \mathbf{0} = \mathbf{B}_{\parallel} \,. \tag{1.36}$$

The first equation (1.23a) connects the longitudinal electric field $\mathscr{C}_{\parallel}(k)$ to the charge distribution $\rho(k)$,

$$\mathscr{E}_{\mathbb{I}}(k) = -\frac{\mathrm{i}}{\varepsilon_0} \rho(k) \frac{k}{k^2}, \qquad (1.37)$$

 $\mathscr{C}_{\parallel}(k)$ appears as the product of two functions of k, the Fourier transforms of which are

$$\rho(\mathbf{k}) \longleftrightarrow \rho(\mathbf{r}),$$

$$-\frac{\mathrm{i}}{\varepsilon_0} \frac{\mathbf{k}}{k^2} \longleftrightarrow \frac{(2\pi)^{3/2}}{4\pi\varepsilon_0} \frac{\mathbf{r}}{\mathbf{r}^3}.$$
(1.38)

Applying relation (1.22) finally gives

$$E_{\parallel}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \,\rho(\mathbf{r}',t) \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$
$$= \frac{1}{4\pi\varepsilon_0} \sum_{\alpha} q_{\alpha} \frac{\mathbf{r}-\mathbf{r}_{\alpha}(t)}{|\mathbf{r}-\mathbf{r}_{\alpha}(t)|^3}.$$
(1.39)

The longitudinal electric field at time t is therefore the Coulomb field calculated as if the charge density ρ was static and equal to its actual value at time t (instantaneous Coulomb field).

It must be emphasized that such a result is independent of any choice of gauge since it has been derived directly from Maxwell equations for the fields E and B without invoking any potential.

The fact that the longitudinal electric field follows instantaneously the charge distribution does not mean that we have electric perturbations travelling faster than light. Only the *total* electric field $E = E_{\parallel} + E_{\perp}$ has a physical meaning and we will see in a following section (section 1.3.6) that the equation of motion of E_{\perp} at a given time t and at a given point r depends on the current j(r', t) at the same time t but in all other points r'. It follows that E_{\perp} also contains instantaneous contributions from the charge distribution, which can be shown to cancel exactly those of E_{\parallel} , so that the total electric field is purely retarded.

Remarks

(i) From the expression (1.10b), or (1.25b), of the electric field in terms of the potentials, it follows that

$$\boldsymbol{E}_{\perp} = -\dot{\boldsymbol{A}}_{\perp}, \qquad (1.40a)$$

$$\boldsymbol{E}_{\parallel} = -\dot{\boldsymbol{A}}_{\parallel} - \nabla \boldsymbol{U}. \tag{1.40b}$$

In Coulomb gauge, we have $A_{\parallel} = 0$, so that

$$\boldsymbol{A}_{\parallel} = \boldsymbol{0} \to \boldsymbol{E}_{\parallel} = -\nabla \boldsymbol{U}. \tag{1.41}$$

In Coulomb gauge, the longitudinal and transverse parts of E are respectively associated with U and A. Comparing (1.41) with (1.39), one deduces that, in Coulomb gauge, U is the Coulomb potential of the charge distribution

$$\mathbf{A}_{\parallel} = \mathbf{0} \to U(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}.$$
 (1.42)

Such a result can also be obtained directly from the potential equation (1.18a) in Coulomb gauge. The solution of such a Poisson equation (tending to zero when $|\mathbf{r}| \rightarrow \infty$) is the same as (1.42).

(ii) From (1.26a), it clearly appears that a gauge transformation changes only A_{\parallel} . Thus, the transverse vector potential A_{\perp} is gauge invariant.

(iii) We decompose here Maxwell equations in two sets: (1.1a) and (1.1b) giving the longitudinal fields, (1.1c) and (1.1d) giving the rate of variation of the transverse fields (see section 1.3.6). Such a decomposition is not the same as in relativity where (1.1b) and (1.1c) on one hand, (1.1a) and (1.1d) on the other hand are combined in two covariant equations

$$\partial_{\mu}F_{\nu\rho} + \partial_{\nu}F_{\rho\mu} + \partial_{\rho}F_{\mu\nu} = 0, \qquad \mu \neq \nu \neq \rho, \qquad (1.43)$$

$$\partial_{\mu}F^{\mu\nu} = \frac{1}{\varepsilon_0 c^2} j^{\mu} , \qquad (1.44)$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{1.45}$$

is the electromagnetic field tensor. A_{μ} the potential 4-vector, j_{μ} the current 4-vector.

1.3.5. Contribution of the longitudinal electric field to the total energy and total momentum

Before studying the last two Maxwell equations (1.23c) and (1.23d), we use now the expression (1.27) of $\mathscr{C}_{\parallel}(k)$ for evaluating the contribution of the longitudinal electric field to a few important physical observables.

Total energy. From the Parseval–Plancherel identity (1.21) it follows that

$$\frac{1}{2}\varepsilon_0 \int \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{E} \cdot \boldsymbol{E} = \frac{1}{2}\varepsilon_0 \int \mathrm{d}^3 \boldsymbol{k} \, \mathcal{E}^* \cdot \mathcal{E} \,. \tag{1.46}$$

Now, replacing \mathscr{E} by $\mathscr{E}_{\parallel} + \mathscr{E}_{\perp}$ and using $\mathscr{E}_{\parallel} \cdot \mathscr{E}_{\perp} = 0$, we get

$$\frac{1}{2}\varepsilon_0 \int \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{E}^2 = \frac{1}{2}\varepsilon_0 \int \mathrm{d}^3 \boldsymbol{k} \, |\mathscr{C}_{\parallel}(\boldsymbol{k})|^2 + \frac{1}{2}\varepsilon_0 \int \mathrm{d}^3 \boldsymbol{k} \, |\mathscr{C}_{\perp}(\boldsymbol{k})|^2 \,. \tag{1.47}$$

The first term of (1.47) is the contribution, H_{long} , of the longitudinal electric field to the total energy H given in (1.7)

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$$H_{\text{long}} = \frac{1}{2}\varepsilon_0 \int d^3k \ |\mathscr{C}_{\parallel}(\boldsymbol{k})|^2 = \frac{1}{2}\varepsilon_0 \int d^3r \, \boldsymbol{E}_{\parallel}^2(\boldsymbol{r}) , \qquad (1.48)$$

whereas the second one, added to the magnetic energy, gives the contribution H_{trans} of the transverse fields E and B,

$$H_{\text{trans}} = \frac{1}{2}\varepsilon_0 \int d^3k \left[|\mathscr{E}_{\perp}(\boldsymbol{k})|^2 + c^2 |\mathscr{B}(\boldsymbol{k})|^2 \right]$$
$$= \frac{1}{2}\varepsilon_0 \int d^3r \left[\boldsymbol{E}_{\perp}^2(\boldsymbol{r}) + c^2 \boldsymbol{B}^2(\boldsymbol{r}) \right].$$
(1.49)

Inserting the expression (1.37) of $\mathscr{C}_{\parallel}(k)$ into (1.48) gives

$$H_{\text{long}} = \frac{1}{2\varepsilon_0} \int d^3k \, \rho^*(\mathbf{k}) \frac{\rho(\mathbf{k})}{k^2}. \tag{1.50}$$

Using relations (1.21) and (1.22), we finally transform (1.50) into

$$H_{\text{long}} = \frac{1}{8\pi\epsilon_0} \int \int d^3r \, d^3r' \, \frac{\rho(r)\rho(r')}{|r-r'|},$$
(1.51)

 H_{long} is nothing but the electrostatic Coulomb energy of the charge distribution. With the expression (1.5a) of ρ , we can also write

$$H_{\text{long}} = V_{\text{Coul}} = \sum_{\alpha} \varepsilon_{\text{Coul}}^{\alpha} + \frac{1}{8\pi\varepsilon_0} \sum_{\alpha\neq\beta} \frac{q_{\alpha}q_{\beta}}{|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|}, \qquad (1.52)$$

where $\varepsilon_{\text{Coul}}^{\alpha}$ is the energy of the Coulomb field of particle α [actually an infinite quantity unless a cut-off is introduced in the Fourier expansion of E_{\parallel} , see section 2.4.5] and the second term represents the electrostatic interaction between different particles.

Finally, the total energy (1.7) can be written

$$H = \sum_{\alpha} \frac{1}{2} m_{\alpha} \dot{\boldsymbol{r}}_{\alpha}^{2} + V_{\text{Coul}} + H_{\text{trans}}, \qquad (1.53)$$

and appears as the sum of three energies: the kinetic energy of the particles (first term), their Coulomb energy (second term), and the energy of the transverse fields (third term).

As for the previous section, all the previous results are independent of any choice of gauge.

Total momentum. Replacing E by $E_{\parallel} + E_{\perp}$ in the second term of (1.8) shows that the total momentum of the field is the sum of two contributions, P_{long} and P_{trans} given by

$$\boldsymbol{P}_{\text{long}} = \varepsilon_0 \int d^3 \boldsymbol{r} \, \boldsymbol{E}_{\parallel}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r}) = \varepsilon_0 \int d^3 \boldsymbol{k} \, \mathscr{C}_{\parallel}^*(\boldsymbol{k}) \times \mathscr{B}(\boldsymbol{k}) \,, \qquad (1.54a)$$

$$\boldsymbol{P}_{\text{trans}} = \varepsilon_0 \int d^3 \boldsymbol{r} \, \boldsymbol{E}_{\perp}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r}) = \varepsilon_0 \int d^3 \boldsymbol{k} \, \mathscr{E}_{\perp}^*(\boldsymbol{k}) \times \mathscr{B}(\boldsymbol{k}) \,. \tag{1.54b}$$

Using the expression (1.37) of \mathscr{C}_{\parallel} , the relation (1.25a) between \mathscr{B} and \mathscr{A} , and the identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{1.55}$$

we transform (1.54a) into

$$\boldsymbol{P}_{\text{long}} = \varepsilon_0 \int d^3k \, \frac{\mathrm{i}\rho^*}{\varepsilon_0} \frac{\boldsymbol{k}}{k^2} \times (\mathrm{i}\boldsymbol{k} \times \mathscr{A})$$
$$= \int d^3k \, \rho^* [\mathscr{A} - \boldsymbol{\kappa}(\boldsymbol{\kappa} \cdot \mathscr{A})] \,. \tag{1.56}$$

We recognize in the bracket of (1.56) the transverse part of \mathcal{A} , so that P_{long} takes the simpler form

$$\boldsymbol{P}_{\text{long}} = \int d^3k \, \rho^* \mathscr{A}_{\perp} = \int d^3r \, \rho^* \boldsymbol{A}_{\perp} = \sum_{\alpha} q_{\alpha} \boldsymbol{A}_{\perp}(\boldsymbol{r}_{\alpha}) \tag{1.57}$$

[we have used the expression (1.5a) of ρ]. As above, such a result is gauge independent.

Finally, the total momentum P given in (1.8) can be written

$$\boldsymbol{P} = \sum_{\alpha} \left[m_{\alpha} \dot{\boldsymbol{r}}_{\alpha} + q_{\alpha} \boldsymbol{A}_{\perp}(\boldsymbol{r}_{\alpha}) \right] + \boldsymbol{P}_{\text{trans}}$$
(1.58)

and appears as the sum of the "mechanical momenta" $m_{\alpha}\dot{r}_{\alpha}$ of the particles, the momentum of the longitudinal field $\sum_{\alpha} q_{\alpha}A_{\perp}(r_{\alpha})$, and the

momentum of the transverse field. Expression (1.58) suggests to introduce for the particle α the quantity

$$\boldsymbol{p}_{\alpha} = \boldsymbol{m}_{\alpha} \dot{\boldsymbol{r}}_{\alpha} + \boldsymbol{q}_{\alpha} \boldsymbol{A}_{\perp}(\boldsymbol{r}_{\alpha}) \,, \tag{1.59}$$

so that P can be written

$$\boldsymbol{P} = \sum_{\alpha} \boldsymbol{p}_{\alpha} + \boldsymbol{P}_{\text{trans}} \,. \tag{1.60}$$

Actually, in Coulomb gauge, p_{α} can be shown to be the "canonical momentum" of r_{α} [2, 9]. It therefore appears that, in Coulomb gauge, the difference between the canonical momentum p_{α} and the mechanical momentum $m_{\alpha}\dot{r}_{\alpha}$ of particle α is the momentum associated with the longitudinal field of particle α .

Remark. Using eq. (1.59), the total energy H given in (1.53) can be written as

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{r}_{\alpha})]^2 + V_{\text{Coul}} + H_{\text{trans}}. \qquad (1.61)$$

It can be shown that H is the Hamiltonian of the system in Coulomb gauge [2, 9].

One could also split E in $E_{\parallel} + E_{\perp}$ in the last term of (1.9), and show that the total angular momentum J is given by

$$\boldsymbol{J} = \sum_{\alpha} \boldsymbol{r}_{\alpha} \times \boldsymbol{p}_{\alpha} + \boldsymbol{J}_{\text{trans}}, \qquad (1.62)$$

where p_{α} is defined in (1.59) and where

$$\boldsymbol{J}_{\text{trans}} = \varepsilon_0 \int d^3 \boldsymbol{r} \, \boldsymbol{r} \times [\boldsymbol{E}_{\perp}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})] \tag{1.63}$$

is the angular momentum of the transverse fields

1.3.6. Maxwell equations for transverse fields

We come back now to the last two Maxwell equations (1.23c) and (1.23d) and we consider first the transverse parts of these two equations which can be written as

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$$\dot{\mathscr{B}} = -i\mathbf{k} \times \mathscr{E} = -i\mathbf{k} \times \mathscr{E}_{\perp}, \qquad (1.64a)$$
$$\dot{\mathscr{E}}_{\perp} = ic^{2}\mathbf{k} \times \mathscr{B} - \frac{1}{\varepsilon_{0}} \mathbf{j}_{\perp}. \qquad (1.64b)$$

The last two Maxwell equations therefore appear as dynamical equations giving the rate of variation of the transverse fields \mathcal{B} and \mathcal{E}_{\perp} .

One can note that the source term appearing in the equation of motion (1.64b) of \mathscr{C}_{\perp} is j_{\perp} and not j. Since in *r*-space the relation between j_{\perp} and j is not local, the rate of variation of $E_{\perp}(r, t)$ in point r and at time *t* depends on the current j(r', t) in all other points r' at the same time *t*. It follows that E_{\perp} contains, as E_{\parallel} , instantaneous contributions from the charge distribution (which can be shown to cancel each other, so that the total field $E = E_{\parallel} + E_{\perp}$ is purely retarded).

It remains to consider the longitudinal part of (1.23c) and (1.23d). The two members of (1.23c) are transverse. The longitudinal part of (1.23d) reads

$$\dot{\mathscr{E}}_{\parallel} + \frac{1}{\varepsilon_0} \, \boldsymbol{j}_{\parallel} = \boldsymbol{0} \,. \tag{1.65}$$

Taking the scalar product of (1.65) with k, using (1.37) and the fact that $\mathbf{k} \cdot \mathbf{j}_{\parallel} = \mathbf{k} \cdot \mathbf{j}$, one gets

$$\dot{\rho} + \mathbf{i}\mathbf{k} \cdot \mathbf{j} = 0. \tag{1.66}$$

This is just the charge conservation equation (1.24) and brings nothing new.

As a conclusion of this section, it is useful to reconsider the definition (1.6) of the "state" of the total system field plus particles at time t_0 . Since the longitudinal field can actually be expressed as a function of the r_{α} 's [see (1.39)], the state of the system is entirely determined by

$$\{\mathscr{E}_{\perp}(\boldsymbol{k}, t_0), \mathscr{B}(\boldsymbol{k}, t_0), \boldsymbol{r}_{\alpha}(t_0), \dot{\boldsymbol{r}}_{\alpha}(t_0)\}$$
(1.67)

for all k and all α .

We will show in the next section (1.4) that it is possible to introduce a further improvement in the choice of the dynamical variables characterizing the state of the field. *Remark.* In this section, we have only considered the field equations (1.23). It is also possible to study the longitudinal and transverse parts of the potential equations (1.27). Since the last term of (1.27b) is longitudinal, the transverse part (1.27b) reads

$$\frac{1}{c^2}\ddot{\mathcal{A}}_{\perp} + k^2 \mathcal{A}_{\perp} = \frac{1}{\varepsilon_0 c^2} j_{\perp}, \qquad (1.68)$$

which becomes in r-space

$$\Box \boldsymbol{A}_{\perp} = \frac{1}{\varepsilon_0 c^2} \boldsymbol{j}_{\perp} \,. \tag{1.69}$$

This looks like (1.14b) except that we have A_{\perp} and j_{\perp} instead of A and j. If one takes the longitudinal part of (1.27b) and if one uses (1.27a) for eliminating \dot{u} , one gets, once more, the charge conservation equation (1.24). As for (1.23d), the longitudinal part of (1.27b) brings nothing new. We are therefore left with a single equation (1.27a), which can be written as

$$k^{2}\mathcal{U} = \frac{1}{\varepsilon_{0}}\rho + \mathbf{i}\boldsymbol{k}\cdot\dot{\boldsymbol{\mathcal{A}}}_{\parallel}$$
(1.70)

(since $\mathbf{k} \cdot \mathbf{A}_{\perp} = 0$), and which is not sufficient for determining the motion of \mathbf{A}_{\perp} and \mathbf{U} . Such a result is actually not surprising since potentials are redundant. In order to determine \mathbf{A}_{\parallel} and \mathbf{U} , we need a supplementary condition, i.e., a gauge condition. If we chose the Coulomb gauge, we fix $\mathbf{A}_{\parallel} = \mathbf{0}$ and (1.70) gives \mathbf{U} [see also (1.18a)]. If we choose the Lorentz gauge, the supplementary condition (1.13) can be written in \mathbf{k} -space,

$$\dot{\mathcal{U}} = -\mathrm{i}c^2 \mathbf{k} \cdot \mathcal{A}_{\parallel} \,. \tag{1.71}$$

Equations (1.70) and (1.71) now form a system of two first-order equations describing the evolution of \mathscr{A}_{\parallel} and \mathscr{U} . Other choices of gauge conditions are of course possible.

1.4. Normal variables

1.4.1. General idea

In *r*-space, the rates of variation $\dot{E}(r)$ and $\dot{B}(r)$ of the fields E and B

depend on the values of E and B in the *neighbourhood* of r. Maxwell equations (1.1) are *partial* differential equations.

Going in k-space, we have first eliminated $\mathscr{E}_{\parallel}(k)$, which is not actually a new dynamical variable since it can be reexpressed in terms of the $r'_{\alpha}s$, and we have then shown that $\mathscr{E}_{\perp}(k)$ and $\mathscr{B}(k)$ only depend on the values of $\mathscr{E}_{\perp}(k)$ and $\mathscr{B}(k)$ [and also $j_{\perp}(k)$] at the same point k. Equations (1.64) form, for each point k, a set of two coupled differential equations.

Looking at the linear differential system (1.64), we can now ask if it is possible to introduce two linear combinations of \mathscr{C}_{\perp} and \mathscr{B} which evolve *independently* from each other, at least when $\mathbf{j}_{\perp} = \mathbf{0}$, i.e. when the field is free.

1.4.2. Definition of normal variables We first rewrite equations (1.64) as

$$\dot{\mathscr{E}}_{\perp} = ic^2 \mathbf{k} \times \mathscr{B} - \frac{1}{\varepsilon_0} \mathbf{j}_{\perp}, \qquad (1.72a)$$

$$\mathbf{k} \times \dot{\mathbf{\mathcal{B}}} = \mathrm{i}k^2 \mathcal{E}_\perp \,. \tag{1.72b}$$

We are looking for the eigenfunction of such a linear system. When $j_{\perp} = 0$, we have from (1.72)

$$\frac{\partial}{\partial t} \left(\mathscr{C}_{\perp} \mp c \kappa \times \mathscr{B} \right) = \mp i \omega \left(\mathscr{C}_{\perp} \mp c \kappa \times \mathscr{B} \right), \tag{1.73}$$

with

$$\boldsymbol{\omega} = ck \,, \qquad \boldsymbol{\kappa} = \boldsymbol{k}/k \,. \tag{1.74}$$

This leads us to *define*, even if $j_{\perp} \neq 0$, two new variables $\alpha(k, t)$ and $\beta(k, t)$ by

$$\boldsymbol{\alpha}(\boldsymbol{k},t) = -\frac{\mathrm{i}}{2\mathcal{N}(\boldsymbol{k})} \left[\mathscr{C}_{\perp}(\boldsymbol{k},t) - c\boldsymbol{\kappa} \times \mathscr{B}(\boldsymbol{k},t) \right], \qquad (1.75a)$$

$$\boldsymbol{\beta}(\boldsymbol{k},t) = -\frac{\mathrm{i}}{2\mathcal{N}(\boldsymbol{k})} \left[\mathscr{C}_{\perp}(\boldsymbol{k},t) + c\boldsymbol{\kappa} \times \mathscr{B}(\boldsymbol{k},t) \right], \qquad (1.75\mathrm{b})$$

where $\mathcal{N}(k)$ is a normalization coefficient which will be chosen later on

in order to have for the total energy H an expression as simple and suggestive as possible.

Before going further, it is important to note that α and β are not actually independent variables. From the reality of E_{\perp} and B, which gives rise to relations such as (1.20) for \mathscr{C}_{\perp} and \mathscr{B} , one deduces

$$\boldsymbol{\beta}(\boldsymbol{k},t) = -\boldsymbol{\alpha}^*(-\boldsymbol{k},t) \,. \tag{1.76}$$

Inverting the linear system (1.75) and using (1.76) gives

$$\mathscr{E}_{\perp}(\boldsymbol{k},t) = i\mathcal{N}(\boldsymbol{k})[\boldsymbol{\alpha}(\boldsymbol{k},t) - \boldsymbol{\alpha}^{*}(-\boldsymbol{k},t)], \qquad (1.77a)$$

$$\mathscr{B}(\boldsymbol{k},t) = \frac{\mathrm{i}\mathcal{N}(\boldsymbol{k})}{c} \left[\boldsymbol{\kappa} \times \boldsymbol{\alpha}(\boldsymbol{k},t) + \boldsymbol{\kappa} \times \boldsymbol{\alpha}^*(-\boldsymbol{k},t)\right].$$
(1.77b)

The knowledge of $\alpha(k, t)$, for all values of k, is therefore equivalent to the knowledge of $\mathscr{C}_{\perp}(k, t)$ and $\mathscr{B}(k, t)$. Furthermore, the $\alpha(k, t)$ are truly independent variables since there are no reality constraints between $\alpha(k, t)$ and $\alpha^*(-k, t)$, as for \mathscr{C}_{\perp} and \mathscr{B} . We can therefore replace the definition (1.67) of the state of the combined system by

$$\left\{\boldsymbol{\alpha}(\boldsymbol{k},t_0),\,\boldsymbol{r}_{\alpha}(t_0),\,\boldsymbol{\dot{r}}_{\alpha}(t_0)\right\}.$$
(1.78)

1.4.3. Equation of evolution of normal variables

From the two Maxwell equations (1.72), and from the definition (1.75a) of α , one deduces

$$\dot{\boldsymbol{\alpha}}(\boldsymbol{k},t) + \mathrm{i}\omega\boldsymbol{\alpha}(\boldsymbol{k},t) = \frac{\mathrm{i}}{2\varepsilon_0 \mathcal{N}(\boldsymbol{k})} \, \boldsymbol{j}_{\perp}(\boldsymbol{k},t) \,. \tag{1.79}$$

It is important to note that, \mathscr{C}_{\perp} and \mathscr{B} being related to α by (1.77), equation (1.79) is *strictly* equivalent to Maxwell equations. Equation (1.79) is however more suggestive than Maxwell equations. It looks like the equation of motion of a (fictitious) harmonic oscillator, having a free evolution frequency ω , and driven by a particle source term, proportional to $j_{\perp}(k, t)$.

When $\mathbf{j} = \mathbf{0}$ (free field), the evolution of the normal variable $\alpha(\mathbf{k}, t)$ is completely decoupled from the others. The solution of (1.79) is then a pure harmonic oscillation, describing the so-called normal modes of vibration of the free field. This is why the $\alpha(\mathbf{k}, t)$'s are called normal variables.

If the sources are imposed from outside (independent of α), the α 's corresponding to different k's still evolve independently from each other, driven by j_{\perp} .

Finally, if the sources are particles interacting with the field, the motion of \dot{j}_{\perp} depends on the $\alpha(k, t)$'s, so that the evolutions of the various $\alpha(k, t)$'s are in general coupled through the current term $\dot{j}_{\perp}(k, t)$. One must add to (1.79) the equation of motion of $\dot{j}_{\perp}(k, t)$ [determined from the Newton-Lorentz equation (1.2) and from the definition(1.5) of the current] and try to solve this set of coupled equations.

Before ending this section, we will introduce a new notation. Since α is (as \mathscr{C}_{\perp} and \mathscr{B}) a transverse vector field, we can, for each value of k, expand $\alpha(k, t)$ along the two mutually perpendicular unit vectors ε and ε' in the plane perpendicular to k (Fig. 1)

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}' \cdot \boldsymbol{\varepsilon}' = \boldsymbol{\kappa} \cdot \boldsymbol{\kappa} = 1,$$

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} \cdot \boldsymbol{\kappa} = \boldsymbol{\varepsilon}' \cdot \boldsymbol{\kappa} = 0.$$
 (1.80)

We thus have

$$\boldsymbol{\alpha}(\boldsymbol{k},t) = \boldsymbol{\varepsilon} \boldsymbol{\alpha}_{\varepsilon}(\boldsymbol{k},t) + \boldsymbol{\varepsilon}' \boldsymbol{\alpha}_{\varepsilon'}(\boldsymbol{k},t)$$
$$= \sum_{\varepsilon} \boldsymbol{\varepsilon} \boldsymbol{\alpha}_{\varepsilon}(\boldsymbol{k},t), \qquad (1.81)$$

where

$$\alpha_{\varepsilon}(k,t) = \varepsilon \cdot \alpha(k,t)$$





Fig. 1. Transverse polarization vectors ε and ε' .

is the component of α along ε . The set $\{\alpha_{\varepsilon}(k, t)\}$ for all k and all ε forms a complete set of independent variables for the transverse field. The equation of motion of $\alpha_{\varepsilon}(k, t)$ is

$$\dot{\alpha}_{\varepsilon}(\mathbf{k},t) + i\omega\alpha_{\varepsilon}(\mathbf{k},t) = \frac{i}{2\varepsilon_{0}\mathcal{N}(\mathbf{k})}\varepsilon \cdot \mathbf{j}(\mathbf{k},t); \qquad (1.83)$$

we have used $\varepsilon \cdot j_{\perp} = \varepsilon \cdot j$.

1.4.4. Expression of the physical observables of the transverse field in terms of the normal variables

Since in the following we will always use the normal variables $\alpha_{\epsilon}(\mathbf{k}, t)$ (and the corresponding quantum operators) for characterizing the transverse field, we give in this section the expression of the various physical observables of the transverse field in terms of the α_{ϵ} 's.

Energy of the transverse field $H_{\text{trans.}}$. We insert in (1.49) the expressions (1.77) of \mathscr{E}_{\perp} and \mathscr{B} in terms of α and α^* [we use the shorter notation α^* for $\alpha^*(-k, t)$]. We also keep the order between α and α^* as it appears in the calculation, even if α and α^* are commuting *c*-numbers. The reason is that similar calculation have to be done in Q.E.D. where α and α^* are replaced by non-commuting operators. The results derived in this section can therefore be extended to the quantum situation without modification.

From (1.77), we deduce

$$\begin{aligned} & \mathscr{E}_{\perp}^{*} \cdot \mathscr{E}_{\perp} = \mathcal{N}^{2} (\boldsymbol{\alpha}^{*} - \boldsymbol{\alpha}_{-}) (\boldsymbol{\alpha} - \boldsymbol{\alpha}_{-}^{*}) \\ & = \mathcal{N}^{2} (\boldsymbol{\alpha}^{*} \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha}_{-} \cdot \boldsymbol{\alpha}_{-}^{*} - \boldsymbol{\alpha}^{*} \cdot \boldsymbol{\alpha}_{-}^{*} - \boldsymbol{\alpha}_{-} \cdot \boldsymbol{\alpha}) , \\ & c^{2} \mathscr{B}^{*} \cdot \mathscr{B} = \mathcal{N}^{2} (\boldsymbol{\alpha}^{*} + \boldsymbol{\alpha}_{-}) (\boldsymbol{\alpha} + \boldsymbol{\alpha}_{-}^{*}) \\ & = \mathcal{N}^{2} (\boldsymbol{\alpha}^{*} \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha}_{-} \cdot \boldsymbol{\alpha}_{-}^{*} + \boldsymbol{\alpha}^{*} \cdot \boldsymbol{\alpha}_{-}^{*} + \boldsymbol{\alpha}_{-} \cdot \boldsymbol{\alpha}) , \end{aligned}$$
(1.84)

so that (1.49) becomes

$$H_{\text{trans}} = \varepsilon_0 \int d^3k \, \mathcal{N}^2 [\boldsymbol{\alpha}^* \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha}_- \cdot \boldsymbol{\alpha}_-^*] \,. \tag{1.85}$$

Changing k into -k in the integral of the second term allows one to replace $\alpha_{-} \cdot \alpha_{-}^{*}$ by $\alpha \cdot \alpha^{*}$. If we now introduce the normalization coefficient $\mathcal{N}(k)$,

$$\mathcal{N}(k) = (\hbar\omega/2\varepsilon_0)^{1/2}, \qquad (1.86)$$

which is chosen in quantum theory in order to have simple commutation relations between the operators corresponding to α_{ε} and α_{ε}^{*} , expression (1.85) takes the more suggestive form

$$H_{\rm trans} = \int d^3k \sum_{\varepsilon} \frac{\hbar\omega}{2} \left[\alpha_{\varepsilon}^*(\mathbf{k}, t) \alpha_{\varepsilon}(\mathbf{k}, t) + \alpha_{\varepsilon}(\mathbf{k}, t) \alpha_{\varepsilon}^*(\mathbf{k}, t) \right].$$
(1.87)

It appears as the sum of the energies of a set of fictitious harmonic oscillators, one oscillator with frequency $\omega = ck$ being associated with each ensemble of vectors k, ε (with ε perpendicular to k). Such an ensemble (k, ε) defines a "mode" of the transverse field.

Momentum of the transverse field P_{trans} . Similar calculations give from (1.54b)

$$\mathbf{P}_{\text{trans}} = \int d^3k \, \sum_{\varepsilon} \frac{\hbar k}{2} \left[\alpha_{\varepsilon}^*(\mathbf{k}, t) \alpha_{\varepsilon}(\mathbf{k}, t) + \alpha_{\varepsilon}(\mathbf{k}, t) \alpha_{\varepsilon}^*(\mathbf{k}, t) \right]. \tag{1.88}$$

Remark. For the angular momentum of the transverse field, given in (1.63), the calculations are a little more tedious than for H_{trans} and P_{trans} . We just give here the result

$$J_{a}^{\text{trans}} = \frac{\hbar}{2i} \sum_{bcd} \int d^{3}k \left[\alpha_{d}^{*} \varepsilon_{abc} k_{b} \partial_{c} \alpha_{d} + \alpha_{b}^{*} \varepsilon_{abc} \alpha_{c} - \alpha_{d} \varepsilon_{abc} k_{b} \partial_{c} \alpha_{d}^{*} - \alpha_{b} \varepsilon_{abc} \alpha_{c}^{*} \right], \qquad (1.89)$$

where a, b, c, d = x, y, or z, $\partial_c = \partial/\partial k_c$ and ε_{abc} is the completely antisymmetric tensor.

Transverse electric and magnetic fields in r-space. The expansions of $E_{\perp}(r, t)$ and B(r, t) are obtained by taking the Fourier transform of (1.77) [In the integral over k of the last term of (1.77), we also change k into -k]. This gives

$$\boldsymbol{E}_{\perp}(\boldsymbol{r},t) = \mathrm{i} \int \mathrm{d}^{3}\boldsymbol{k} \sum_{\varepsilon} \mathscr{E}_{\omega}[\alpha_{\varepsilon}(\boldsymbol{k},t)\boldsymbol{\varepsilon} \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - \alpha_{\varepsilon}^{*}(\boldsymbol{k},t)\boldsymbol{\varepsilon} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}], \qquad (1.90)$$
$$\boldsymbol{B}(\boldsymbol{r},t) = \mathrm{i} \int \mathrm{d}^{3}\boldsymbol{k} \sum \mathscr{B}_{\omega}[\alpha_{\varepsilon}(\boldsymbol{k},t)\boldsymbol{\kappa} \times \boldsymbol{\varepsilon} \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - \alpha_{\varepsilon}^{*}(\boldsymbol{k},t)\boldsymbol{\kappa} \times \boldsymbol{\varepsilon} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}],$$

(1.91)

with

$$\mathscr{E}_{\omega} = [\hbar\omega/2\varepsilon_0(2\pi)^3]^{1/2}, \qquad \mathscr{B}_{\omega} = \mathscr{E}_{\omega}/c.$$
(1.92)

In the expansions (1.90) and (1.91) the reality of E_{\perp} and **B** is manifest^{*}.

For a free field, the solution of the equation of motion (1.83) of α_{ε} is

$$\alpha_{\varepsilon}(\mathbf{k},t) = \alpha_{\varepsilon}(\mathbf{k}) e^{-i\omega t} \,. \tag{1.93}$$

Inserting (1.93) into (1.90) and (1.91) then gives for E_{\perp} and **B** an expansion in *plane progressive waves* $\varepsilon \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$. For example,

$$\boldsymbol{E}_{\perp}^{\text{free}}(\boldsymbol{r},t) = \mathrm{i} \int \mathrm{d}^{3}\boldsymbol{k} \sum_{\varepsilon} \mathscr{C}_{\omega} \alpha_{\varepsilon}(\boldsymbol{k}) \boldsymbol{\varepsilon} \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] + \mathrm{c.c.}$$
(1.94)

We also check with (1.93) that, for a free field, H_{trans} and P_{trans} given in (1.87) and (1.88) are constants of motion (independent of t).

It must be emphasized that the definition (1.75) of the normal variables, and the expansions (1.87) to (1.91) which follow from this definition, are valid in presence or in absence of sources. On the other hand, the simple solution (1.93) for $\alpha_{\varepsilon}(\mathbf{k}, t)$ only holds for a free field. In the presence of sources, the solution of eq. (1.83) is more complicated than (1.93) and the expansions (1.90) and (1.91) of E_{\perp} and B are no longer superpositions of plane progressive waves. Similarly, the transverse energy given in (1.87) is no longer a constant of motion. There are exchanges of energy between the transverse fields and the particles, and only the total energy given in (1.53) is conserved. Similar considerations hold for P_{trans} .

It is usual to call "positive frequency part" of E_{\perp} , and to denote $E_{\perp}^{(+)}(\mathbf{r}, t)$, the part of the expansion (1.90) containing only the $\alpha_{\varepsilon}(\mathbf{k}, t)$ [and not $\alpha_{\varepsilon}^{*}(\mathbf{k}, t)$],

$$\boldsymbol{E}_{\perp}^{(+)}(\boldsymbol{r},t) = \mathrm{i} \int \mathrm{d}^{3}\boldsymbol{k} \sum_{\varepsilon} \mathscr{C}_{\omega} \alpha_{\varepsilon}(\boldsymbol{k},t) \varepsilon \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} ; \qquad (1.95)$$

the remaining part of (1.90) being called the "negative frequency part" and denoted $E_{\perp}^{(-)}(r, t)$,

$$\boldsymbol{E}_{\perp}^{(-)}(\boldsymbol{r},t) = [\boldsymbol{E}_{\perp}^{(+)}(\boldsymbol{r},t)]^* .$$
(1.96)

* The polarization vector ε is real. It would be possible to introduce complex polarization vectors for describing circularly or elliptically polarized modes of the transverse field. In such a case the second ε in (1.90) and (1.91) has to be replaced by ε^* . Such a denomination is due to the fact that, for a free field, $E_{\perp}^{(+)}$ is a superposition of waves in $e^{-i\omega t}$.

Transverse vector potential $A_{\perp}(r, t)$. For the following, it will be also useful to give the expansion of the transverse part A_{\perp} of the vector potential A in terms of the normal variables.

We first note that the transverse fields E_{\perp} and **B** are related only to A_{\perp} . From (1.10a) and (1.40a), we have

$$\boldsymbol{E}_{\perp}(\boldsymbol{r},t) = -\frac{\partial}{\partial t} \boldsymbol{A}_{\perp}(\boldsymbol{r},t), \qquad (1.97a)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}_{\perp}(\boldsymbol{r},t) \tag{1.97b}$$

(since $\nabla \times A_{\parallel} = 0$). Relations (1.97) are of course independent of any choice of gauge.

We show now that $A_{\perp}(r, t)$ can be expressed as

$$\mathbf{A}_{\perp}(\mathbf{r},t) = \int \mathrm{d}^{3}k \, \sum_{\varepsilon} \, \mathscr{A}_{\omega}[\alpha_{\varepsilon}(\mathbf{k},t)\varepsilon \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \mathrm{c.c.}] \,, \tag{1.98}$$

with

$$\mathscr{A}_{\omega} = \mathscr{B}_{\omega}/k = \mathscr{E}_{\omega}/\omega \,. \tag{1.99}$$

Since ε is perpendicular to k, the vector field written in (1.98) is transverse. If we calculate the curl of the expansion (1.98), we find the expansion (1.91) of **B**, so that (1.98) satisfies (1.97b). We finally apply $-\partial/\partial t$ to the expansion (1.98), using eq. (1.83) for reexpressing the time derivative of the two functions $\alpha_{\varepsilon}(k, t)$ and $\alpha_{\varepsilon}^{*}(k, t)$ appearing in (1.98). The first term, $i\omega\alpha_{\varepsilon}(k, t)$, appearing in $-\dot{\alpha}_{\varepsilon}(k, t)$, and its complex conjugate, give, when inserted in (1.98), a contribution which coincides with the expansion (1.90) of $E_{\perp}(r, t)$, since we have chosen in (1.99) $\mathcal{A}_{\omega} = \mathcal{E}_{\varepsilon}/\omega$. The contribution of the last term of (1.83) (and its complex conjugate) vanishes [the two terms k, ε and $-k, \varepsilon$ cancel each other in the integral over k]. It follows that (1.98) also satisfies (1.97a) and can therefore be considered as the transverse vector potential expansion.

We can also calculate the Fourier transform $\mathscr{A}_{\perp}(k, t)$ of $A_{\perp}(r, t)$. Changing k into -k in the second integral of (1.98) gives Introduction to quantum electrodynamics

$$\mathscr{A}_{\perp\varepsilon}(\mathbf{k},t) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega}} \left[\alpha_{\varepsilon}(\mathbf{k},t) + \alpha_{\varepsilon}^*(-\mathbf{k},t) \right], \qquad (1.100)$$

with

$$\mathscr{A}_{\perp\varepsilon} = \varepsilon \cdot \mathscr{A}_{\perp}.$$

Combining (1.100) and (1.77a), we can finally express α_{ε} in terms of $\mathscr{A}_{\perp \varepsilon}$ and $\mathscr{C}_{\perp \varepsilon}$ [instead of $\mathscr{C}_{\perp \varepsilon}$ and $\mathscr{B}_{\varepsilon}$ as in (1.75a)]

$$\alpha_{\varepsilon}(\mathbf{k},t) = \sqrt{\frac{\varepsilon_0}{2\hbar\omega}} \left[\omega \mathscr{A}_{\perp\varepsilon}(\mathbf{k},t) - \mathrm{i} \mathscr{E}_{\perp\varepsilon}(\mathbf{k},t) \right].$$
(1.101)

1.4.5. Analogies and differences between the normal variables and the wave function of a spin 1 particle in k-space

Consider first the case of a free field. Equation (1.79) can be written as

$$i\hbar\dot{\alpha}(\mathbf{k},t) = \hbar\omega\alpha(\mathbf{k},t),$$
 (1.102)

and then looks like the Schrödinger equation for a "vector wave function" $\alpha(k, t)$, the corresponding "Hamiltonian" being diagonal in *k*-space with matrix elements $\hbar\omega\delta(k-k')$. Equation (1.87) can be also interpreted as the average value of such an Hamiltonian in the transverse wave function $\alpha(k, t)$. Similarly, since in quantum mechanics, the momentum operator of a particle is diagonal in *k*-space with matrix elements $\hbar k \delta(k-k')$, eq. (1.88) can be interpreted as the average value of the momentum operator in the wave function $\alpha(k, t)$. Finally, one can show that expression (1.89), giving the angular momentum of the transverse field, coincides with the average value of J = L + S, where *L* and *S* are the usual quantum mechanical orbital and spin angular momentum in the transverse wave function $\alpha(k, t)$ [the first term of the bracket of (1.89) corresponds to *L*, the second one to *S*].

All these results would suggest to interpret $\alpha(k, t)$ as the wave function in k-space of a spin 1 particle^{*}, which would be the photon ([10] chapter I). Such an analogy however cannot be pushed too far. First, one can show that the spatial Fourier transform of $\alpha(k, t)$ cannot be interpreted as a wave function for the photon in r-space, and more

^{*} The value 1 of the spin is related to the vector character of α .

generally that one cannot construct a position operator for the photon [11]. Secondly, the equation of evolution (1.79) of α in the presence of sources has not the structure of a Schrödinger equation. It is not homogeneous. Such a result is actually not surprising. Schrödinger equation conserves the norm of the wave function, and therefore the number of particles. But, it is well known that, in the presence of sources, photons can be absorbed or emitted. It is therefore impossible to introduce a Schrödinger equation for a single photon in the presence of sources. Actually, the quantization must concern the electromagnetic field itself, photons then appearing as elementary excitations of the quantized field. We will see in the subsequent chapters that the "wave function", more precisely the "state vector" of the quantized field is a vector of a Fock space where the number of photons can vary from zero (vacuum) to infinity.

