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Bose-Einstein Condensation: An Introduction

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Abstract. The goal of this first lecture is to introduce the notion of Bose-Einstein condensation in the simple case of a perfect Bosonic gas, which means a set of Bosons trapped in an external potential and without mutual interactions. After briefly recalling some statistical physics results, we study the case of a perfect gas of Bosons, first trapped in a harmonic potential (section 2), then in a box (section 3). We underline the singularity discovered by Einstein [1], which appears when the density in phase space exceeds a critical value.

1 Statistical Mechanics Reminders

1.1 Grand-canonical Partition Function

In order to describe a set of indistinguishable quantum particles, the most convenient statistical ensemble is the grand-canonical ensemble [2, 3, 4]. It is obtained by assuming that the considered system can exchange energy and particles with a much bigger reservoir [5]. The presence of the reservoir fixes the mean number of particles N and the mean energy U. The equilibrium state is then determined by choosing the system's density operator $\hat{\rho}$ to maximize the missing information, or *statistical entropy*:

$$S(\hat{\rho}) = -k_B \mathrm{Tr}\left(\hat{\rho}\ln(\hat{\rho})\right) \ , \tag{1}$$

given the two constraints:

$$\langle \hat{N} \rangle = N \qquad \qquad \langle \hat{H} \rangle = U \ . \tag{2}$$

This maximization under constraints can be easily solved using Lagrange multipliers. One obtains:

$$\hat{\rho} = \frac{e^{-\alpha\hat{N}-\beta\hat{H}}}{Z_G} \qquad \text{avec} \qquad Z_G = \mathrm{Tr}\left(e^{-\alpha\hat{N}-\beta\hat{H}}\right) \ . \tag{3}$$

The function Z_G is called the *grand canonical partition function*. Lagrange multipliers α and β are associated to the constraints on $\langle \hat{N} \rangle$ and $\langle \hat{H} \rangle$. The β parameter is linked to the temperature T by the formula $\beta = (k_B T)^{-1}$. The α parameter is linked to the chemical potential μ (energy needed to add a particle) by $\alpha = -\beta\mu$, and to the fugacity $z = e^{-\alpha} = e^{\beta\mu}$. Determining the values of z and β which satisfy the constraints (2) can be done through:

$$N = z\frac{\partial}{\partial z}\ln Z_G(z,\beta,V)\ , \tag{4}$$

$$U = -\frac{\partial}{\partial \beta}\ln Z_G(z,\beta,V)\ . \tag{5}$$

It is sufficient to invert in order to get z and β as functions of N and U.

Once Z_G set, all the thermodynamic variables can be obtained through simple derivation. Thus pressure is determined from:

$$P = k_B T\frac{\partial}{\partial V}\ln Z_G(z,\beta,V)\ . \tag{6}$$

1.2 The Perfect Quantum Gas

Computing Z_G explicitly, which allows to derive explicit results from expressions such as (6), is very harsh in the general case of an interacting fluid. However, the perfect gas case can be exactly solved in a very simple way, as we shall see now.

For a system of N particles which do not interact, the complete Hamiltonian is the sum of one-body Hamiltonians:

$$\hat{H} = \hat{h}_1 + \hat{h}_2 + \ldots + \hat{h}_N\ . \tag{7}$$

Let us write $\{|\lambda\rangle\}$ for a basis of the one-body Hamiltonian $\hat{h}$ eigenvectors, and ϵ_λ for the energy associated to $|\lambda\rangle$:

$$\hat{h}\,|\lambda\rangle = \epsilon_\lambda\,|\lambda\rangle\ . \tag{8}$$

Let us now perform second quantization, and introduce the operators a_λ for destruction and $a_\lambda^\dagger$ for creation of a particle in the individual state λ. The complete Hamiltonian and the *number of particles* operator can be written as:

$$\hat{H} = \sum_\lambda \epsilon_\lambda\, a_\lambda^\dagger\, a_\lambda \qquad\qquad \hat{N} = \sum_\lambda a_\lambda^\dagger\, a_\lambda\ . \tag{9}$$

A basis of eigenstates in the Fock space is $\{|N_\lambda, N_{\lambda'}, N_{\lambda"}, \ldots\rangle\}$ where the occupation numbers N_λ of the individual quantum states (i) are equal to 0 or 1 in the case of fermions, (ii) are any positive or null integers in the case of bosons. For convenience, we write ℓ a given set $\{N_\lambda\}$: $|\ell\rangle \equiv |N_\lambda, N_{\lambda'}, N_{\lambda"}, \ldots\rangle$. We thus get

$$\hat{N}\,|\ell\rangle = N_\ell\,|\ell\rangle \qquad \text{with} \qquad N_\ell = \sum_\lambda N_\lambda\ , \tag{10}$$

$$\hat{H}\,|\ell\rangle = E_\ell\,|\ell\rangle \qquad \text{with} \qquad E_\ell = \sum_\lambda N_\lambda\,\epsilon_\lambda\ . \tag{11}$$

The grand-canonical partition function Z_G given by (3) is easily derived in the basis $|\ell\rangle$:

$$Z_G = \sum_{\ell} e^{-\alpha N_\ell - \beta E_\ell} = \sum_{N_\lambda, N_{\lambda'}, \ldots} e^{-(\alpha+\beta\epsilon_\lambda)N_\lambda} \times e^{-(\alpha+\beta\epsilon_{\lambda'})N_{\lambda'}} \times e^{-(\alpha+\beta\epsilon_{\lambda''})N_{\lambda''}} \times \ldots$$
$$= \prod_\lambda \zeta_\lambda \ , \tag{12}$$

where we wrote:

$$\zeta_\lambda = \sum_{N_\lambda} e^{-(\alpha+\beta\epsilon_\lambda)N_\lambda} \ . \tag{13}$$

This factorization of Z_G as a product of partition functions, each of them related to an individual quantum state λ, is the major advantage in using the grand-canonical formalism.

The Fermionic Case: Fermi-Dirac Statistics

For Fermions, the possible values of N_λ in the sum (13) are $N_\lambda = 0$ or $N_\lambda = 1$. We thus have:

$$\zeta_\lambda^{(F)} = 1 + e^{-\alpha-\beta\epsilon_\lambda} = 1 + z e^{-\beta\epsilon_\lambda} \ . \tag{14}$$

The partition function satisfies:

$$\ln Z_G = \sum_\lambda \ln\left(1 + z e^{-\beta\epsilon_\lambda}\right) \ . \tag{15}$$

By looking at (4), the expression of the total number of particles in the system can be obtained:

$$N = \sum_\lambda N_\lambda \qquad \text{with} \qquad N_\lambda = \frac{1}{e^{\beta(\epsilon_\lambda-\mu)}+1} \ . \tag{16}$$

For a system at fixed temperature, the chemical potential can take any value, positive or negative. A big negative