The previous analogy can however be useful. It suggests for example to study the transverse eigenfunctions of J^2 and J_z in k-space. This leads to the multipolar expansion of the transverse field which is more convenient than the expansion in plane waves given in the previous section for studying problems where the angular momentum plays an important role [10].

Periodic boundary conditions - introduction of simpler notations

One sometimes considers that the fields are contained in a cube with volume L^3 , and that they satisfy periodic boundary conditions on the faces of the cube. At the end of the calculation, one lets L tend to infinity. All physical predictions (cross sections, transition probabilities...) must of course be independent of L.

The advantage of such a procedure is to replace Fourier transforms by Fourier series. In other words, integrals over k are replaced by discrete summations over

$$k_{x,y,z} = 2\pi n_{x,y,z}/L \,, \tag{1.103}$$

where $n_{x,y,z}$ are integers (positive, negative or zero). The notation $\alpha_{\varepsilon}(\mathbf{k}, t)$ is replaced by

$$\alpha_{\varepsilon}(k,t) \to \alpha_{k\varepsilon}(t) \,. \tag{1.104}$$

One can use even shorter notations

$$\alpha_{k_i\varepsilon_i} \to \alpha_i \,, \tag{1.105}$$

where the index *i* labels mode $k_i \varepsilon_i$.

As a recapitulation of this section, we give now the expansion of H_{trans} , P_{trans} , A_{\perp} , E_{\perp} , B in α_i and α_i^*

$$H_{\text{trans}} = \sum_{i} \frac{1}{2} \hbar \omega_i (\alpha_i^* \alpha_i + \alpha_i \alpha_i^*), \qquad (1.106)$$

$$\boldsymbol{P}_{\text{trans}} = \sum_{i} \frac{1}{2} \hbar \boldsymbol{k}_{i} (\boldsymbol{\alpha}_{i}^{*} \boldsymbol{\alpha}_{i} + \boldsymbol{\alpha}_{i} \boldsymbol{\alpha}_{i}^{*}), \qquad (1.107)$$

$$\mathbf{A}_{\perp} = \sum_{i} \mathcal{A}_{\omega_{i}}(\alpha_{i} \boldsymbol{\varepsilon}_{i} \ \mathrm{e}^{\mathrm{i} \boldsymbol{k}_{i} \cdot \boldsymbol{r}} + \alpha_{i}^{*} \boldsymbol{\varepsilon}_{i} \ \mathrm{e}^{-\mathrm{i} \boldsymbol{k}_{i} \cdot \boldsymbol{r}}), \qquad (1.108)$$

$$\boldsymbol{E}_{\perp} = \mathrm{i} \sum_{i} \mathscr{E}_{\omega_{i}} (\alpha_{i} \boldsymbol{\varepsilon}_{i} \; \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{i} \cdot \boldsymbol{r}} - \alpha_{i}^{*} \boldsymbol{\varepsilon}_{i} \; \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_{i} \cdot \boldsymbol{r}}), \qquad (1.109)$$

$$\boldsymbol{B} = i \sum_{i} \mathcal{B}_{\omega_{i}}(\alpha_{i}\boldsymbol{\kappa}_{i} \times \boldsymbol{\varepsilon}_{i} e^{i\boldsymbol{k}_{i}\cdot\boldsymbol{r}} - \alpha_{i}^{*}\boldsymbol{\kappa}_{i} \times \boldsymbol{\varepsilon}_{i} e^{-i\boldsymbol{k}_{i}\cdot\boldsymbol{r}}), \qquad (1.110)$$

with

$$\mathscr{E}_{\omega_i} = (\hbar \omega_i / 2\varepsilon_0 L^3)^{1/2}, \qquad \mathscr{B}_{\omega_i} = \mathscr{E}_{\omega_i} / c, \qquad \mathscr{A}_{\omega_i} = \mathscr{E}_{\omega_i} / \omega_i. \quad (1.111)$$

In these expressions, Σ_i means sum over modes $k_i \varepsilon_i$. Note also that, when going from Fourier integrals to Fourier series, $1/(2\pi)^{3/2}$ in equations (1.19) is replaced by $1/L^{3/2}$. This explains why \mathscr{E}_{ω_i} contains L^3 instead of $(2\pi)^3$ [compare (1.92) and (1.111)].

Finally, the equation of evolution of α_i is

$$\dot{\alpha}_i + \mathbf{i}\omega_i\alpha_i = \frac{\mathbf{i}}{\sqrt{2\varepsilon_0\hbar\omega_i}}\,\boldsymbol{j}_i\,,\tag{1.112}$$

with

$$j_i = \frac{1}{\sqrt{L^3}} \int d^3 r \, \mathrm{e}^{-\mathrm{i}k_i \cdot r} \, \varepsilon_i \cdot j(r) \,. \tag{1.113}$$

2. Quantum electrodynamics in Coulomb gauge - general framework

2.1. Introduction

After the brief survey of classical electrodynamics presented in the previous chapter, the problem is now to quantize such a theory. We first review in this introduction a few possible approaches to such a problem.

(i) Elementary approach

We have shown in section 1 that the total system electromagnetic field plus particles is formally equivalent to a set of interacting particles and harmonic oscillators. The simplest possible idea for quantizing such a system is therefore to quantize the corresponding particles and oscillators. For the particles, the position r_{α} and the momentum p_{α} of particle α become, as usual, operators (with a commutator equal to $i\hbar$). For the field, the normal variables α_i and α_i^* of oscillator *i* are replaced by the annihilation and creation operators a_i and a_i^+ , well known for a quantum harmonic oscillator (and with a commutator equal to 1). All physical observables which, as shown in section 1, can be expressed in terms of r_{α} , p_{α} , α_i , α_i^* , thus become operators acting in the Hilbert space of the whole system.

Actually, what is lacking in such an approach is a proof of the fact that r_{α} and p_{α} can be considered as canonically conjugate variables, as well as a_i and a_i^+ [more precisely $(a_i + a_i^+)/\sqrt{2}$ and $i(a_i^+ - a_i)/\sqrt{2}$]. An explicit expression of the Hamiltonian in terms of these variables is also needed. We have indeed given in section 1 the expression of the total energy of the system but we have not shown under what conditions such an expression can be considered as the Hamiltonian.

A possible solution to this difficulty is to postulate the expression of the Hamiltonian [actually eq. (1.61) in Coulomb gauge, with H_{trans} replaced by (1.106)] and to check a posteriori that the Heisenberg equations deduced from such an Hamiltonian, and from the basic commutation relations, are correct [Maxwell–Lorentz equations between operators].

(ii) Lagrangian and Hamiltonian approach

Rather than postulating the expression of the Hamiltonian and the basic commutation relations and checking afterwards that they lead to the correct equations of motion, the usual procedure is to start from a Lagrangian formulation of classical electrodynamics.

From the Lagrangian leading to the Maxwell-Lorentz equations (which thus appear as Lagrange equations deduced from a variational principle), it is first possible to define clearly the generalized coordinates of the system and their conjugate momenta (derivatives of the Lagrangian with respect to the generalized velocities). This leads to a clear identification of the pairs of conjugate physical observables, which, in the canonical quantization, will become operators with a commutator equal to $i\hbar$. The expression of the Hamiltonian also follows directly from the expressions of the Lagrangian and from the definition of the conjugate momenta.

It turns out that the standard Lagrangian of classical electrodynamics is a function of the potentials A and U, and not of the fields. Furthermore, \dot{U} does not appear in this Lagrangian, so that U has no conjugate momentum. This raises some difficulties for the quantization. One possible solution is to eliminate U from the Lagrangian and all other redundant variables (such a redundancy is in particular due to the presence of the potentials in the Lagrangian, rather than the fields). In this perspective, the choice of the Coulomb gauge appears to be particularly convenient. A second advantage of the Lagrangian approach is therefore to provide a better understanding of gauge problems in both classical and quantum electrodynamics.

(iii) Full relativistic approach

The two previous approaches treat matter and radiation in a quite asymmetrical way. Radiation is described by a relativistic field, matter by a fixed number of nonrelativistic particles. It is clear however that particles, such as electrons, can become relativistic and that their number can vary (creation of pairs). A satisfactory approach to Q.E.D. must therefore start from a more symmetrical description of radiation and matter where both systems are described by a relativistic field.

Such a program is achieved by starting from a Lagrangian for the coupled Dirac and Maxwell fields considered as classical fields, and giving Maxwell equations in the presence of the Dirac current on the one hand, and Dirac equations in the presence of Maxwell fields on the other hand. Such a theory is then quantized by using commutators for the Maxwell field, anticommutators for the Dirac field. Photons, electrons and positrons appear in such an approach as elementary excitations of the quantized Maxwell and Dirac fields.

In this course, because of lack of time, we will follow mainly the elementary approach described in (i). The Lagrangian and Hamiltonian approach is presented elsewhere [2, 9]. It can be shown that it leads to the same expression for the Hamiltonian in Coulomb gauge, as the one derived here in a heuristic way, and to the same quantum theory. In the last lecture (section 6), we will introduce by heuristic arguments the full Q.E.D. Hamiltonian describing the interacting

quantized Dirac and Maxwell fields and we will consider to what extent the single electron nonrelativistic Q.E.D. Hamiltonian derived from the elementary approach (i) can be related to such a many-particle relativistic Hamiltonian.

In the next subsection (section 2.2), we present the main lines of the quantization in Coulomb gauge. We then consider the problem of the evolution in time in both Heisenberg and Schrödinger pictures (section 2.3). Section 2.4 is devoted to a detailed analysis of the structure of the Hamiltonian. Finally we discuss in section 2.5 the electric dipole approximation for localized systems of charges.

2.2. Quantization in Coulomb gauge - elementary approach

2.2.1. Basic dynamical variables – commutation relations Each particle α is described by two conjugate operators r_{α} and p_{α} satisfying

$$[r_{\alpha i}, r_{\beta j}] = [p_{\alpha i}, p_{\beta j}] = 0, \qquad [r_{\alpha i}, p_{\beta j}] = i\hbar \delta_{\alpha\beta}\delta_{ij}, \qquad i, j = x, y, z.$$

$$(2.1)$$

The $\delta_{\alpha\beta}$ expresses that the variables of two different particles commute.

For quantizing the field, we replace the normal variables α_i and α_i^* by the well-known annihilation and creation operators a_i and a_i^+ satisfying

$$[a_i, a_j] = [a_i^+, a_j^+] = 0, \qquad [a_i, a_j^+] = \delta_{ij}.$$
(2.2)

The δ_{ij} expresses that the variables of two different modes of the transverse field commute.

Remark. We implicitly suppose here that we are working in a Schrödinger picture where operators are time independent. The basic commutation relations (2.1) and (2.2) remain however valid in the Heisenberg picture provided that the two operators appearing in the commutator are taken at the *same time* (equal time commutators).

2.2.2. Space of states

The space of states \mathscr{C} is the tensor product of \mathscr{C}_P , space of states of the particles, by \mathscr{C}_R space of states of the transverse field
$$\mathscr{E} = \mathscr{E}_P \otimes \mathscr{E}_R \tag{2.3}$$

 \mathscr{C}_R itself is the tensor product of the spaces of states \mathscr{C}_i of the various oscillators *i*

$$\mathscr{E}_{R} = \mathscr{E}_{1} \otimes \mathscr{E}_{2} \dots \otimes \mathscr{E}_{i} \otimes \dots$$
(2.4)

A possible orthonormal basis of \mathscr{C}_i is $\{|n_i\rangle\}$ where $n_i = 0, 1, 2, 3...$ labels the energy levels of oscillator *i*. If $\{|s\rangle\}$ is an orthonormal basis of \mathscr{C}_p , we can therefore take for the whole space \mathscr{C} the following basis:

$$\{|s\rangle|n_1\rangle|n_2\rangle\dots|n_i\rangle\dots\}=\{|s;n_1,n_2,\dots,n_i\dots\rangle\}.$$
(2.5)

2.2.3. Expression of the various field observables

The transverse fields $E_{\perp}(r)$, B(r), $A_{\perp}(r)$ in each point r of space become operators obtained by replacing the normal variables α_i and α_i^* of the classical expansions by the corresponding operators a_i and a_i^+ . We get in this way,

$$\boldsymbol{E}_{\perp}(\boldsymbol{r}) = \sum_{i} \mathrm{i} \mathscr{E}_{\omega_{i}}(a_{i}\boldsymbol{\varepsilon}_{i} \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{i}\cdot\boldsymbol{r}} - a_{i}^{+}\boldsymbol{\varepsilon}_{i} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_{i}\cdot\boldsymbol{r}})\,, \qquad (2.6)$$

$$\boldsymbol{B}(\boldsymbol{r}) = \sum_{i} \mathrm{i} \mathcal{B}_{\omega_{i}}(a_{i}\boldsymbol{\kappa}_{i} \times \boldsymbol{\varepsilon}_{i} \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{i} \cdot \boldsymbol{r}} - a_{i}^{\dagger}\boldsymbol{\kappa}_{i} \times \boldsymbol{\varepsilon}_{i} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_{i} \cdot \boldsymbol{r}}), \qquad (2.7)$$

$$\mathbf{A}_{\perp}(\mathbf{r}) = \sum_{i} \mathscr{A}_{\omega_{i}}(a_{i}\varepsilon_{i} e^{i\mathbf{k}_{i}\cdot\mathbf{r}} + a_{i}^{+}\varepsilon_{i} e^{-i\mathbf{k}_{i}\cdot\mathbf{r}}), \qquad (2.8)$$

with

$$\mathscr{E}_{\omega_i} = \left(\frac{\hbar\omega_i}{2\varepsilon_0 L^3}\right)^{1/2}, \qquad \mathscr{B}_{\omega_i} = \mathscr{E}_{\omega_i}/c, \qquad \mathscr{A}_{\omega_i} = \mathscr{E}_{\omega_i}/\omega_i.$$
(2.9)

The total electric field E is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\perp}(\boldsymbol{r}) + \boldsymbol{E}_{\parallel}(\boldsymbol{r}), \qquad (2.10)$$

with

$$\boldsymbol{E}_{\parallel}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\alpha} q_{\alpha} \frac{\boldsymbol{r} - \boldsymbol{r}_{\alpha}}{|\boldsymbol{r} - \boldsymbol{r}_{\alpha}|^3}.$$
(2.11)

The position r_{α} of particle α is now an operator in (2.11).

Since we have kept the order between α_i and α_i^* in the calculations leading to (1.87) and (1.88), it is not necessary to do such calculations again in the quantum case and we can write for the quantum operators H_{trans} and P_{trans}

$$H_{\text{trans}} = \varepsilon_0 \int d^3 r \left[E_{\perp}^2(r) + c^2 B^2(r) \right]$$

= $\sum_i \frac{\hbar \omega_i}{2} \left[a_i^+ a_i + a_i a_i^+ \right] = \sum_i \hbar \omega_i \left[a_i^+ a_i + \frac{1}{2} \right],$ (2.12)

[we have used (2.2) to replace $a_i a_i^+$ by $a_i^+ a_i + 1$]

$$\boldsymbol{P}_{\text{trans}} = \varepsilon_0 \int d^3 \boldsymbol{r} \, \boldsymbol{E}_{\perp}(\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r})$$
$$= \sum_i \frac{\hbar \boldsymbol{k}_i}{2} \left[a_i^+ a_i + a_i a_i^+ \right] = \sum_i \hbar \boldsymbol{k}_i a_i^+ a_i$$
(2.13)

[we have used $\sum_i \hbar k_i/2 = 0$].

Since we are in Coulomb gauge, we can finally write

$$\boldsymbol{A}_{\parallel} = \boldsymbol{0} , \qquad (2.14)$$

$$U(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\alpha} \frac{q_{\alpha}}{|\mathbf{r} - \mathbf{r}_{\alpha}|},$$
(2.15)

U is simply the electrostatic potential of the charge distribution

Remark. From the expansions of the various field observables in a_i and a_i^+ [(2.6) to (2.8)], and from the basic commutation relations (2.2), one can derive the following field commutators:

$$[A_{\perp i}(\mathbf{r}), A_{\perp j}(\mathbf{r}')] = 0, \qquad (2.16)$$

$$[A_{\perp i}(\mathbf{r}), E_{\perp j}(\mathbf{r}')] = \frac{1}{\varepsilon_0} \frac{\hbar}{i} \delta^{\perp}_{ij}(\mathbf{r} - \mathbf{r}'), \qquad (2.17)$$

where i, j = x, y, z and δ_{ij}^{\perp} is the "transverse delta function" defined in (1.35),

$$[E_{\perp x}(\mathbf{r}), B_{y}(\mathbf{r}')] = \frac{1}{\varepsilon_{0}} \frac{\hbar}{i} \frac{\partial}{\partial z} \,\delta(\mathbf{r} - \mathbf{r}') \,. \tag{2.18}$$

Since $E_{\parallel}(\mathbf{r})$ can be reexpressed in terms of the r_{α} 's [see eq. (2.11)] which commute with a_i and a_i^+ , eqs. (2.17) and (2.18) remain valid if E_{\perp} is replaced by the total electric field $E(\mathbf{r})$. All these field commutators are taken in the Schrödinger picture. They also remain valid as equal time field commutators in the Heisenberg picture (see the remark at the end of section 2.2.1)

2.2.4. Hamiltonian and total momentum in Coulomb gauge We postulate for the Hamiltonian of the total system the following expression in Coulomb gauge:

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} |[\mathbf{P}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{r}_{\alpha})]^{2} + \sum_{\alpha} \varepsilon_{\text{Coul}}^{\alpha} + \frac{1}{8\pi\varepsilon_{0}} \sum_{\alpha\neq\beta} \frac{q_{\alpha}q_{\beta}}{|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|} + \sum_{i} \hbar\omega_{i}[a_{i}^{+}a_{i} + \frac{1}{2}].$$
(2.19)

The choice of such an expression is actually suggested by the form (1.61) of the classical energy of the total system. Taking the Hamiltonian (2.19) and using the basic commutation rules of section 2.2.1, we will demonstrate in section 2.3 that $m_{\alpha}\dot{r}_{\alpha}$ and p_{α} are linked by relation (1.59). Such a relation therefore appears here as an equation of motion (as well as the relation $E_{\perp} = -\dot{A}_{\perp}$ between E_{\perp} and A_{\perp}). We will also show in section 2.3 that the basic equations of classical electrodynamics (Maxwell-Lorentz equations) remain valid between operators and can be deduced as Heisenberg equations from the Hamiltonian (2.19). This will validate the elementary approach followed here and the choice (2.19) made for H.

Finally from (1.60) and (2.13), we get for the total momentum in Coulomb gauge

$$\boldsymbol{P} = \sum_{\alpha} \boldsymbol{p}_{\alpha} + \sum_{i} \hbar \boldsymbol{k}_{i} a_{i}^{\dagger} a_{i} \,. \tag{2.20}$$

2.3. Evolution in time

2.3.1. Heisenberg picture; the quantum Maxwell-Lorentz equations In the Heisenberg picture, the state vector $|\psi\rangle$ is time independent and the various observable G(t) evolve according to Heisenberg equations

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) = \frac{1}{\mathrm{i}\hbar} \left[G(t), H(t)\right],\tag{2.21}$$

where H(t) is the Hamiltonian

Heisenberg equations for the particles. Consider first the Heisenberg equation for r_{α}

$$\dot{\boldsymbol{r}}_{\alpha} = \frac{1}{\mathrm{i}\hbar} [\boldsymbol{r}_{\alpha}, \boldsymbol{H}] = \frac{1}{\mathrm{i}\hbar} \left[\boldsymbol{r}_{\alpha}, \frac{1}{2m_{\alpha}} (\boldsymbol{p}_{\alpha} - q_{\alpha}\boldsymbol{A}_{\perp}(\boldsymbol{r}_{\alpha}))^{2} \right]$$
$$= \frac{1}{m_{\alpha}} [\boldsymbol{p}_{\alpha} - q_{\alpha}\boldsymbol{A}_{\perp}(\boldsymbol{r}_{\alpha})] .$$
(2.22)

Such an equation is nothing but the well-known relation (1.59) between the mechanical momentum $\pi_{\alpha} = m_{\alpha}\dot{r}_{\alpha}$ and the canonical momentum p_{α} .

For the following calculations, it will be useful to evaluate the commutator between the two components $\pi_{\alpha j}$ and $\pi_{\alpha l}$ of π_{α} (j, l = x, y, z)

$$[\pi_{\alpha j}, \pi_{\alpha l}] = -q_{\alpha}[p_{\alpha j}, A_{\perp l}(\boldsymbol{r}_{\alpha})] - q_{\alpha}[A_{\perp j}(\boldsymbol{r}_{\alpha}), p_{\alpha l}]$$

= $i\hbar q_{\alpha}[\partial_{j}A_{\perp l}(\boldsymbol{r}_{\alpha}) - \partial_{l}A_{\perp j}(\boldsymbol{r}_{\alpha})]$
= $i\hbar q_{\alpha}\sum_{k} \varepsilon_{jlk}B_{k}(\boldsymbol{r}_{\alpha}).$ (2.23)

We consider now the Heisenberg equation for $\pi_{\alpha j}$,

$$\dot{\pi}_{\alpha j} = m_{\alpha} \ddot{r}_{\alpha j} = \frac{1}{\mathrm{i}\hbar} \left[\pi_{\alpha j}, H \right], \qquad (2.24)$$

and calculate the contributions of the four terms appearing in the expression (2.19) of H. The contribution of the first term of H (kinetic energy) is, according to (2.23),

$$\frac{1}{\mathrm{i}\hbar} \left[\left. \pi_{\alpha j}, \sum_{l} \left. \pi_{\alpha l}^{2} \right/ 2m_{\alpha} \right] = \frac{1}{2\mathrm{i}\hbar m_{\alpha}} \sum_{l} \left\{ \left[\pi_{\alpha j}, \pi_{\alpha l} \right] \pi_{\alpha l} + \pi_{\alpha l} \left[\pi_{\alpha j}, \pi_{\alpha l} \right] \right\}$$
$$= \frac{q_{\alpha}}{2m_{\alpha}} \sum_{k} \sum_{l} \varepsilon_{j l k} \left\{ \pi_{\alpha l} B_{k}(\mathbf{r}_{\alpha}) + B_{k}(\mathbf{r}_{\alpha}) \pi_{\alpha l} \right\}, \qquad (2.25)$$

and can be considered as the *j*-component of the symmetrized Lorentz magnetic force

$$\frac{1}{2}q_{\alpha}\{\boldsymbol{r}_{\alpha}\times\boldsymbol{B}(\boldsymbol{r}_{\alpha})-\boldsymbol{B}(\boldsymbol{r}_{\alpha})\times\dot{\boldsymbol{r}}_{\alpha}\}.$$
(2.26)

The second and third terms of H (Coulomb energy) give

$$\frac{1}{\mathrm{i}\hbar} \left[\pi_{\alpha j}, V_{\mathrm{Coul}} \right] = \frac{1}{\mathrm{i}\hbar} \left[p_{\alpha j}, V_{\mathrm{Coul}} \right] = -\frac{\partial}{\partial r_{\alpha j}} V_{\mathrm{Coul}} \,, \tag{2.27}$$

i.e., the *j*-component of the longitudinal electric force $q_{\alpha}E_{\parallel}(r_{\alpha})$. Finally, we have for the last term of *H* (energy of the transverse field)

$$\frac{1}{\mathrm{i}\hbar} \left[\pi_{\alpha j}, \sum_{i} \hbar \omega_{i} (a_{i}^{+} a_{i}^{+} + \frac{1}{2}) \right] = \mathrm{i} q_{\alpha} \sum_{i} \omega_{i} \left[A_{\perp j}(\mathbf{r}_{\alpha}), a_{i}^{+} a_{i} \right].$$
(2.28)

Using $[a_i, a_i^{\dagger} a_i] = a_i$, and the expansions (2.6) and (2.8), one can transform (2.28) into the *j*-component of the transverse electric force $q_{\alpha}E_{\perp}(\mathbf{r}_{\alpha})$.

Combining all the previous results, we finally get

$$m_{\alpha}\ddot{r}_{\alpha} = q_{\alpha}E(r_{\alpha}) + \frac{1}{2}q_{\alpha}[\dot{r}_{\alpha} \times B(r_{\alpha}) - B(r_{\alpha}) \times \dot{r}_{\alpha}], \qquad (2.29)$$

where E is the total electric field. Equation (2.29) is the quantum form of the Newton-Lorentz equation.

Heisenberg equations for the fields. The same linear relations exist between the classical transverse fields and $\{\alpha_i, \alpha_i^*\}$ on the one hand, the quantum transverse fields and $\{a_i, a_i^*\}$ on the other hand. If we show that $\dot{\alpha}_i$ and \dot{a}_i satisfy similar equations, this will prove that all field equations are the same in classical and quantum theories. Instead of writing Heisenberg equations for E_{\perp} , B, A_{\perp} , it is therefore simpler to consider such an equation for a_i ,

$$\dot{a}_i = \frac{1}{\mathrm{i}\hbar} \left[a_i, H \right]. \tag{2.30}$$

As for π_{α} , we calculate now the contributions of the four terms of H. The last term gives $-i\omega_i a_i$. The second and third terms of H commute with a_i and do not contribute to \dot{a}_i . Finally, the first term gives

$$\frac{1}{i\hbar} \left[a_{i}, \sum_{\alpha} \frac{\pi_{\alpha}^{2}}{2m_{\alpha}} \right] = \sum_{\alpha} \frac{1}{2i\hbar m_{\alpha}} \left\{ \pi_{\alpha} \cdot \frac{\partial \pi_{\alpha}}{\partial a_{i}^{+}} + \frac{\partial \pi_{\alpha}}{\partial a_{i}^{+}} \cdot \pi_{\alpha} \right\}$$
$$= \sum_{\alpha} \frac{iq_{\alpha}}{2\hbar} \mathscr{A}_{\omega_{i}} \varepsilon_{i} \left[\frac{\pi_{\alpha}}{m_{\alpha}} \exp(-ik_{i} \cdot r_{\alpha}) + \exp(-ik_{i} \cdot r_{\alpha}) \frac{\pi_{\alpha}}{m_{\alpha}} \right].$$
(2.31)

We have used $[a_i, f(a_i^+)] = \partial f / \partial a_i^+$ which follows from (2.2). Introducing the symmetrized current

$$j(\mathbf{r}) = \frac{1}{2} \sum_{\alpha} q_{\alpha} [\dot{\mathbf{r}}_{\alpha} \delta(\mathbf{r} - \mathbf{r}_{\alpha}) + \delta(\mathbf{r} - \mathbf{r}_{\alpha}) \dot{\mathbf{r}}_{\alpha}], \qquad (2.32)$$

we transform (2.31) into

$$\frac{\mathrm{i}}{(2\varepsilon_0 \hbar \omega_i)^{1/2}} j_i, \qquad (2.33)$$

where

$$x_i = \frac{1}{\sqrt{L^3}} \int d^3r \ e^{-ik_i \cdot r} \ \varepsilon_i \cdot j(r)$$
(2.34)

is the Fourier component of j. Finally, we get

$$\dot{a}_i + \mathrm{i}\omega_i a_i = \frac{\mathrm{i}}{(2\varepsilon_0 \hbar \omega_i)^{1/2}} j_i.$$
(2.35)

The quantum equation of motion of a_i in (2.35) has therefore exactly the same form as the classical equation of motion of α_i [see eq. (1.112)]. We conclude that Maxwell equations remain valid between field operators.

All basic equations of classical electrodynamics can therefore be extended to Q.E.D. (with a proper symmetrization of the products of non-commuting Hermitian operators, such as the magnetic Lorentz force, or the charge current).

Advantages of the Heisenberg picture. A first advantage of the Heisenberg picture is that it provides a convenient framework for a discussion of the analogies and differences between classical and quantum theories. It leads to similar equations of motion, but for operators instead of *c*-numbers.

A second advantage of such a picture is that it gives the possibility of defining "2-time averages", i.e., the average value in the time independent state vector $|\psi\rangle$ of products of two operators F(t) and G(t') taken at two different times

$$\langle \psi | F(t) G(t') | \psi \rangle$$
. (2.36)

Important examples of 2-time averages are statistical functions such as correlation functions or linear susceptibilities which can be introduced for a quantum system and which respectively describe the dynamics of the fluctuations taking place in such a system, or the linear response of the system to small perturbations. We will consider these statistical functions in more detail in section 4, in connection with a discussion of the physical mechanisms responsible for radiative corrections.

2.3.2. Schrödinger picture

In such a picture, the observables G are time independent and the state vector $|\psi(t)\rangle$ evolves according to Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle.$$
(2.37)

If $|\psi(t)\rangle$ is expanded on the orthonormal basis (2.5),

$$|\psi(t)\rangle = \sum_{sn_1n_2...} C_{sn_1n_2}...(t)|s; n_1, n_2, ...\rangle,$$
 (2.38)

and if the expansion (2.38) is introduced in (2.37), one gets a set of linear differential equations for the coefficients $C_{sn,n_2}...(t)$.

The Schrödinger picture is very convenient for introducing transition amplitudes

$$\langle s'; n'_1, n'_2, \ldots | U(t) | s; n_1, n_2, \ldots \rangle,$$
 (2.39)

where

$$U(t) = e^{-iHt/\hbar}$$
(2.40)

is the evolution operator. From a physical point of view, (2.39)

represents the amplitude that the combined system field plus particles, starting from the initial state $|s; n_1, n_2, \ldots\rangle$, ends up, after a time *t*, in the final state $|s'; n'_1, n'_2, \ldots\rangle$. Perturbation techniques are usually used for calculating such amplitudes, which can be introduced for various important processes where photons are emitted, absorbed or scattered by systems of charged particles.

2.4. Structure of the Hamiltonian

2.4.1. Hamiltonian in the presence of static fields

We first slightly generalize the expressions (2.19) of H to situations where an external static field, described by the potentials A_0 , U_0 , is applied to the particles (such a static field is not considered as a dynamical system and the values of A_0 and U_0 in a given point r are fixed *c*-numbers). We must now take

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{r}_{\alpha}) - q_{\alpha} \mathbf{A}_{0}(\mathbf{r}_{\alpha})]^{2} + V_{\text{Coul}} + \sum_{\alpha} q_{\alpha} U_{0}(\mathbf{r}_{\alpha}) + \sum_{i} \hbar \omega_{i}(\alpha_{i}^{+}a_{i} + \frac{1}{2}). \qquad (2.41)$$

The justification for such an expression is that Heisenberg equations for the particles give the Newton-Lorentz equations in the fields $E + E_0$ and $B + B_0$, where E and B are the quantum field operators already introduced, and where $E_0 = -\nabla U_0$ and $B_0 = \nabla \times A_0$ are the static electric and magnetic fields.

Note that the gauge used for A_0 , U_0 is not necessarily the Coulomb gauge.

We will frequently use the notations

$$\boldsymbol{\pi}_{\alpha}^{0} = \boldsymbol{p}_{\alpha} - \boldsymbol{q}_{\alpha} \boldsymbol{A}_{0}(\boldsymbol{r}_{\alpha}), \qquad (2.42a)$$

$$\boldsymbol{\pi}_{\alpha} = \boldsymbol{p}_{\alpha} - \boldsymbol{q}_{\alpha} \boldsymbol{A}_{0}(\boldsymbol{r}_{\alpha}) - \boldsymbol{q}_{\alpha} \boldsymbol{A}_{\perp}(\boldsymbol{r}_{\alpha}) \,. \tag{2.42b}$$

2.4.2. Hamiltonian of the particles – Hamiltonian of the radiation field Interaction Hamiltonian

It is interesting to divide the Hamiltonian H given in (2.19) into three parts

$$H = H_{\rm P} + H_{\rm R} + H_{\rm I} \,, \tag{2.43}$$

where $H_{\rm P}$ only depends on the particle observables r_{α} and p_{α} (Hamiltonian of the particles), $H_{\rm R}$ only depends on the field operators a_i and a_i^+ (Hamiltonian of the radiation field), $H_{\rm I}$ depends on both r_{α} , p_{α} and a_i , a_i^+ (interaction Hamiltonian). From (2.19) one derives

$$H_{\rm P} = \sum_{\alpha} \frac{p_{\alpha}^2}{2m_{\alpha}} + V_{\rm Coul} , \qquad (2.44)$$

$$H_{\rm R} = \sum \hbar \omega_i (a_i^+ a_i + \frac{1}{2}), \qquad (2.45)$$

$$H_{\rm I} = H_{\rm 11} + H_{\rm 12} \,, \tag{2.46}$$

where H_{11} is linear in the fields

$$H_{\rm II} = -\sum_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} p_{\alpha} \cdot A_{\perp}(r_{\alpha}) \tag{2.47}$$

[we have used the transversality of A_{\perp} which implies $p_{\alpha} \cdot A_{\perp}(r_{\alpha}) = A_{\perp}(r_{\alpha}) \cdot p_{\alpha}$], and H_{12} quadratic

$$H_{12} = \sum_{\alpha} \frac{q_{\alpha}^2}{2m_{\alpha}} [A_{\perp}(\boldsymbol{r}_{\alpha})]^2 .$$
(2.48)

In the presence of static fields, one must start from (2.41) and $H_{\rm P}$ and $H_{\rm II}$ are replaced by

$$H_{\rm P} = \sum_{\alpha} \frac{\pi_{\alpha}^{02}}{2m_{\alpha}} + V_{\rm Coul} + \sum_{\alpha} q_{\alpha} U_0(\mathbf{r}_{\alpha}), \qquad (2.49)$$

$$H_{\rm II} = -\sum_{\alpha} \frac{g_{\alpha}}{m_{\alpha}} \,\boldsymbol{\pi}_{\alpha}^{0} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r}_{\alpha}) \,, \tag{2.50}$$

 π^0_{α} , defined in (2.42a), is a pure atomic operator.

Up to now, we have considered charged particles without internal degrees of freedom. We can remove this restriction by adding to be observables r_{α} and p_{α} of particle α the spin operator S_{α} . Because of the magnetic moment associated with such a spin,

$$M_{S}^{\alpha} = g_{\alpha} \frac{q_{\alpha}}{2m_{\alpha}} S_{\alpha} , \qquad (2.51)$$

where g is the g-factor, new terms must be added to the Hamiltonian.

The magnetic coupling of S_{α} with the static field B_0

$$-\sum_{\alpha} \boldsymbol{M}_{S}^{\alpha} \cdot \boldsymbol{B}_{0}(\boldsymbol{r}_{\alpha})$$
(2.52)

must be added to $H_{\rm P}$, and a new interaction Hamiltonian $H'_{\rm 11}$, linear in the field operator **B**, appears

$$H'_{11} = -\sum_{\alpha} \boldsymbol{M}_{\boldsymbol{S}}^{\alpha} \cdot \boldsymbol{B}(\boldsymbol{r}_{\alpha}) \,. \tag{2.53}$$

Note that, since $\mathscr{B}_{\omega} = \mathscr{E}_{\omega}/c$ [see (2.9)], the coupling of the spin S_{α} with **B** is of order 1/c in comparison with the electric coupling of the charge q_{α} .

All these spin-dependent terms have been introduced here in a heuristic way. For electrons, they can be derived directly from Dirac equation (see section 6).

2.4.3. Relative magnitude of the various interaction terms for bound particles

Consider first the ratio H_{12}/H_{11} . The order of magnitude of A and p is taken equal to the square root of their mean-square value in the state considered.

$$\frac{H_{12}}{H_{11}} \simeq \frac{q^2 A^2 / m}{q A p / m} = \frac{q A p / m}{p^2 / m} \simeq \frac{H_{11}}{H_{\rm P}}.$$
(2.54)

If $H_{II} \ll H_P$, which is always the case for small radiation intensities (since $H_{II} \sim A$), H_{I2} is smaller than H_{I1} . At very high intensities (when the incident field becomes of the order of the intra-atomic field), H_{I2} can become larger than H_{II} .

Note however that in some scattering processes, such as Rayleigh scattering, H_{12} can play a role at order 1 in perturbation theory (H_{12} can destroy the incoming photon and create the outgoing one, so that a single matrix element of H_{12} is needed), whereas H_{11} only appears at order 2 (H_{11} can only create or destroy one photon at a time, so that two matrix elements of H_{11} are needed). Even if H_{12} is smaller than H_{11} , the contribution of H_{12} to order 1 can be of the same order as the one of H_{11} at order 2.

Remark. For a free particle, or for a weakly bound particle, H_{12} (more precisely the diagonal elements of H_{12}) can be interpreted as a kinetic

energy of vibration of the electron in the radiation field. We will come back on this physical picture in section 4.

We consider now the ratio H'_{11}/H_{11} . Using (2.51), (2.53) and the fact that $B \sim kA$ (since $B = \nabla \times A$), we get

$$\frac{H'_{\rm II}}{H_{\rm II}} \simeq \frac{q\hbar B/m}{qAp/m} \simeq \frac{\hbar kA}{pA} = \frac{\hbar k}{p}, \qquad (2.55)$$

i.e., the ratio between the photon momentum $\hbar k$ and the particle momentum p. For low-energy radiation, i.e., for optical or microwave radiation, such a ratio is much smaller than 1.

2.4.4. Selection rules

From the basic commutation relations (2.1), (2.2), one can show that, in absence of static fields (or in presence of static fields invariant in a space translation)

$$[P, H] = 0 = [P, H_{\rm I}]$$
(2.56)

where P is the total momentum given in (2.20).

It follows that the total momentum is, as in classical theory, a constant of motion (in Heisenberg picture)

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{P}(t) = \mathbf{0} \ . \tag{2.57}$$

From (2.56), it also follows that $H_{\rm I}$ can only induce transitions between states of the combined field plus particles system having the same total momentum. As a consequence of this momentum conservation (combined with energy conservation) one can derive wellknown effects in emission and absorption processes, such as the Doppler effect or the recoil shift.

2.4.5. Introduction of a cut off

The Hamiltonians (2.19) or (2.41) are of course only valid for slow (nonrelativistic) particles. They cannot therefore describe correctly the interaction of the particle with relativistic modes of the radiation field (for which $\hbar \omega \ge mc^2$), since such interactions would give high velocities

to the particles, or create new particles (such as electron-positron pairs).

We prefer therefore to eliminate the coupling with relativistic modes, which is present in the interaction Hamiltonian H_{I} , and which is certainly inaccurate. Such an elimination is achieved by introducing a cut off in the expansions of the field operators appearing in H_{I} . All summations over k_i are limited to

$$|\mathbf{k}_i| = k_i \le k_c \,, \tag{2.58}$$

where

$$\hbar c k_{\rm c} = \hbar \omega_{\rm c} \ll m_{\alpha} c^2 \,. \tag{2.59}$$

The cut-off frequency ω_c is however chosen large compared to the characteristic frequencies ω_0 of the particles

$$\omega_0 \ll \omega_c \ll m_a c^2 / \hbar , \qquad (2.60)$$

so that a broad spectral range is kept in H_1 for describing the important electromagnetic interactions of the particles, in particular the resonant interactions which give rise to real emission and absorption processes. Actually, with the cut off (2.58), we renounce describing the effect on the particles of virtual emission and reabsorption of high-frequency photons.*

Remarks

(i) For consistency, the same cut-off k_c must be also introduced in the interaction with the longitudinal electric field which gives rise to the Coulomb interaction term. If the longitudinal electric field created by particle α is expanded into longitudinal modes, and if a cut-off k_c is introduced in this expansion, the energy ε_{α} of the Coulomb field associated with particle α becomes finite and equal to

$$\varepsilon_{\text{Coul}}^{\alpha} = q_{\alpha}^2 k_c / 4\pi^2 \varepsilon_0 \,. \tag{2.61}$$

The electrostatic interaction between different particles is also

^{*} In the subsequent seminar [1], the effect of such virtual processes on the electron-spin anomaly g-2 is evaluated to order 1 in the fine structure constant α . The calculations are based on the full relativistic Q.E.D. Hamiltonian.



Fig. 2. Some important photon energies and wavelengths for the hydrogen atom (figure not at scale).

modified, but such a modification can be neglected if the average distance between two different particles is larger than $1/k_c$ (a cut off at k_c in k-space is equivalent to an averaging over a length $1/k_c$ in r-space). For $\hbar\omega_c = mc^2$ (where m is the electron mass), the characteristic length is \hbar/mc , i.e., the Compton wavelength which is much smaller than the atomic dimensions.

(ii) The modes which are selected by (2.58) have their wave vectors inside a sphere in k-space. For a different observer, moving with a velocity v, these modes will be Doppler shifted and will no longer form a sphere. It follows that the cut off introduced in this section is not relativistically invariant.

(iii) We can consider some important photon energies $\hbar\omega$ and the corresponding wavelengths $\lambda = \hbar c/\hbar \omega$ for the simplest atomic system, the hydrogen atom (see fig. 2 which is not at scale). A first important energy is the electron rest mass energy mc^2 , the corresponding wavelength being the Compton wavelength $\lambda_c = \hbar/mc$. The cut-off energy $\hbar\omega_c$ must be smaller than mc^2 . The characteristic atomic energies (noted $\hbar\omega_0$ above) are of the order of the ionization energy of the atom, i.e., of the order of $\alpha^2 mc^2$, where α is the fine structure constant. Since $\alpha = 1/137$, such an energy is at least four orders of magnitude smaller than mc^2 . It is therefore easy to find a cut-off frequency ω_c fulfilling condition (2.60). Finally, we can note that the Bohr radius a_0 is equal to π_c/α , so that a wavelength of the order of a_0 corresponds to photon energies of the order of αmc^2 , much smaller than mc^2 , but also much larger than $\alpha^2 mc^2$ (see fig. 2). For an atom such as hydrogen, the energy range between 0 and αmc^2 , which contains all important atomic frequencies, corresponds therefore to wavelengths large compared to atomic dimensions. This leads us to introduce now the electric dipole approximation.

2.5. Electric dipole approximation

When the charged particles interacting with the radiation field form

one (or several) localized systems such as atoms or molecules, with spatial extensions a smaller than the wavelengths λ of the field, one can neglect the spatial variation of the field over each localized system of charges. Such a long wavelength approximation allows one to transform the Hamiltonian H studied in the previous section into a more suggestive one involving electric dipole couplings. We analyze in this section such an electric dipole approximation.

2.5.1. A few results concerning dipole moments

For the following discussion, it will be useful to review first a few simple results of electromagnetism.

Charge density for a system of localized charges. Consider a set of charges q_{α} , with positions r_{α} localized near a point \mathbf{R}_{A} , in a volume with linear spatial extension a

$$|\mathbf{r}_{\alpha} - \mathbf{R}_{\mathsf{A}}| \lesssim a \quad \forall \alpha \,. \tag{2.62}$$

The corresponding charge density in r-space and k-space can be written,

$$\rho_{\rm A}(\boldsymbol{r}) = \sum q_{\alpha} \delta(\boldsymbol{r} - \boldsymbol{r}_{\alpha}), \qquad (2.63)$$

$$\rho_{\rm A}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{\alpha} q_{\alpha} \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_{\alpha}} \,. \tag{2.64}$$

The localization assumption can be used to transform (2.63) or (2.64). It is simpler to work in k-space. For all wave vectors k such that $ka \ll 1$, we can write

$$(2\pi)^{3/2} \rho_{A}(k) = \sum_{\alpha} q_{\alpha} e^{-ik \cdot R_{A}} e^{ik(R_{A} - r_{\alpha})}$$
$$= \sum_{\alpha} q_{\alpha} e^{-ik \cdot R_{A}} \left[1 + ik(R_{A} - r_{\alpha}) + \ldots \right].$$
(2.65)

Introducing the "total charge" Q_A ,

$$Q_{\rm A} = \sum_{\alpha} q_{\alpha} \,, \tag{2.66}$$

and the "dipole moment" d_A with respect to R_A ,

$$\boldsymbol{d}_{\mathrm{A}} = \sum_{\alpha} q_{\alpha} (\boldsymbol{r}_{\alpha} - \boldsymbol{R}_{\mathrm{A}}) , \qquad (2.67)$$

we get

$$(2\pi)^{3/2}\rho_{\mathrm{A}}(\mathbf{k}) = Q_{\mathrm{A}} \,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{R}_{\mathrm{A}}} - \mathrm{i}\mathbf{k}\cdot\mathbf{d}_{\mathrm{A}} \,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{R}_{\mathrm{A}}} + \dots \qquad (2.68)$$

The Fourier transform of (2.68) then gives

$$\rho_{\rm A}(\boldsymbol{r}) = Q_{\rm A}\delta(\boldsymbol{r} - \boldsymbol{R}_{\rm A}) - \nabla \cdot \boldsymbol{d}_{\rm A}\delta(\boldsymbol{r} - \boldsymbol{R}_{\rm A}) + \dots \qquad (2.69)$$

Such a simple result means that the charge distribution of a localized system of charges can be expanded into a total charge Q_A localized in R_A , a polarization density $P_A(r)$ corresponding to a dipole moment d_A localized in R_A ,

$$\boldsymbol{P}_{\mathrm{A}}(\boldsymbol{r}) = \boldsymbol{d}_{\mathrm{A}}\delta(\boldsymbol{r} - \boldsymbol{R}_{\mathrm{A}}), \qquad (2.70)$$

and so on . . . ,

$$\rho_{\rm A}(\boldsymbol{r}) = Q_{\rm A}\delta(\boldsymbol{r} - \boldsymbol{R}_{\rm A}) - \nabla \cdot \boldsymbol{P}_{\rm A}(\boldsymbol{r}) + \dots \qquad (2.71)$$

For the following calculations, it is useful to introduce the Fourier transform of (2.70)

$$\mathscr{P}_{A}(k) = \left(\frac{1}{2\pi}\right)^{3/2} d_{A} e^{-ik \cdot R_{A}}, \qquad (2.72)$$

which can be used to transform (2.68) into

$$\rho_{\mathsf{A}}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^{3/2} Q_{\mathsf{A}} \,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{R}_{\mathsf{A}}} - \mathrm{i}\mathbf{k}\cdot\mathscr{P}_{\mathsf{A}}(\mathbf{k}) + \dots \qquad (2.73)$$

Remark. Strictly speaking, expansions (2.65) and (2.68) have a meaning only if $ka \ll 1$. Consequently, the Fourier transform of (2.68), leading

* The definition (2.67) of the dipole moment d_A depends on the point of reference R_A , except if the total charge Q_A is equal to zero.

to (2.69), should be limited to the corresponding values of k ($ka \ll 1$). It follows that the $\delta(r - r_A)$ functions appearing in (2.69), (2.70) and (2.71) are not true delta functions but have actually a width of the order of a. They can be, however, considered as true delta functions for physical effects involving characteristic lengths much larger than a (for example, the interaction of the atomic system with a radiation with wavelength $\lambda \gg a$).

All previous results can be generalized to several distinct systems of charges \mathscr{G}_A , \mathscr{G}_B ... localized near \mathbf{R}_A , \mathbf{R}_B ... with total charges Q_A , Q_B ... and dipole moment \mathbf{d}_A , \mathbf{d}_B ... (with respect to \mathbf{R}_A , \mathbf{R}_B ...). One can show that

$$\rho(\mathbf{r}) = \rho_{\rm A}(\mathbf{r}) + \rho_{\rm B}(\mathbf{r}) + \dots, \qquad (2.74)$$

$$\rho(\mathbf{k}) = \rho_{\mathrm{A}}(\mathbf{k}) + \rho_{\mathrm{B}}(\mathbf{k}) + \dots, \qquad (2.75)$$

where $\rho_A(\mathbf{r})$ and $\rho_A(\mathbf{k})$ are given by (2.71) and (2.73), $\rho_B(\mathbf{r})$ and $\rho_B(\mathbf{k})$ by similar expressions with A replaced by B.

Electric induction in the presence of globally neutral localized systems of charges. From now on, we suppose that the various systems of localized charges \mathcal{G}_A , \mathcal{G}_B ... are globally neutral

$$Q_{\rm A} = 0 = Q_{\rm B} = \dots,$$
 (2.76)

and we neglect all higher-order terms after the dipole one in (2.71) and (2.73).

From the electric field E and the total polarization density

$$\boldsymbol{P}(\boldsymbol{r}) = \boldsymbol{P}_{\mathrm{A}}(\boldsymbol{r}) + \boldsymbol{P}_{\mathrm{B}}(\boldsymbol{r}) + \dots$$
(2.77)

one can introduce the electric induction D

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r}) \,. \tag{2.78}$$

From Maxwell equation (1.1a), $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$, and from eq. (2.71) with $Q_A = 0$ and the corresponding equation for $\mathcal{S}_B \dots$, one deduces that

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = 0 , \qquad (2.79)$$

which means that **D** is a pure transverse field, so that we can also write

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}_\perp(\boldsymbol{r}) + \boldsymbol{P}_\perp(\boldsymbol{r}) \,. \tag{2.80}$$

Remarks

(i) Equations (2.79) and (2.80) are actually valid only after an averaging of the fields over a volume large compared to a^3 (see remark of the previous section).

(ii) Since the polarization density P(r) is localized near R_A , R_B ... [see (2.70) and (2.77)], it follows from (2.78) that, outside \mathcal{S}_A , \mathcal{S}_B ... the electric induction coincides with the *total* electric field E(r) [within the multiplicative factor ε_0],

$$D(r) = \varepsilon_0 E(r)$$
 if r is outside $\mathscr{G}_A, \mathscr{G}_B \dots$ (2.81)

Using the equivalent expression (2.80) of **D**, we also deduce from (2.81) that, outside \mathscr{G}_{A} , \mathscr{G}_{B} ..., $P_{\perp}(r)$ coincides with $\varepsilon_{0}E_{\parallel}(r)$.

Electrostatic interaction between two globally neutral localized systems of charges. The electrostatic energy of a charge distribution is given in terms of the charge density $\rho(k)$ by

$$V_{\text{Coul}} = \frac{1}{2\varepsilon_0} \int d^3k \, \frac{\rho^*(\boldsymbol{k})\rho(\boldsymbol{k})}{k^2} \tag{2.82}$$

[see also eq. (1.50) and its Fourier transform (1.51)].

If we replace ρ by $\rho_A + \rho_B + ...$ in (2.82), and if we keep the crossed terms in $\rho_A^* \rho_B$ (or $\rho_A \rho_B^*$), we obtain the electrostatic interaction energy between the two localized systems \mathcal{S}_A and \mathcal{S}_B . Using the expression (2.73) of ρ_A (with $Q_A = 0$), and the corresponding expression for ρ_B , we get

$$V_{\text{Coul}}^{\text{AB}} = \frac{1}{2\varepsilon_0} \int d^3k \, [\boldsymbol{\kappa} \cdot \boldsymbol{\mathscr{P}}_{\text{A}}^*(\boldsymbol{k})] [\boldsymbol{\kappa} \cdot \boldsymbol{\mathscr{P}}_{\text{B}}(\boldsymbol{k})] + \mathbf{A} \rightleftharpoons \mathbf{B}$$
$$= \frac{1}{\varepsilon_0} \int d^3k \, \boldsymbol{\mathscr{P}}_{\text{A}\parallel}^*(\boldsymbol{k}) \cdot \boldsymbol{\mathscr{P}}_{\text{B}\parallel}(\boldsymbol{k}) \,. \tag{2.83}$$

From the reality condition of \mathcal{P}_A and \mathcal{P}_B , one can show that the

 $A \rightleftharpoons B$ term of the first line of (2.83) is equal to the first term.] Using the Parseval–Plancherel identity (1.21), we can also write

$$V_{\text{Coul}}^{\text{AB}} = \frac{1}{\varepsilon_0} \int d^3 \boldsymbol{r} \, \boldsymbol{P}_{\text{A}\parallel}(\boldsymbol{r}) \cdot \boldsymbol{P}_{\text{B}\parallel}(\boldsymbol{r}) \,. \tag{2.84}$$

Actually, since the two dipole moments d_A and d_B are supposed to be localized in two different points of space R_A and R_B , (with $|R_A - R_B| \ge a$), we have

$$\int d^3 \boldsymbol{r} \, \boldsymbol{P}_{\rm A}(\boldsymbol{r}) \cdot \boldsymbol{P}_{\rm B}(\boldsymbol{r}) = \boldsymbol{d}_{\rm A} \cdot \boldsymbol{d}_{\rm B} \int d^3 \boldsymbol{r} \, \delta(\boldsymbol{r} - \boldsymbol{R}_{\rm A}) \, \delta(\boldsymbol{r} - \boldsymbol{R}_{\rm B}) = 0 \,, \qquad (2.85)$$

so that (2.84) can also be written

$$V_{\text{Coul}}^{\text{AB}} = -\frac{1}{\varepsilon_0} \int d^3 \boldsymbol{r} \, \boldsymbol{P}_{\text{A}\perp}(\boldsymbol{r}) \cdot \boldsymbol{P}_{\text{B}\perp}(\boldsymbol{r}) \,. \tag{2.86}$$

The electrostatic energy between two dipole moments is therefore simply related to the scalar product of the corresponding longitudinal or transverse polarization densities.

2.5.2. Long wavelength approximation

Suppose we have two globally neutral localized systems of charges \mathscr{G}_A and \mathscr{G}_B , for example two neutral atoms localized in \mathbf{R}_A and \mathbf{R}_B . The index α labels the particles of \mathscr{G}_A , the index β the particles of \mathscr{G}_B .

The total Hamiltonian H in Coulomb gauge can be written,

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{r}_{\alpha})]^{2} + \sum_{\beta} \frac{1}{2m_{\beta}} [\mathbf{p}_{\beta} - q_{\beta} \mathbf{A}_{\perp}(\mathbf{r}_{\beta})]^{2} + V_{\text{Coul}}^{\text{AA}} + V_{\text{Coul}}^{\text{BB}} + V_{\text{Coul}}^{\text{AB}} + \sum_{i} \hbar \omega_{i} (a_{i}^{+}a_{i} + \frac{1}{2}).$$
(2.87)

The first sum represents the kinetic energy of the particles α of \mathscr{G}_A , the second one the kinetic energy of the particles β of \mathscr{G}_B . The second line of (2.87) is the total Coulomb energy of the system, which has been divided in three terms, V_{Coul}^{AA} which represents the Coulomb energy of \mathscr{G}_A considered alone (sum of the Coulomb self-energies of each parti-

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cle α of \mathscr{G}_A and of the Coulomb interaction energy between different particles of \mathscr{G}_A), $V_{\text{Coul}}^{\text{BB}}$ which has the same meaning for \mathscr{G}_B , $V_{\text{Coul}}^{\text{AB}}$ which represents the Coulomb interaction energy between \mathscr{G}_A and \mathscr{G}_B . Finally, the third line of (2.87) is the energy of the transverse field.

Suppose that the fields irradiating these systems of charges, or radiated by them, have a wavelength λ much larger than the spatial extension *a* of \mathcal{G}_A and \mathcal{G}_B ,

$$\lambda \ge a \,. \tag{2.88}$$

For the corresponding modes of the transverse field, we can write

$$e^{i\mathbf{k}\cdot\mathbf{r}_{a}} \simeq e^{i\mathbf{k}\cdot\mathbf{R}_{A}}, \qquad e^{i\mathbf{k}\cdot\mathbf{r}_{\beta}} \simeq e^{i\mathbf{k}\cdot\mathbf{R}_{B}}.$$
 (2.89)

The long wavelength approximation consists in replacing $A_{\perp}(r_{\alpha})$ by $A_{\perp}(\mathbf{R}_{\rm A})$ and $A_{\perp}(r_{\beta})$ by $A_{\perp}(\mathbf{R}_{\rm B})$ in the first line of (2.87) which thus becomes

$$\sum_{\alpha} \frac{1}{2m_{\alpha}} [\boldsymbol{p}_{\alpha} - q_{\alpha} \boldsymbol{A}_{\perp}(\boldsymbol{R}_{\mathrm{A}})]^{2} + \sum_{\beta} \frac{1}{2m_{\beta}} [\boldsymbol{p}_{\beta} - q_{\beta} \boldsymbol{A}_{\perp}(\boldsymbol{R}_{\mathrm{B}})]^{2}.$$
(2.90)

Remark. Actually, one must not forget that the expansions of $A_{\perp}(r_{\alpha})$ and $A_{\perp}(r_{\beta})$ in plane waves also contain high-frequency modes for which the approximations (2.89) are certainly not valid [even if we introduce a cut-off k_c as in the previous section (2.4.5), k_c is generally larger than 1/a. For example, the energy range between αmc^2 and mc^2 in fig. 2 corresponds to modes which are not relativistic but which have a wavelength shorter than the Bohr radius]. Of course, if \mathcal{S}_A and \mathcal{S}_B are atomic or molecular systems, their typical internal Bohr frequencies ω_0 are much smaller than c/a,

$$\omega_0 \ll c/a \,, \tag{2.91}$$

so that the resonant interactions of \mathscr{S}_A and \mathscr{S}_B take place with modes of the radiation field for which $ka \ll 1$ (see for example the energy range between 0 and αmc^2 on fig. 2). Nonresonant interactions with high-frequency modes cannot however be neglected since they give rise to virtual processes contributing for example to radiative corrections (atoms or molecules can virtually emit and reabsorb photons of any frequency). If we are interested in a precise calculation of such effects, we cannot use the approximate expression (2.90). We must come back to a more precise Hamiltonian, even more precise than the ones considered in this chapter (see section 2.4), since it must also include relativistic effects (see seminar following this course for example [1]).

2.5.3. Unitary transformation ([9] and references in [12])

The simple form of the kinetic energy term (2.90), after the long wavelength approximation, suggests to make a unitary transformation T translating all p_{α} by an amount $q_{\alpha}A_{\perp}(\mathbf{R}_{\rm A})$ and all p_{β} by an amount $q_{\beta}A_{\perp}(\mathbf{R}_{\rm B})$. Such a translation operator can be written

$$T = \exp\left\{-\frac{\mathrm{i}}{\hbar}\left[\sum_{\alpha} q_{\alpha} \boldsymbol{r}_{\alpha} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{R}_{\mathrm{A}}) + \sum_{\beta} q_{\beta} \boldsymbol{r}_{\beta} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{R}_{\mathrm{B}})\right]\right\}$$
$$= \exp\left\{-\frac{\mathrm{i}}{\hbar}\left[\boldsymbol{d}_{\mathrm{A}} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{R}_{\mathrm{A}}) + \boldsymbol{d}_{\mathrm{B}} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{R}_{\mathrm{B}})\right]\right\}.$$
(2.92)

and we have

$$T p_{\alpha} T^{+} = p_{\alpha} + q_{\alpha} A_{\perp} (\mathbf{R}_{A}) ,$$

$$T p_{\beta} T^{+} = p_{\beta} + q_{\beta} A_{\perp} (\mathbf{R}_{B}) ,$$
(2.93)

so that the kinetic energy term (2.90) takes the much simpler form

$$\sum_{\alpha} \frac{1}{2m_{\alpha}} p_{\alpha}^{2} + \sum_{\beta} \frac{1}{2m_{\beta}} p_{\beta}^{2} .$$
 (2.94)

We have used the fact that, because of (2.16), $A_{\perp}(\mathbf{R}_{A})$ and $A_{\perp}(\mathbf{R}_{B})$ commute with T.

Remark. In the derivation of (2.93), we have implicitly supposed that \mathbf{R}_{A} and \mathbf{R}_{B} are two fixed points of space (*c*-numbers commuting with p_{α} and p_{β}). In other words, we neglect the translational motion of \mathcal{S}_{A} and \mathcal{S}_{B} . We could also consider that \mathbf{R}_{A} and \mathbf{R}_{B} are the centers of mass of \mathcal{S}_{A} and \mathcal{S}_{B} . In such a case, \mathbf{R}_{A} and \mathbf{R}_{B} would be functions of the r_{α} 's and r_{β} 's, and new terms would appear in the right side of (2.93), which can be shown to introduce corrections of the same order of magnitude as the magnetic dipole and electric quadrupole couplings [which have been neglected here by stopping the expansions (2.71) and (2.73) after the dipole term].

The unitary operator T given in (2.92) can also be written

$$T = \exp\left[\sum_{i} \left(\lambda_{i}^{*} a_{i} - \lambda_{i} a_{i}^{+}\right)\right], \qquad (2.95)$$

with

$$\lambda_i = \lambda_{iA} + \lambda_{iB} \,, \tag{2.96}$$

$$\lambda_{iA} = \frac{i}{(2\varepsilon_0 \hbar \omega_i L^3)^{1/2}} (\varepsilon_i \cdot d_A) e^{-ik_i \cdot R_A},$$

$$\lambda_{iB} = \frac{i}{(2\varepsilon_0 \hbar \omega_i L^3)^{1/2}} (\varepsilon_i \cdot d_B) e^{-ik_i \cdot R_B}.$$
 (2.97)

It follows that T is also a translation operator for a_i and a_i^+ ,

$$Ta_i T^+ = a_i^+ + \lambda_i^*,$$

$$Ta_i^+ T^+ = a_i^+ + \lambda_i^*.$$
(2.98)

(See for example Glauber's lectures in [13].)

2.5.4. Expression of the observables in the new representation

It is well known that a given physical quantity is generally represented by different mathematical operators in two different representations. More precisely, we can obtain the "new" representation G' of a physical quantity by applying T to its old representation G,

$$G' = TGT^+ , (2.99)$$

corresponding old operator $G \leftarrow$ physical quantity \rightarrow corresponding "new" operator $G' = TGT^+$

A similar result holds for physical states: corresponding "old" ket $|\psi\rangle \leftarrow$ physical state \rightarrow corresponding "new" ket $|\psi'\rangle = T|\psi\rangle$.

We can, for instance, calculate the operators associated with several field quantities in the new representation. For the transverse electric field, its new representation $E'_{\perp}(r) = TE_{\perp}(r)T^+$ can be calculated using (2.98) and (2.72). We get

$$E'_{\perp}(\mathbf{r}) = \mathbf{i} \sum_{i} \mathscr{E}_{\omega_{i}}[(a_{i} + \lambda_{i})\varepsilon_{i} \ e^{\mathbf{i}\mathbf{k}_{i}\cdot\mathbf{r}} - (a_{i}^{+} + \lambda_{i}^{*})\varepsilon_{i} \ e^{-\mathbf{i}\mathbf{k}_{i}\cdot\mathbf{r}}]$$

$$= E_{\perp}(\mathbf{r}) - \sum_{\mathbf{k}_{i}\varepsilon_{i}} \left[\frac{1}{2\varepsilon_{0}L^{3}} \left(\varepsilon_{i} \cdot d_{A}\right) e^{\mathbf{i}\mathbf{k}_{i}\cdot(\mathbf{r}-\mathbf{R}_{A})} + \mathrm{c.c.} \right] + \mathbf{A} \rightleftharpoons \mathbf{B}$$

$$= E_{\perp}(\mathbf{r}) - \frac{1}{\varepsilon_{0}} \mathbf{P}_{\perp}(\mathbf{r}) . \qquad (2.100)$$

We can also evaluate the operator describing the electric induction in the new representation. The electric induction D(r) is equal to $\varepsilon_0 E_{\perp}(r) + P_{\perp}(r)$ [see (2.80)]. Since $P_{\perp}(r)$ commutes with T, we obtain, using (2.100), the new representation $D'(r) = TD(r)T^+$ of the electric induction

$$\mathbf{D}'(\mathbf{r}) = \varepsilon_0 \mathbf{E}_{\perp}(\mathbf{r})$$

= $\varepsilon_0 \sum_i i \left(\frac{\hbar \omega_i}{2\varepsilon_0 L^3}\right)^{1/2} [\varepsilon_i a_i e^{i\mathbf{k}_i \cdot \mathbf{r}} - \varepsilon_i a_i^+ e^{-i\mathbf{k}_i \cdot \mathbf{r}}].$ (2.101)

It therefore appears that the mathematical operator written in (2.6), and noted in the old representation $E_{\perp}(r)$, is now in the new representation associated with the electric induction in r.

Remark. A similar discussion could be done for atomic quantities. For example, the velocity of a particle α is a physical quantity which is represented in the old representation by the operator $[p_{\alpha} - q_{\alpha}A_{\perp}(r_{\alpha})]/m_{\alpha}$, approximated by $[p_{\alpha} - q_{\alpha}A_{\perp}(R_{\Lambda})]/m_{\alpha}$ in the long wavelength limit. Using (2.93) we find that such a velocity is represented in the new representation by

$$\boldsymbol{v}_{\alpha}' = T \frac{1}{m_{\alpha}} \left[\boldsymbol{p}_{\alpha} - \boldsymbol{q}_{\alpha} \boldsymbol{A}_{\perp}(\boldsymbol{R}_{\mathrm{A}}) \right] T^{+} = \frac{\boldsymbol{p}_{\alpha}}{m_{\alpha}} \,. \tag{2.102}$$

This explains the simple form (2.94) of the kinetic energy in the new representation.

2.5.5. Expression of the Hamiltonian in the new representation

We can evaluate the Hamiltonian H' in the new representation by applying to H the same unitary transformation.

The transformation of the kinetic energy term (2.90) has already been done in section 2.5.3 and leads to the simple expression (2.94).

The second line of (2.87) is a pure atomic operator depending only on the r_{α} 's and r_{β} 's and which therefore remains unchanged in the transformation *T*. Finally, the transformation of the third line of (2.87) leads to

$$TH_{R}T^{+} = T \sum_{i} \left[\hbar \omega_{i}(a_{i}^{+}a_{i} + \frac{1}{2}) \right]T^{+}$$

$$= \sum_{i} \hbar \omega_{i} \left[(a_{i}^{+} + \lambda_{i}^{*})(a_{i} + \lambda_{i}) + \frac{1}{2} \right]$$

$$= H_{R} + \sum_{i} \hbar \omega_{i} (\lambda_{i}a_{i}^{+} + \lambda_{i}^{*}a_{i}) + \sum_{i} \hbar \omega_{i} \lambda_{i}^{*} \lambda_{i} . \qquad (2.103)$$

In addition to $H_{\rm R}$, we get a term linear in λ_i or λ_i^* , and a term bilinear in λ_i and λ_i^* . Consider first the linear term. Using (2.96) and (2.97), we can transform this term into

$$-d_{A} \cdot \sum_{i} i \sqrt{\frac{\hbar \omega_{i}}{2\varepsilon_{0}L^{3}}} \left(a_{i}\varepsilon_{i} e^{ik_{i} \cdot R_{A}} - a_{i}^{+}\varepsilon_{i} e^{-ik_{i} \cdot R_{B}} \right) + A \Longrightarrow B.$$
(2.104)

We get the scalar product (with the minus sign) of d_A (or d_B) with a field operator which, according to (2.101), is equal in the new representation to $(1/\varepsilon_0)D'(\mathbf{R}_A)$ [or $(1/\varepsilon_0)D'(\mathbf{R}_B)$]

$$-d_{\rm A} \cdot \frac{D'(R_{\rm A})}{\varepsilon_0} - d_{\rm B} \cdot \frac{D(R_{\rm B})}{\varepsilon_0}.$$
(2.105)

Thus, the interaction term involves the electric induction rather than the electric field.

Consider now the last term of (2.103). We have first two "square" terms in $\lambda_{iA}^* \lambda_{iA}$ and $\lambda_{iB}^* \lambda_{iB}$. Using the expression (2.97) of λ_{iA} , we get

$$\sum_{i} \hbar \omega_{i} \lambda_{iA}^{*} \lambda_{iA} = \sum_{k, \varepsilon_{i}} \frac{1}{2\varepsilon_{0} L^{3}} (\varepsilon_{i} \cdot d_{A})^{2} .$$
(2.106)

If we introduce a cut-off $k_{\rm M}$ in the sum over $|\mathbf{k}_i|$ (with $k_{\rm M}a \ll 1$) such a term (and the corresponding one for $\lambda_{i\rm B}^*\lambda_{i\rm B}$) becomes

$$\varepsilon_{\rm A}' = \sum_{i} \hbar \omega_i \lambda_{i{\rm A}}^* \lambda_{i{\rm A}} = \frac{k_{\rm M}^3}{18\varepsilon_0 \pi^2} d_{\rm A}^2 , \qquad (2.107)$$

 $\varepsilon_{\rm A}'$ and $\varepsilon_{\rm B}'$ are pure atomic operators related to $\mathscr{G}_{\rm A}$ and $\mathscr{G}_{\rm B}$ and

representing dipole self-energies. Finally we have a "crossed" term between A and B

$$\sum_{i} \hbar \omega_{i} (\lambda_{iB}^{*} \lambda_{iA} + \lambda_{iA}^{*} \lambda_{iB})$$

$$= \frac{1}{2\varepsilon_{0} L^{3}} \sum_{i} (\varepsilon_{i} \cdot d_{A}) (\varepsilon_{i} \cdot d_{B}) e^{ik_{i}(R_{A} - R_{B})} + A \rightleftharpoons B. \qquad (2.108)$$

The A \rightleftharpoons B term of (2.108) doubles the first one. Replacing the discrete sum by an integral and using (2.72), we transform (2.108) into

$$\frac{1}{\varepsilon_0} \int d^3k \sum_{\boldsymbol{\varepsilon}} (\boldsymbol{\varepsilon} \cdot \boldsymbol{\mathscr{P}}_{\mathbf{A}}^*) (\boldsymbol{\varepsilon} \cdot \boldsymbol{\mathscr{P}}_{\mathbf{B}}) = \frac{1}{\varepsilon_0} \int d^3k \, \boldsymbol{\mathscr{P}}_{\mathbf{A}\perp}^*(k) \cdot \boldsymbol{\mathscr{P}}_{\mathbf{B}\perp}(k)$$
$$= \frac{1}{\varepsilon_0} \int d^3r \, \boldsymbol{P}_{\mathbf{A}\perp}(r) \cdot \boldsymbol{P}_{\mathbf{B}\perp}(r) \,. \tag{2.109}$$

According to (2.86) such a term is nothing but $-V_{\text{Coul}}^{\text{AB}}$ and therefore cancels the electrostatic interaction between A and B, appearing in the second line of (2.87).

Combining all the previous results, we get for $H' = THT^+$

$$H' = \sum_{\alpha} \frac{p_{\alpha}^{2}}{2m_{\alpha}} + V_{\text{Coul}}^{\text{AA}} + \varepsilon_{A}' + \sum_{\beta} \frac{p_{\beta}^{2}}{2m_{\beta}} + V_{\text{Coul}}^{\text{BB}} + \varepsilon_{B}'$$
$$+ H_{\text{R}} - d_{\text{A}} \cdot \frac{D'(R_{\text{A}})}{\varepsilon_{0}} - d_{\text{B}} \cdot \frac{D(R_{\text{B}})}{\varepsilon_{0}}. \qquad (2.110)$$

In the new representation, the Hamiltonian has a very simple structure. We have first two atomic Hamiltonians, for \mathscr{S}_A and \mathscr{S}_B , representing for each system, the sum of the kinetic energy, the Coulomb energy (inside \mathscr{S}_A or \mathscr{S}_B), the dipole self-energy (ε'_A or ε'_B). Then we have the field energy H_R and finally the interaction Hamiltonian representing the coupling of d_A (and d_B) with the electric induction in \mathbb{R}_A (and \mathbb{R}_B).

Two advantages of the new representation clearly appear on (2.110). First, the interaction Hamiltonian is linear in field operators and not quadratic as it was the case for *H*. Second, the instantaneous Coulomb interaction between A and B has disappeared.

This last result has a simple interpretation: the new Hamiltonian contains the coupling of the electric dipole d_A with the total induction

in \mathbf{R}_{A} , in particular the induction produced by \mathcal{S}_{B} . But, the induction produced in \mathbf{R}_{A} by \mathcal{S}_{B} coincides with the *total* electric field produced by \mathcal{S}_{B} , since the polarization density $\mathbf{P}_{B}(\mathbf{r})$ associated with \mathcal{S}_{B} is localized in \mathbf{R}_{B} and since \mathbf{R}_{A} is outside \mathcal{S}_{B} (see remark (ii) on page 53). It follows that the new interaction Hamiltonian contains the coupling of \mathbf{d}_{A} with both the transverse and longitudinal electric field produced by \mathcal{S}_{B} . This also explains why the new Hamiltonian (2.110) is more suitable than the old one (2.87) for studying retardation effects in the electromagnetic interaction between two neutral atoms.

3. Quantum electrodynamics in Coulomb gauge - physical discussion

3.1. Introduction

The purpose of this lecture is to discuss some important features of the quantum theory of radiation introduced in the previous section 2.

We first review (section 3.2) some observables of the quantized field. Most experiments actually correspond to photoelectric measurements and we give the expression of the field observables which are measured in these experiments.

We then analyze (section 3.3) the eigenstates of the observables corresponding to the total energy and total momentum of the field, which clearly exhibit the corpuscular features of the field. These states can be analyzed in terms of *elementary excitations* of the field, or *photons*, having a well-defined energy $\hbar\omega$ and momentum $\hbar k$. By superposing one-photon states, with different energies, it is possible to construct nonstationary states, which propagate with the speed of light. The ground state of the quantized free field, i.e., the vacuum of photons, exhibits important quantum features, which are analyzed in section 3.4.

The wave aspects of the field are also very important. We introduce in section 3.5 the quasiclassical states, or coherent states, which realize the best compromise between the complementary photon and wave aspects. The last section 3.6 is devoted to detailed discussion of interference phenomena. We show how the quantum theory of radiation provides a very convenient framework for analyzing these effects, and how the notion of interference can be extended to processes involving more than one photon.

3.2. Free field observables

In order to focus the discussion on the field observables, we will consider here the free field, in absence of sources.

3.2.1. Energy and momentum

We have already given in section 2 the expression of the corresponding observables $H_{\rm R}$ and $P_{\rm R}$,

$$H_{\rm R} = \sum \hbar \omega_i (a_i^+ a_i + \frac{1}{2}), \qquad (3.1)$$

$$\boldsymbol{P}_{\mathrm{R}} = \sum_{i} \hbar \boldsymbol{k}_{i} a_{i}^{\dagger} a_{i} \,. \tag{3.2}$$

 $H_{\rm R}$ and $P_{\rm R}$ are "global" observables, since they are related to integrals of the electric and magnetic fields over the whole space.

3.2.2. Fields in a given point r

Contrarily to H_R and P_R , these observables are "local". The expression of E(r), B(r), A(r) is given in (2.6), (2.7), (2.8)*.

If we consider the contribution of mode *i* to A(r=0), we find something proportional to $a_i + a_i^+$. Similarly, the contribution of mode *i* to E(r=0) is proportional to $i(a_i - a_i^+)$. This gives a physical meaning to the "position" $x_i = (a_i + a_i^+)/\sqrt{2}$ and to the momentum $p_i = i(a_i^+ - a_i)/\sqrt{2}$ of the fictitious harmonic oscillator associated with mode *i* and to its wave functions. For example, it is well known that the ground state of an harmonic oscillator has a Gaussian wave function in *x*-space and *p*-space. It follows that the distribution of the possible values of the contribution of mode *i* to E(0) is given by a Gaussian curve in the ground state of this oscillator (no photon in mode *i*).

3.2.3. Observables corresponding to photoelectric measurements

In the optical domain, most local measurements of the field are based on the photoelectric effect. One puts an atom in the field at point r and one records the photoelectron produced by the photoionization of this atom. Such measurements are *destructive*, since the photon giving rise to the photoelectric effect disappears.

^{*} The free field considered in this chapter is transverse so that we can write E(r) and A(r) instead of $E_{\perp}(r)$ and $A_{\perp}(r)$.

Single countings signals. Suppose that a broad band photodetector is put at point r in a radiation field. It can be shown (see for example Glauber's lectures in [13]) that the probability of observing a photoionization in this detector between t and t + dt is proportional to $w_1(r, t) dt$ where

$$w_{1}(\mathbf{rt}) = \langle \psi | \mathbf{E}^{(-)}(\mathbf{rt}) \cdot \mathbf{E}^{(+)}(\mathbf{rt}) | \psi \rangle, \qquad (3.3)$$

 $E^{(+)}$ and $E^{(-)}$ are the positive and negative frequency components of the electric field defined in (1.95) and (1.96). The observation time is supposed sufficiently short so that perturbation theory can be applied.

The "single counting rate" w_1 is the average in the state $|\psi\rangle$ of the field observable (in the Heisenberg picture)

$$I(rt) = E^{(-)}(rt) \cdot E^{(+)}(rt), \qquad (3.4)$$

I(r, t) is an Hermitian operator, normally ordered (annihilation operators at right, creation operators at left) which can be called the *light intensity* in r at time t.

Remark. It is also possible to give a semiclassical treatment of the photoelectric effect, where only the detector is quantized but not the field (see for example [14, 15]). For the single counting rate, one finds instead of (3.3)

$$w_{\rm I}^{\rm cl}(rt) = E_{\rm cl}^{(-)}(rt) \cdot E_{\rm cl}^{(+)}(rt) = I_{\rm cl}(rt), \qquad (3.5)$$

where $E_{cl}^{(+)}$ and $E_{cl}^{(-)}$ are the positive and negative frequency components of the classical electric field and $I_{cl} = |E_{cl}^{(+)}|^2$ the classical intensity.

Double counting signals. We consider now two photodetectors in r and r'. The probability to have a photoionization in r' between t' and t' + dt' and another one in r between t and t + dt can be shown [13] to be proportional to $w_{II}(rt, r't') dt dt'$ where

$$w_{\rm II}(\mathbf{rt}, \mathbf{r}'t') = \sum_{m,n} \langle \psi | E_m^{(-)}(\mathbf{r}'t') E_n^{(-)}(\mathbf{rt}) E_n^{(+)}(\mathbf{rt}) E_m^{(+)}(\mathbf{r}'t') | \psi \rangle, \qquad (3.6)$$

with m, n = x, y, z. The "double counting rate" w_{II} is the average of

the normally ordered observable

$$\sum_{m,n} E_m^{(-)}(\mathbf{r}'t') E_n^{(-)}(\mathbf{r}t) E_n^{(+)}(\mathbf{r}t) E_m^{(+)}(\mathbf{r}'t') .$$
(3.7)

Since $E^{(-)}(rt)$ and $E^{(+)}(rt)$ do not commute, such an observable cannot be written as I(rt)I(r't'), i.e., as the product of the two light intensities in rt and r't'.

Remark. The semiclassical expression of the double counting rate is equal to

$$w_{\rm II}^{\rm cl}(\mathbf{r}t, \, \mathbf{r}'t') = I_{\rm cl}(\mathbf{r}t)I_{\rm cl}(\mathbf{r}'t') \,. \tag{3.8}$$

For a fluctuating classical field, the average of (3.8) must be taken over all possible realizations of the field. In such a case, the double counting rate is given by the correlation function of the light intensity.

3.3. Elementary excitations of the quantized free field – photons

3.3.1. Eigenstates of H_R and P_R

Consider first the *i*th oscillator (mode i). We have the well-known results

$$a_i^+ a_i |n_i\rangle = n_i |n_i\rangle, \qquad n_i = 0, 1, 2, \dots,$$
 (3.9)

$$a_i^+|n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle, \qquad (3.10a)$$

$$a_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle, \qquad (3.10b)$$

$$a_i|0_i\rangle = 0, \qquad (3.10c)$$

$$|n_i\rangle = \frac{(a_i^+)^{n_i}}{\sqrt{n_i!}} |0_i\rangle . \tag{3.11}$$

Since $a_i^{\dagger}a_i$ commutes with $a_j^{\dagger}a_j$, the eigenstates of H_R and P_R are (tensorial) products of the eigenstates $|n_i\rangle$ of $a_i^{\dagger}a_i$,

$$H_{\mathsf{R}}|n_1\ldots n_i\ldots\rangle = \sum_i (n_i + \frac{1}{2})\hbar\omega_i|n_1\ldots n_i\ldots\rangle, \qquad (3.12a)$$

$$\boldsymbol{P}_{\mathsf{R}}|\boldsymbol{n}_{1}\ldots\boldsymbol{n}_{i}\ldots\rangle = \sum_{i} \boldsymbol{n}_{i} \hbar \boldsymbol{k}_{i} |\boldsymbol{n}_{1}\ldots\boldsymbol{n}_{i}\ldots\rangle . \qquad (3.12b)$$

The ground state of the field corresponds to all n_i equal to zero and will be noted $|0\rangle$ (vacuum state),

$$|0\rangle = |0_1 \dots 0_i \dots\rangle. \tag{3.13}$$

From (3.11) it follows that the eigenstates $|n_1n_2...n_i...\rangle$ can be obtained by applying on the vacuum a certain number of creation operators,

$$|n_1 \dots n_i \dots\rangle = \frac{(a_1^+)^{n_1}}{\sqrt{n_1!}} \dots \frac{(a_i^+)^{n_i}}{\sqrt{n_i!}} \dots |0\rangle.$$
 (3.14)

3.3.2. Interpretation in terms of photons

With respect to the vacuum, the state $|n_1 \dots n_i \dots\rangle$ has an energy $\sum_i n_i \hbar \omega_i$ and a momentum $\sum_i n_i \hbar k_i$. It behaves as an ensemble of n_1 particles with an energy $\hbar \omega_1$ and a momentum $\hbar k_1 \dots$ and n_i particles with an energy $\hbar \omega_i$ and a momentum $\hbar k_1 \dots$

These particles are called "photons". They describe the elementary excitations of the various modes of the quantized field.

From (3.10), it follows that a_i^+ creates a photon *i*, whereas a_i annihilates a photon *i*. The total number of photons is described by the operator

$$N = \sum_{i} a_i^+ a_i \,. \tag{3.15}$$

Finally, since the field has been quantized with commutators, photons are bosons. Actually, the number of photons i, n_i , can be larger than 1.

3.3.3. One photon states - propagation

The creation operator a_k^+ acting upon the vacuum $|0\rangle$ gives a state $a_k^+|0\rangle$ with one photon k. These states can be linearly superposed to give

$$|\psi\rangle = \sum_{k} c_k a_k^+ |0\rangle \,. \tag{3.16}$$

Such a linear combination is an eigenstate of the operator N given in (3.15),

$$N|\psi\rangle = |\psi\rangle, \qquad (3.17)$$

but not of H_R and P_R (since several values of k appear in (3.16)). It follows that, in general, $|\psi\rangle$ describes a nonstationary one-photon state.

In order to discuss the propagation of such a state, we will consider a simple one-dimensional problem. All the modes appearing in the expansion (3.16) are supposed to have their wave vectors parallel to the x-axis and the same polarization, so that $E^{(+)}(rt)$ will be simply noted $E^{(+)}(xt)$,

$$E^{(+)}(xt) = \sqrt{\frac{\hbar c}{2\varepsilon_0 L^3}} \sum_k \sqrt{k} a_k e^{i(kx - \omega t)}. \qquad (3.18)$$

The single counting rate $w_{I}(xt)$ in the state (3.16) is then given by

$$w_{I}(xt) = \frac{\hbar c}{2\varepsilon_{0}L^{3}} \bigg| \sum_{k} \sqrt{k} c_{k} e^{ik(x-ct)} \bigg|^{2}.$$
(3.19)

It clearly appears on (3.19) that $w_I(x, t)$, which only depends on x - ct, propagates without deformation with a speed c.

Remarks

(i) A measurement of P_R on the field in the state (3.16) gives the value $\hbar k$ with the probability $|c_k|^2$ [we suppose $\langle \psi | \psi \rangle = \sum_k |c_k|^2 = 1$]. $|c_k|^2$ can therefore be considered as the probability distribution for P_R .

(ii) $w_I(xt)$ represents the probability of obtaining a photoelectron at the point x. It could be tempting to consider $w_I(xt)$, in the one-photon subspace, as the probability for the photon to be at point x. This would introduce the idea of the "position" of the photon. To support such an interpretation, it would be necessary to show that it is possible to construct a complete set of localized states for the photon, i.e., a complete set of states for which $w_I(rt)$ is zero everywhere except in one point. This is actually impossible because of the transverse character of the photon field. For example, to localize a photon with a polarization parallel to the z-axis, the transversality imposes to use only plane waves with their wave vectors in the xy-plane, and $w_I(rt)$ would be completely delocalized in the z direction. In fact, it has been shown by general arguments, that it is impossible to define a position operator for the photon [11].

3.4. Some properties of the vacuum

3.4.1. Qualitative discussion

For a real harmonic oscillator, it is well known that the basic commutation relation $[x, p] = i\hbar$ prevents the *simultaneous* cancellation of the potential energy x^2 and the kinetic energy p^2 . The lowest energy level results from a "compromise" between these two energies which cannot be made both equal to zero. This explains why the ground state has a non-zero energy (zero-point energy $\hbar\omega/2$), and why in this ground state $\langle x^2 \rangle \neq 0$ and $\langle p^2 \rangle \neq 0$.

A similar situation occurs for the quantized field. The basic commutation relation $[a_i, a_j^+] = \delta_{ij}$ [see also (2.18)] prevents the simultaneous cancellation of E^2 and B^2 , i.e., the electric and magnetic energies. It follows that the ground state of the field, i.e., the vacuum $|0\rangle$, has a nonzero energy ($E_0 = \sum_i \hbar \omega_i/2$) and that, in this state, $\langle E^2 \rangle \neq 0$ and $\langle B^2 \rangle \neq 0$. This is a pure quantum effect.

3.4.2. Vacuum fluctuations

Using the expansion of E in a_i and a_i^+ and

$$a_i|0\rangle = 0; \qquad \langle 0|a_i a_i^+|0\rangle = \delta_{ii}, \qquad (3.20)$$

one derives

$$\langle 0|\boldsymbol{E}(\boldsymbol{r}\boldsymbol{t})|0\rangle = 0, \qquad (3.21)$$

$$\langle 0|(\boldsymbol{E}(\boldsymbol{r}\boldsymbol{t}))^2|0\rangle = \frac{\hbar c}{2\varepsilon_0 \pi^2} \int_0^{k_{\rm M}} k^3 \,\mathrm{d}\boldsymbol{k} \,. \tag{3.22}$$

The average value of $E^2(r)$ in the vacuum is proportional to \hbar (pure quantum effect) and diverges if the upper bound k_M of the integral tends to infinity.

It follows that, even in the vacuum of photons, there is an electric field everywhere in space, with a zero average value, but with a non-zero variance.

The dynamics of these vacuum fluctuations is described by the symmetric correlation function

$$C_{mn}(\mathbf{r}; t+\tau, t) = \frac{1}{2} \langle 0 | E_m(\mathbf{r}, t+\tau) E_n(\mathbf{r}, t) + E_n(\mathbf{r}, t) E_m(\mathbf{r}, t+\tau) | 0 \rangle,$$
(3.23)

where m, n = x, y, z. Using the expansion of the free field Heisenberg operators, one derives

$$C_{mn}(\mathbf{r}; t+\tau, t) = \delta_{mn} \frac{\hbar c}{6\varepsilon_0 \pi^2} \operatorname{Re} \int_0^{k_{\rm M}} k^3 \, \mathrm{e}^{\mathrm{i} c k \tau} \, \mathrm{d}k \,. \tag{3.24}$$

The correlation function C_{mn} is real and only depends on τ (this is due to the fact that the vacuum is a stationary state). The width in τ of $C_{mn}(\tau)$ is of the order of $1/ck_{\rm M}$. This means that vacuum fluctuations have a very short correlation time. It also appears in (3.24) that the spectral power density of vacuum fluctuations is proportional to ω^3 .

An atomic electron in the vacuum of photons interacts with vacuum fluctuations. Can spontaneous emission by an excited atom be considered as an emission "triggered" by vacuum fluctuations? We will come back to this problem in the next lecture devoted to a physical discussion of radiative processes. We will see that vacuum fluctuations play an important role in spontaneous emission of radiation and in radiative corrections, but that another physical mechanism is also present, the interaction of the electron with its own field (radiation reaction).

3.5. Quasiclassical states

3.5.1. General idea

Consider a *classical* free field. Its state is, according to the results of section 1, characterized by the set $\{\alpha_i\}$ of normal variables. Once the set $\{\alpha_i\}$ is known, all field quantities are known. For example,

$$H_{R}^{cl}(\{\alpha_{i}\}) = \sum \hbar \omega_{i} \alpha_{i}^{*} \alpha_{i}, \qquad (3.25)$$

$$\boldsymbol{P}_{R}^{cl}(\{\alpha_{i}\}) = \sum \hbar k_{i} \alpha_{i}^{*} \alpha_{i}, \qquad (3.26)$$

$$\mathbf{A}_{cl}(\{\alpha_i\}, rt) = \sum_i \mathcal{A}_{\omega_i} \alpha_i \varepsilon_i \, e^{i(k_i \cdot r - \omega_i t)} + c.c. , \qquad (3.27)$$

and so on.

For a *quantum* free field, the situation is more complex. Since the various quantum field observables (reviewed in section 3.2) do not commute, it is impossible to find common eigenstates of these observables with eigenvalues equal to the values of the corresponding classical observables.

In this section, we try to find the quantum state $|\{\alpha_i\}\rangle$ which "reproduces" in the best possible way the properties of the classical state $\{\alpha_i\}$. The general idea is to look for a quantum state $|\{\alpha_i\}\rangle$ such that, for all important observables, the quantum average values in this state $|\{\alpha_i\}\rangle$ coincide with the corresponding classical values. More precisely we want to have

$$\langle \{\alpha_i\} | H_{\mathsf{R}} | \{\alpha_i\} \rangle - E_{\mathsf{vac}} = H_{\mathsf{R}}^{\mathsf{cl}}(\{\alpha_i\}) \,. \tag{3.28}$$

(We have substracted the energy E_{vac} of the vacuum because all energies are measured with respect to the vacuum.)

$$\langle \{\alpha_i\} | \boldsymbol{P}_{\mathsf{R}} | \{\alpha_i\} \rangle = \boldsymbol{P}_{\mathsf{R}}^{\mathsf{cl}}(\{\alpha_i\}), \qquad (3.29)$$

$$\langle \{\alpha_i\} | \mathbf{A}(\mathbf{rt}) | \{\alpha_i\} \rangle = \mathbf{A}_{cl}(\{\alpha_i\}, \mathbf{rt}), \qquad (3.30)$$

for all r and t, plus similar equations for E, B.

3.5.2. Characterization of quasiclassical states

If the expansions of $H_{\rm R}$, $P_{\rm R}$, A(r, t) in a_i and a_i^+ are inserted in the left part of (3.28), (3.29), (3.30), and if the expressions (3.25), (3.26), (3.27) of $H_{\rm R}^{\rm cl}(\{\alpha_i\})$, $P_{\rm R}^{\rm cl}(\{\alpha_i\})$, $A_{\rm cl}(\{\alpha_i\}, rt)$ are used, one finds that conditions (3.28), (3.29), (3.30) are equivalent to

$$\langle \{\alpha_i\} | a_i | \{\alpha_i\} \rangle = \alpha_i \quad \forall i.$$
(3.31)

$$\langle \{\alpha_i\} | a_i^+ a_i | \{\alpha_i\} \rangle = \alpha_i^* \alpha_i \quad \forall i.$$
(3.32)

If we then introduce

$$b_i = a_i - \alpha_i \mathbb{1} \tag{3.33}$$

where 1 is the unit operator, we transform (3.31) and (3.32) into

$$\langle \{\alpha_i\} | b_i | \{\alpha_i\} \rangle = 0 \quad \forall i , \tag{3.34}$$

$$\langle \{\alpha_i\} | b_i^+ b_i | \{\alpha_i\} \rangle = 0 \quad \forall i , \tag{3.35}$$

Equation (3.35) expresses that the norm of $b_i |\{\alpha_i\}\rangle$ is equal to zero. The solution of (3.34) and (3.5) is therefore

$$b_i |\{\alpha_i\}\rangle = 0 , \qquad (3.36)$$

i.e.,

$$a_i |\{\alpha_i\}\rangle = \alpha_i |\{\alpha_i\}\rangle \,. \tag{3.37}$$

It follows that

$$|\{\alpha_i\}\rangle = |\alpha_1\rangle|\alpha_2\rangle\dots|\alpha_i\rangle\dots, \tag{3.38}$$

with

$$a_i |\alpha_i\rangle = \alpha_i |\alpha_i\rangle \,. \tag{3.39}$$

The quasiclassical, or coherent, state $|\{\alpha_i\}\rangle$ is thus the product of the eigenstates of the various annihilation operators a_i , the eigenvalues α_i being just the corresponding classical normal variables.

From (3.39) it follows that

$$\langle \alpha_i | a_i^+ = \alpha_i^* \langle \alpha_i | , \qquad (3.40)$$

and also that

$$\boldsymbol{E}^{(+)}(\boldsymbol{rt})|\{\alpha_i\}\rangle = \boldsymbol{E}_{cl}^{(+)}(\{\alpha_i\}, \boldsymbol{rt})|\{\alpha_i\}\rangle, \qquad (3.41)$$

$$\langle \{\alpha_i\} | \boldsymbol{E}^{(-)}(\boldsymbol{r}t) = \boldsymbol{E}^{(-)}_{cl}(\{\alpha_i\}, \boldsymbol{r}t) \langle \{\alpha_i\} | .$$
(3.42)

More generally, all normally ordered observables have an average value in $|\{\alpha_i\}\rangle$ equal to the corresponding classical value in the classical state α_i .

3.5.3. Some properties of quasiclassical states [13], [16] Projecting (3.39) on the bra $\langle n-1 |$ gives the following relation

$$\sqrt{n}\langle n|\alpha\rangle = \alpha\langle n-1|\alpha\rangle, \qquad (3.43)$$

(we consider a single mode of the radiation field and we skip the index i) from which it follows that

$$|\alpha\rangle = e^{-|\alpha|^{2/2}} \sum_{n=0}^{\infty} \left(\alpha^{n} / \sqrt{n!} \right) |n\rangle .$$
(3.44)

The probability $\mathcal{P}(n)$ for having *n* photons in the quasiclassical state $|\alpha\rangle$ is therefore given by a Poisson law

$$\mathscr{P}(n) = \mathrm{e}^{-|\alpha|^2} |\alpha|^{2n} / n! , \qquad (3.45)$$

with a mean value

$$\langle n \rangle = |\alpha|^2 \,, \tag{3.46}$$

and a variance

$$\Delta n^2 = \langle n \rangle = |\alpha|^2 \,. \tag{3.47}$$

From (3.44), one can also derive the following normalization relation

$$|\langle \beta | \alpha \rangle|^2 = e^{-|\beta - \alpha|^2}, \qquad (3.48)$$

and closure relation

$$\frac{1}{\pi} \int d^2 \alpha \, |\alpha\rangle \langle \alpha| = 1 \,, \tag{3.49}$$

where $d^2 \alpha = d \operatorname{Re}(\alpha) d \operatorname{Im}(\alpha)$.

Finally, it can be shown that, in the x-representation, a quasiclassical state is represented by a minimum Gaussian wave packet which oscillates without deformation.

Remark. It is possible to use the quasiclassical states $|\alpha\rangle$ as an (overcomplete) basis for expanding the density operator ρ of the mode. A particularly interesting situation occurs when ρ can be written as

$$\rho = \int d^2 \alpha \, P(\alpha) |\alpha\rangle \langle \alpha| \,, \tag{3.50}$$

 $P(\alpha)$ is then the "P-representation" of the density operator ρ [13].

 $P(\alpha)$ is real and normalized (this follows from $\rho^+ = \rho$ and from Tr $\rho = 1$). Furthermore, from (3.39) and (3.40), one can show that

$$\langle (a^+)^m a^l \rangle = \int d^2 \alpha \, P(\alpha) \alpha^{*m} \alpha^l \,. \tag{3.51}$$

For all normally ordered products of a^+ and a, $P(\alpha)$ appears as a density of probability for α . Actually $P(\alpha)$ is not a true probability, but rather a quasiprobability, since one can show that it can take negative values. The *P*-representation provides a clear discussion of pure quantum effects. Such quantum effects appear for example for states of the quantum field leading to violation of semiclassical inequalities established with positive definite classical densities $\mathcal{P}(\alpha)$. Also, the master equation describing the damping of ρ due to relaxation processes often becomes a Fokker–Planck equation for $P(\alpha)$ (see for example [17] §8 or [18] §17.1).

3.6. Analysis of interference phenomena

3.6.1. Motivations of a Q.E.D. approach

The wave-particle duality has been first introduced in physics for light. As soon as in 1909, Einstein introduced the idea that the wave aspect and the particle aspect of light were inseparable. The wave-particle duality was then extended to matter and led to the spectacular development of quantum mechanics.

In nonrelativistic quantum mechanics, the connection between the wave aspect and the particle aspect is well known. The wave function $\psi(r)$ (which contains the wave aspect) gives the probability $|\psi(r)|^2$ to find the particle in r.

If we come back to light, we are faced with the problem that there is no wave function in r-space for the photon. It is not correct (as this is usually done in elementary introductions to quantum physics) to consider classical Maxwell waves as wave functions for the photon.

It is therefore interesting to try to understand how Q.E.D. can explain interference fringes observed with photoelectric devices without using any wave function for the photon. What are the "objects" which interfere? Is it possible to observe interference effects on double counting signals? With two independent laser beams? With photons arriving one by one?
3.6.2. A simple model

In order to keep the calculations as simple as possible, we will use a very simple model.

We consider a free field in a state $|\psi\rangle$ such that only two modes 1 and 2 are non-empty, all other modes *i* being in the lowest state $|0_i\rangle$.

$$|\psi\rangle = |\psi_{12}\rangle \bigotimes \prod_{i \neq 1,2} |0_i\rangle.$$
(3.52)

Examples of such situations can be realized, by reflecting a parallel beam on two mirrors (Fresnel mirrors) or by using two independent laser beams. The two modes are taken with the same polarization so that the vector character of the field can be ignored in the following.

The most general form of the state vector $|\psi_{12}\rangle$ in (3.52) is

$$|\psi_{12}\rangle = \sum_{n_1n_2} c_{n_1n_2} |n_1, n_2\rangle.$$
 (3.53)

It may happen that $|\psi_{12}\rangle$ is factorized (this is the case for two independent laser beams)

$$|\psi_{12}\rangle = |\psi_1\rangle|\psi_2\rangle. \tag{3.54}$$

A quasiclassical state is a particular case of (3.54),

$$|\psi_{12}\rangle = |\alpha_1\rangle|\alpha_2\rangle. \tag{3.55}$$

It will be also useful to consider single-photon states for which

$$|\psi_{12}\rangle = c_1|1_10_2\rangle + c_2|0_11_2\rangle. \tag{3.56}$$

The single photon has an amplitude c_1 to be in mode 1, c_2 to be in mode 2.

In the following, we will discuss single and double counting rates. Since w_{I} and w_{II} are average values of normally ordered products of field operators, and since all modes $i \neq 1$, 2 are empty, we can write

$$E^{(+)}(\mathbf{r}t) = E_1^{(+)}(\mathbf{r}t) + E_2^{(+)}(\mathbf{r}t)$$

= $\mathscr{E}_1 a_1 \exp[i(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)] + \mathscr{E}_2 a_2 \exp[i(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t)].$ (3.57)

(The contribution of modes $i \neq 1$, 2 in the expansion of $E^{(+)}$ vanishes for w_{I} and w_{II} in the state (3.52).)

3.6.3. Interference effects observable on single counting signals

General case. Inserting (3.57) into the expression (3.3) of w_{I} , we get

$$W_{I} = \langle \psi_{12} | (E_{1}^{(-)} + E_{2}^{(-)})(E_{1}^{(+)} + E_{2}^{(+)}) | \psi_{12} \rangle$$

= $\langle \psi_{12} | E_{1}^{(-)} E_{1}^{(+)} | \psi_{12} \rangle + \langle \psi_{12} | E_{2}^{(-)} E_{2}^{(+)} | \psi_{12} \rangle$
+ $\langle \psi_{12} | E_{1}^{(-)} E_{2}^{(+)} | \psi_{12} \rangle + \langle \psi_{12} | E_{2}^{(-)} E_{1}^{(+)} | \psi_{12} \rangle$, (3.58)

which can be also written,

$$w_{1}(\mathbf{r}t) = |\mathscr{C}_{1}|^{2} \langle \psi_{12} | a_{1}^{+} a_{1} | \psi_{12} \rangle + 1 \rightleftharpoons 2 + 2 \operatorname{Re} \mathscr{C}_{1}^{*} \mathscr{C}_{2} \times \langle \psi_{12} | a_{1}^{+} a_{2} | \psi_{12} \rangle \exp\{\mathrm{i}[(\mathbf{k}_{2} - \mathbf{k}_{1}) \cdot \mathbf{r} - (\omega_{2} - \omega_{1})t]\}.$$
(3.59)

If $\langle \psi_{12} | a_1^{\dagger} a_2 | \psi_{12} \rangle \neq 0$, we see on (3.59) that there is a sinusoidal r dependence in $w_1(rt)$. The single counting rate exhibits interference fringes.

Quasiclassical states. Suppose that $|\psi_{12}\rangle$ is a quasiclassical state (3.55). Using (3.57), (3.41) and (3.42), we obtain

$$w_{\rm I}(rt) = |E_{\rm 1cl}^{(+)}(\{\alpha_1\}, rt)|^2 + |E_{\rm 2cl}^{(+)}(\{\alpha_2\}, rt)|^2.$$
(3.60)

For a quasiclassical state, $w_I(rt)$ therefore appears as the square of the sum of two classical Maxwell waves. In this particular case, it is not incorrect to consider the interference between two classical electromagnetic waves for calculating the probability that the photon manifests its presence in rt.

Factorized states. If $|\psi_{12}\rangle$ is given by (3.54), $w_{I}(rt)$ becomes

$$w_{1}(\mathbf{r}t) = |\mathscr{E}_{1}|^{2} \langle \psi_{1} | a_{1}^{+} a_{1} | \psi_{1} \rangle + 1 \rightleftharpoons 2 + 2 \operatorname{Re} \mathscr{E}_{1}^{*} \mathscr{E}_{2} \langle \psi_{1} | a_{1}^{+} | \psi_{1} \rangle$$
$$\times \langle \psi_{2} | a_{2} | \psi_{2} \rangle \exp\{\mathrm{i}[(\mathbf{k}_{2} - \mathbf{k}_{1}) \cdot \mathbf{r} - (\omega_{2} - \omega_{1})t]\}.$$
(3.61)

Fringes exist only if $\langle \psi_1 | a_1^{\dagger} | \psi_1 \rangle$ and $\langle \psi_2 | a_2 | \psi_2 \rangle$ do not vanish, i.e., if $\langle E_1 \rangle$ and $\langle E_2 \rangle$ are different from zero.

In particular, if

$$|\psi_{12}\rangle = |n_1\rangle|n_2\rangle, \qquad (3.62)$$

i.e., if the number of photons in each mode is well defined, there are no fringes since $\langle n_1 | a_1 | n_1 \rangle = \langle n_2 | a_2 | n_2 \rangle = 0$. A state $|n\rangle$ is the quantum analogue of a classical monomode field with a given energy but a random phase.

Finally, one can note that, since in general

$$\langle \psi_1 | a_1^{\dagger} a_1 | \psi_1 \rangle \neq \langle \psi_1 | a_1^{\dagger} | \psi_1 \rangle \langle \psi_1 | a_1 | \psi_1 \rangle, \qquad (3.63)$$

we have

$$w_{\mathbf{I}}(\mathbf{rt}) \neq |\langle E_1 \rangle + \langle E_2 \rangle|^2, \qquad (3.64)$$

 w_1 cannot therefore be considered as resulting from the interference between two classical waves given by $\langle E_1 \rangle$ and $\langle E_2 \rangle$.

Single-photon states. We finally consider the single-photon states given in (3.56). $E_1^{(+)}(rt)$ operating on this state gives

$$E^{(+)}(\mathbf{r}t)|\psi_{12}\rangle = c_1 \mathscr{E}_1 \exp[\mathrm{i}(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)]|0_1 0_2\rangle.$$
(3.65)

Destroying a photon in a single-photon state gives the vacuum. A similar expression can be written for $E_2^{(+)}(rt)$, so that we finally get

$$w_{\mathbf{I}}(\mathbf{r}t) = |c_1 \mathscr{E}_1 \exp[\mathbf{i}(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)] + c_2 \mathscr{E}_2 \exp[\mathbf{i}(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t)]|^2. \quad (3.66)$$

Fringes appear on $w_{I}(rt)$. Actually, since the single photon is destroyed in the measurement, the experiment must be repeated several times with the same initial conditions, in order to observe the fringes.

Remarks

(i) For the single-photon states (3.56), one can show that

$$\langle \psi_{12} | E_1 | \psi_{12} \rangle = \langle \psi_{12} | E_2 | \psi_{12} \rangle = 0 .$$
(3.67)

It follows that the wave $c_1 \mathscr{C}_1 \exp[i(k_1 \cdot r - \omega_1 t)]$ appearing in (3.66) is *not* the average electric field in mode 1 which is equal to zero (and similarly for the other term of (3.66)). The two waves which interfere in (3.66) are therefore not related to average electric fields.

(ii) Fringes would not appear for a statistical mixture of $|1_10_2\rangle$ and $|0_11_2\rangle$ with weights $|c_1|^2$ and $|c_2|^2$. The phase relation which exists

between the two states appearing in (3.56), and which is related to $c_1^*c_2$, is crucial for the appearance of fringes.

3.6.1. Interference effects observable on double counting signals

General case. If one replaces the four operators $E^{(\pm)}$ appearing in the expression (3.6) of w_{II} by $E_{1}^{(\pm)} + E_{2}^{(\pm)}$, one gets $2^{4} = 16$ terms with various sinusoidal dependences with respect to the variables (r, t), (r, t'), (r + r', t + t'), (r - r', t - t'). We conclude that interference effects are also observable on double counting signals.

Quasiclassical state. Using (3.41) and (3.42) we can write for such states

$$w_{\rm II} = I_{\rm cl}(\{\alpha_1 \alpha_2\}, \, rt) I_{\rm cl}(\{\alpha_1 \alpha_2\}, \, r't') \,, \tag{3.68}$$

where $I_{cl}(\{\alpha_1\alpha_2\}, rt)$ is the classical intensity

$$I_{\rm cl}(\{\alpha_1\alpha_2\}, rt) = |E_{\rm lcl}^{(+)}(\{\alpha_1\}, rt)|^2 + |E_{\rm 2cl}^{(+)}(\{\alpha_2\}, rt)|^2.$$
(3.69)

We get the same result as in the semiclassical theory: w_{II} is the product of the two classical intensities in rt and r't'.

Single-photon states. In such states, one finds that

$$w_{\rm II}(\mathbf{rt}, \mathbf{r}'t') = 0 \quad \forall \quad \mathbf{rt}, \, \mathbf{r}'t' \,. \tag{3.70}$$

Such a result is easy to understand. The first $E^{(+)}$ at the right of expression (3.6) acting upon the single-photon state (3.56) gives the vacuum $|0_10_2\rangle$. The second $E^{(+)}$ acting upon the vacuum gives 0. In other words, it is impossible to detect two photons in a state which contains only one.

The quantized field in the single-photon state (3.56), gives single counting rates $w_{\rm I}$ which are different from zero (see next subsection), but the double counting rates $w_{\rm II}$ are identical to zero for all rt and r't'. Such a result can never occur with a classical field. It is impossible to find a classical field $E_{\rm cl}^{(+)}$ such that $w_{\rm I}^{\rm cl} \neq 0$ and $w_{\rm II}^{\rm cl} = 0$ for all rt and r't'.

Another example of a typical quantum situation can be given (fig. 3). A single photon emitted by an excited atom is sent towards a beam splitter and the signals given by two photodetectors A and B put behind the beam splitter are recorded. One can get a photoelectron in



Fig. 3. Example of a typical quantum situation.

A or B, but never in A and B. A semiclassical theory would predict possible coincidences between A and B.

Two-photon states. We will consider here a very simple example of a two-photon state,

$$|\psi_{12}\rangle = |1_1\rangle|1_2\rangle, \qquad (3.71)$$

i.e., a state where there is one photon in mode 1 and another one in mode 2.

When (3.71) is inserted in the expressions (3.3) and (3.6) of w_1 and w_{II} , we find that

$$w_{\rm I}(\mathbf{r}t) = |\mathscr{E}_1|^2 + |\mathscr{E}_2|^2, \qquad (3.72)$$

$$w_{\mathrm{II}}(\boldsymbol{r}t,\,\boldsymbol{r}'t)=2|\mathscr{E}_1|^2|\mathscr{E}_2|^2$$

× [1 + Re exp{i[
$$(k_1 - k_2) \cdot (r - r') - (\omega_1 - \omega_2)(t - t')$$
]}]. (3.73)

Such a result shows that it is possible to observe interference effects on double counting rates (associated with the last term of (3.73)) in conditions where they are not observable on single counting rates (as shown in the previous subsection, this is due to the fact that, in the state (3.71), $\langle E_1 \rangle$ and $\langle E_2 \rangle$ vanish because of the random character of the phase).

Interference effects on double counting rates (or higher-order counting rates) are very useful from an experimental point of view. They give the possibility of exploring the space time coherence properties of various types of fields. For example, in radioastronomy, intensity correlations are less sensitive than field correlations to the phase fluctuations introduced by the atmosphere [19]. To observe the interferences between two independent laser beams [20], it is also simpler to use w_{II} than w_{I} . The relative phase of the two lasers cannot be easily reproduced from one run to another when the data are averaged. w_{I} is sensitive to the relative phase whereas w_{II} is not.

3.6.5. Interpretation in terms of interferences between transition amplitudes

We first write w_1 in a different way. Inserting the closure relation

$$\sum_{f} |\psi_{f}\rangle\langle\psi_{f}| = 1$$
(3.74)

over a complete set of field states between the two operators $E^{(-)}$ and $E^{(+)}$ appearing in the expression (3.3) of w_1 , one gets

$$w_{I}(rt) = \sum_{f} \langle \psi | E^{(-)}(rt) | \psi_{f} \rangle \langle \psi_{f} | E^{(+)}(rt) | \psi \rangle$$

= $\sum_{f} | \langle \psi_{f} | E^{(+)}(rt) | \psi \rangle |^{2}$
= $\sum_{f} | \langle \psi_{f} | [E^{(+)}(rt) + E^{(+)}_{2}(rt)] | \psi \rangle |^{2}.$ (3.75)

It is then possible to interpret expression (3.75) in the following way. There are two different paths for going from the "initial" state $|\psi\rangle$ of the field to the "final" one (fig. 4). The first path corresponds to the absorption of one photon of the mode 1 in *rt*, the second path to the absorption of one photon of the mode 2 in *rt*.



Fig. 4. Two different paths for going from the initial state $|\psi\rangle$ to the final state $|\psi_f\rangle$ in a single counting experiment.



Fig. 5. Four different paths for going from the initial state $|\psi\rangle$ to the final state $|\psi_f\rangle$ in a double counting experiment.

Two amplitudes are associated with these two paths, $\langle \psi_f | E_1^{(+)}(rt) | \psi \rangle$ and $\langle \psi_f | E_2^{(+)}(rt) | \psi \rangle$. The total amplitude for going from $|\psi\rangle$ to $|\psi_f\rangle$ is the sum of these two amplitudes

$$\langle \psi_f | E_1^{(+)}(rt) | \psi \rangle + \langle \psi_f | E_2^{(+)}(rt) | \psi \rangle.$$
(3.76)

The transition probability $|\psi\rangle \rightarrow |\psi_f\rangle$ is obtained by squaring the modulus of the amplitude (3.76). Since we do not observe the final state of the field, we must sum the probabilities over the nonobserved final states* $|\psi_f\rangle$.

The same discussion also applies to w_{II} . By inserting the closure relation (3.74) between the two $E^{(-)}$ and the two $E^{(+)}$ appearing in the expression (3.6) of w_{II} , we can show that the amplitude $|\psi\rangle \rightarrow |\psi_f\rangle$, in a double counting process, is the sum of four different amplitudes corresponding to the four different paths represented in fig. 5. Each of these paths involves two photons of the same mode 1 or 2, or of two different modes 1 and 2, one of them being absorbed in r't', the other one in *rt*. These amplitudes interfere when the total amplitude is

^{*} Eventually, if the initial state is not a pure state $|\psi\rangle$, but a statistical mixture of states, we must average (3.75) over the possible initial states.

squared, in order to obtain the probability $|\psi\rangle \rightarrow |\psi_f\rangle$, and summed over nonobserved final states.

As a conclusion of this section, we can summarize a few results concerning the description given by Q.E.D. of the wave-particle duality.

(i) The *wave* aspect is related to the fact that the operators of the various modes are linearly superposable and have a sinusoidal dependence in r and t. It is because $E^{(+)} = E_1^{(+)} + E_2^{(+)}$ that we get in $w_1 = \langle E^{(-)}E^{(+)} \rangle$ "square" terms and "crossed" terms. The calculations are formally very similar to the corresponding classical ones, but now $E_1^{(\pm)}$ are noncommuting operators and not c-numbers.

(ii) The *particle* aspect is contained in the *state* vector $|\psi\rangle$ which describes the excitation of the field, i.e., the number of photons in the various modes.

(iii) State vectors are also linearly superposable, and this also is important for interference effects since a linear superposition of states has not the same physical content as a statistical mixture.

4. Vacuum fluctuations and radiation reaction: identification of their respective contributions

This section presents the results discussed in the following reprint of Journal de Physique 43 (1982) 1617, an article by J. Dalibard, J. Dupont-Roc and C. Cohen-Tannoudji.

4.1. Introduction

Understanding the physical mechanisms responsible for spontaneous emission of radiation by an excited atom, or for radiative corrections such as radiative line shifts, electron's self-energy or magnetic moment... is a very stimulating problem which has received a lot of attention $[1, 2]^*$.

The quantitative results for these corrections are of course well established. The physical interpretations remain however more controversial. Two extreme points of view have been investigated. In the first one, the interaction of the electron with the quantum fluctuations of the vacuum field, the so-called "vacuum fluctuations", is considered

^{*} For references in this section please consult the reference list at the end of this section on p. 124.

as playing the central role. One tries to interpret spontaneous emission as an emission "triggered" by vacuum fluctuations. The most famous example of such an approach is the interpretation of the Lamb shift as being due to the averaging of the Coulomb potential of the nucleus by the electron vibrating in vacuum fluctuations [3]. One must not forget however that such a picture leads to the wrong sign for the electron's spin anomaly g-2: the vibration of the electron's spin in vacuum fluctuations does not increase the effective magnetic moment but reduces it [3, 4]. In the second point of view, the basic physical mechanism is identified as the interaction of the electron with its own field, the so-called "radiation reaction" although it would be proper to call it the electromagnetic self-interaction since it includes the interaction of the electron with its Coulomb field as well as with its radiation field [5-8]. We will use in the following the shorter denomination "self-reaction" for this process. In such an approach, one tries to interpret O.E.D. radiative corrections along the same lines as the radiative damping and the radiative shift of an oscillating classical dipole moment. We should note however that the vacuum field cannot be completely forgotten in the interpretation of finer details of spontaneous emission, such as fluorescence spectrum or intensity correlations, which are related to higherorder correlation functions [9, 10].

Actually, it is now generally accepted that vacuum fluctuations and self-reaction are "two sides of the same quantum mechanical coin" [11], and that their respective contributions to each physical process cannot be unambiguously determined [11–14]. Such an opinion is based on the following analysis, carried out in the Heisenberg picture which provides a very convenient theoretical framework since it leads, for the relevant dynamical variables, to equations of motion very similar to the corresponding classical ones. The calculations [11–14] can be summarized by the general scheme of fig. 6.

Heisenberg's equations of motion for field and atomic variables are derived from the Hamiltonian of the combined atom + field system. The equation for the field looks like the equation of motion of an harmonic oscillator driven by an atomic source term and is readily integrated. This leads to an expression for the total field E which is a sum of two terms:

$$E = E_{\rm f} + E_{\rm s} \,. \tag{4.1}$$

The "free field" $E_{\rm f}$ corresponds to the solution of the homogeneous



Fig. 6. Principle of the derivation of the atomic dynamical equation.

field equation (without atomic source term), and coincides with the "vacuum field" when no photons are initially present. The "source field" E_s is the field generated by the atomic source (solution of the inhomogeneous field equation). Consider now the atomic equation. The rate of variation, dG(t)/dt, of a given atomic observable G(t) appears to be proportional to the product of atomic and field operators, N(t) and E(t), taken at the same time:

$$\frac{\mathrm{d}G(t)}{\mathrm{d}t} \sim N(t) E(t) \,. \tag{4.2}$$

The final step of the calculation consists in inserting in (4.2) the solution (4.1) obtained for E(t), which leads to a dynamical equation for the atomic system (fig. 6). The contributions of E_t and E_s to dG/dt can be interpreted as rates of variation

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sf}} \sim N(t) E_{\mathrm{f}}(t), \qquad (4.3a)$$

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}} \sim N(t) E_{\mathrm{s}}(t),$$
 (4.3b)

respectively due to vacuum fluctuations and self-reaction. This interpretation directly follows from the physical origin of $E_{\rm f}$ and $E_{\rm s}$. The ambiguity mentioned above for this separation comes from the fact that the two atomic and field operators N(t) and E(t) appearing in (4.2) commute [they commute at the initial time $t = t_0$, when they act in different spaces, and the Hamiltonian evolution between t_0 and t preserves this commutation]. They can therefore be taken in any order, N(t) E(t) as in (4.2), or E(t) N(t). However, $E_t(t)$ and $E_s(t)$ do not commute separately with N(t), as their sum does. Consequently, $N(t) E_{\rm f}(t)$ and $E_{\rm f}(t) N(t)$ generally differ. The two rates of variation (4.3a) and (4.3b) therefore depend on the initial order between the two commuting operators N(t) and E(t), the total rate (4.2) being of course independent of this order. In particular, if the normal order has been chosen in (4.2) [with all field annihilation operators at right, all field creation operators at left], the contribution of vacuum fluctuations vanishes when the average is taken over the vacuum state of the field, and all radiative corrections appear to come from self-reaction. Different orders taken in (4.2) would lead to different conclusions. Thus, it seems that the relative contributions of vacuum fluctuations and self-reaction cannot be unambiguously identified.

Motivations of this paper. In this paper, we would like to present some arguments supporting the choice of a particular order in (4.2) leading, in our opinion, to a physically well-defined separation between the contributions of vacuum fluctuations and self-reaction. We do not question of course the mathematical equivalence of all possible initial orders in (4.2). Our argument rather concerns the physical interpretation of the two rates of variation appearing when (4.1) is inserted in (4.2). If G is an atomic observable (Hermitian operator), the two rates of variation contributing to (d/dt) G(t), which is also Hermitian, must be separately Hermitian, if we want them to have a physical meaning. Furthermore, the field and atomic operators appearing in the different rates of variation must also be Hermitian if we want to be able to analyse these rates in terms of well defined physical quantities. We show in this paper that these hermiticity requirements restrict the possible initial orders in (4.2) to only one, the completely symmetrical order.

A second motivation of this paper is to point out that, with such a symmetrical order, a clear connection can be made with a statistical mechanics point of view which appears to be in complete agreement with the usual physical pictures associated with vacuum fluctuations and self-reaction. For example, the radiative corrections can be expressed as products of correlation functions by linear susceptibilities. For the vacuum fluctuations part of these corrections, one gets the correlation function of the field multiplied by the linear susceptibility of the atom, which supports the picture of a fluctuating vacuum field polarizing the atomic system and interacting with this induced polarization, whereas for the self-reaction part, the reverse result is obtained: product of the correlation function of the atomic system by the linear "susceptibility" of the field which corresponds to the picture of a fluctuating dipole moment "polarizing" the field, i.e., producing a field, and interacting with this field.

Organization of the paper. In section 4.2 we introduce our notations and the basic concepts (vacuum field, source field, radiation reaction, . . .) by applying the general theoretical scheme of fig. 6 to the derivation of the quantum generalization of the Abraham-Lorentz equation [17] describing the dynamics of an atomic electron interacting with a static potential and with the quantized radiation field. We discuss the physical content of this equation and the difficulties associated with the quantum nature of field variables. We explain also why it is necessary to extend the calculations of section 4.2 (dealing with the position r and the momentum p of the electron) to more general atomic observables G.

The calculation of dG/dt, which is presented in section 4.3, raises the problem of the order between commuting observables, mentioned above in connection with eq. (4.2) (such a difficulty does not appear for r and p). We show how it is possible, by the physical considerations mentioned above, to single out the completely symmetrical order in (4.2). We then extend in section 4.4 the discussion to the more general case of a "small system" \mathcal{G} (playing the role of the atomic system) interacting with a "large reservoir". \mathcal{R} (playing the role of the electromagnetic field with its infinite number of degrees of freedom). The advantage of such a generalization is to provide a deeper insight in the problem. We point out in particular that the expressions giving $\langle (dG/dt)_{vf} \rangle$ and $\langle (dG/dt)_{sr} \rangle$, averaged in the vacuum state of the field and calculated to first order in the fine structure constant α , can be expressed in terms of simple statistical functions of the two interacting systems (correlation functions and linear susceptibilities). We discuss the mathematical structure of these expressions and their physical content.

Finally, the general results of sections 4.3 and 4.4 are applied in section 4.5 to the physical discussion of the relative contributions of

vacuum fluctuations and self-reaction to the dynamics of an atomic electron. Two types of effects are considered: the shift of atomic energy levels, described by the Hamiltonian part of $\langle (dG/dt)_{vt} \rangle$ and $\langle (dG/dt)_{sr} \rangle$, and the dissipative effects associated with the exchange of energy between the electron and the radiation field.

4.2. The quantum form of the Abraham-Lorentz equation

A few basic concepts are introduced in this section, by considering a very simple system formed by an electron bound near the origin by an external potential and interacting with the electromagnetic field.

We first introduce the Hamiltonian of the combined system "bound electron + electromagnetic field" (subsection 4.2.1). We then establish, in the Heisenberg representation, the quantum dynamical equation for the electron (subsection 4.2.2). This equation appears to be very similar to the corresponding classical one, known as the Abraham-Lorentz equation. This close analogy is however misleading and we will try to explain the difficulties hidden in the quantum equation (subsection 4.2.3).

4.2.1. Basic Hamiltonian in Coulomb gauge

Field variables. The electric field is divided into two parts: the longitudinal field E_{\parallel} and the transverse field E_{\perp} . The longitudinal field at point **R** is the instantaneous Coulomb field created by the electron at this point. It is expressed as a function of the electron position operator **r**,

$$E_{\parallel}(\mathbf{R}) = -\nabla_{\mathbf{R}} \frac{e}{4\pi\varepsilon_0 |\mathbf{R} - \mathbf{r}|}.$$
(4.4)

The transverse field $E_{\perp}(\mathbf{R})$, the vector potential $A(\mathbf{R})$ and the magnetic field $B(\mathbf{R})$ are expanded in a set of transverse plane waves, normalized in a cube of volume L^3 :

$$\boldsymbol{E}_{\perp}(\boldsymbol{R}) = \sum_{k\epsilon} \left(\mathscr{C}_k \boldsymbol{\varepsilon} \, \mathrm{e}^{\mathrm{i} k \boldsymbol{R}} \right) \, \boldsymbol{a}_{k\epsilon} + \mathrm{hc} \,, \tag{4.5a}$$

$$A(\mathbf{R}) = \sum_{k\varepsilon} (\mathscr{A}_k \varepsilon e^{ik\mathbf{R}}) a_{k\varepsilon} + hc, \qquad (4.5b)$$

$$\boldsymbol{B}(\boldsymbol{R}) = \sum_{k\varepsilon} \left(\mathscr{B}_{k} \boldsymbol{\kappa} \times \boldsymbol{\varepsilon} \, \mathrm{e}^{\mathrm{i}k\boldsymbol{R}} \right) a_{k\varepsilon} + \mathrm{hc} \,, \tag{4.5c}$$

with

$$\mathscr{E}_{k} = i\sqrt{\frac{\hbar\omega}{2\varepsilon_{0}L^{3}}}, \qquad \mathscr{A}_{k} = \sqrt{\frac{\hbar}{2\varepsilon_{0}L^{3}\omega}}, \qquad \mathscr{B}_{k} = i\frac{\omega\mathscr{A}_{k}}{c}, \qquad (4.6)$$

$$\boldsymbol{\kappa} = \boldsymbol{k}/\boldsymbol{k} \,, \tag{4.7}$$

 $a_{k\varepsilon}$ and $a_{k\varepsilon}^+$ are the annihilation and creation operators for a photon with wave vector k and polarization ε . The summation concerns all the wave vectors k with components multiple of $2\pi/L$ and, for a given k, two transverse orthogonal polarizations ε_1 and ε_2 .

In classical theory, expansions similar to (4.5) can be written, the operators $a_{k\varepsilon}$ and $a^+_{k\varepsilon}$ being replaced by *c*-numbers $\alpha_{k\varepsilon}(t)$ and $\alpha^*_{k\varepsilon}(t)$ which are actually "normal" variables for the field.

In order to calculate the energy of the Coulomb field of the particle, it is also convenient to take the Fourier transform of the longitudinal field (4.4) (for a given value of r):

$$\boldsymbol{E}_{\parallel}(\boldsymbol{R}) = \sum_{\boldsymbol{k}} -i \frac{e}{2\varepsilon_0 L^3 k} \,\boldsymbol{\kappa} \, e^{i\boldsymbol{k}\cdot(\boldsymbol{R}-\boldsymbol{r})} + \text{hc} \,. \tag{4.8}$$

Electron variables. The electron motion is described by the position operator r and the conjugate momentum p:

$$p = \frac{\hbar}{i} \nabla_r \,. \tag{4.9}$$

The velocity operator, v, is given by

$$mv = p - eA(r), \qquad (4.10)$$

where *m* is the electron mass. Note that *v* is not an electronic operator since it includes field variables through A(r). The electron is bound near the origin by an external static potential $V_0(\mathbf{R})$. If spin is taken into account, the electron variables are supplemented by the spin operator *S*. Magnetic and spin effects will be briefly discussed in subsection 4.5.2. They are neglected elsewhere.

The Hamiltonian. In the nonrelativistic approximation, the Hamiltonian is the sum of five terms: the rest mass energy of the electron, its

kinetic energy, its potential energy in $V_0(\mathbf{R})$, the energy of the longitudinal field and the energy of the transverse fields:

$$H = mc^{2} + \frac{1}{2m} \left[\mathbf{p} - e\mathbf{A}(\mathbf{r}) \right]^{2} + V_{0}(\mathbf{r}) + \frac{\varepsilon_{0}}{2} \int d^{3}\mathbf{R} \, \mathbf{E}_{\parallel}^{2}(\mathbf{R}) + \sum_{k\epsilon} \frac{\hbar\omega}{2} \left(a_{k\epsilon}^{+} a_{k\epsilon} + a_{k\epsilon} a_{k\epsilon}^{+} \right).$$
(4.11)

The energy of the longitudinal field appears to be a constant. representing the energy of the electrostatic field associated with the charge. This constant can be written as

$$\frac{1}{2}\varepsilon_0 \int \mathrm{d}^3 R \, \boldsymbol{E}_{\parallel}^2(\boldsymbol{R}) = \delta m_1 c^2 \,, \tag{4.12}$$

 δm_1 can be considered as a correction to the mechanical rest mass *m* of the electron. The same correction appears in classical theory.

Introduction of a cut off. It is well known that divergences appear in the computation of various physical quantities (such as energy, momentum, ...) associated with a charged point particle interacting with the electromagnetic field. These divergences are due to the contribution of the modes with large wave vectors. In order to deal with finite expressions, we will consider only the coupling of the electron with modes k such that

$$|\mathbf{k}| < k_{\rm M} \,. \tag{4.13}$$

This cut-off $k_{\rm M}$ is chosen not too large so that the nonrelativistic approximation is correct for all the modes which are taken into account ($\hbar\omega_{\rm M} \ll mc^2$ with $\omega_{\rm M} = ck_{\rm M}$). On the other hand, $\omega_{\rm M}$ must be large compared to the characteristic resonance frequencies ω_0 of the bound electron. This gives two bounds for $k_{\rm M}$:

$$\frac{\omega_0}{c} \ll k_{\rm M} \ll \frac{mc}{\hbar} \,. \tag{4.14}$$

It is well known that theories using such a cut off are no longer relativistic invariant [15]. The modes selected by condition (4.13) are not the same in two different reference frames, because of the Doppler effect. It is possible to restore relativistic invariance, by using some more sophisticated cut-off procedures [16]. However, we are not concerned here with the relativistic aspects of radiative problems and we will ignore these difficulties. To summarize, all the sums over k appearing here after must be understood as limited by condition (4.13). The same restriction also applies to the expansion (4.8) of the longitudinal field. The energy of the longitudinal field is then finite and equal to

$$\delta m_1 c^2 = \sum_k \frac{e^2}{2\varepsilon_0 L^3 k^2} = \frac{e^2 k_{\rm M}}{4\pi^2 \varepsilon_0},\tag{4.15}$$

which can be written as $(\alpha/\pi)\hbar\omega_{\rm M}$, where α is the fine structure constant.

Electric dipole approximation. We also suppose in this paper that the binding potential localizes the electron in a volume centred on the origin, with a linear dimension a much smaller than the wavelength of the modes interacting with the particle. (The cut-off $k_{\rm M}$ introduced above is supposed to satisfy $k_{\rm M} a \ll 1$.) Such an assumption which is justified for an atomic electron, allows us to neglect the spatial variation of the fields interacting with the electron. We will then replace the fields at the electron position E(r), A(r) by the fields at the origin E(0), A(0).

The electric dipole approximation is not essential for the derivation of the results presented in this paper. But the calculations are much simpler and the physical conclusions remain unchanged^{*}.

To summarize the previous discussion, we will use hereafter the following Hamiltonian:

$$H_{\rm ED} = (m + \delta m_1) c^2 + \frac{1}{2m} (p - eA(0))^2 + V_0(r) + \sum_{\substack{ke \\ |k| \le k_{\rm M}}} \frac{\hbar \omega}{2} (a_{ke}^+ a_{ke} + a_{ke} a_{ke}^+), \qquad (4.16)$$

* Corrections to the electric dipole approximation are of higher order in 1/c. They have to be considered when relativistic corrections are included in the Hamiltonian (see for example [4]).

with

$$A(0) = \sum_{\substack{k\varepsilon \\ |k| < k_{\rm M}}} \mathscr{A}_k \varepsilon \ a_{k\varepsilon} + {\rm hc} \ . \tag{4.17}$$

4.2.2. Dynamics of the electron interacting with the electromagnetic field. Principle of the calculation. The rate of variation of electron and field variables can be determined from the Hamiltonian (4.16). The corresponding two sets of equations are of course coupled; the field evolution depends on the charge motion and, conversely, the electron experiences a force due to the field.

The derivation of a dynamical equation for the electron from these two sets of coupled equations is well known [8, 13, 14] and follows the general scheme of fig. 6. One first integrates the field equations in presence of the particle. The solution obtained for the field is then inserted in the electron equation. This leads to a quantum dynamical equation describing the motion of the electron interacting with the free field as well as with its own field.

The electromagnetic field in presence of the particle. Since all field operators are expressed in terms of $a_{k\varepsilon}$ and $a^+_{k\varepsilon}$, we start with the Heisenberg equation for $a_{k\varepsilon}(t)$:

$$\dot{a}_{k\varepsilon}(t) = \frac{\mathrm{i}}{\hbar} \left[H(t), \, a_{k\varepsilon}(t) \right] = -\mathrm{i}\omega a_{k\varepsilon}(t) + \frac{\mathrm{i}e}{m\hbar} \, \mathscr{A}_k \varepsilon^* \cdot \pi(t) \,, \tag{4.18}$$

where

$$\pi(t) = mv(t) = p(t) - eA(0, t).$$
(4.19)

Equation (4.18) is then formally integrated and gives

$$a_{k\varepsilon}(t) = a_{k\varepsilon}(t_0) \operatorname{e}^{-\mathrm{i}\omega(t-t_0)} + \frac{\mathrm{i}e}{m\hbar} \mathscr{A}_k \int_{t_0}^t \mathrm{d}t' \operatorname{e}^{-\mathrm{i}\omega(t-t')} \varepsilon^* \cdot \boldsymbol{\pi}(t') \,. \tag{4.20}$$

The evolution of $a_{ke}(t)$ appears to be the superposition of a free evolution [first term of (4.20)] and a forced evolution driven by the motion of the charge [second term of (4.20)]. We finally insert (4.20) in the expansions (4.5) of the transverse field. The contributions of the

two terms of (4.20) correspond respectively to the free fields $(A_f, E_{\perp f})$ and to the source fields $(A_s, E_{\perp s})$. Actually, we need only for the following to know the fields for $\mathbf{R} = \mathbf{0}$ (because of the electric dipole approximation). From (4.5) and (4.20), one easily derives (see appendix A for the details of the calculation):

$$A(0, t) = A_{\rm f}(0, t) + A_{\rm s}(0, t), \qquad (4.21)$$

with

$$A_{\mathbf{f}}(\mathbf{0}, t) = \sum_{k\varepsilon} \left(\mathscr{A}_k \varepsilon \, \mathrm{e}^{-\mathrm{i}\omega(t-t_0)} \right) \, a_{k\varepsilon}(t_0) + \mathrm{hc} \,, \tag{4.22a}$$

$$A_{s}(0, t) = \frac{4}{3} \frac{\delta m_{1}}{me} \pi(t) - \frac{2}{3} \frac{e}{4\pi\epsilon_{0}mc^{3}} \dot{\pi}(t); \qquad (4.22b)$$

similarly,

$$E_{\perp}(0, t) = E_{\perp f}(0, t) + E_{\perp s}(0, t), \qquad (4.23)$$

with

$$E_{\perp f}(\mathbf{0}, t) = \sum_{k\varepsilon} \left(\mathscr{C}_k \varepsilon \, \mathrm{e}^{-\mathrm{i}\omega(t-t_0)} \right) \, a_{k\varepsilon}(t_0) + \mathrm{hc} \,, \tag{4.24a}$$

$$E_{\perp s}(0, t) = -\frac{4}{3} \frac{\delta m_1}{me} \,\dot{\pi}(t) + \frac{2}{3} \frac{e}{4\pi\varepsilon_0 mc^3} \,\ddot{\pi}(t) \tag{4.24b}$$

The quantum Abraham-Lorentz equation. The Heisenberg equations for the operators r and π are

$$m\dot{r}(t) = \frac{\mathrm{i}m}{\hbar} [H, r] = \pi(t), \qquad (4.25)$$

$$\dot{\pi}(t) = \frac{\mathrm{i}}{\hbar} [H, \pi] = -\nabla V_0(r) + eE_\perp(0, t)$$

$$+ \frac{e}{2m} (\pi \times B(0, t) - B(0, t) \times \pi). \qquad (4.26)$$

The last term of the right member of eq. (4.26) is smaller than the second one by a factor v/c [see eq. (4.6)]. It will be neglected hereafter. On the other hand, we notice that $E_{\perp}(0, t)$ is not multiplied by any

electronic operator so that the problem of order raised in the introduction does not appear here. Replacing in (4.26) the total transverse electric field by the sum (4.23) of the free field and the source field and using (4.25) to eliminate π , one gets:

$$m\ddot{r}(t) = -\nabla V_0(r) + eE_{\perp s}(0, t) + eE_{\perp f}(0, t)$$

= $-\nabla V_0(r) - \frac{4}{3}\delta m_1 \ddot{r}(t) + \frac{2}{3}\frac{e^2}{4\pi\varepsilon_0 c^3} \ddot{r}(t) + eE_{\perp f}(0, t).$ (4.27)

This equation is very similar to the classical Abraham-Lorentz equation [17]. This is not surprising since the classical Hamiltonian is similar to (4.16). The general scheme of fig. 6 is valid for both quantum and classical theories, and the Hamilton-Jacobi equations have the same structure as the quantum Heisenberg ones. Since there is no problem of order, the physical interpretation of this equation is clear. Apart from the external potential $V_0(r)$, two fields act on the electron: its own field and the free field. The coupling of the electron with its own field is described by two terms: the first one, proportional to \ddot{r} , corresponds to a mass renormalization from m to $m + \frac{4}{3}\delta m_1^*$. The second one proportional to \vec{r} , is the quantum analogue of the force which produces the radiative damping of the classical particle. The last term of (4.27) describes the coupling of the electron with the free field, i.e., the field which would exist if the particle was not there. This free field may include an incident radiation field. Classically, the description of the electron free motion is obtained by taking $E_{1t}(0, t) = 0$. In quantum mechanics on the contrary, E_{if} is an operator. Although its average value can be zero (in the vacuum state for example), its quadratic average value is always strictly positive. The modifications of the electron dynamics originating from this term correspond to the effect of vacuum fluctuations.

To summarize, it is possible to derive a quantum form of the Abraham-Lorentz equation. The self-reaction terms appear in a natural and unambiguous way and are formally identical in quantum and classical theories. In the quantum equation, the term describing the interaction of the particle with the free field operator cannot be

^{*} As in classical theory, the fact that the mass correction in the Abraham–Lorentz equation and the mass correction in the rest mass energy (4.12) differ by a factor $\frac{4}{3}$ is due to the lack of covariance of the cut-off procedure.

considered as a *c*-number equal to zero in the vacuum. We discuss now some consequences of the quantum nature of this last term.

4.2.3. The difficulties of the quantum dynamical equation

In its traditional form, the classical Abraham–Lorentz equation suffers from a well-known defect: the existence of preacceleration and selfaccelerated solutions. The discussion of the same problem in quantum theory is undoubtedly interesting [18]. We prefer here to focus on some more fundamental difficulties inherent in the quantum formalism and which are hidden behind the formal analogy between the classical and the quantum dynamical equations.

First, it is worth noting that eq. (4.27) relates noncommuting operators. This of course complicates the resolution of the equation, but is unavoidable in a quantum theory of the electron dynamics.

Another difficulty lies in the fact that such an equation includes both particle and field operators, respectively r, p and E_t . This problem does not appear in the classical treatment where the free field, taken equal to zero, does not contribute to the Abraham-Lorentz equation. In quantum mechanics, E_t cannot be cancelled in the same way: physically, this means that the electron cannot escape the vacuum fluctuations. To estimate the two contributions of vacuum fluctuations and self-reaction, we then have to integrate the quantum Abraham-Lorentz equation with a source term; this introduces further complications. To avoid this problem, one may try to deal only with electron operators averaged over the state of the field. Suppose that the radiation field is in the vacuum state at the initial time t_0 : Let $\langle S(t) \rangle_R$ be the average in this radiation state of the particle operator S(t). $\langle S(t) \rangle_R$ is still an operator, acting only in the electron Hilbert space. The average of equation (4.27) gives

$$m\langle \ddot{r}\rangle_{\rm R} = -\langle \nabla V_0(r)\rangle_{\rm R} - \frac{4}{3}\,\delta m_1 \langle \ddot{r}\rangle_{\rm R} + \frac{2}{3}\frac{e^2}{4\pi\varepsilon_0 c^3}\,\langle \ddot{r}\rangle_{\rm R}\,. \tag{4.28}$$

We have used the fact that the average value of E_t is zero in the vacuum state. It seems in this last equation that vacuum fluctuations have disappeared and do not play any role in the evolution of $\langle r \rangle_R$. Actually, the simplicity of eq. (4.28) is misleading; the averaged operators $\langle r \rangle_R$, $\langle p \rangle_R$ do not have the same properties as the original operators r, p. For example, their commutation relations are not the canonical ones $([\langle r \rangle_R, \langle p \rangle_R] \neq i\hbar)$ and their evolution is not unitary. So,

we are no longer able to draw a parallel between the classical Abraham-Lorentz equation and the evolution of $\langle r \rangle_{R}$ given by (4.28).

Furthermore, all the dynamical aspects of the electron motion cannot be described only by the two operators $\langle r \rangle_{R}$ and $\langle p \rangle_{R}$. The value of the product $\langle r \cdot p \rangle_{R}$, for example, cannot be calculated as a function of $\langle r \rangle_{R}$ and $\langle p \rangle_{R}$. Similarly, equation (4.28) is not a true differential equation since $\langle \nabla V_{0}(r) \rangle_{R}$ cannot be expressed in terms of $\langle r \rangle_{R}$ and $\langle p \rangle_{R}$. This equation is then not "closed": it links $\langle r \rangle_{R}$ and its derivatives to another operator $\langle \nabla V_{0}(r) \rangle_{R}$ for which we have to find the evolution equation (the vacuum fluctuations will probably contribute to this equation, which proves that their disappearance in (4.28) was only superficial).

The previous discussion clearly shows that we cannot avoid to study now the evolution of electron observables other than r and p and to ask about their rate of variation the same type of questions concerning the respective contributions of vacuum fluctuations and self-reaction. This problem will be dealt with in the next section. Note that the simplifications which occurred above for the evolution of r (no order problem in (4.27) and nullity of the vacuum average of $E_{\rm f}$ in (4.28)) will not occur for the evolution of a general particle observable.

There is a supplementary reason for studying the evolution of operators other than r and p. Very few experiments are dealing with the position or the momentum of an atomic electron. One rather measures the population of an energy level, the frequency or the damping of some atomic oscillations associated with off-diagonal elements of the density matrix. This suggests that operators such as $|i\rangle\langle i|$ or $|i\rangle\langle j|$ (where $|i\rangle$ and $|j\rangle$ are eigenstates of the electron in the potential V_0) are more directly connected to experiment than r and p.

4.3. Identification of the contributions of vacuum fluctuations and selfreaction to the rate of variation of an arbitrary atomic observable

In this section, we first evaluate the contributions of the various terms of the interaction Hamiltonian to the rate of variation, dG/dt, of an arbitrary atomic observable G (subsection 4.3.1). We then discuss the problem of order which arises when the total field appearing in this rate is split into its free part and its source part (subsection 4.3.2). We solve this problem by introducing hermiticity conditions associated with the requirement of physical meaning (subsections 4.3.3 and 4.3.4).

Finally, we discuss the problem of the vacuum average of the various rates which requires a perturbative calculation (subsection 4.3.5).

4.3.1. Contribution of the various terms of the interaction Hamiltonian It will be convenient to divide the total Hamiltonian (4.16) into three parts, the Hamiltonian

$$H_{\rm s} = \frac{p^2}{2m} + V_0(r) \tag{4.29}$$

of the electron in the static potential $V_0(r)$, the Hamiltonian

$$H_{\rm R} = \sum_{k\varepsilon} \hbar \omega \left(a_{k\varepsilon}^+ a_{k\varepsilon} + \frac{1}{2} \right) \tag{4.30}$$

of the transverse radiation field, and the Hamiltonian

$$V = -\frac{e}{m} \boldsymbol{p} \cdot \boldsymbol{A}(\boldsymbol{0}) + \frac{e^2}{2m} \boldsymbol{A}^2(\boldsymbol{0}) + \frac{e^2}{4\pi\varepsilon_0} \frac{k_{\rm M}}{\pi}$$
(4.31)

of the electron-field coupling, characterized by the electric charge e and including the energy of the longitudinal field of the electron (4.15).

The rate of variation of an atomic observable G can then be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}G = \frac{\mathrm{i}}{\hbar}[H_{\mathrm{s}},G] + \frac{\mathrm{i}}{\hbar}[V,G].$$
(4.32)

We discuss now the contributions of the three terms of V to the second commutator (to order 2 in e).

(i) The last term of V is a c-number which commutes with G and which therefore does not produce any dynamical evolution. This term corresponds to an overall displacement of electronic energy levels which we have already interpreted in section 4.2 as due to the contribution $\delta m_1 c^2$ of the Coulomb field of the electron to the electron rest mass energy. This effect must obviously be associated with self-reaction since it originates from the longitudinal field created by the electron itself. The same situation exists in classical theory.

(ii) The second term of V does not depend on atomic variables and

thus commutes with G. It has no dynamical consequences. It nevertheless contributes to the total energy. Let us calculate its average value. Since we limit the calculation to order 2 in e, we can replace A(0) by the free field $A_f(0)$. The term then becomes independent of the atomic state and can be interpreted as an overall shift of the electron energy levels. The value of this shift for the vacuum state of the field is given by

$$\left\langle 0 \left| \frac{e^2}{2m} A_{\rm f}^2(\mathbf{0}) \right| 0 \right\rangle = \sum_{\mathbf{k}\varepsilon} \frac{e^2 \mathscr{A}_{\mathbf{k}^2}}{2m} = \frac{e^2 \hbar k_{\rm M}^2}{8\pi^2 \varepsilon_0 mc} = \delta m_2 c^2 \,. \tag{4.33}$$

This shift can be interpreted as a new contribution, $\delta m_2 c^2$, to the electron rest mass energy. It is proportional to the vacuum average of the square of the free field and thus is clearly a vacuum fluctuation effect, the interpretation of which is well known [19]: it is the kinetic energy associated with the electron vibrations produced by the vacuum fluctuations of the electric field.

(iii) Finally, only the first term of (4.31) contributes to the dynamical evolution of G. The corresponding term of (4.32) can be written as

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{coupling}} = -\frac{\mathrm{i}e}{\hbar m} \left[\mathbf{p} \cdot \mathbf{A}(\mathbf{0}), G \right] = e\mathbf{N} \cdot \mathbf{A}(\mathbf{0}), \qquad (4.34)$$

where N is an atomic operator given by

$$N = -\frac{\mathrm{i}}{\hbar m} \left[p, G \right]. \tag{4.35}$$

If G coincides with p or r, N is equal to 0 or to a constant and (4.34) reduces to 0 or to A(0). We find again that the evolution of r and p is very simple.

Finally, combining (4.32) and (4.34) and reintroducing the time explicitly in the operators, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) = \frac{\mathrm{i}}{\hbar} \left[H_{\mathrm{s}}(t), G(t)\right] + eN(t) \cdot A(0, t) \,. \tag{4.36}$$

4.3.2. The problem of order

In expression (4.36), we split, as in section 4.2, the field A(0, t) in two parts, $A_{\rm f}(0, t)$ representing the free field and $A_{\rm s}(0, t)$ representing the

source field. If the atomic operator N(t) does not reduce to 0 or to a constant (as it is the case for r and p), we are immediately faced with the problem of order mentioned in the introduction. Since N(t) and A(0, t) commute, we can start in eq. (4.36) with any order,

$$N(t) \cdot A(0, t)$$
, or $A(0, t) \cdot N(t)$.

More generally, we can write the last term of (4.36) as

$$e\lambda \mathbf{N}(t) \cdot \mathbf{A}(\mathbf{0}, t) + e(1 - \lambda)\mathbf{A}(\mathbf{0}, t) \cdot \mathbf{N}(t), \qquad (4.37)$$

with λ arbitrary. Replacing A by $A_f + A_s$ leads to

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{coupling}} = \left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{vf}} + \left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}},\tag{4.38}$$

where the two rates

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{vf}} = e\lambda \mathbf{N}(t) \cdot \mathbf{A}_{\mathrm{f}}(\mathbf{0}, t) + e(1-\lambda)\mathbf{A}_{\mathrm{f}}(\mathbf{0}, t) \cdot \mathbf{N}(t), \qquad (4.39)$$

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}} = e\lambda \mathbf{N}(t) \cdot \mathbf{A}_{\mathrm{s}}(\mathbf{0}, t) + e(1-\lambda)\mathbf{A}_{\mathrm{s}}(\mathbf{0}, t) \cdot \mathbf{N}(t), \qquad (4.40)$$

depend on λ since A_f and A_s do not commute separately with N(t).

 λ being arbitrary, the splitting (4.38) of the total rate is not uniquely defined [11–13].

4.3.3. Physical interpretation and hermiticity conditions

In order to remove this indetermination, we introduce now some simple physical considerations.

Suppose that G is a physical observable, represented by a Hermitian operator. The rate of variation of G due to the coupling is also a Hermitian operator [this clearly appears on (4.36) since N(t) and A(0, t) are commuting Hermitian operators]. Our purpose is to split this rate of variation in two rates, involving A_f and A_s respectively, and having separately a well-defined physical interpretation in terms of vacuum fluctuations and self-reaction. This interpretation requires that (4.39) and (4.40) should have separately a physical meaning, and

consequently should be separately Hermitian. This condition determines λ which must be equal to 1/2. Thus the splitting of dG/dt is unique and given by

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{vf}} = e_2^1 [\mathbf{N}(t) \cdot \mathbf{A}_{\mathrm{f}}(\mathbf{0}, t) + \mathbf{A}_{\mathrm{f}}(\mathbf{0}, t) \cdot \mathbf{N}(t)], \qquad (4.41)$$

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}} = e_2^1 [\mathbf{N}(t) \cdot \mathbf{A}_{\mathrm{s}}(\mathbf{0}, t) + \mathbf{A}_{\mathrm{s}}(\mathbf{0}, t) \cdot \mathbf{N}(t)]. \qquad (4.42)$$

This could have been obtained by choosing the completely symmetrical order in (4.37).

4.3.4. Generalization to more complicated situations

It may happen that the total rate of variation of G does not appear as simple as in (4.34), i.e., as the product of an atomic observable by a field observable. For example, if we had not made the electric dipole approximation, the electron position operator r would appear in A. Another example is the appearance of non-Hermitian operators in (4.34) when the total field A is decomposed into its positive and negative frequency components which are not Hermitian. We extend now the previous treatment to these more complex situations.

We first note that, in the most general case, the total rate of variation of a physical observable G (due to the coupling with the field) can always be written as

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{coupling}} = \sum_{i} e(A_{i}N_{i} + A_{i}^{\dagger}N_{i}^{\dagger}), \qquad (4.43)$$

where the A_i are field operators and the N_i atomic operators which commute, but which are not necessarily Hermitian. For example, in simple models dealing with two-level atoms and using the "rotating wave approximation", the coupling Hamiltonian is taken of the form

$$V = -(E^{(+)}D^{+} + E^{(-)}D^{-}), \qquad (4.44)$$

where D^+ and D^- are the raising and lowering components of the dipole moment operator, and $E^{(+)}$ and $E^{(-)}$ the positive and negative frequency components of the field [20].

In such a case, we get

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{coupling}} = E^{(+)}F^{+} + E^{(-)}F^{-}, \qquad (4.45)$$

with

$$F^{+} = \frac{i}{\hbar} [G, D^{+}], \qquad (4.46)$$

(4.45) has a structure similar to (4.43).

G being Hermitian, the right side of (4.43) is of course also Hermitian, but since the atomic and field operators commute, it could be written as well as

$$\sum_{i} (N_i A_i + N_i^{\dagger} A_i^{\dagger}) \quad \text{or} \quad \sum_{i} (N_i A_i + A_i^{\dagger} N_i^{\dagger}),$$

or

$$\sum_i \left(A_i N_i + N_i^{\dagger} A_i^{\dagger}\right),\,$$

or any combination of these forms. When A_i is replaced by $A_{if} + A_{is}$, it is easy to see that the hermiticity condition imposed on $(dG/dt)_{vf}$ and $(dG/dt)_{sr}$ is no longer sufficient for removing the indetermination. For example

$$\sum_{i} (A_{it}N_i + N_i^{\dagger}A_{it}^{\dagger}) \text{ and } \sum_{i} (N_iA_{it} + A_{it}^{\dagger}N_i^{\dagger})$$

are two Hermitian rates of variation which could be attributed to vacuum fluctuations and which generally do not coincide since A_{if} and A_{if}^{+} do not commute with N_i and N_i^{+} . For the simple model considered above [see (4.44) and (4.45)] these two rates respectively correspond to the anti normal and normal orders. So, when the A_i and the N_i are not Hermitian, we must introduce a new requirement.

Coming back to the expression (4.43) of the total rate, we first re-express this rate in terms of physically well-defined atomic and field quantities, i.e., in terms of Hermitian operators. The physical justification for such a transformation is that we want to be able to analyse the total physical rate in terms of physical quantities. For example, it would be difficult to elaborate a physical picture from an expression involving only the positive frequency part of the field which is not observable. Introducing the real and the imaginary part of the various operators appearing in (4.43), and using the fact that field and atomic operators commute, we transform (4.43) into the strictly equivalent expression

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{coupling}} = e \sum_{i} \left(\frac{A_{i} + A_{i}^{+}}{2}\right) \left(N_{i} + N_{i}^{+}\right) + e \sum_{i} \left(\frac{A_{i} - A_{i}^{+}}{2\mathrm{i}}\right) \\ \times \left(\frac{N_{i} - N_{i}^{+}}{-\mathrm{i}}\right).$$

$$(4.47)$$

But now the total rate appears as a sum of products of observables of the field by observables of the particle as in (4.34) and the procedure of the previous section can be applied to each of these products and singles out the completely symmetric order

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{coupling}} = e \, \frac{1}{2} \sum_{i} \left[\left(\frac{A_{i} + A_{i}^{+}}{2}\right) (N_{i} + N_{i}^{+}) + (N_{i} + N_{i}^{+}) \left(\frac{A_{i} + A_{i}^{+}}{2}\right) \right] + e \, \frac{1}{2} \sum_{i} \left[\left(\frac{A_{i} - A_{i}^{+}}{2\mathrm{i}}\right) \left(\frac{N_{i} - N_{i}^{+}}{-\mathrm{i}}\right) + \left(\frac{N_{i} - N_{i}^{+}}{-\mathrm{i}}\right) \left(\frac{A_{i} - A_{i}^{+}}{2\mathrm{i}}\right) \right]$$
(4.48)

when A_i is replaced by $A_{if} + A_{is}$ in (4.47)

To summarize the previous discussion, a unique well-defined order is singled out by the following two conditions:

(i) The two rates $(dG/dt)_{vf}$ and $(dG/dt)_{sr}$ must have separately a physical meaning.

(ii) Before replacing A_i by $A_{if} + A_{is}$, the total rate must be expressed in terms of physical field and particle quantities.

4.3.5. Vacuum average of the various rates

To progress further, we must now take the average of the two rates (4.41) and (4.42) over the vacuum state of the field. The calculation of such an average is not trivial (as it was the case in the previous section for r and p). This is due to the presence of products of field and atomic operators in the right side of the equations. For example, when we average the product $eA_{\rm f}(0, t) \cdot N(t)$, we must not forget that these two operators are correlated since the atomic operator N(t) depends on the free field which has perturbed its evolution from the initial time t_0 to t. Consequently, before taking the vacuum average, we have first to

calculate, to a given order in e, N(t) as a function of unperturbed (free) atomic and field operators. Since we limit our calculation to order 2 in e (i.e., to order 1 in the fine structure constant α), we must solve the Heisenberg equation for N(t) up to order e [e already appears in (4.41) and A_t is of order e^0]. When we insert the perturbative expansion of N(t), which contains zero or one field operator taken at a time t' such that $t_0 < t' < t$, in the product $A_t(0, t) \cdot N(t)$, and when we take the vacuum average, we get one-time averages $\langle 0|A_t(t)|0 \rangle$ which are equal to zero, and two-time averages such as

 $\langle 0|A_{\mathrm{fi}}(\mathbf{0},t)A_{\mathrm{fi}}(\mathbf{0},t')|0\rangle$

(with i, j = x, y, z), i.e., vacuum averages of products of two components of free field operators taken at two different times. Similar considerations can be made about the other products of (4.41) and (4.42).

Actually, such perturbative calculations are not specific of our choice of the symmetrical order in (4.37) and they can be found in other papers where other choices are investigated [11, 13]. Rather than duplicating these calculations, we prefer in the next section to reconsider our problem of the separation between vacuum fluctuations and self-reaction from a more general point of view where one asks the same type of questions for a small system \mathcal{G} (generalizing the atom) interacting with a large reservoir \mathcal{R} (generalizing the field). The extension of the previous treatment to this more general situation is straightforward. It leads to mathematical expressions which, because of their generality, have a more transparent structure. In particular, since we don't use, in the intermediate steps of the calculation, simplifications specific to a particular choice of \mathcal{S} and \mathcal{R} , we find that some important statistical functions of \mathcal{S} and \mathcal{R} appear explicitly in the final expressions and this provides a deeper physical insight in the problem.

4.4. Extension of the previous treatment to a system \mathcal{S} interacting with a large reservoir \mathcal{R}

4.4.1. Introduction - outline of the calculation

It is well known that spontaneous emission, and all associated effects such as radiative corrections or radiative damping, can be considered as a problem which can be studied in the general framework of the quantum theory of relaxation in the motional narrowing limit [21, 22]. Such a theory deals with the damping and energy shifts of a small system \mathscr{S} coupled to a large reservoir \mathscr{R} . Large means that \mathscr{R} has many degrees of freedom so that the correlation time τ_c of the observables of \mathscr{R} is very short, allowing a perturbative treatment of the effect of the \mathscr{S} - \mathscr{R} coupling during a time τ_c . For spontaneous emission, the atom plays the role of \mathscr{S} , the vacuum field, with its infinite number of modes, plays the role of \mathscr{R} , and the correlation time of vacuum fluctuations is short enough for having the motional narrowing condition well fulfilled.

This point of view suggests that we can extend to any \mathscr{G} - \mathscr{R} system the same type of questions we have asked about the atom field system. Is it possible to understand the evolution of \mathscr{G} as being due to the effect of the reservoir fluctuations acting upon \mathscr{G} , or should we invoke a kind of self-reaction, \mathscr{G} perturbing \mathscr{R} which reacts back on \mathscr{G} ? Is it possible to make a clear and unambiguous separation between the contributions of these two effects?

The extension of the treatment of section 4.3 to this more general case is straightforward. We first note that, although most presentations of the quantum theory of relaxation use the Schrödinger picture (one derives a master equation for the reduced density operator of \mathcal{S}), we have to work here in the Heisenberg picture. Actually, the Heisenberg picture is also used in the derivation of the Langevin-Mori equations describing the evolution of the observables of \mathcal{S} as being driven by a Langevin force (having a zero reservoir average) and a friction force (producing not only a damping but also a shift of energy levels) [21, 23, 24]. Our problem here is to identify in the friction force the contribution of reservoir fluctuations and self-reaction. Following the general scheme of fig. 6, we start with the Hamiltonian of the \mathcal{S} - \mathcal{R} system

$$H = H_{\rm s} + H_{\rm R} + V, \tag{4.49}$$

where

$$V = -\sum_{i} R_{i}S_{i} \tag{4.50}$$

is the interaction Hamiltonian, and R_i and S_i are Hermitian observables of \mathcal{R} and \mathcal{S} [we can always suppose that V has been put in this form, eventually after a transformation analogous to the one changing (4.43) into (4.47)]. We then write the Heisenberg equation for the reservoir observable R_i appearing in (4.50). The solution of this equation can be written as the sum of a free unperturbed part R_{if} (solution to order 0 in V), and of a "source part" R_{is} due to the presence of \mathcal{S} (solution to order 1 and higher in V)

$$R_i = R_{if} + R_{is} \,. \tag{4.51}$$

Expression (4.51) is finally inserted in the last term of the Heisenberg equation for an arbitrary system observable G,

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar} \left[G, H_{\mathrm{s}} \right] + \frac{1}{\mathrm{i}\hbar} \left[G, -\sum_{i} R_{i} S_{i} \right], \qquad (4.52)$$

in order to indentify the contribution of reservoir fluctuations and self-reaction. The problem of order between the commuting observables R_i and

$$N_i = \frac{-1}{i\hbar} \left[G, S_i \right] \tag{4.53}$$

in the last term of (4.52) arises in the same way as in section 4.3 and is solved by the same physical considerations which impose the completely symmetric order. We thus get

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{rf}} = \frac{1}{2} \sum_{i} \left(N_i R_{i\mathrm{f}} + R_{i\mathrm{f}} N_i\right), \qquad (4.54\mathrm{a})$$

$$\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)_{\mathrm{sr}} = \frac{1}{2} \sum_{i} \left(N_i R_{i\mathrm{s}} + R_{i\mathrm{s}} N_i\right). \tag{4.54b}$$

It remains to perform the average of $(dG/dt)_{rf}$ and $(dG/dt)_{sr}$ in the reservoir state (reservoir average). As explained in subsection 4.3.5, this requires a perturbative calculation leading, to order 2 in V, to two time operator averages which can be expressed in terms of correlation functions and linear susceptibilities. This is precisely where the advantage of working with a general \mathscr{P} - \mathscr{R} system appears. As already explained in subsection 4.3.5, the intermediate steps of the calculation remain general. For example, when we solve perturbatively the Heisenberg equation for R_{i} , we get for the source part, R_{is} , a

perturbative expansion where, at the lowest order, the linear susceptibility of the reservoir appears. In the particular case of the atom field system, the calculation of the source field has been done exactly and the result expressed in terms of atomic operators and time derivatives of these operators (see eq. (4.24)). In such an intermediate calculation, the fact that the susceptibility of the electromagnetic field is involved remains hidden, and thus this important function does not appear explicitly in the final result for $\langle (dG/dt)_{sr} \rangle$.

In order not to increase too much the length of this paper, we will not give here the detailed calculations of $\langle (dG/dt)_{rf} \rangle$ and $\langle (dG/dt)_{sr} \rangle$ following the general scheme outlined above. These calculations will be presented in a forthcoming paper [25], together with a discussion of the various approximations used in the derivation. We just give in this section the results of these calculations which will be useful for the discussion of section 4.5. We first give the expression of the correlation functions and linear susceptibilities in terms of which we then discuss the structure of the terms describing the effect of reservoir fluctuations and self-reaction.

4.4.2. Correlation functions and linear susceptibilities [26]

When the reservoir average is calculated up to order 2 in V, the reservoir only appears in the final result through two statistical functions.

The first one

$$C_{ii}^{(R)}(\tau) = \frac{1}{2} \langle \{ R_{if}(t) R_{jf}(t-\tau) + R_{jf}(t-\tau) R_{if}(t) \} \rangle_{R}$$
(4.55)

is the symmetric correlation function of the two free reservoir observables R_{it} and R_{jt} . The average is taken over the initial state of the reservoir which is supposed to be stationary, so that $C_{ij}^{(R)}$ only depends on τ . $C_{ij}^{(R)}(\tau)$ is a real function of τ which describes the dynamics of the fluctuations of R_{if} and R_{jt} in the reservoir state.

The second statistical function

$$\chi_{ij}^{(\mathsf{R})}(\tau) = \frac{\mathrm{i}}{\hbar} \langle [R_{i\mathrm{f}}(t), R_{j\mathrm{f}}(t-\tau)] \rangle_{\mathrm{R}} \theta(\tau) , \qquad (4.56)$$

where $\theta(\tau)$ is the Heaviside function, is the linear susceptibility of the reservoir. It generally depends on the reservoir state. $\chi_{ij}^{(R)}(\tau)$ is also a real function of τ , which describes the linear response of the averaged observable $\langle R_{if}(t) \rangle_{R}$ when the reservoir is acted upon by a perturbation

proportional to R_{jf} . Note that both C and χ have a classical limit (if this is case for \mathcal{R}): this is obvious for C, and for χ , the commutator divided by $i\hbar$ becomes the Poisson bracket.

Similar functions can of course be introduced for the small system \mathscr{S} in an energy level $|a\rangle$, with energy E_a . We will denote them by

$$C_{ij}^{(S,a)}(\tau) = \frac{1}{2} \langle a | S_{if}(t) S_{jf}(t-\tau) + S_{jf}(t-\tau) S_{if}(t) | a \rangle, \qquad (4.57)$$

$$\chi_{ij}^{(\mathrm{S},a)}(\tau) = \frac{\mathrm{i}}{\hbar} \langle a | [S_{i\mathrm{f}}(t), S_{j\mathrm{f}}(t-\tau)] | a \rangle \theta(\tau) , \qquad (4.58)$$

where the upper indices (S, a) mean that \mathscr{S} is in $|a\rangle$, and where the lower index f on S_{it} and S_{jt} means that these operators are unperturbed system operators evolving only under the effect of H_s (as for R_{if} and R_{jf} which evolve only under the effect of H_R).

Finally, we will note $\hat{C}_{ij}^{(R)}(\omega)$, $\hat{\chi}_{ij}^{(R)}(\omega)$, $\hat{C}_{ij}^{(S,a)}(\omega)$, $\hat{\chi}_{ij}^{(S,a)}(\omega)$ the Fourier transforms of (4.55)–(4.58), the Fourier transform $f(\omega)$ of $f(\tau)$ being defined by

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\tau) \,\mathrm{e}^{-\mathrm{i}\omega\tau} \,\mathrm{d}\tau \,. \tag{4.59}$$

4.4.3. Structure of the results describing the effect of reservoir fluctuations. The first important result concerning the reservoir averaged rate of variation $\langle (dG/dt)_{rf} \rangle_{R}$ is that only $C_{ij}^{(R)}(\tau)$ appears in its expression, and not $\chi_{ij}^{(R)}(\tau)$. Furthermore, the corresponding relaxation equations have exactly the same structure as the ones which would be obtained if the reservoir observables R_i were replaced in the interaction Hamiltonian (4.50) by fluctuating *c*-numbers $r_i(t)$ having the same correlation functions $C_{ii}^{(R)}(\tau)$,

$$\overline{r_i(t)r_j(t-\tau)} = C_{ij}^{(\mathsf{R})}(\tau) \,. \tag{4.60}$$

We conclude that, with our choice of the symmetric order in (4.54), the effect of reservoir fluctuations is the same as the one of a classical random field having the same symmetric correlation function as the quantum one.

We show also in reference [25] that the average rate of variation $\langle (dG/dt)_{rf} \rangle_{R}$, and also $\langle (dG/dt)_{sr} \rangle_{R}$, can be decomposed into a Hamiltonian part and a non-Hamiltonian part. The Hamiltonian part des-

cribes (in the so-called secular approximation) a shift of the energy levels of \mathscr{S} due to the $\mathscr{G}-\mathscr{R}$ coupling. The non-Hamiltonian part describes, among other things, the exchange of energy between \mathscr{S} and \mathscr{R} .

The shift, $(\delta E_a)_{rt}$, of the level $|a\rangle$ of \mathscr{S} due to reservoir fluctuations is found to be

$$(\delta E_a)_{\rm rf} = -\frac{1}{2} \sum_{ij} \int_{-\infty}^{+\infty} \mathrm{d}\tau \, C_{ij}^{(\rm R)}(\tau) \chi_{ij}^{(\rm S,a)}(\tau) \,. \tag{4.61}$$

Such a result has a very simple structure and a very clear physical meaning (fig. 7a). One can consider that the fluctuations of \mathcal{R} , characterized by $C_{ij}^{(\mathcal{R})}(\tau)$, polarize \mathscr{S} which responds to this perturbation in a way characterized by $\chi_{ij}^{(S,a)}(\tau)$. The interaction of the fluctuations of \mathscr{R} with the polarization to which they give rise in \mathscr{S} has a nonzero value because of the correlations which exist between the fluctuations of \mathscr{R} and the induced polarization in \mathscr{S} . The factor 1/2 in (4.61) is even somewhat similar to the factor 1/2 appearing in the polarization energy of a dielectric. Finally, it is shown in [25] (by parity arguments) that only the reactive part of $\chi_{ij}^{(S,a)}(\tau)$ contributes to the integral (4.61). To summarize this discussion, we can say that the energy shift $(\delta E_a)_{\rm rf}$ can be interpreted as resulting from the polarization of \mathscr{S} by the fluctuations of \mathscr{R} .



Fig. 7. Physical pictures for the effect of reservoir fluctuations and self-reaction. (a) Reservoir fluctuations: the reservoir fluctuates and interacts with the polarization induced in the small system. (b) Self-reaction: the small system fluctuates and polarizes the reservoir which reacts back on the small system.

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We now turn to the discussion of the non-Hamiltonian part of $\langle (dG/dt)_{rf} \rangle_{R}$. A very suggestive result concerns the absorption of energy by \mathscr{S} when \mathscr{S} is in $|a\rangle$. The effect is described by $\langle (dH_s/dt)_{rf} \rangle_{R,a}$ (G is replaced by H_s and the average is taken over both the state of the reservoir and the state $|a\rangle$ of \mathscr{S}). One finds

$$\left\langle \left(\frac{\mathrm{d}H_{\mathrm{s}}}{\mathrm{d}t}\right)_{\mathrm{rf}}\right\rangle_{\mathrm{R},a} = -\pi \sum_{ij} \int_{-\infty}^{+\infty} \mathrm{d}\omega \ \hat{C}_{ij}^{(\mathrm{R})}(\omega) \,\mathrm{i}\omega [\hat{\chi}_{ij}^{(\mathrm{S},a)^*}(\omega) - \hat{\chi}_{ji}^{(\mathrm{S},a)}(\omega)] \,.$$

$$(4.62)$$

This result is identical with the one which would be obtained if a classical random perturbation with a spectral power density $\hat{C}_{ij}^{(R)}(\omega)$ was acting upon \mathscr{S} (see reference [27], §124; see also [28]). The term inside the brackets is actually the dissipative part of the susceptibility of \mathscr{S} at frequency ω . This dissipative part is multiplied by the spectral power density of the perturbation produced by \mathscr{R} . Here again we get a result in agreement with the picture of \mathscr{S} responding to the fluctuations of \mathscr{R} .

4.4.4. Structure of the terms describing the effect of self-reaction

As expected, the reservoir appears in $\langle (dG/dt)_{sr} \rangle_{R}$ only through the linear susceptibility $\chi_{ij}^{(R)}(\tau)$. Thus, it appears that \Re is now polarized by \mathscr{G} . We can interpret the rate of variation $\langle (dG/dt)_{sr} \rangle_{R}$ as being due to the reaction back on \mathscr{G} of the polarization of \Re by \mathscr{G} (fig. 7b).

As in the previous section, it will be interesting now to discuss the shift $(\delta E_a)_{sr}$ of $|a\rangle$ due to self-reaction. This shift is found to be

$$(\delta E_a)_{\rm sr} = -|\frac{1}{2} \sum_{ij} \int_{-\infty}^{+\infty} \mathrm{d}\tau \,\chi_{ij}^{(\rm R)}(\tau) C_{ij}^{(\rm S,a)}(\tau) \,. \tag{4.63}$$

The same comments can be made as for (4.61), the roles of \mathscr{S} and \mathscr{R} being interchanged. Here also, only the reactive part of $\chi_{ij}^{(R)}(\tau)$ contributes to (4.63).

Finally, we can study the equation corresponding to (4.62) for self-reaction

$$\left\langle \left(\frac{\mathrm{d}H_{\rm s}}{\mathrm{d}t}\right)_{\rm sr}\right\rangle_{\rm R,a} = \pi \sum_{ij} \int_{-\infty}^{+\infty} \mathrm{d}\omega \ \hat{C}_{ij}^{\rm (S,a)}(\omega) \,\mathrm{i}\omega \left[\hat{\chi}_{ij}^{\rm (R)^*}(\omega) - \hat{\chi}_{ji}^{\rm (R)}(\omega)\right]. \tag{4.64}$$

Here also the same comments can be made, the roles of \mathscr{S} and \mathscr{R} being interchanged. Note however the difference of sign between (4.62)

and (4.64). This is due to the fact that (4.62) describes a transfer of energy from \mathcal{R} to \mathcal{S} (gain for \mathcal{S}), whereas (4.64) describes a transfer from \mathcal{S} to \mathcal{R} (loss for \mathcal{S}). Actually (4.62) can also describe a loss for \mathcal{S} , and (4.64) a gain, if there are adequate population inversions in \mathcal{S} for (4.62), in \mathcal{R} for (4.64), responsible for an amplifying behaviour of the susceptibility (instead of a dissipative one).

It must be emphasized that all the results derived in this section follow from the choice of the symmetric order in the total rate dG/dtbefore replacing R_i by $R_{if} + R_{is}$. They can be all interpreted in terms of two simple physical pictures: \Re fluctuates and polarizes \mathcal{G} , \mathcal{G} fluctuates and polarizes \Re . The clear physical structure of the results which have been obtained in this way, and the coherence of the physical interpretation can be considered as a confirmation *a posteriori* of the pertinence of the method of separation we propose in this paper. The privileged character of the symmetric order for physical interpretation is thus confirmed.

Remark. The previous treatment allows an easy and clear discussion of the consequences of the fluctuation dissipation theorem [26]. Note first that this theorem holds only for systems in thermal equilibrium (populations of the various levels varying according to the Boltzmann factor corresponding to a given temperature). The above treatment is more general, and is valid for an arbitrary stationary state of the reservoir (the energy levels may have any population). For a reservoir at thermal equilibrium which is the case for the electromagnetic field in the vacuum state, the fluctuation dissipation theorem states that the correlation function $\hat{C}_{ij}^{(R)}(\omega)$ is proportional to the dissipative part of the corresponding reservoir susceptibility. Thus, in this case, one could formally replace in (4.61) and (4.62) the correlation function of the reservoir by the dissipative part of the reservoir susceptibility and make the reservoir fluctuations to apparently disappear from formulae (4.61) and (4.62). But it is also clear that, after such a formal transformation, these two expressions have lost their physical meaning since they appear as the product of two susceptibilities.

4.5. Physical discussion. Contributions of vacuum fluctuations and selfreaction to the radiative corrections and radiative damping of an atomic electron

We now come back to our initial problem concerning the respective

contributions of vacuum fluctuations and self-reaction for an atomic electron.

We have given in the previous section very simple and general expressions for important physical effects such as the shifts of the energy levels of \mathcal{S} , or the energy exchanges between \mathcal{S} and \mathcal{R} , these expressions involving only correlation functions or linear susceptibilities of \mathcal{S} and \mathcal{R} .

What we have to do now is to calculate first these correlation functions and linear susceptibilities in the case where \mathcal{S} is an atom and \mathcal{R} the vacuum electromagnetic field (subsection 4.5.1). We will then be able, using (4.61)–(4.64), to discuss the respective contributions of vacuum fluctuations and self-reaction to the radiative corrections for an atomic electron (subsection 4.5.2) and the rate of exchange of energy between the atom and the field (subsection 4.5.3).

4.5.1. Correlation functions and linear susceptibilities for the vacuum field and for an atomic electron

Comparing (4.50) and the first term of (4.31) (which is the only one to produce a dynamical evolution of atomic observables, see subsection 4.3.1), we get, for the atom field problem

$$R_{i}(t) = A_{i}(0, t),$$

$$S_{i}(t) = \frac{e}{m} p_{i}(t),$$
(4.65)

with i = x, y, z.

According to (4.55) and (4.56), the relevant statistical functions for the field are:

$$C_{ij}^{(\mathrm{R})}(\tau) = \frac{1}{2} \langle 0 | A_{if}(0, t) A_{jf}(0, t-\tau) + A_{jf}(0, t-\tau) A_{if}(0, t) | 0 \rangle, \quad (4.66)$$

$$\chi_{ij}^{(\mathsf{R})}(\tau) = \frac{1}{\hbar} \langle 0 | [A_{if}(\mathbf{0}, t), A_{jf}(\mathbf{0}, t-\tau)] | 0 \rangle \,\theta(\tau) \,, \tag{4.67}$$

where $|0\rangle$ is the vacuum state of the field and the index f means a free evolution for the operators. The calculation of these two functions is straightforward and given in the appendix B. One gets:

$$C_{ij}^{(\mathsf{R})}(\tau) = \frac{\hbar \delta_{ij}}{12\pi^2 \varepsilon_0 c^3} \int_{-\omega_{\mathsf{M}}}^{\omega_{\mathsf{M}}} |\omega| \, \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\omega \,, \tag{4.68}$$
$$\chi_{ij}^{(\mathsf{R})}(\tau) = \frac{\delta_{ij}}{3\pi^2 \varepsilon_0 c^3} \left(\omega_{\mathsf{M}} \delta(\tau) - \frac{\pi}{2} \, \delta'(\tau) \right). \tag{4.69}$$

The Fourier transforms of (4.68) and (4.69) are also useful:

$$\hat{C}_{ij}^{(\mathsf{R})}(\omega) = \frac{\hbar \delta_{ij}}{12\pi^2 \varepsilon_0 c^3} |\omega| , \qquad (4.70)$$

$$\hat{\chi}_{ij}^{(R)}(\omega) = \frac{\delta_{ij}}{6\pi^3 \varepsilon_0 c^3} \left(\omega_{\rm M} - i\frac{\pi}{2}\omega \right).$$
(4.71)

Because of the cut-off (4.13) expressions (4.70) and (4.71) hold only for $|\omega| < \omega_{\rm M}$, \hat{C} and $\hat{\chi}$ being equal to zero elsewhere. It follows that the δ and δ' functions in (4.69) have actually a width $1/\omega_{\rm M}$.

Remarks

(i) The linear susceptibility of the field relates the linear response of the field, at point **0** and at time *t*, to the perturbation associated with the motion of the electron at earlier times. This response is nothing but the source field produced by the electron (and calculated to lowest order in *e*). Going back to the precise definition of χ [26], and using (4.69), we get for the "linear response" $\langle 0|A_i(t)|0\rangle$:

$$\langle 0|A_i(t)|0\rangle = \sum_j \int_{-\infty}^{+\infty} d\tau \Big\langle 0 \Big| \chi_{ij}^{(\mathbb{R})}(\tau) \Big(\frac{e}{m}\Big) p_j(t-\tau) \Big| 0 \Big\rangle$$

$$= \frac{e\omega_{\mathrm{M}}}{3\pi^2 \varepsilon_0 c^3 m} \langle 0| p_i(t)|0\rangle - \frac{e}{6\pi \varepsilon_0 c^3 m} \langle 0| \dot{p}_i(t)|0\rangle , \qquad (4.72)$$

which coincides, to order 1 in e, with the expression given in (4.22b) for the source field. This clearly shows that, in the derivation of (4.22b), we have implicitly calculated the susceptibility of the field. Rather than using this intermediate result, we have preferred in sections 4.4 and 4.5 to keep general expressions such as (4.61)–(4.64), which have a clear physical meaning, and to specify the values of C and χ for the field only in these final expressions.

(ii) The free field commutator of (4.67) is a *c*-number ([a, a^+] = 1), proportional to \hbar [see expression (4.6) of \mathcal{A}_k]. It follows that the linear susceptibility $\chi^{(R)}$ of the field is independent of the state of the field, and independent of \hbar . Therefore the classical and quantum linear susceptibilities coincide. Since the source field is directly related to $\chi^{(R)}$

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(see previous remark), it has the same expression in both classical and quantum theories, and this explains why self-reaction forces are formally identical in classical and quantum Abraham–Lorentz equations.

We consider now the atomic statistical functions. Their calculation is also straightforward. Using (4.65) in (4.57) and (4.58), replacing $p_{it}(t)$ by $\exp(iH_st/\hbar)p_i \exp(-iH_st/\hbar)$ and introducing some closure relations, we get:

$$C_{ij}^{(S,a)}(\tau) = \frac{1}{2} \frac{e^2}{m^2} \sum_{b} \left\{ \langle a | p_i | b \rangle \langle b | p_j | a \rangle e^{i\omega_{ab}\tau} + \langle a | p_j | b \rangle \langle b | p_i | a \rangle e^{-i\omega_{ab}\tau} \right\},$$

$$\chi_{ij}^{(S,a)}(\tau) = \frac{i}{\hbar} \frac{e^2}{m^2} \theta(\tau) \sum_{b} \left\{ \langle a | p_i | b \rangle \langle b | p_j | a \rangle e^{i\omega_{ab}\tau} - \langle a | p_j | b \rangle \langle b | p_i | a \rangle e^{-i\omega_{ab}\tau} \right\},$$

$$(4.73)$$

where $\hbar \omega_{ab} = E_a - E_b$.

The Fourier transforms of (4.73) and (4.74) are:

$$\hat{C}_{ij}^{(S,a)}(\omega) = \frac{1}{2} \frac{e^2}{m^2} \sum_{b} \left\{ \langle a | p_i | b \rangle \langle b | p_j | a \rangle \delta(\omega - \omega_{ab}) + \langle a | p_j | b \rangle \langle b | p_i | a \rangle \delta(\omega + \omega_{ab}) \right\}, \qquad (4.75)$$

$$\hat{\chi}_{ij}^{(S,a)}(\omega) = -\frac{e^2}{2\pi\hbar m^2} \sum_{b} \left\{ \langle a | p_j | b \rangle \langle b | p_i | a \rangle \mathscr{P}\left(\frac{1}{\omega + \omega_{ab}}\right) - \langle a | p_i | b \rangle \langle b | p_j | a \rangle \mathscr{P}\left(\frac{1}{\omega - \omega_{ab}}\right) \right\}$$

$$- i \frac{e^2}{2\hbar m^2} \sum_{b} \left\{ \langle a | p_j | b \rangle \langle b | p_i | a \rangle \delta(\omega + \omega_{ab}) - \langle a | p_i | b \rangle \langle b | p_j | a \rangle \delta(\omega - \omega_{ab}) \right\} \qquad (4.76)$$

where P means principal part.

The first term of (5.12), which contains only principal parts, is the reactive part χ' of the susceptibility, whereas the second one, which contains only δ -functions, is the dissipative part $i\chi''$ [26–27].

4.5.2. Radiative corrections for an atomic electron

Calculations of $(\delta E_a)_{\rm vf}$ and $(\delta E_a)_{\rm sr}$. We can now use the results of the

previous section for evaluating the two integrals appearing in the expressions (4.61) and (4.63) giving the energy shifts of the atomic level a respectively due to vacuum fluctuations and self-reaction. We must not forget to add $\delta m_2 c^2$ to (δE_a)_{vf} and $\delta m_1 c^2$ to (δE_a)_{sr} where $\delta m_2 c^2$ and $\delta m_1 c^2$ are given by (4.33) and (4.15) and represent overall energy shifts respectively due to vacuum fluctuations and self-reaction (see subsection 4.3.1).

Using (4.63), (4.69) and (4.73), we first calculate

$$-\frac{1}{2}\sum_{ij}\int_{-\infty}^{+\infty} \mathrm{d}\tau \chi_{ij}^{(\mathrm{R})}(\tau) C_{ij}^{(\mathrm{S},a)}(\tau) = -\frac{\omega_{\mathrm{M}}}{6\pi^{2}\varepsilon_{0}c^{3}}\sum_{ij}\delta_{ij}C_{ij}^{(\mathrm{S},a)}(0)$$
$$= -\frac{\omega_{\mathrm{M}}}{6\pi^{2}\varepsilon_{0}c^{3}}\frac{e^{2}}{m^{2}}\langle a|p^{2}|a\rangle, \qquad (4.77)$$

which gives

$$(\delta E_a)_{\rm sr} = \delta m_1 c^2 - \frac{4}{3} \frac{\delta m_1}{m} \left\langle a \left| \frac{p^2}{2m} \right| a \right\rangle.$$
(4.78)

For $(\delta E_a)_{vf}$, we first use the Parseval–Plancherel identity

$$-\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\tau \, C_{ij}^{(\mathsf{R})}(\tau)\chi_{ij}^{(\mathsf{S},a)}(\tau) = -\pi \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \hat{C}_{ij}^{(\mathsf{R})*}(\omega)\hat{\chi}_{ij}^{(\mathsf{S},a)}(\omega) \,. \tag{4.79}$$

The integral over ω is then performed. Using (4.70) and (4.76), we get for (4.79)

$$-\frac{1}{2} \int_{-\infty}^{+\infty} d\tau C_{ij}^{(\mathbf{R})}(\tau) \chi_{ij}^{(\mathbf{S},a)}(\tau) = \frac{e^2}{24\pi^2 \varepsilon_0 m^2 c^3} \sum_b |\langle a| \mathbf{p} | b \rangle|^2$$
$$\times \int_{-\omega_{\mathbf{M}}}^{\omega_{\mathbf{M}}} d\omega |\omega| \left\{ \mathscr{P}\left(\frac{1}{\omega + \omega_{ab}}\right) - \mathscr{P}\left(\frac{1}{\omega - \omega_{ab}}\right) \right\}$$
$$= -\frac{e^2}{6\pi^2 \varepsilon_0 m^2 c^3} \sum_b \omega_{ab} |\langle a| \mathbf{p} | b \rangle|^2 \log \frac{\omega_{\mathbf{M}}}{|\omega_{ab}|}.$$
(4.80)

(Terms in $1/\omega_{\rm M}$ have been neglected in (4.80).)

As in similar calculations [29], we introduce an average atomic frequency $\bar{\omega}$ defined by:

$$\sum_{b} \omega_{ab} |\langle a | \mathbf{p} | b \rangle|^2 \log \frac{\omega_{\mathrm{M}}}{|\omega_{ab}|} = \sum_{b} \omega_{ab} |\langle a | \mathbf{p} | b \rangle|^2 \log \frac{\omega_{\mathrm{M}}}{\bar{\omega}}.$$
(4.81)

The summation over b in (4.81) can then be easily done:

$$\sum_{b} \omega_{ab} |\langle a | \mathbf{p} | b \rangle|^{2} = \frac{1}{2\hbar} \langle a | [[H_{s}, \mathbf{p}], \mathbf{p}] | a \rangle$$
$$= -\frac{\hbar}{2} \langle a | \Delta V_{0}(\mathbf{r}) | a \rangle.$$
(4.82)

Finally, one gets for $(\delta E_a)_{vf}$,

$$(\delta E_a)_{\rm vf} = \frac{e^2\hbar}{12\pi^2\varepsilon_0 m^2 c^3} \log \frac{\omega_{\rm M}}{\bar{\omega}} \langle a|\Delta V_0(\mathbf{r})|a\rangle + \delta m_2 c^2 \,. \tag{4.83}$$

It is important to note that, in the derivation of (4.78) and (4.83), we have not used approximations such as the two-level approximation, or the rotating wave approximation. The energy level shifts are due to virtual transitions involving nonresonant couplings. Consequently, a correct derivation of these shifts must take into account all atomic states and both positive and negative frequency components of the field.

Main effect of self-reaction: modification of kinetic energy due to a mass renormalization. The first term of (4.78) has already been interpreted as the increase of the rest mass energy of the electron due to its Coulomb field. The last term can be considered as the first-order correction to the average kinetic energy of the electron when m is replaced by $m + 4\delta m_1/3$:

$$\left\langle a \left| \frac{p^2}{2(m + \frac{4}{3}\delta m_1)} \right| a \right\rangle = \left\langle a \left| \frac{p^2}{2m} \right| a \right\rangle \times \left(1 - \frac{4}{3} \frac{\delta m_1}{m} + \cdots \right).$$
(4.84)

The electron is surrounded by its Coulomb field, and when one pushes the electron, one has also to push its Coulomb field (electromagnetic mass).

The mass corrections appearing in the two terms of (4.78) are not the same. This discrepancy is due to the noncovariant cut-off (see discussion in subsection 4.2.1), and also exists in classical theory.

Finally it must be noted that, since the $2s_{1/2}$ and $2p_{1/2}$ states of hydrogen have the same average kinetic energy, a mass correction produces equal shifts for the two levels and cannot remove their degeneracy. Self-reaction alone cannot therefore explain the Lambshift.

Main effect of vacuum fluctuations: modification of potential energy. The first term of (4.83) coincides with the standard nonrelativistic expression for the Lamb-shift [29]. It appears as a correction to the potential $V_0(\mathbf{r})$ which becomes $V_0(\mathbf{r}) + \delta V_0(\mathbf{r})$ where

$$\delta V_0(\mathbf{r}) = \frac{e^2 \hbar}{12\pi^2 \varepsilon_0 m^2 c^3} \log \frac{\omega_{\rm M}}{\bar{\omega}} \Delta V_0(\mathbf{r}) \,. \tag{4.85}$$

If $V_0(\mathbf{r})$ is the Coulomb potential of a nucleus located at the origin, $\Delta V_0(\mathbf{r})$ is proportional to $\delta(\mathbf{r})$, and therefore only s states are shifted by such a correction, which explains in particular how the degeneracy between $2s_{1/2}$ and $2p_{1/2}$ can be removed.

Welton has pointed out [3] that a correction of the same type as (4.85) would be obtained, if the electron was submitted to a fluctuating classical field, with frequencies large compared to the atomic frequencies. The electron, vibrating in such a fluctuating field, averages the external static potential over a finite volume. If the spectral density of this random perturbation is identified with the one of vacuum fluctuations, one gets for the coefficient of $\Delta V_0(\mathbf{r})$ the same value as in (4.85), $\bar{\omega}$ being simply replaced by a low-frequency cut off. Welton's analysis establishes a connection between Lamb-shifts of atomic levels and vacuum fluctuations and provides a clear and simple physical picture.

Our choice of the symmetric order in (4.37) ascribes corrections such as (4.85) to vacuum fluctuations and entirely legitimates Welton's interpretation for the Lamb-shift.

We have already seen (subsection 4.3.1. ii) that vacuum fluctuations are also responsible for a correction δm_2 to the electron mass (last term of (4.83)).

Remarks

(i) It may appear surprising that our calculation doesn't give any correction to the kinetic energy associated with the mass correction δm_2 due to vacuum fluctuations. One would expect to find, as in the previous section, a term of the order of

$$-\frac{\delta m_2}{m} \left\langle a \left| \frac{\mathbf{p}^2}{2m} \right| a \right\rangle. \tag{4.86}$$

Actually, coming back to the expressions (4.15) and (4.33) of δm_1 and δm_2 , and introducing the fine structure constant $\alpha = e^2/4\pi\varepsilon_0\hbar c$, one can write

$$\frac{\delta m_1}{m} = \frac{\alpha}{\pi} \frac{\hbar \omega_{\rm M}}{mc^2}, \qquad (4.87a)$$

$$\frac{\delta m_2}{m} = \frac{\alpha}{2\pi} \left(\frac{\hbar \omega_{\rm M}}{mc^2}\right)^2. \qquad (4.87b)$$

Therefore, it clearly appears that $\delta m_2/m$ is of higher order in 1/c than $\delta m_1/m$. This explains why the correction to the kinetic energy associated with δm_2 is not given by our calculation which is limited to the lowest order in 1/c. The basic Hamiltonian (4.16) does not contain any relativistic correction. A more precise calculation including in the Hamiltonian relativistic corrections up to order $1/c^2$ [4] (and using an effective Hamiltonian method for evaluating radiative corrections) actually gives the expected correction (4.86).

(ii) The present calculation (as well as the one of reference [4]) does not include of course any multiparticle effect (virtual pair creation). It is well known [19] that many-particle effects reduce the divergence of the electron self-energy $(\delta m_1 + \delta m_2)c^2$, with respect to the cut-off $\omega_{\rm M}$. Instead of having a linear and quadratic divergence (see (4.87)), one gets a logarithmic one. Also, new correction terms, associated with vacuum polarization effects, appear.

Classical versus quantum effects. A striking difference can be pointed out between the contributions of self-reaction and vacuum fluctuations to radiative corrections: \hbar does not appear in $(\delta E_a)_{\rm sr}$ [see (4.78) and the expression of δm_1], whereas \hbar does appear in both terms of $(\delta E_a)_{\rm vf}$ [see (4.83) and the expression (4.33) of δm_2].

The fact that self-reaction corrections are purely classical (independent of \hbar) is not surprising. We have already explained (see remark (ii) of subsection 4.5.1) why self-reaction terms are identical in both classical and quantum theories.

On the other hand, vacuum fluctuation corrections have an essentially quantum nature since they are due to the nonzero mean square value of the fields in the vacuum, which is a pure quantum effect. It must be noted however that, once the correlation function of vacuum fluctuations is computed from the quantum theory of radiation, their effect on the atom (to the lowest order in α) may be evaluated semi-classically, since we have shown in subsection 4.4.3 that reservoir fluctuations have the same effect (to the lowest order) as a classical random field having the same correlation function. This explains why pure quantum effects, such as those produced by the vacuum fluctuations of the quantized radiation field, can be calculated as if a classical random field, with a power spectral density equal to $\hbar\omega/2$ per mode, was acting upon the atom [30].

To summarize, our choice of the symmetric order in (4.37) leads to self-reaction corrections which are strictly equivalent to the corresponding classical ones, whereas vacuum fluctuations appear to be responsible for pure quantum effects which can be however computed semi-classically, once the correlation function of vacuum fluctuations is given.

Spin and magnetic effects. Interpretation of the spin anomaly g - 2. In this section, we take into account the spin S of the electron and the corresponding magnetic moment

$$\boldsymbol{M}_{\mathrm{s}} = \frac{e}{m} \boldsymbol{S} \,. \tag{4.88}$$

Even in the absence of any external static magnetic field B_0 , M_s interacts with the magnetic field **B** of the transverse radiation field. We should add to the interaction Hamiltonian V given in (4.31) a term

$$-\boldsymbol{M}_{\mathrm{s}} \cdot \boldsymbol{B}(\boldsymbol{0}) = -\frac{e}{m} \boldsymbol{S} \cdot \boldsymbol{B}(\boldsymbol{0}) \tag{4.89}$$

describing such a coupling. This would introduce in the final expressions of radiative corrections new correlation functions and new linear susceptibilities involving two components of **B**, or one component of **B** and one component of **A**. Since an extra 1/c factor appears in the expansion of **B** in plane waves [see expression (4.6)], we conclude that the new radiative corrections associated with (4.89) would be at least one order in 1/c higher than those calculated previously, and which, according to (4.77) and (4.83), are in e^2/c^3 (or α/mc^2). If we restrict our calculations to the lowest order in 1/c, as we do in the nonrelativistic approach used in this paper, we can therefore ignore the magnetic couplings of the spin with the radiation field and neglect (4.89)*.

The same argument does not apply of course to the interaction of S

^{*} If we would like to go to higher orders in 1/c, we should include relativistic corrections in the Hamiltonian and retardation effects.

with an external static magnetic field B_0 deriving from the static vector potential A_0 :

$$\boldsymbol{B}_0(\boldsymbol{R}) = \nabla \times \boldsymbol{A}_0(\boldsymbol{R}) \,. \tag{4.90}$$

We must add to the atomic Hamiltonian H_s a new term describing the interaction of M_s with the static magnetic field B_0 at the electron position

$$-\boldsymbol{M}_{s} \cdot \boldsymbol{B}_{0}(\boldsymbol{r}) = -\frac{e}{m} \boldsymbol{S} \cdot \boldsymbol{B}_{0}(\boldsymbol{r}) \,. \tag{4.91}$$

We must also replace the electron momentum p by:

$$\pi_0 = p - eA_0(r) \,. \tag{4.92}$$

To summarize, if, at the lowest order in 1/c, i.e., at order e^2/c^3 , we want to include spin and magnetic effects, we must use

$$H_{\rm s} = \frac{\pi_0^2}{2m} + V_0(\boldsymbol{r}) - \frac{e}{m} \boldsymbol{S} \cdot \boldsymbol{B}_0(\boldsymbol{r}) \tag{4.93}$$

instead of (4.29), and replace p by π_0 in the first term of (4.31):

$$-\frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{0}) \rightarrow -\frac{e}{m}\,\pi_{0}\cdot\boldsymbol{A}(\boldsymbol{0})\,. \tag{4.94}$$

What are the corresponding changes in $(\delta E_a)_{sr}$ and $(\delta E_a)_{vf}$? Since the field operators remain unchanged in (4.94), we still use (4.68) and (4.69) for $C^{(R)}$ and $\chi^{(R)}$. On the other hand, we must change p into π_0 in the expressions (4.73) and (4.74) of $C^{(S)}$ and $\chi^{(S)}$.

Consider first the modifications occurring for $(\delta E_a)_{\rm sr}$. The only change in (4.78) is that we have $\pi_0^2/2m$ instead of $p^2/2m$. Since $\pi_0^2/2m$ has the physical meaning of a kinetic energy in presence of the static vector potential A_0 , we conclude that the main effect of self-reaction is, as before, to change the mass appearing in the kinetic energy

$$\frac{\pi_0^2}{2m} \to \frac{\pi_0^2}{2(m + \frac{4}{3}\delta m_1)}.$$
(4.95)

It must be emphasized that, at this order in 1/c, the mass renormalization due to self-reaction does not affect the last term of (4.93). The mass *m* which appears in the spin magnetic moment eS/m remains unchanged. We don't get any term of the form

$$+\frac{\delta m_1}{m}\frac{e}{m}\boldsymbol{S}\cdot\boldsymbol{B}_0.$$
(4.96)

We will come back later on this important point, when discussing the origin of the spin anomaly g - 2.

We now discuss the modifications for $(\delta E_a)_{vt}$. The calculations are very similar to previous ones, the only difference being that, in the double commutator of (4.82), we must use the new expression (4.93) of H_s and replace p by π_0 . We have therefore to calculate:

$$\frac{1}{2\hbar} \left\langle a \left| \left[\left[\frac{\pi_0^2}{2m} + V_0(\boldsymbol{r}) - \frac{e}{m} \, \boldsymbol{S} \cdot \boldsymbol{B}_0(\boldsymbol{r}), \, \pi_0 \right] \right| \, a \right\rangle. \tag{4.97}$$

If we suppose that $B_0(r)$ is homogeneous (independent of r) and if we keep only terms linear in B_0 , expression (4.97) reduces to (4.82). Thus, for homogeneous weak static magnetic fields, vacuum fluctuations do not introduce any new radiative correction related to spin and magnetic effects.

We have now at our disposal all what is needed for discussing the contribution of self-reaction and vacuum fluctuations to the electron dynamics in presence of a weak homogeneous static magnetic field. Combining the previous results, the corrected atomic Hamiltonian (including radiative corrections) can be written:

$$\frac{\pi_0^2}{2(m+\frac{4}{3}\delta m_1)} + V_0(r) + \delta V_0(r) - \frac{e}{m} S \cdot \mathbf{B}_0(r) , \qquad (4.98)$$

where corrections including δm_1 are due to self-reaction and $\delta V_0(r)$ to vacuum fluctuations.

The spin magnetic moment appearing in the last term of (4.98) can be written as

$$\boldsymbol{M}_{\mathrm{s}} = \frac{e}{m} \boldsymbol{S} = 2 \frac{e}{2m} \boldsymbol{S} \,. \tag{4.99}$$

In terms of the "bare" (uncorrected) mass, the g-factor of the electron spin is 2. But, the mass which is measured experimentally, in deflection experiments, is the renormalized mass, i.e., the mass which appears in the corrected kinetic energy

$$\bar{m} = m + \frac{4}{3}\delta m_1, \qquad (4.100)$$

so that, if we reexpress M_s in terms of \bar{m} , we have from (4.99)

$$M_{\rm s} = \frac{e}{m} S = g \frac{e}{2\bar{m}} S, \qquad (4.101)$$

with

$$g = 2\frac{\bar{m}}{m} = 2\left(1 + \frac{4}{3}\frac{\delta m_1}{m}\right) > 2.$$
(4.102)

So, it clearly appears that the positive sign of g - 2 is due to the fact that self-reaction corrects only to lowest order the kinetic energy and not the magnetic coupling between S and B_0 . The motion of the charge is slowed down but not the precession of the spin. This is easy to understand. In the nonrelativistic limit we are considering in this paper, electric effects predominate over magnetic ones and self-reaction is stronger for a charge than for a magnetic moment. We therefore arrive at the same conclusions as other treatments [4, 31].

If the calculation was pushed to higher orders in 1/c as in [4], we would get corrections to the spin magnetic moment, especially those due to the vacuum fluctuations of the magnetic field B(0) of the radiation field which exert a fluctuating torque on M_s , producing an angular vibration of the spin and, consequently, a decrease of the effective magnetic moment. This is the equivalent of Welton's picture for g-2 which would produce a negative spin anomaly if this was the only mechanism. We understand now the failure of such a picture. For g-2, the predominant physical mechanism is self-reaction which slows down the motion of the electric charge.

4.5.3. Rate of exchange of energy between the electron and the radiation field.

Contribution of self-reaction. We start from (4.64) and we use the expressions (4.71) of χ^{R} and (4.75) of C^{S} . Because of the δ functions

appearing in (4.75), the integral over ω is readily done, and we get for the rate of energy loss due to self-reaction by the electron in state *a*

$$\left\langle 0, a \left| \left(\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{s}} \right)_{\mathrm{sr}} \right| 0, a \right\rangle = -\frac{e^2}{6\pi m^2 \varepsilon_0 c^3} \sum_{b} \sum_{i} |\langle a| p_i | b \rangle|^2 \omega_{ab}^2 \,. \tag{4.103}$$

Now, we write*

$$\frac{1}{m} \langle a | p_i | b \rangle \omega_{ab} = \frac{1}{m\hbar} \langle a | [H_s, p_i] | b \rangle$$
$$= -\frac{i}{m} \langle a | \dot{p}_i | b \rangle$$
$$= -i \langle a | \ddot{r}_i | b \rangle.$$
(4.104)

Finally, by using (4.104) and the closure relation over *b*, we transform (4.103) into

$$\left\langle 0, a \left| \left(\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{s}} \right)_{\mathrm{sr}} \right| 0, a \right\rangle = -\frac{2}{3} \frac{e^2}{4\pi\varepsilon_0 c^3} \langle a | (\ddot{r})^2 | a \rangle$$

$$(4.105)$$

Such a result is extremely simple and exactly coincides with what is found in classical radiation theory. The rate of radiation of electromagnetic energy is proportional to the square of the acceleration of the radiating charge, the proportionality coefficient being just the one appearing in (4.105). We note also that, if self-reaction was alone, the atomic ground state would not be stable, since the square of the acceleration has a nonzero average value in such a state.

Contribution of vacuum fluctuations. We now use (4.62) and the expressions (4.70) of $C^{(R)}$ and (4.76) of $\chi^{(S)}$. This gives

$$\left\langle 0, a \left| \left(\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{s}} \right)_{\mathrm{vf}} \right| 0, a \right\rangle = \frac{e^2}{12\pi\varepsilon_0 m^2 c^3} \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \omega |\omega| \sum_b \sum_i |\langle a|p_i|b\rangle|^2 \times [\delta(\omega + \omega_{ab}) - \delta(\omega - \omega_{ab})]$$
$$= -\frac{e^2}{6\pi\varepsilon_0 m^2 c^3} \sum_b \sum_i |\langle a|p_i|b\rangle|^2 \omega_{ab} |\omega_{ab}| . \tag{4.106}$$

* The atomic operators appearing in $\chi^{(S)}$ are free atomic operators. This is why their time derivative is given by the commutator with H_s (and not with total Hamiltonian H).

Using (4.104), and distinguishing the terms $\omega_{ab} > 0$ ($E_a > E_b$) and the terms $\omega_{ab} < 0$ ($E_a < E_b$), we get

$$\left\langle 0, a \left| \left(\frac{\mathrm{d}H_{\mathrm{s}}}{\mathrm{d}t} \right)_{\mathrm{vf}} \right| 0, a \right\rangle = \frac{2}{3} \frac{e^2}{4\pi\varepsilon_0 c^3} \left\{ \sum_{\substack{b \\ E_b > E_a}} \langle a | \vec{r} | b \rangle \cdot \langle b | \vec{r} | a \rangle - \sum_{\substack{b \\ E_b < E_a}} \langle a | \vec{r} | b \rangle \cdot \langle b | \vec{r} | a \rangle \right\}.$$
(4.107)

The first line describes an absorption of energy by the electron which jumps from a to a higher state b, whereas the second line describes an emission of energy by jumps to lower states. This is in agreement with the picture of a random field inducing in the atomic system both downwards and upwards transitions.

Now, coming back to (4.105), we can reintroduce the closure relation over b between \ddot{r} and \ddot{r} , which gives:

$$\left\langle 0, a \left| \left(\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{s}} \right)_{\mathrm{sr}} \right| 0, a \right\rangle = -\frac{2}{3} \frac{e^2}{4\pi\varepsilon_0 c^3} \sum_{b} \langle a | \vec{r} | b \rangle \cdot \langle b | \vec{r} | a \rangle$$

$$= -\frac{2}{3} \frac{e^2}{4\pi\varepsilon_0 c^3} \left\{ \sum_{\substack{b \\ E_b > E_a}} \langle a | \vec{r} | b \rangle \cdot \langle b | \vec{r} | a \rangle$$

$$+ \sum_{\substack{b \\ E_b < E_a}} \langle a | \vec{r} | b \rangle \cdot \langle b | \vec{r} | a \rangle \right\}.$$

$$(4.108)$$

Adding (4.107) and (4.108), we get for the total rate of energy loss by the electron in state a

$$\left\langle 0, a \left| \frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{s}} \right| 0, a \right\rangle = -\frac{4}{3} \frac{e^2}{4\pi\varepsilon_0 c^3} \sum_{\substack{b\\E_b < E_a}} \left\langle a | \vec{r} | b \right\rangle \cdot \left\langle b | \vec{r} | a \right\rangle.$$
(4.109)

This satisfactory result means that the electron in the vacuum can only loose energy by cascading downwards to lower energy levels. In particular, the ground state is stable since it is the lowest state.

The previous discussion clearly shows that the ground state cannot be stable in absence of vacuum fluctuations which exactly balance the energy loss due to self-reaction [28]. In other words, if self-reaction was alone, the ground state would collapse and the atomic commutation relation $[x, p] = i\hbar$ would not hold. Such a collapse is prevented by vacuum fluctuations which actually originate from the quantum nature of the field, i.e., from the commutation relation $[a, a^+] = 1$. We have here an illustration of a very general principle of quantum mechanics. When two isolated systems interact (here the atom and the field), treating one of them quantum mechanically and the other one semi-classically leads to consistencies [32]. The field commutation relations are necessary for preserving the atomic ones and vice versa.

4.6. Conclusion

We have removed the apparent indetermination in the separation of vacuum fluctuations and self-reaction by imposing to the corresponding rates of variation to have a well-defined physical meaning (hermiticity requirements).

Such a procedure is very general and can be extended to the case of a small system \mathscr{S} interacting with a large reservoir \mathscr{R} . The results of the calculation can be expressed* in terms of simple statistical functions of the two interacting systems, leading to simple physical pictures: \mathscr{R} fluctuates and polarizes \mathscr{S} (reservoir fluctuations effects); \mathscr{S} fluctuates and polarizes \mathscr{R} (self-reaction effects).

When applied to the case of an atomic electron interacting with the vacuum field, such a procedure gives results in complete agreement with the usual pictures associated with vacuum fluctuations and self-reaction. All self-reaction effects, which are independent of \hbar , are strictly identical to those derived from classical radiation theory. All vacuum fluctuation effects, which are proportional to \hbar , can be interpreted by considering the vibration of the electron induced by a random field having a spectral power density equal to $\hbar\omega/2$ per mode.

Appendix A: Calculation of the source fields $A_s(0, t)$ and $E_{\perp s}(0, t)$

Equations (4.5a) and (4.5b) give the expressions of A and E_{\perp} in terms of the creation and annihilation operators

$$A(0, t) = \sum_{\substack{k\varepsilon \\ |k| < k_{\rm M}}} \mathscr{A}_k \varepsilon a_{k\varepsilon}(t) + {\rm hc} , \qquad (A.1a)$$

* It must be kept in mind that all the calculations have been limited to order 2 in the coupling constant. At higher orders, cross terms would appear between reservoir fluctuations and self-reaction.

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$$\boldsymbol{E}_{\perp}(\boldsymbol{0}, t) = \sum_{\substack{k\varepsilon \\ |k| < k_{\mathrm{M}}}} \mathscr{E}_{k} \varepsilon a_{k\varepsilon}(t) + \mathrm{hc} \;. \tag{A.1b}$$

Inserting (4.20) into these two equations, one gets the expression of $A_s(0, t)$ and $E_{\perp s}(0, t)$:

$$A_{s}(\mathbf{0}, t) = \sum_{\substack{k\varepsilon \\ |k| < k_{M}}} i \frac{\varepsilon \mathscr{A}_{k}^{2}}{m\hbar} \int_{t_{0}}^{t} dt' e^{-i\omega(t-t')} \varepsilon [\varepsilon^{*} \cdot \boldsymbol{\pi}(t')] + hc, \qquad (A.2a)$$

$$\boldsymbol{E}_{\perp s}(\boldsymbol{0}, t) = \sum_{\substack{k \varepsilon \\ |k| < k_{\mathrm{M}}}} \mathrm{i} \, \frac{e \mathcal{A}_{k} \mathcal{E}_{k}}{m \hbar} \int_{t_{0}}^{t} \mathrm{d}t' \, \mathrm{e}^{-\mathrm{i}\omega(t-t')} \, \boldsymbol{\varepsilon} \left(\boldsymbol{\varepsilon}^{*} \cdot \boldsymbol{\pi}(t')\right) + \mathrm{hc} \,. \tag{A.2b}$$

We now permute the summation over k, ε and the integration on t', the angular summation is easily performed and we get

$$A_{s}(0, t) = -\frac{e}{3\pi\varepsilon_{0}c^{3}m} \int_{t_{0}}^{t} dt' \,\pi(t')\delta'_{M}(t-t'), \qquad (A.3a)$$

$$\boldsymbol{E}_{\perp s}(\boldsymbol{0}, t) = \frac{e}{3\pi\varepsilon_0 c^3 m} \int_{t_0}^t \mathrm{d}t' \ \boldsymbol{\pi}(t') \delta_{\mathrm{M}}''(t-t') , \qquad (A.3b)$$

where the function $\delta_{M}(\tau)$ is given by

$$\delta_{\rm M}(\tau) = \frac{1}{2\pi} \int_{-\omega_{\rm M}}^{+\omega_{\rm M}} d\omega \ e^{-i\omega\tau} \,. \tag{A.4}$$

This function $\delta_M(\tau)$ is symmetric, centred on $\tau = 0$, has a width equal to $1/\omega_M$ and satisfies the equation

$$\int_{-\infty}^{+\infty} d\tau \,\delta_{\rm M}(\tau) = 1 \,. \tag{A.5}$$

Equations (A.3a) and (A.3b) can be written, by putting $\tau = t - t'$ and taking t_0 equal to $-\infty$:

$$A_{s}(0, t) = -\frac{e}{3\pi\varepsilon_{0}c^{3}m} \int_{0}^{+\infty} \mathrm{d}\tau \,\pi(t-\tau)\delta'_{\mathsf{M}}(\tau)\,, \qquad (A.6a)$$

$$\boldsymbol{E}_{\perp s}(\boldsymbol{0}, t) = \frac{e}{3\pi\varepsilon_0 c^3 m} \int_0^{+\infty} \mathrm{d}\tau \,\boldsymbol{\pi}(t-\tau) \delta_{\mathrm{M}}''(\tau) \,. \tag{A.6b}$$

Using an integration by parts, one gets

$$\boldsymbol{A}_{s}(\boldsymbol{0},t) = \frac{e}{3\pi\varepsilon_{0}c^{3}m} \,\delta_{M}(0)\boldsymbol{\pi}(t) - \frac{e}{3\pi\varepsilon_{0}c^{3}m} \int_{0}^{+\infty} \mathrm{d}\tau \,\dot{\boldsymbol{\pi}}(t-\tau)\delta_{M}(\tau) \,, \tag{A.7a}$$

$$\boldsymbol{E}_{\perp s}(\boldsymbol{0}, t) = -\frac{e}{3\pi\varepsilon_0 c^3 m} \,\delta_{\mathsf{M}}(0)\dot{\boldsymbol{\pi}}(t) + \frac{e}{3\pi\varepsilon_0 c^3 m} \int_0^{+\infty} \mathrm{d}\tau \ddot{\boldsymbol{\pi}}(t-\tau)\delta_{\mathsf{M}}(\tau) \,. \tag{A.7b}$$

The characteristic times for the evolution of $\pi(t)$ are very long compared to the width $1/\omega_{\rm M}$ of $\delta_{\rm M}(\tau)$ [see Eq. (4.14)]. We can therefore replace in (A.7a) and (A.7b) $\dot{\pi}(t-\tau)$ and $\ddot{\pi}(t-\tau)$ by $\dot{\pi}(t)$ and $\ddot{\pi}(t)$. The remaining integral of $\delta_{\rm M}(\tau)$ from $\tau = 0$ to $\tau = \infty$ is equal to 1/2, as a consequence of the symmetry of $\delta_{\rm M}(\tau)$. One finally gets

$$\mathbf{A}_{s}(\mathbf{0},t) = \frac{e\omega_{M}}{3\pi^{2}\varepsilon_{0}c^{3}m} \,\boldsymbol{\pi}(t) - \frac{e}{6\pi\varepsilon_{0}c^{3}m} \,\dot{\boldsymbol{\pi}}(t) \,, \tag{A.8a}$$

$$\boldsymbol{E}_{\perp s}(\boldsymbol{0}, t) = -\frac{e\omega_{\rm M}}{3\pi^2\varepsilon_0 c^3 m} \, \dot{\boldsymbol{\pi}}(t) + \frac{e}{6\pi\varepsilon_0 c^3 m} \, \ddot{\boldsymbol{\pi}}(t) \,, \tag{A.8b}$$

(A.8a) and (A.8b) are nothing but (4.22) and (4.24b) using the expression of δm_1 given in eq. (4.15).

Appendix B: Correlation function and linear susceptibility of the field

The correlation function of the field is given [cf. eq. (4.66)]:

$$C_{ij}^{(\mathsf{R})}(\tau) = \frac{1}{2} \langle 0 | A_{if}(0, t) A_{jf}(0, t-\tau) + A_{jf}(0, t-\tau) A_{if}(0, t) | 0 \rangle, \qquad (B.1)$$

where the operator $A_{f}(0, t)$ is the free vector potential. Using its expansion in plane waves, one gets

$$C_{ij}^{(R)}(\tau) = \frac{1}{2} \sum_{k\varepsilon} \mathscr{A}_{k}^{2} \varepsilon_{i} \varepsilon_{j} \langle 0 | a_{kef}(t) a_{kef}^{+}(t-\tau) + a_{kef}(t-\tau) a_{kef}^{+}(t) | 0 \rangle$$

$$= \frac{1}{2} \sum_{k\varepsilon} \mathscr{A}_{k}^{2} \varepsilon_{i} \varepsilon_{j} (e^{-i\omega\tau} + e^{i\omega\tau}) . \qquad (B.2)$$

Replacing the sum by an integral and using the expression (4.6) of \mathcal{A}_k , one gets

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$$C_{ij}^{(\mathbf{R})}(\tau) = \delta_{ij} \frac{\hbar}{12\pi^2 \varepsilon_0 c^3} \int_0^{\omega_{\mathrm{M}}} \mathrm{d}\omega \,\omega (\mathrm{e}^{\mathrm{i}\omega\tau} + \mathrm{e}^{-\mathrm{i}\omega\tau}) \,. \tag{B.3}$$

This can also be written:

$$C_{ij}^{(R)}(\tau) = \delta_{ij} \frac{\hbar}{12\pi^2 \varepsilon_0 c^3} \int_{-\omega_M}^{\omega_M} d\omega |\omega| e^{i\omega\tau}$$
(B.4)

The linear susceptibility is calculated in the same way. Starting from

$$\chi_{ij}^{(\mathsf{R})} = \frac{\mathrm{i}}{\hbar} \langle 0 | [A_{it}(\mathbf{0}, t), A_{jt}(\mathbf{0}, t-\tau)] | 0 \rangle \theta(\tau) , \qquad (B.5)$$

one gets

$$\chi_{ij}^{(R)}(\tau) = \frac{i}{\hbar} \,\delta_{ij} \,\frac{\hbar}{6\pi^2 \varepsilon_0 c^3} \int_0^{\omega_M} d\omega \,\,\omega (e^{-i\omega\tau} - e^{+i\omega\tau})\theta(\tau)$$
$$= -\frac{\delta_{ij}}{3\pi\varepsilon_0 c^3} \,\delta_M'(\tau)\theta(\tau) \,. \tag{B.6}$$

In this paper, the susceptibility of the field always appears in expressions such as

$$\int_{-\infty}^{+\infty} \chi_{ij}^{(R)}(\tau) f_{ij}^{(s)}(\tau) \, \mathrm{d}\tau \,, \tag{B.7}$$

where $f_{ij}^{(s)}(\tau)$ is a function concerning the small system \mathscr{S} . The characteristic times of evolution of $f_{ij}^{(s)}$ are then much larger than $1/\omega_{\rm M}$ so that one can proceed in the same way as for (A.6a). Using an integration by parts, one finds that

$$\chi_{ij}^{(R)}(\tau) = \frac{\delta_{ij}}{3\pi^2 \varepsilon_0 c^3} \left[\omega_M \delta(\tau) - \frac{\pi}{2} \,\delta'(\tau) \right],\tag{B.8}$$

where δ here acts on the slowly varying functions $f_{ij}^{(s)}(\tau)$ as a true delta function.

References to section 4

 A discussion of early works on Q.E.D. can be found in: Weisskopf, V.F., Physics in the Twentieth Century-Selected essays (The M.I.T. Press, Cambridge, 1972) p. 96 to 129.

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- [2] Foundations of Radiation Theory and Quantum Electrodynamics, edited by A.O. Barut (Plenum Press, New York, 1980). This book contains several papers reviewing recent contributions to these problems.
- [3] Welton, T.A., Phys. Rev. 74 (1948) 1157.
- [4] Dupont-Roc, J., Fabre, C. and Cohen-Tannoudji, C., J. Phys. B11 (1978) 563.
- [5] Series, G.W., in: Optical Pumping and Atomic Line Shapes, edited by T. Skalinski (Panstwowe Wdawnictwo Naukowe, Warszawa, 1969) p. 25.
- [6] Bullough, R.K., in: Coherence and Quantum Optics, edited by L. Mandel and E. Wolf (Plenum Press, New York, 1973) p. 121.
- [7] Ackerhalt, J.R., Knight, P.L. and Eberly, J.H., Phys. Rev. Lett. 30 (1973) 456.
- [8] Ackerhalt, J.R. and Eberly, J.H., Phys. Rev. D10 (1974) 3350.
- [9] Kimble, H.J. and Mandel, L., Phys. Rev. Lett. 34 (1975) 1485.
- [10] Kimble, H.J. and Mandel, L., Phys. Rev. A13 (1976) 2123.
- [11] Senitzky, I.R., Phys. Rev. Lett. 31 (1973) 955.
- [12] Milonni, P.W., Ackerhalt, J.R. and Smith, W.A., Phys. Rev. Lett. 31 (1973) 958.
- [13] Milonni, P.W. and Smith, W.A., Phys. Rev. A11 (1975) 814.
- [14] Millonni, P.W., Phys. Rep. 25 (1976) 1.
- [15] French, J.B., Weisskopf, V.F., Phys. Rev. 75 (1949) 1240.
- [16] Feynman, R.P., Phys. Rev. 74 (1948) 1430.
- [17] Jackson, J.D., Classical Electrodynamics, 2nd edition (Wiley, New York, 1975) section 17.
- [18] Moniz, E.J. and Sharp, D.H., Phys. Rev. D10 (1974) 1133; Phys. Rev. D15 (1977) 2850 and their contribution in reference [2].
- [19] Weisskopf, V.F., Phys. Rev. 56 (1939) 81.
- [20] Allen, L. and Eberly, J.H., Optical Resonance and Two-Level Atoms (John Wiley and Sons, New York) 1975.
- [21] Agarwal, G.S., Springer Tracts in Modern Physics, Volume 70 (Springer Verlag, Berlin) 1974.
- [22] Cohen-Tannoudji, C., in Frontiers in Laser Spectroscopy, Volume 1, Les Houches 1975 Session XXVII, edited by Balian R., Haroche S. and Liberman S. (North Holland, Amsterdam, 1977) p. 3.
- [23] Louisell, W.H., Quantum Statistical Properties of Radiation (Wiley, New York, 1973).
- [24] Grabert, H., Z. Phys. B26 (1977) 79.
- [25] Dalibard, J. Dupont-Roc, J. and Cohen-Tannoudji, C., J. Physique, 45 (1984) 123.
- [26] Martin, P. in Many Body Physics, Les Houches 1967, edited by De Witt C. and Balian R. (Gordon and Breach, New York, 1968) p. 39.
- [27] Landau, L. and Lifchitz, E.M., Statistical Physics (Pergamon Press, 1958) section 124.
- Fain, V.M., Sov. Phys. J.E.T.P. 23 (1966) 882.
 Fain, V.M. and Khanin, Y.I., Quantum Electronics (M.I.T. Press, Cambridge) 1969.
 Fain, B., Il Nuovo Cimento 68B (1982) 73.
- [29] Bethe, H.A. and Salpeter, E.E., Quantum Mechanics of One and Two-Electron Atoms (Plenum Press, New York) 1977.
- [30] See contribution of Boyer, T.H. in reference [2] and references quoted in.
- [31] Grotch, H. and Kazes, E., Am. J. Phys. 45 (1977) 618 and their contribution in reference [2].
- [32] Senitzky, I.R., Phys. Rev. Lett. 20 (1968) 1062.

5. The effective Hamiltonian method

5.1. Introduction - general idea

Suppose we have an unperturbed Hamiltonian H_0 the energy levels of which are bunched in manifolds \mathscr{C}^0_{α} , \mathscr{C}^0_{β} ... well separated from each other (fig. 8). We note $|i, \alpha\rangle$ the eigenstates of H_0 with eigenvalues $E_{i\alpha}$,

$$H_{0}|i, \alpha\rangle = E_{i\alpha}|i, \alpha\rangle,$$

$$|E_{j\beta} - E_{i\alpha}| \gg |E_{j\alpha} - E_{i\alpha}| \quad \text{if } \beta \neq \alpha.$$
(5.1)

They are labelled by two types of quantum numbers, α corresponding to high-frequency degrees of freedom (α also labels the manifolds), and *i* corresponding to low-frequency degrees of freedom.

Suppose now that we add a perturbation λV , where λ is a sufficiently small dimensionless parameter so that

$$\lambda \langle i, \alpha | V | j, \beta \rangle \ll |E_{i\alpha} - E_{j\beta}| \quad \text{if} \quad \beta \neq \alpha .$$
(5.2)

Consequently, the spectrum of $H = H_0 + \lambda V$ is also formed by wellseparated manifolds. The coupling λV introduces two types of effects. First, a modification of the wave functions. The wave functions of \mathscr{C}^0_{α} are "contaminated" by those of \mathscr{C}^0_{β} ($\beta \neq \alpha$). Secondly, a modification of the energies. In particular, the virtual transitions induced by λV between different manifolds produce energy shifts proportional to λ^2 .

In this section, we will be mainly interested in the modification of the energies inside a given manifold. In other words, we would like to study the modification of the slow motion of the system due to the





Fig. 8. Manifolds of the unperturbed Hamiltonian.

coupling λV . We will show (section 5.2) that it is possible to find an effective Hamiltonian, H_{eff}^{α} , acting only inside \mathscr{C}_{α}^{0} , and such that its eigenvalues coincide with the corresponding eigenvalues of $H = H_0 + \lambda V$ to a given order in λ . This effective Hamiltonian describes the new slow motion in \mathscr{C}_{α}^{0} . It includes the effect on the slow degrees of freedom of their coupling with the fast ones. We will then apply (section 5.3) this effective Hamiltonian method to the problem of radiative corrections in the non-relativistic limit by studying how the slow motion of a weakly bound electron is modified by its coupling with a mode of the quantized radiation field. The effective Hamiltonian method is actually quite general and has been applied to various physical problems (see for example [21] and references therein).

5.2. Calculation of the effective Hamiltonian

5.2.1. Unitary transformation

If we apply a unitary transformation e^{iS} (with $S = S^+$) to the Hamiltonian H, the transformed Hamiltonian

$$\tilde{H} = e^{iS} H e^{-iS} \tag{5.3}$$

has the same spectrum as H. We try now to choose e^{iS} in such a way that the restriction of \tilde{H} to \mathscr{C}^{0}_{α} can be considered as an effective Hamiltonian, H^{α}_{eff} , to a given order in λ (in the sense defined in section 5.1).

The first condition to be imposed on e^{iS} is that the off-diagonal elements of \tilde{H} between \mathscr{C}^0_{α} and \mathscr{C}^0_{β} (with $\beta \neq \alpha$) vanish as λ^n if λ tends to zero,

$$\langle i, \alpha | \hat{H} | j, \beta \rangle = C \lambda^n + D \lambda^{n+1} + \dots \quad \text{if} \quad \beta \neq \alpha \,.$$
 (5.4)

If this is achieved, it will follow that the eigenvalues of the restriction of \tilde{H} to \mathscr{C}^{0}_{α} ,

$$P^{0}_{\alpha}\tilde{H}P^{0}_{\alpha}, \text{ with } P^{0}_{\alpha} = \sum_{i} |i, \alpha\rangle\langle i, \alpha|, \qquad (5.5)$$

give the correct perturbed energies associated with \mathscr{C}^0_{α} , at least to order λ^n , and consequently, that (5.5) is a correct effective Hamiltonian to this order.

Condition (5.4) is not sufficient for determining S. If it is satisfied by e^{iS} , it is also satisfied by $e^{iS} e^{iT}$ where e^{iT} does not connect \mathscr{C}^0_{α} and \mathscr{C}^0_{β} . We will reduce this indetermination by imposing to S to be purely off diagonal in α ,

$$\langle i, \alpha | S | j, \alpha \rangle = 0.$$
(5.6)

To summarize, we look for an Hermitian matrix S, purely off diagonal [condition (5.6)], such that the matrix elements of $e^{iS} H e^{-iS}$ between \mathscr{C}^0_{α} and \mathscr{C}^0_{β} with $\beta \neq \alpha$ vanish as λ^n if $\lambda \to 0$ [conditions (5.4)].

Remark. The diagonalization of $P^0_{\alpha}\tilde{H}P^0_{\alpha}$ gives the correct perturbed energies (up to order λ^n) associated with \mathscr{C}^0_{α} . But it must be emphasized that the eigenstates of $P^0_{\alpha}\tilde{H}P^0_{\alpha}$ are *not* the correct eigenstates, since they are linear superpositions of the unperturbed levels of \mathscr{C}^0_{α} , without any contamination from the other manifolds. If we are also interested in the eigenstates of $H = H_0 + \lambda V$, we must apply e^{-iS} to the eigenstates of $P^0_{\alpha}\tilde{H}P^0_{\alpha}$.

5.2.2. Power series expansion

The transformed Hamiltonian \tilde{H} can be written

$$\tilde{H} = e^{iS} H e^{-iS}$$

= $H + i[S, H] + \frac{i^2}{2!} [S, [S, H]] + \dots$ (5.7)

If we expand S in a power series of λ

$$S = \lambda S_1 + \lambda^2 S_2 + \lambda^3 S_3 + \dots, \tag{5.8}$$

and if we introduce the expansion (5.8) into (5.7), replacing also H by $H_0 + \lambda V$, we get the following power series expansion of \tilde{H} :

$$\tilde{H} = \tilde{H}_0 + \lambda \tilde{H}_1 + \lambda^2 \tilde{H}_2 + \lambda^3 \tilde{H}_3 + \dots, \qquad (5.9)$$

with

$$\tilde{H}_0 = H_0$$
, (5.10)
 $\tilde{H}_1 = i[S_1, H_0] + V$. (5.11)

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$$\tilde{H}_{2} = i[S_{2}, H_{0}] + i[S_{1}, V] + \frac{i^{2}}{2!} [S_{1}, [S_{1}, H_{0}]], \qquad (5.12)$$

$$\tilde{H}_{3} = i[S_{3}, H_{0}] + i[S_{2}, V] + \frac{i^{2}}{2!} [S_{1}, [S_{2}, H_{0}]] + \frac{i^{2}}{2!} [S_{2}, [S_{1}, H_{0}]] + \frac{i^{2}}{2!} [S_{1}, [S_{1}, V]] + \frac{i^{3}}{3!} [S_{1}, [S_{1}, [S_{1}, H_{0}]]], \qquad (5.13)$$

and so on.

We use now the two conditions (5.4) and (5.6) on S to determine S and the effective Hamiltonian H^{α}_{eff} inside \mathscr{C}^{0}_{α} to a given order in λ . Before, we note that condition (5.6) must be satisfied for all λ , which implies

$$\langle i, \alpha | S_1 | j, \alpha \rangle = 0 = \langle i, \alpha | S_2 | j, \alpha \rangle = \dots$$
 (5.14)

5.2.1. Order 1

To order 1 in λ , \tilde{H} can be approximated by $\tilde{H}_0 + \lambda \tilde{H}_1$. According to (5.10), \tilde{H}_0 coincides with H_0 and has no off-diagonal elements. It follows that the off-diagonal elements of $\tilde{H}_0 + \lambda \tilde{H}_1 = H_0 + \lambda \tilde{H}_1$ between \mathscr{C}^0_{α} and \mathscr{C}^0_{β} (with $\alpha \neq \beta$) vanish as λ if $\lambda \to 0$. The restriction of $\tilde{H}_0 + \lambda \tilde{H}_1$ to \mathscr{C}^0_{α} can therefore be considered as an effective Hamiltonian in \mathscr{C}^0_{α} , correct up to order 1 in λ

$$P^{0}_{\alpha}(H_{0} + \lambda \dot{H}_{1})P^{0}_{\alpha} = H^{\alpha}_{\text{eff}} \quad \text{up to order } \lambda .$$
(5.15)

The matrix element of such an effective Hamiltonian between two states $|i, \alpha\rangle$ and $|j, \alpha\rangle$ of \mathscr{C}^0_{α} can be written, according to (5.11),

$$E_{i\alpha}\delta_{ij} + \lambda \langle i, \alpha | S_1 | j, \alpha \rangle (E_{j\alpha} - E_{i\alpha}) + \lambda \langle i, \alpha | V | j, \alpha \rangle$$

= $E_{i\alpha}\delta_{ij} + \lambda \langle i, \alpha | V | j, \alpha \rangle$. (5.16)

Because of condition (5.6), the knowledge of S_1 is not necessary for determining the effective Hamiltonian to order 1. Equation (5.16) expresses that the effective Hamiltonian is just the restriction of $H_0 + \lambda V$ to \mathscr{C}^0_{α} .

5.2.4. Order 2 To order 2 in λ , \tilde{H} coincides with $H_0 + \lambda \tilde{H}_1 + \lambda^2 \tilde{H}_2$. Let us write [condition (5.4)] that the off-diagonal part ($\alpha \neq \beta$) of $H_0 + \lambda \tilde{H}_1 + \lambda^2 \tilde{H}_2$ vanishes as λ^2 if $\lambda \to 0$. This implies that the off-diagonal part of \hat{H}_1 is equal to zero. Taking the matrix element of \tilde{H}_1 [given by (5.11)] between $|i, \alpha\rangle$ and $|j, \beta\rangle$ ($\beta \neq \alpha$), we get

$$\langle i, \alpha | \tilde{H}_1 | j, \beta \rangle = 0$$

= $i \langle i, \alpha | S_1 | j, \beta \rangle (E_{j\beta} - E_{i\alpha}) + \langle i, \alpha | V | j, \beta \rangle .$ (5.17)

This condition determines the off-diagonal elements of S_1 ,

$$\langle i, \alpha | S_1 | j, \beta \rangle = -i \frac{\langle i, \alpha | V | j, \beta \rangle}{E_{i\alpha} - E_{j\beta}}, \qquad (5.18)$$

which are the only ones to be different from zero because of condition (5.6).

Having cancelled the off-diagonal part of $H_0 + \lambda \tilde{H}_1 + \lambda^2 \tilde{H}_2$ up to order 1 in λ , we can consider now the restriction of $H_0 + \lambda \tilde{H}_1 + \lambda^2 \tilde{H}_2$ to \mathcal{E}^0_{α} as an effective Hamiltonian correct up to order λ^2 ,

$$P^{0}_{\alpha}(H_{0} + \lambda \dot{H}_{1} + \lambda^{2}\dot{H}_{2})P^{0}_{\alpha} = H^{\alpha}_{\text{eff}} \quad \text{up to order } \lambda^{2}.$$
(5.19)

In order to calculate the matrix elements of (5.19) inside \mathscr{C}_{α}^{0} , we need in principle S_2 , since S_2 appears in the expression (5.12) of \tilde{H}_2 . But S_2 only appears in a commutator with H_0 , and the matrix element of this commutator between two states $|i, \alpha\rangle$ and $|j, \alpha\rangle$ of \mathscr{C}_{α}^{0} ,

$$\langle i, \alpha | [S_2, H_0] | j, \alpha \rangle = \langle i, \alpha | S_2 | j, \alpha \rangle (E_{j\alpha} - E_{i\alpha}) = 0, \qquad (5.20)$$

vanishes because of condition (5.14) on S_2 . Since we know S_1 [eq. (5.18)], and the expression of \tilde{H}_1 and \tilde{H}_2 in terms of S_1 and V [eqs. (5.11) and (5.12)], we know everything for calculating the matrix elements of (5.19) inside \mathscr{C}^0_{α} . A straightforward calculation gives

$$\langle i, \alpha | H_{\text{eff}}^{\alpha} | j, \alpha \rangle = E_{i\alpha} \delta_{ij} + \lambda \langle i, \alpha | V | j, \alpha \rangle$$

+ $\lambda^{2} \sum_{\beta \neq \alpha} \sum_{k} \langle i, \alpha | V | k, \beta \rangle \langle k, \beta | V | j, \alpha \rangle$
 $\times \frac{1}{2} \left(\frac{1}{E_{i\alpha} - E_{k\beta}} + \frac{1}{E_{j\alpha} - E_{k\beta}} \right) + O(\lambda^{3}).$ (5.21)

If i = j, one gets, for the λ^2 term of the diagonal matrix element of

 H_{eff}^{α} , a result quite similar to the second-order perturbation shift. If $i \neq j$, one gets the half sum of two energy denominators, one for $|i, \alpha\rangle$, the other one for $|j, \alpha\rangle$.

This method of calculation can be easily extended to higher orders. By cancelling the off-diagonal matrix elements of \tilde{H}_2 , we get an equation which determines S_2 . The restriction of $H_0 + \lambda \tilde{H}_1 + \lambda^2 \tilde{H}_2 + \lambda^3 \tilde{H}_3$ to \mathcal{E}^0_{α} , i.e., the effective Hamiltonian up to order λ^3 , can then be entirely calculated, since the contribution of $[S_3, H_0]$ vanishes because of condition (5.14).

5.3. Application to radiative corrections in the nonrelativistic limit

As an application of the effective Hamiltonian method, we present now an approach to radiative corrections in the nonrelativistic limit following closely the treatment presented in references [4] and [5]. We will just give here the general idea and the principle of the calculations, referring the reader to [4] and [5] for more details. Such a method will be then extended to the relativistic domain in the subsequent seminar [1].

5.3.1. Unperturbed Hamiltonian H_0 and coupling V

Consider the system formed by a single non-relativistic electron interacting with the quantized radiation field. The Hamiltonian H of the total system can be written

$$H = H_{\rm e} + H_{\rm R} + H_{\rm I} \,, \tag{5.22}$$

where H_e is the electronic Hamiltonian, H_R the Hamiltonian of the radiation field, H_I the interaction Hamiltonian.

To order 2 in the electronic charge q, the effects of the various modes add independently. They correspond to virtual emissions and reabsorptions of photons of a given mode. We can therefore consider the simpler system formed by a single electron interacting with a *single mode* with frequency ω . At the end of the calculation, we will have to make an integration over ω .

The unperturbed Hamiltonian for the electron-mode ω system is

$$H_0 = H_e + \hbar\omega (a^+ a + \frac{1}{2}), \qquad (5.23)$$

and the coupling V is, according to the results of section 2.4, of the

form

$$V = V_1 + V_2, (5.24)$$

where V_1 is linear in a and a^+ and of order q,

$$V_1 = V_+ a^+ + V_- a \,, \tag{5.25}$$

and V_2 is quadratic in a and a^+ and of order q^2 ,

$$V_2 = V_{++}(a^+)^2 + V_{--}a^2 + V_{+-}a^+a + V_{-+}aa^+, \qquad (5.26)$$

 V_1 and V_2 are actually the contributions of mode ω to the interaction Hamiltonians $H_{I1} + H'_{I1}$ and H_{I2} of section 2.4 in the case where there is a single electron. The explicit expression of the electronic operators V_+ , V_- , V_{++} , ... V_{-+} could therefore be deduced from the results of section 2.4 (see also [4] and [5]). For the following discussion, we need only to consider the general structure of V_1 and V_2 .

5.3.2. Unperturbed manifolds of the electron-mode ω system

We will suppose now that we consider a weakly bound electron, more precisely that the electron Bohr frequencies ω_e are small compared to the frequency ω of the mode,

$$\omega_{\rm e} \ll \omega \,. \tag{5.27}$$

The energy levels of the unperturbed Hamiltonian H_0 are labelled by two quantum numbers: N for the number of photons in the mode, *i* for the electronic energy level. As a consequence of (5.27), the energy levels of H_0 bunch in well separated manifolds \mathscr{C}_N , corresponding to the different values of N (N plays the role of the quantum number α used in section 5.2). \mathscr{C}_N for example corresponds to the energy levels of the electron in presence of N photons [fig. 9]. The distance between energy levels of the same manifold, of the order of $\hbar \omega_e$, is small compared to the distance $\hbar \omega$ between adjacent manifolds.

The various terms of V_1 and V_2 introduce various types of couplings between these manifolds. V_+a^+ couples \mathscr{C}_N to \mathscr{C}_{N+1} and V_-a couples \mathscr{C}_N to \mathscr{C}_{N-1} . $V_{+-}a^+a$ and $V_{-+}a a^+$ only act inside \mathscr{C}_N . $V_{++}(a^+)^2$ and $V_{--}a^2$ respectively couple \mathscr{C}_N to \mathscr{C}_{N+2} and \mathscr{C}_{N-2} .

The idea is now to describe the effect of these various couplings on the energy levels of \mathscr{C}_N by an effective Hamiltonian H^N_{eff} acting inside \mathscr{C}_N . This effective Hamiltonian will describe the modification of the slow motion of the electron due to its high-frequency vibration in the mode ω , in terms of a new effective mass, a new effective magnetic moment, new electric or magnetic form factors. Comparing such

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Fig. 9. Unperturbed manifolds of H_0 and various couplings between them.

corrections for $N \neq 0$ and for N = 0 will allow us to compare "stimulated" corrections (depending on the incident photons) and "spontaneous" corrections which exist even in the vacuum of photons and which are usually called radiative corrections.

5.3.3. Expression of the effective Hamiltonian inside \mathscr{C}_N ; operatorial form

According to the results of section 5.2, the matrix element of the effective Hamiltonian H_{eff}^N between $|iN\rangle$ and $|jN\rangle$ is given by

$$\langle j, N | H_{\text{eff}}^{N} | i, N \rangle = \langle j, N | (V_{+-}a^{+}a + V_{-+}aa^{+}) | i, N \rangle$$

$$+ \frac{1}{2} \sum_{k} \langle j, N | V_{-}a | k, N + 1 \rangle \langle k, N + 1 | V_{+}a^{+} | i, N \rangle$$

$$\times \left[\frac{1}{E_{i} - E_{k} - \hbar\omega} + \frac{1}{E_{j} - E_{k} - \hbar\omega} \right]$$

$$+ \frac{1}{2} \sum_{l} \langle j, N | V_{+}a^{+} | l, N - 1 \rangle \langle l, N - 1 | V_{-}a | i, N \rangle$$

$$\times \left[\frac{1}{E_{i} - E_{l} + \hbar\omega} + \frac{1}{E_{j} - E_{l} + \hbar\omega} \right]$$
(5.28)

 $(V_{++} \text{ and } V_{--} \text{ are already of order } q^2 \text{ and do not appear in } H^N_{\text{eff}} \text{ because}$ they are associated with off-diagonal couplings between \mathscr{C}_N and $\mathscr{C}_{N\pm 2}$).

Using $a|N\rangle = \sqrt{N}|N-1\rangle$ and $a^+|N\rangle = \sqrt{N+1}|N+1\rangle$, one can get rid of the field operators a and a^+ in (5.28) and transform (5.28) into

$$\langle j, N | H_{\text{eff}}^{N} | i, N \rangle = (N+1) \langle j | R | i \rangle + N \langle j | S | i \rangle, \qquad (5.29)$$

where R and S are pure electronic operators given by

$$\langle j|R|i\rangle = \langle j|V_{-+}|i\rangle + \frac{1}{2}\sum_{k} \langle j|V_{-}|k\rangle\langle k|V_{+}|i\rangle \left[\frac{1}{E_{i} - E_{k} - \hbar\omega} + \frac{1}{E_{j} - E_{k} - \hbar\omega}\right], \quad (5.30)$$

and a similar expression for S with the following substitutions $V_{-+} \rightarrow V_{+-}$, $V_{-} \rightarrow V_{+}$, $V_{+} \rightarrow V_{-}$, $\omega \rightarrow -\omega$.

Up to now, R and S are only defined by their matrix elements. It is possible to use condition (5.27) for deriving an explicit operatorial expression for H_{eff}^N . Let us illustrate this transformation on a given term, for example,

$$\frac{1}{2}\sum_{k}\frac{\langle j|V_{-}|k\rangle\langle k|V_{+}|i\rangle}{E_{i}-E_{k}-\hbar\omega},$$
(5.31)

appearing in (5.30). Since the electronic Bohr frequency $(E_i - E_k)/\hbar$ is small compared to ω , we can expand the energy denominator of (5.30) into

$$\frac{1}{E_i - E_k - \hbar\omega} = -\frac{1}{\hbar\omega} - \frac{1}{\hbar^2\omega^2} (E_i - E_k) + \dots$$
(5.32)

The term in $1/\hbar\omega$, inserted in (5.31), gives

$$-\frac{1}{2\hbar\omega}\sum_{k}\langle j|V_{-}|k\rangle\langle k|V_{+}|i\rangle = -\frac{1}{2\hbar\omega}\langle j|V_{-}V_{+}|i\rangle.$$
(5.33)

The term in $1/\hbar^2 \omega^2$, on the other hand, gives

$$-\frac{1}{2\hbar^{2}\omega^{2}}\sum_{k}\langle j|V_{-}|k\rangle\langle k|V_{+}|i\rangle\langle E_{i}-E_{k}\rangle$$

$$=-\frac{1}{2\hbar^{2}\omega^{2}}\langle j|V_{-}V_{+}H_{e}-V_{-}H_{e}V_{+}|i\rangle$$

$$=-\frac{1}{2\hbar^{2}\omega^{2}}\langle j|V_{-}[V_{+},H_{e}]|i\rangle. \qquad (5.34)$$

Finally, (5.31) can be considered as the matrix element between $|i\rangle$ and $|j\rangle$ of the electronic operator

$$-\frac{1}{2\hbar\omega}V_{-}V_{+}-\frac{1}{2\hbar^{2}\omega^{2}}V_{-}[V_{+},H_{e}]+\dots$$
(5.35)

We can in this way transform all other terms of (5.30) and obtain an explicit operatorial expression for R.

5.3.4. Stimulated corrections

The effective Hamiltonian H_{eff}^N can be written

$$H_{\text{eff}}^{N} = (N+1)R + NS$$

= $N(R+S) + R$. (5.36)

The term proportional to N, N(R + S), represents corrections proportional to the number of incident photons, which describe the modification of the dynamical properties of the weakly bound electron due to a high-frequency irradiation.

These corrections are calculated and discussed in detail in reference [4]. The important point is that they can be all interpreted semiclassically in terms of the vibration of the charge and the spin in the incident wave. For example, the kinetic energy of vibration of the charge increases the rest mass energy of the electron, the vibrating charge averages the static potentials over a finite volume. Let us in particular mention that, if the calculations are pushed far enough in 1/c, crossed charge spin effects appear. For example, the vibration of the charge in the electric field of the incident wave gives rise to a static motional magnetic field which interacts with the spin magnetic moment. We will not enter here in more details and refer the reader to [4].

5.3.5. Spontaneous corrections

The last term, R, of (5.36) exists even in the vacuum of photons (N = 0) and describes radiative corrections. It is calculated and discussed in detail in reference [5]. In order to interpret it, we can write it as

$$R = \frac{1}{2}(R+S) + \frac{1}{2}(R-S).$$
(5.37)

The first term, (R + S)/2, has exactly the same structure as the term

N(R+S) describing stimulated corrections, except that N is replaced by 1/2. It has therefore the same physical interpretation, the real incident field being replaced by vacuum fluctuations (corresponding to a zero point energy $\hbar \omega/2$ per mode instead of $N\hbar\omega$). The corrections associated with this term also exactly coincide with those calculated in section 4, from an Heisenberg equation approach, and attributed to the effect of vacuum fluctuations.

The second term of (5.37), (R - S)/2, describes radiative corrections due to the interaction of the electron with the field radiated by this electron in the mode ω . This term also exactly coincides with the results of the previous lecture. If one takes the radiative corrections attributed to self-reaction in lecture 4, and if one evaluates the contribution of mode ω of to these corrections, one finds exactly the term (R - S)/2 of the effective Hamiltonian approach.

There is therefore a complete agreement between the two approaches to radiative corrections presented in this course, the Heisenberg equation approach of section 4, with the separation of vacuum fluctuations and self-reaction effects based on Hermiticity requirements for the corresponding forces, and the effective Hamiltonian approach of this section, based on a comparison between stimulated effects and spontaneous effects.

Concerning the spin anomaly g-2, the effective Hamiltonian approach can actually be extended to include all relativistic corrections and many particle effects (to lowest order in q^2). This will be done in the subsequent seminar [1]. We must first introduce in the next section (section 6) the second quantized Hamiltonian describing the coupling between the quantized Dirac and Maxwell fields.

6. Simple introduction to interacting quantized Dirac and Maxwell fields

6.1. Introduction

The simple treatment presented in the previous chapters has a limited domain of validity. It is restricted to slow nonrelativistic particles $(v \ll c)$ and cannot describe processes where the total number of charged particles varies (pair creations and annihilations). Such limitations also appear on the formalism which treats radiation and matter in a quite asymmetrical way, using a quantized relativistic field for

radiation, with an *arbitrary* number of elementary excitations (photons), but considering only a *fixed* number of charged particles described by nonrelativistic wave functions.

The purpose of this chapter is to try to give to the reader an idea of what a more satisfactory approach to Q.E.D. can be, describing not only radiation but also matter by a quantized relativistic field.

We start in section 6.2 by a very brief review of Dirac equation considered as a relativistic wave equation for a single electron. For further details, we refer the reader to standard text books [22, 23, 24]. We then (section 6.3) second quantize such an equation and introduce the quantized Dirac field, the elementary excitations of which describe electrons (e^-) and positrons (e^+). The most important result of section 6.3 is the expression of the Hamiltonian of the interacting quantized Dirac and Maxwell fields, which can be expressed in terms of creation and annihilation operators of electrons, positrons and photons and which is the starting point for any Q.E.D. calculation.

Such an Hamiltonian seems quite different from those used in the previous chapters and we try in section 6.4 to explain the connection which exists between the two approaches. We show that the single electron nonrelativistic Hamiltonian used in the first part of this course can be considered as an effective Hamiltonian acting inside the single electron manifold of the full second quantized Hamiltonian of section 6.3.

Finally, the results of this chapter will be used in the subsequent seminar [1] for deepening the simple discussion of the spin anomaly g-2 presented above in sections 4 and 5. Starting from the full Hamiltonian of section 6.3, we will try to compute the contribution of relativistic modes ($\hbar \omega \ge mc^2$) to such a radiative correction.

6.2. A brief review of the Dirac equation

6.2.1. Dirac matrices

A simple procedure for deriving a single-particle wave equation is to start from the dispersion relation E = f(p) between the energy E and the momentum p of the particle and to replace E and p by

$$E \to i\hbar \frac{\partial}{\partial t}, \quad p \to -i\hbar \nabla.$$
 (6.1a)

In the presence of a Maxwell field described by the potential A, ϕ , one

uses the following (minimal coupling) substitution:

$$i\hbar \frac{\partial}{\partial t} \rightarrow i\hbar \frac{\partial}{\partial t} - q\phi, \quad -i\hbar \nabla \rightarrow -i\hbar \nabla - qA, \quad (6.1b)$$

where q is the charge of the particle.

If one wants to get a first-order differential equation for the wave function ψ , one must start from a relation linear in both E and p (since, in relativity, t and r play a symmetric role). Such a relation can be written

$$E = \beta m c^2 + c \boldsymbol{\alpha} \cdot \boldsymbol{p} \,, \tag{6.2}$$

where β and α are dimensionless and real. Furthermore, (6.2) must be consistent with the well-known relativistic dispersion relation

$$E^2 = m^2 c^4 + p^2 c^2 \,. \tag{6.3}$$

Squaring (6.2) gives

$$E^{2} = m^{2}c^{4}\beta^{2} + mc^{3}\sum_{i} (\alpha_{i}\beta + \beta\alpha_{i})p_{i} + c^{2}\sum_{i}\sum_{j} p_{i}p_{j}\alpha_{i}\alpha_{j}, \qquad (6.4)$$

with *i*, *j* = *x*, *y*, *z*. Using the fact that $p_ip_j = p_jp_i$ and comparing (6.3) and (6.4) leads to the following conditions:

$$\beta^{2} = 1 ,$$

$$\alpha_{i}\beta + \beta\alpha_{i} = 0 ,$$

$$\alpha_{i}\alpha_{j} + \alpha_{j}\alpha_{i} = 2\delta_{ij} ,$$
(6.5)

which clearly show that β and α cannot be *c*-numbers, but must necessarily be matrices (with a rank at least equal to 4). The wave function ψ must necessarily be a spinor with several components (at least 4), and we must consider for the particle both external (r, p, ...) and internal degrees of freedom $(\beta, \alpha, \text{spin}, ...)$.

Dirac equation corresponds to the realization of dimension 4 of eqs. (6.6). One can check that the four Dirac matrices

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbb{1} & 0\\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \boldsymbol{\alpha}_i = \begin{pmatrix} 0 & \sigma_i\\ \sigma_i & 0 \end{pmatrix}, \tag{6.6}$$

where 1 is the unit 2×2 matrix and σ_i the three Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.7}$$

satisfy conditions (6.5).

Finally, we give a few relations satisfied by Dirac matrices and which are useful for subsequent calculations. From the well-known commutation relations of Pauli matrices

$$\sigma_i \sigma_j - \sigma_j \sigma_i = 2i \sum_k \varepsilon_{ijk} \sigma_k , \qquad (6.8)$$

where ε_{ijk} is the completely antisymmetric tensor, and from (6.6), one derives

$$\alpha_i \alpha_j - \alpha_j \alpha_i = 2i \sum_k \varepsilon_{ijk} \sigma_k \,. \tag{6.9}$$

Combining this equation with the last equation (6.5) gives

$$\alpha_i \alpha_j = \delta_{ij} + i \sum_k \varepsilon_{ijk} \sigma_k \,. \tag{6.10}$$

It follows that, if A and B are two vectors (not acting upon internal degrees of freedom)

$$(\boldsymbol{\alpha} \cdot \boldsymbol{A})(\boldsymbol{\alpha} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{B} + \mathrm{i} \boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \,. \tag{6.11}$$

6.2.2. Dirac Hamiltonian – Dirac current The Dirac equation can be written

$$i\hbar \frac{\partial}{\partial t}\psi = \mathcal{H}_{\rm D}\psi, \qquad (6.12)$$

where \mathscr{H}_{D} is the Dirac Hamiltonian. From (6.1) and (6.2), it follows that, for a free electron,

$$\mathscr{H}_{\mathrm{D}} = \beta m c^2 + c \boldsymbol{\alpha} \cdot \boldsymbol{p} \,, \tag{6.13}$$

whereas, for an electron interacting with a Maxwell field,

$$\mathcal{H}_{\rm D} = \beta m c^2 + c \boldsymbol{\alpha} \cdot \boldsymbol{\pi} + q \boldsymbol{\phi}(\boldsymbol{r}) , \qquad (6.14a)$$
$$\boldsymbol{\pi} = \boldsymbol{p} - q \boldsymbol{A}(\boldsymbol{r}) . \qquad (6.14b)$$

From (6.12), one can show that, if

$$\rho = q\psi^{+}(\boldsymbol{r})\psi(\boldsymbol{r}), \qquad (6.15a)$$

$$\mathbf{j} = qc\psi^{\dagger}(\mathbf{r})\boldsymbol{\alpha}\psi(\mathbf{r}), \qquad (6.15b)$$

then

$$\frac{\partial}{\partial t}\boldsymbol{\rho} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 , \qquad (6.16)$$

 ρ and *j* can therefore be interpreted as a conserved charge density and charge current (Dirac current). Note that (6.15) and (6.16) keep the same form in presence or in absence of a Maxwell field.

6.2.3. Energy spectrum for a free electron

For a free electron, p and \mathcal{H}_D commute, so that one can find common eigenstates to p and \mathcal{H}_D (plane waves). For each value of p, there are two eigenvalues of \mathcal{H}_D ,

$$E = \pm (m^2 c^4 + p^2 c^2)^{1/2} \,. \tag{6.17}$$

The energy spectrum of a free electron is therefore formed by two continuums, one above $+mc^2$ and one below $-mc^2$.

The form of the eigenstates is particularly simple for p = 0, since then $\mathcal{H}_D = \beta mc^2$. One finds two eigenspinors

$$\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix} \text{ for } E = +mc^2, \qquad (6.18)$$

and two eigenspinors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{for } E = -mc^2 \,. \tag{6.19}$$

It follows that a free electron, at rest and with a positive energy $+mc^2$, can exist in two different internal states (spin 1/2). Such a result remains valid for $p \neq 0$ [a Lorentz transformation is applied to (6.18)], but, for $p \neq 0$, the four components of ψ are generally different from zero.

6.2.4. Negative energy states - hole theory

The existence of negative eigenvalues for \mathcal{H}_D raises problems of physical interpretation.

One would first try to consider the corresponding states as extra mathematical solutions, without physical meaning, and keep only positive energy states as physical states. The difficulty which then appears is that the interaction with a quantized radiation field couples positive and negative energy states. An electron, in a positive energy state, would spontaneously emit a photon and "fall" into a negative energy state. Positive energy states are unstable with respect to spontaneous emission.

This difficulty suggested to Dirac to consider that all negative energy states are filled. Since electrons are fermions, the Pauli exclusion principle then prevents a positive energy electron from falling into an already occupied negative energy state. The stability of positive energy states is thus restored. Furthermore, such a point of view suggests new interesting predictions. The absence of an electron with energy E < 0, charge q, momentum p, spin μ is equivalent to the presence of a particle with energy -E > 0, charge -q, momentum -p, spin $-\mu$. Such a particle is nothing but the positron, or the antiparticle of the electron, which thus appears as a "hole" in the continuum of negative energy states. Other interesting predictions directly follow from such a point of view. By absorption of a photon, a negative energy electron can be promoted to a positive energy state, leaving a hole in the continuum of negative energy states. Such a process corresponds to the transformation of a photon into a pair electron positron $(e^{-} - e^{+})$. The reverse process (pair annihilation) also exists.

All the previous considerations clearly show that the Dirac equation cannot be maintained as a single-particle relativistic equation. Such a result is actually not surprising since the number of particles is not conserved in relativity. We are thus led to quantize the Dirac wave function $\psi(x)$ (second quantization), in order to get a quantized Dirac field $\Psi(x)$ describing an indefinite number of particles.

6.3. Second quantization of the Dirac equation [25]

6.3.1. Quantized Dirac field

We follow here the general second quantization procedure used in nonrelativistic quantum mechanics for describing an ensemble of identical fermions in the occupation number representation [26].

We start from the expansion of a single-particle Dirac wave function $\psi(\mathbf{r})$ on an orthonormal basis. Such a basis can be for example the basis of the free electron Dirac Hamiltonian \mathcal{H}_D (6.13). We will call $u_p(\mathbf{r})$ the eigenspinors of \mathbf{p} and \mathcal{H}_D with eigenvalues \mathbf{p} , $E_p = +[m^2c^2 + p^2c^2]^{1/2}$, $v_{\bar{p}}(\mathbf{r})$, the eigenspinors of \mathbf{p} and \mathcal{H}_D with eigenvalues $-\mathbf{p}$, $-E_p$. [In order to simplify the notations, we skip the spin quantum numbers.] $\psi(\mathbf{r})$ can be written

$$\psi(\mathbf{r}) = \sum_{p} \left[\gamma_{p} u_{p}(\mathbf{r}) + \gamma_{\bar{p}} v_{\bar{p}}(\mathbf{r}) \right], \qquad (6.20)$$

 $\psi(\mathbf{r})$ is then quantized by replacing the coefficients γ_p and $\gamma_{\bar{p}}$ of this expansion by operators c_p and $c_{\bar{p}}$ annihilating one electron in the corresponding state

$$\Psi(\mathbf{r}) = \sum_{p} \left[c_p u_p(\mathbf{r}) + c_{\bar{p}} v_{\bar{p}}(\mathbf{r}) \right].$$
(6.21)

Since electrons are fermions, we must use anticommutation relations for these operators,

$$[c_{p}, c_{q}]_{+} = 0 = [c_{\bar{p}}, c_{\bar{q}}]_{+} = [c_{p}, c_{\bar{q}}]_{+} ,$$

$$[c_{p}, c_{q}^{+}]_{+} = \delta_{pq}, \qquad [c_{\bar{p}}, c_{\bar{q}}^{+}]_{+} = \delta_{pq} .$$
(6.22)

Following the general ideas of section 6.2.4, we now reinterpret $c_{\bar{p}}$ and $c_{\bar{p}}^+$. Since annihilating (creating) one electron -p, $-E_p$ is equivalent to creating (annihilating) one positron +p, $+E_p$, it will be convenient to write

$$c_{\bar{p}} = b_{p}^{+}, \qquad c_{\bar{p}}^{+} = b_{p},$$
 (6.23)

 b_p^+ and b_p can therefore be considered as creation and annihilation operators for a positron p, E_p , and the quantized Dirac fields $\Psi(r)$ and $\Psi^+(r)$ are given by

$$\Psi(\mathbf{r}) = \sum \left[c_p u_p(\mathbf{r}) + b_p^+ v_{\bar{p}}(\mathbf{r}) \right], \qquad (6.24a)$$

$$\Psi^{+}(\mathbf{r}) = \sum_{p} \left[c_{p}^{+} u_{p}^{+}(\mathbf{r}) + b_{p} v_{\bar{p}}^{+}(\mathbf{r}) \right].$$
(6.24b)

The anticommutation rules for the c's and b's directly follow from (6.22) and (6.23), the only nonzero anticommutators being

$$[c_{p}, c_{q}^{+}]_{+} = \delta_{pq}, \qquad [b_{p}, b_{q}^{+}]_{+} = \delta_{pq}.$$
(6.25)

6.3.2. Hamiltonian of the quantized Dirac field

The average of the single-particle Dirac Hamiltonian (6.13) in the wave function $\psi(\mathbf{r})$ is

$$\langle \mathscr{H}_{\mathsf{D}} \rangle = \int \mathrm{d}^3 r \, \psi^+(r) \, \mathscr{H}_{\mathsf{D}} \psi(r) \,.$$
 (6.26)

The second quantized Hamiltonian H_D is obtained from (6.26) by replacing the wave function $\psi(r)$ by the quantized Dirac field $\Psi(r)$,

$$H_{\rm D} = \int d^3 r \, \Psi^+(\mathbf{r}) \mathcal{H}_{\rm D} \Psi(\mathbf{r})$$
$$= \int d^3 r \, \Psi^+(\mathbf{r}) [\beta m c^2 + c \boldsymbol{\alpha} \cdot \boldsymbol{p}] \Psi(\mathbf{r}) \,. \tag{6.27}$$

Using the expansions (6.24) and the fact that \mathcal{H}_{D} is diagonal in the basis $\{u_{p}, v_{\bar{p}}\}$, one gets

$$H_{\rm D} = \sum_{p} E_{p} c_{p}^{+} c_{p} - \sum_{p} E_{p} b_{p} b_{p}^{+} .$$
(6.28)

But, from (6.25), $b_p b_p^+ = 1 - b_p^+ b_p$, so that

$$H_{\rm D} = E_0 + \sum_p E_p c_p^+ c_p + \sum_p E_p b_p^+ b_p \,. \tag{6.29}$$

The physical interpretation of (6.29) is very clear. $E_0 = \sum_p (-E_p)$ is the energy of the vacuum and is not observable (only deviations with respect to the vacuum are observable). $c_p^+ c_p$ is the number of electrons with energy E_p , which contribute to the total energy by an amount $E_p c_p^+ c_p$. Similarly, $b_p^+ b_p$ is the number of positrons with energy E_p . One can note that the anticommutation relation between b_p and b_p^+ is

essential for preventing the field energy from becoming infinitely negative.

In the presence of an external static field A_0 , ϕ_0 we just replace the expression (6.13) of \mathcal{H}_D by (6.14) [A and ϕ being replaced by A_0 and ϕ_0], and we get

$$H_{\rm D} = \int {\rm d}^3 r \, \Psi^+(r) \{\beta m c^2 + c \alpha \cdot [p - q A_0(r)] + q \phi_0(r)\} \Psi(r) \,. \tag{6.30}$$

It is then more convenient to expand Ψ and Ψ^+ on the orthonormal basis of \mathcal{H}_D including the static fields A_0 , ϕ_0 .

6.3.3. Hamiltonian of the interacting quantized Dirac and Maxwell fields

We start now from the Hamiltonian \mathscr{H} of a single Dirac particle interacting with the quantized Maxwell field. We use the Coulomb gauge for such a quantized field and denote by A(r) the (transverse) vector potential. The contribution of the longitudinal field E_{\parallel} to the Maxwell field energy is, as shown in section 1.3.5, the electrostatic energy \mathscr{V}_{Coul} of the charge distribution ρ . For a single particle, \mathscr{V}_{Coul} reduces to a *c*-number, ε_{Coul} , given in (2.61). As in the previous sections, we denote by $H_{\rm R}$ the energy of the transverse Maxwell field. The contribution of an eventual external static field A_0 , ϕ_0 is also included in \mathscr{H} which can thus be written

$$\mathcal{H} = H_{\mathrm{R}} + \varepsilon_{\mathrm{Coul}} + \beta m c^2 + c \alpha \cdot [p - q A_0(r) - q A(r)] + q \phi_0(r) \,. \tag{6.31}$$

In order the get the second quantized form of \mathcal{H} , we first note that $H_{\rm R}$ is a pure photon operator and remains unchanged. In the Coulomb energy of the charge distribution ρ ,

$$\mathcal{V}_{\text{Coul}} = \frac{q^2}{8\pi\varepsilon_0} \int \int d^3r \, d^3r' \, \frac{\rho(r)\rho(r')}{|r-r'|}, \qquad (6.32)$$

we replace the charge densities $\rho(\mathbf{r})$ and $\rho(\mathbf{r}')$ by operators obtained by replacing in (6.15a) the Dirac wave functions by the corresponding field operators. And, for the remaining part of \mathcal{H} , we take its average value in $\psi(\mathbf{r})$ and replace ψ by Ψ . We thus get

$$H = H_{\rm R} + V_{\rm Coul} + \int d^3 r \, \Psi^+(r) \\ \times \{\beta m c^2 + c \boldsymbol{\alpha} \cdot [p - q \boldsymbol{A}_0(r) - q \boldsymbol{A}(r)] + q \phi_0(r)\} \Psi(r), \qquad (6.33)$$
with

$$V_{\text{Coul}} = \frac{q^2}{8\pi\varepsilon_0} \int \int d^3r \, d^3r' \, \frac{\Psi^+(r)\Psi(r)\Psi^+(r')\Psi(r')}{|r-r'|} \tag{6.34}$$

The Hamiltonian H can also be written

$$H = H_{\rm R} + H_{\rm D} + H_{\rm I}, \tag{6.35}$$

where $H_{\rm R}$ is the Hamiltonian of the quantized Maxwell field, $H_{\rm D}$ the Hamiltonian (6.30) of the quantized Dirac field (in the presence of the external static fields) and

$$H_{\rm I} = -\int d^3r j(r) \cdot A(r) + V_{\rm Coul}$$
(6.36)

the interaction Hamiltonian. In (6.36),

$$\mathbf{j}(\mathbf{r}) = qc \int d^3 \mathbf{r} \, \Psi^+(\mathbf{r}) \boldsymbol{\alpha} \Psi(\mathbf{r}) \tag{6.37}$$

is the second quantized Dirac current.

H acts in the Fock space of photons, electrons and positrons $[\Psi$ destroys e⁻ or creates e⁺, Ψ^+ creates e⁻ or destroys e⁺, *A* creates or destroys photons], and describes elementary Q.E.D. interaction processes.

Finally, it will be interesting to write the Heisenberg equations for the Maxwell and Dirac fields. Consider first the Maxwell field. As explained in the first part of this course, it is more convenient to calculate \dot{a}_i , where a_i is the annihilation operator for a photon of the mode i,

$$\dot{a}_{i} = \frac{1}{\mathrm{i}\hbar} [a_{i}, H]$$
$$= \frac{1}{\mathrm{i}\hbar} [a_{i}, H_{\mathrm{R}}] - \frac{1}{\mathrm{i}\hbar} \left[a_{i}, \int \mathrm{d}^{3}r j(r) \cdot A(r) \right].$$
(6.38)

The contribution of the first commutator is just $-i\omega a_i$. Taking the expansion of A(r) in a_j and a_j^+ , one gets for the second commutator of (6.38)

$$\frac{\mathrm{i}}{\sqrt{2\varepsilon_0 \hbar \omega_i}} j_i \,, \tag{6.39}$$

where

$$j_i = \frac{1}{\sqrt{L^3}} \int d^3 r \, e^{-ik_i \cdot r} \, \varepsilon_i \cdot j(r) \,. \tag{6.40}$$

The equation giving \dot{a}_i is therefore the same as in section 2.3.1 [see eq. (2.35)], the only difference being the expression of the current j(r) which is now given by (6.37) instead of (2.32). This shows that Maxwell equations remain valid between Maxwell field operators, the source term being the second quantized Dirac current (6.37). We consider now the Heisenberg equation for the Dirac field operator $\Psi(r)$. It will be useful first to note that, as a consequence of (6.25), we have

$$[\Psi(\mathbf{r}), \Psi(\mathbf{r}')]_{+} = 0 = [\Psi^{+}(\mathbf{r}), \Psi^{+}(\mathbf{r}')]_{+},$$

$$[\Psi(\mathbf{r}), \Psi^{+}(\mathbf{r}')]_{+} = \delta(\mathbf{r} - \mathbf{r}').$$
(6.41)

Using such anticommutation relations, it is then possible to show that

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}) = [H_{\rm D}, \Psi(\mathbf{r})] + [H_{\rm I}, \Psi(\mathbf{r})]$$
$$= \{\beta mc^2 + c\boldsymbol{\alpha} \cdot [\boldsymbol{p} - qA_0(\boldsymbol{r})] + q\phi_0(\boldsymbol{r})\}\Psi(\boldsymbol{r})$$
$$-qc\boldsymbol{\alpha} \cdot A(\boldsymbol{r})\Psi(\boldsymbol{r}) + \frac{1}{2}q[\phi(\boldsymbol{r})\Psi(\boldsymbol{r}) + \Psi(\boldsymbol{r})\phi(\boldsymbol{r})], \quad (6.42)$$

where

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \int d^3r' \, \frac{\Psi^+(\mathbf{r}')\Psi(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{6.43}$$

is the electrostatic potential of the second quantized charge distribution. Equation (6.42) shows that the Dirac field operator $\Psi(r)$ satisfies the Dirac equation in presence of the static and quantized Maxwell fields. [The last term of (6.42) represents the symmetrized form of the interaction with $\phi(r)$.]

The fact that Maxwell and Dirac equations remain valid between field operators can be considered as a justification a posteriori of the basic Hamiltonian H given in (6.33).

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6.4. Justification of the single particle Hamiltonians used for nonrelativistic electrons

We try now to explain the relation which exists between the basic Q.E.D. Hamiltonian (6.33) and those introduced in the first part of this course.

6.4.1. Splitting of the total Hamiltonian into an unperturbed part H_0 and a coupling V

If we consider only slow particles and nonrelativistic modes of the radiation field ($\hbar \omega \ll mc^2$), the largest energy appearing in the equation is the rest mass energy mc^2 . This suggests to isolate in the total Hamiltonian H given in (6.33) an unperturbed part

$$H_0 = \int d^3 r \, \Psi^+(\mathbf{r}) \beta m c^2 \Psi(\mathbf{r}) \,, \tag{6.44}$$

and to treat the remaining part

$$V = H - H_0 \tag{6.45}$$

as a small perturbation.

Since

$$\mathcal{H}_0 = \beta m c^2 \tag{6.46}$$

appears in the expression of H_0 , it will now be more convenient to expand Dirac field operators on the orthonormal basis of \mathcal{H}_0 . The spectrum of \mathcal{H}_0 is actually very simple. It is formed (fig. 10) by two degenerate manifolds \mathscr{E}_{\pm}^0 , with energies $\pm mc^2$.



Fig. 10. Unperturbed manifolds of \mathcal{H}_0 .

The eigenstates of \mathscr{C}_{+}^{0} are spinors with only the first two components different from zero. We will call them $u_{0p}(\mathbf{r})$, where p is a labelling quantum number (for example, the eigenvalue of \mathbf{p}). Similarly, we will call $v_{0p}(\mathbf{r})$ the eigenstates of \mathscr{C}_{-}^{0} which have only their last two components different from zero. As in (6.24), we have

$$\Psi(\mathbf{r}) = \sum_{\mathbf{r}} \left[c_{0p} u_{0p}(\mathbf{r}) + b_{0p}^+ v_{0\bar{p}}(\mathbf{r}) \right], \tag{6.47a}$$

$$\Psi^{+}(\mathbf{r}) = \sum_{p} \left[c_{0p}^{+} u_{0p}^{+}(\mathbf{r}) + b_{0p} v_{0p}^{+}(\mathbf{r}) \right], \qquad (6.47b)$$

where c_{0p} (c_{0p}^+) is the annihilation (creation) operator for a particle with energy $+mc^2$ and a momentum p, b_{0p}^+ (b_{0p}) the creation (annihilation) operator for an antiparticle with energy $+mc^2$ and momentum p.

6.4.2. Manifolds of H_0 corresponding to different numbers of particles and antiparticles

Inserting (6.47) in (6.44) gives, as in section 6.3.2,

$$H_0 = E_0 + mc^2 (N_e + N_p), \qquad (6.48)$$

where E_0 is the energy of the vacuum (non-observable),

$$N_{\rm e} = \sum_{p} c_{0p}^{+} c_{0p} \tag{6.49}$$

is the total number of particles e^- , and

$$N_{\rm p} = \sum_{p} b_{0p}^{+} b_{0p} \tag{6.50}$$

the total number of antiparticles e^+ .

Since the eigenvalues of N_e and N_p are integers (0, 1, 2, ...), it follows that the spectrum of H_0 is formed by a series of manifolds, represented in fig. 11, where we have plotted vertically the energy $(mc^2$ times the total number of e^+ and e^-), and horizontally the total charge $(N_p - N_e)$. The lowest manifold is the vacuum of e^+ and e^- , with energy 0 and charge 0. Then, we have two single particle manifolds at $+mc^2$ (the single e^- manifold with charge -1, the single e^+ manifold with charge +1), then three two-particles manifolds (two e^- , one e^- and one e^+ , two e^+ , with charges -2, 0, +2 and energies 2 mc^2), and so on Introduction to quantum electrodynamics



Fig. 11. Unperturbed manifolds of H₀.

Because of charge conservation, V only couples manifolds belonging to the same vertical column. Such couplings correspond to the creation or to the annihilation of one or several pairs e⁻, e⁺. Since the distance between two manifolds of the same column is at least $2mc^2$, we can, in the nonrelativistic domain, treat the effect of V on each manifold by the effective Hamiltonian method introduced in section 5. For example, we determine, in the next section (section 6.4.3), the effective Hamiltonian in the manifold 1 e⁻, and show that it coincides with the single particle Hamiltonian used in nonrelativistic quantum mechanics and acting upon two components spinors. We could also study the effective Hamiltonian corresponding to other manifolds, such as 1e-, 1e⁺ for example. The coupling with the vacuum, which appears in fig. 11 for (1e⁻, 1e⁺), but not for 2e⁻ or 2e⁺, suggests that an additional force must exist between a particle and its antiparticle, and which is not present between two particles or two antiparticles. This is the "annihilation force".

6.4.3. Effective Hamiltonian in the one-particle manifold

Selection rules for the various terms of V. We consider now the part \mathcal{U} of \mathcal{H}

$$\mathcal{U} = c\boldsymbol{\alpha} \cdot [\boldsymbol{p} - q\boldsymbol{A}_0(\boldsymbol{r}) - q\boldsymbol{A}(\boldsymbol{r})], \qquad (6.51)$$

which contains the Dirac matrices α_i and which gives rise to the second quantized operator

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$$U = \int d^3 r \, \Psi^+(r) \, \mathcal{U} \Psi(r) \,. \tag{6.52}$$

According to (6.6), the α_i 's are "odd" matrices which have zero matrix elements between two u_0 's or two v_0 's

$$\langle u_{0p}|\mathcal{U}|u_{0q}\rangle = \langle v_{0\bar{p}}|\mathcal{U}|v_{0\bar{q}}\rangle = 0.$$
(6.53)

It follows that

$$U = \sum_{qr} \langle v_{0\bar{q}} | \mathcal{U} | u_{0r} \rangle b_{0q} c_{0r} + \sum_{st} \langle u_{0s} | \mathcal{U} | v_{0\bar{t}} \rangle c_{0s}^+ b_{0t}^+ .$$
(6.54)

U creates a pair (term $c_0^+ b_0^+$) or destroys a pair (term $b_0 c_0$), and therefore couples the manifold (1e⁻) to the manifold (2e⁻, 1e⁺).

We note

$$\mathcal{W} = q\phi_0(\mathbf{r}) \,, \tag{6.55}$$

and

$$W = \int d^3 r \, \Psi^+(r) \, \mathcal{W} \Psi(r) \,, \tag{6.56}$$

the terms of \mathcal{H} and H involving the external static potential. The only nonzero matrix elements of \mathcal{W} are between two u_0 's or two v_0 's, so that

$$W = \sum_{qr} \langle u_{0q} | \mathcal{W} | u_{0r} \rangle c_{0q}^{+} c_{0r} + \sum_{st} \langle v_{0\bar{s}} | \mathcal{W} | v_{0\bar{t}} \rangle b_{0s} b_{0\bar{t}}^{+}.$$
(6.57)

W does not change the total number of e^- and e^+ and acts only inside each manifold.

It remains to consider V_{Coul} given in (6.34). Using the expansion (6.47) of Ψ and Ψ^+ , one can show that V_{Coul} changes the total number of e^- and e^+ by $\Delta N = 0$, ± 2 , ± 4 . Since we restrict ourselves in the following to order q^2 , and since V_{Coul} is already of order q^2 , only the restriction of V_{Coul} to the manifold (1e⁻) will be needed.

Expression of the effective Hamiltonian. Subtraction of the vacuum shift. If we note \mathscr{C}_1^0 and \mathscr{C}_3^0 the manifolds (1e⁻) an (2e⁻, 1e⁺), P_1^0 and P_3^0 the corresponding projectors, $|1_{0p}\rangle$ the state of \mathscr{C}_1^0 corresponding to one e⁻ in the state u_{0p} , we have, according to the results of section 5,

$$\langle 1_{0p} | H_{\text{eff}} | 1_{0p'} \rangle = mc^2 \delta_{pp'} + H_{\text{R}} \delta_{pp'} + \varepsilon_{\text{Coul}} \delta_{pp'} + \langle 1_{0p} | W | 1_{0p'} \rangle - \frac{\langle 1_{0p} | UP_3^0 U | 1_{0p'} \rangle}{2mc^2} .$$
 (6.58)

Since the only nonzero matrix elements of U are between \mathscr{C}_1^0 and \mathscr{C}_3^0 , we can replace P_3^0 by the unit operator in the last term of (6.58) which then becomes

$$-\frac{\langle 1_{0p}|U^2|1_{0p'}\rangle}{2mc^2}.$$
(6.59)

This term is of order $(1/c)^0$, since U is proportional to c [see eq. (6.51)]. To lowest order in q (q^2), and to lowest order in 1/c (order 0), there are no other terms in H_{eff} .

All energies are measured with respect to the vacuum of e^- and e^+ . So, we must substract from the previous effective Hamiltonian a unit operator ($\delta_{pp'}$) multiplied by the shift of the vacuum. This finally gives

$$\langle 1_{0p} | H_{\text{eff}} | 1_{0p'} \rangle = mc^2 \delta_{pp'} + H_{\text{R}} \delta_{pp'} + \langle 1_{0p} | W | 1_{0p'} \rangle - \delta_{pp'} \langle 0 | W | 0 \rangle + \langle 1_{0p} | V_{\text{Coul}} | 1_{0p'} \rangle - \delta_{pp'} \langle 0 | V_{\text{Coul}} | 0 \rangle - \frac{\langle 1_{0p} | U^2 | 1_{0p'} \rangle}{2mc^2} + \delta_{pp'} \frac{\langle 0 | U^2 | 0 \rangle}{2mc^2}.$$
(6.60)

Calculation of the effective Hamiltonian. We consider first the terms in W of (6.60). From the expression (6.57) of W, we get for these terms

$$\sum_{qr} \langle u_{0q} | \mathcal{W} | u_{0r} \rangle [\langle 1_{0p} | c_{0q}^{+} c_{0r} | 1_{0p'} \rangle - \delta_{pp'} \langle 0 | c_{0q}^{+} c_{0r} | 0 \rangle] + \sum_{st} \langle v_{0\bar{s}} | \mathcal{W} | v_{0\bar{t}} \rangle [\langle 1_{0p} | b_{0s} b_{0t}^{+} | 1_{0p'} \rangle - \delta_{pp'} \langle 0 | b_{0s} b_{0t}^{+} | 0 \rangle] .$$
(6.61)

Now, using

$$c_{0r}|1_{0p'}\rangle = \delta_{rp'}|0\rangle, \qquad c_{0r}|0\rangle = 0 \tag{6.62}$$

and the adjoint relations, we find that the bracket of the first line of (6.61) reduces to $\delta_{rp'}\delta_{qp}$, which gives for this line $\langle u_{0p}|\mathcal{W}|u_{0p'}\rangle$. The

calculation of the bracket of the second line of (6.61) gives

$$\langle 1_{0p} | b_{0s} b_{0t}^{+} | 1_{0p'} \rangle = \delta_{st} \langle 1_{0p} | 1_{0p'} \rangle = \delta_{st} \delta_{pp'} ,$$

$$\langle 0 | b_{0s} b_{0t}^{+} | 0 \rangle = \delta_{st} .$$
(6.63)

The two terms of the second line of (6.61) therefore cancel. Finally, the contribution of W to H_{eff} is just

$$\langle u_{0p}|\mathcal{W}|u_{0p'}\rangle,\tag{6.64}$$

which is the same result as if we had used the single particle operator W.

We will not give here the calculation of the contribution of V_{Coul} , since a similar calculation will be done in the subsequent seminar [1]. We just give the result which is quite simple. As expected, one gets $\varepsilon_{\text{Coul}}\delta_{pp'}$, where $\varepsilon_{\text{Coul}}$, which is the energy of the Coulomb field of a single electron, has been already discussed in the previous sections (see section 1.3.5).

It remains to consider the last line of (6.60). From the expression (6.54) of U, four types of operators appear in U^2 :

| $b_{0q}c_{0r}b_{0q'}c_{0r'},$ | $b_{0q}c_{0r}c_{0s}^{+}b_{0t}^{+}$, | |
|-----------------------------------|---|--------|
| $c_{0s}^+ b_{0t}^+ b_{0q} c_{0r}$ | $c_{0s}^+ b_{0t}^+ c_{0s'}^+ b_{0t'}^+$. | (6.65) |

In the manifold \mathscr{C}_1^0 , as well as in the vacuum, only the second term of (6.65) has nonzero matrix elements. [The operators of the first and fourth terms change the total number of e^- and e^+ by $\Delta N = \mp 4$. The operator of the third term destroys one e^+ which is not present in the initial state.] This gives for the last line of (6.60)

$$-\frac{1}{2mc^{2}}\sum_{qrst} \langle v_{0\bar{q}}|\mathcal{U}|u_{0r}\rangle\langle u_{0s}|\mathcal{U}|v_{0\bar{t}}\rangle \times [\langle 1_{0p}|b_{0q}c_{0r}c_{0s}^{+}b_{0t}^{+}|1_{0p'}\rangle - \delta_{pp'}\langle 0|b_{0q}c_{0r}c_{0s}^{+}b_{0t}^{+}|0\rangle].$$
(6.66)

Now, using the anticommutation relations (6.25), we transform the first term of the bracket of (6.66) into

$$\langle 1_{0p} | b_{0q} [\delta_{rs} - c_{0s}^+ c_{0r}] b_{0t}^+ | 1_{0p'} \rangle = \delta_{rs} \delta_{pp'} \delta_{qt} - \delta_{sp} \delta_{qt} \delta_{rp'} , \qquad (6.67)$$

and the second one into

$$\delta_{pp'}\langle 0|b_{0q}c_{0r}c_{0s}^{+}b_{0t}^{+}|0\rangle = \delta_{pp'}\delta_{rs}\delta_{qt}.$$
(6.68)

Inserting (6.67) and (6.68) into (6.66) gives

$$+\frac{1}{2mc^2}\sum_{q}\langle v_{0\bar{q}}|\mathcal{U}|u_{0p'}\rangle\langle u_{0p}|\mathcal{U}|v_{0\bar{q}}\rangle.$$
(6.69)

We can now commute the two matrix elements of (6.69), since A(r), which appears in the expression (6.51) of \mathcal{U} , commutes with itself, and use the closure relation between the two operators \mathcal{U} , since the matrix elements of \mathcal{U} between two u_0 's are zero. This gives for the last line of (6.60)

$$+\frac{1}{2mc^2}\langle u_{0p}|\mathcal{U}^2|u_{0p'}\rangle.$$
 (6.70)

Combining all the previous results, we get

$$\langle \mathbf{1}_{0p} | H_{\text{eff}} | \mathbf{1}_{0p'} \rangle = \langle u_{0p} | \mathcal{H}_{\text{eff}} | u_{0p'} \rangle, \qquad (6.71)$$

where \mathcal{H}_{eff} is a single particle operator, acting upon two components spinors and given by

$$\mathcal{H}_{\text{eff}} = mc^2 + H_{\text{R}} + \varepsilon_{\text{Coul}} + q\phi_0(\mathbf{r}) + \mathcal{U}^2/2mc^2 \,. \tag{6.72}$$

We finally calculate \mathcal{U}^2 . Using relation (6.11) and putting

$$\boldsymbol{\pi} = \boldsymbol{p} - q\boldsymbol{A}_0(\boldsymbol{r}) - q\boldsymbol{A}(\boldsymbol{r}) \tag{6.73}$$

we have

$$\frac{\mathscr{U}^2}{2mc^2} = \frac{\boldsymbol{\pi}^2}{2m} + \frac{\mathrm{i}}{2m} \boldsymbol{\sigma} \cdot (\boldsymbol{\pi} \times \boldsymbol{\pi}) \,. \tag{6.74}$$

But,

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = -q[\boldsymbol{p} \times (\boldsymbol{A}_0 + \boldsymbol{A}) + (\boldsymbol{A}_0 + \boldsymbol{A}) \times \boldsymbol{p}]$$

= $iq\hbar \nabla \times [\boldsymbol{A}_0(\boldsymbol{r}) + \boldsymbol{A}(\boldsymbol{r})] = iq\hbar [\boldsymbol{B}_0(\boldsymbol{r}) + \boldsymbol{B}(\boldsymbol{r})],$ (6.75)

where $B_0(r)$ is the static magnetic field and B(r) the magnetic component of the quantized Maxwell field. We can thus give the final expression for the effective Hamiltonian,

$$\mathscr{H}_{\text{eff}} = mc^2 + H_{\text{R}} + \varepsilon_{\text{Coul}} + q\phi_0(\mathbf{r}) + \frac{\pi^2}{2m} - \frac{q\hbar}{2m} \,\boldsymbol{\sigma} \cdot \left[\boldsymbol{B}_0(\mathbf{r}) + \boldsymbol{B}(\mathbf{r})\right].$$
(6.76)

6.4.4. Physical discussion

The expression (6.76) coincides with the Q.E.D. Hamiltonian introduced in the first part of this course when there is only one electron. We get the same kinetic energy term $(\pi^2/2m)$, and the magnetic couplings of the spin (last term) are deduced here from the full Hamiltonian (6.33) and not introduced from "outside", as we did in section 2. By comparing (6.76) and (2.51), we see also that the electron g-factor is equal to 2.

The results obtained in this section represent therefore a justification of the nonrelativistic approach used in sections 1 and 2. They clearly show that the Hamiltonians used in these sections are not "true" Hamiltonians, but rather effective ones.

The calculation of the effective Hamiltonian in \mathscr{C}_1^0 could be pushed to higher orders in 1/c. This would give the expression of relativistic corrections such as the spin-orbit coupling, the Darwin term, the velocity-mass correction, etc.

Remark. Going back to expression (6.72), we see that the same result would be obtained if we start from the single particle Dirac Hamiltonian and treat by the effective Hamiltonian method the effect of virtual transitions induced by \mathscr{U} between \mathscr{C}_{+}^{0} and \mathscr{C}_{-}^{0} (manifolds of $\mathscr{H}_{0} = \beta mc^{2}$). The unitary transformation applied to the single particle Dirac Hamiltonian could be also considered as a transformation leading to an "even" Hamiltonian (Foldy–Wouthuysen transformation) [22], acting only upon two-component spinors. Actually, one can show that the single-particle theory would not give correct results to higher orders in 1/c, in presence of a quantized Maxwell field. This is why we have preferred to derive (6.76) from the full many-particle Hamiltonian (6.33).

References*

- [1] J. Dupont-Roc and C. Cohen-Tannoudji, Seminar following this course, Present volume p. 157.
- [2] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, Introduction à la Théorie Quantique du Rayonnement (a paraître).
- [3] J. Dalibard, J. Dupont-Roc and C. Cohen-Tannoudji, J. Physique 43 (1982) 1617.
- [4] P. Avan, C. Cohen-Tannoudji, J. Dupont-Roc and C. Fabre, J. Physique 37 (1976) 993.
- [5] J. Dupont-Roc, C. Fabre and C. Cohen-Tannoudji, J. Phys. B11 (1978) 563.
- [6] J.D. Jackson, Classical Electrodynamics (2nd ed., Wiley, New York, 1975).
- [7] L. Landau and E. Lifshitz, The Classical Theory of Fields (Addison-Wesley, Reading, Mass., 1951; Pergamon Press, London, 1951).
- [8] A. Messiah, Mécanique Quantique, Vols. 1 et 2 (Dunod, Paris 1964); English translation: Quantum Mechanics (McGraw-Hill, New York, 1968), chapter XXI, §III.
- [9] E.A. Power, Introductory Quantum Electrodynamics (Longmans, London, 1964).
- [10] A.I. Akhiezer and V.B. Berestetskii, Quantum Electrodynamics (Wiley, New York, 1965).
- [11] E. Wigner and T.D. Newton, Rev. Mod. Phys. 21 (1949) 400; M.H.L. Pryce, Proc.
 Roy. Soc. 195A (1948) 62.
- [12] E.A. Power and T. Thirunamachandran, Am. J. Phys. 46 (1978) 370 and references therein.
- [13] R.J. Glauber, Optical Coherence and Photon Statistics, in: Quantum Optics and Electronics, Les Houches 1964, eds. C. de Witt, A. Blandin and C. Cohen-Tannoudji (Gordon and Breach, New York, 1965) p. 63.
- [14] W.E. Lamb and M.O. Scully, in: Polarisation, Matière et Rayonnement (Presses Universitaires de France, Paris, 1969) p. 363; see also M.O. Scully and M. Sargent III, Physics Today, March 1972, p. 38.
- [15] L. Mandel, E.C.G. Sudarshan and E. Wolf, Proc. Phys. Soc. 84 (1964) 435.
- [16] C. Cohen-Tannoudji, B. Diu and F. Laloë, Mécanique Quantique, Vols. 1 and 2, (Hermann, Paris, 1973); English translation: Quantum Mechanics (Wiley and Hermann, Paris, 1977), Complement G_v.
- [17] R.J. Glauber, Coherence and Quantum Detection, in: Quantum Optics, Proc. Int. Sch. of Phys. Enrico Fermi, Course XLII, ed. R.J. Glauber (Academic Press, New York, 1969) p. 15.
- [18] M. Sargent III, M.O. Scully and W.E. Lamb, Laser Physics (Addison-Wesley, Reading, Mass., 1974).
- [19] R. Hanbury Brown and R.Q. Twiss, Proc. Roy. Soc. A248 (1958) 235.
- [20] R.L. Pfleegor and L. Mandel, Phys. Rev. 159 (1967) 1084; J. Opt. Soc. Am. 58 (1968) 946.
- [21] I. Shavitt and L.T. Redmon, J. Chem. Phys. 73 (1980) 5711; C.E. Soliverez, Phys. Rev. A24 (1981) 4.

* References of chapter 4, which is a reprint of an article of Journal de Physique, are given on page 124.

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- [22] Reference [8], chapter XX.
- [23] J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).
- [24] S.S. Schweber, An Introduction to Relativistic Quantum Field Theory (Harper and Row, New York, 1961).
- [25] J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields (McGraw-Hill, New York, 1965).
- [26] L.D. Landau and E.M. Lifshitz, Quantum Mechanics, Nonrelativistic Theory (Pergamon Press, Oxford, 1965).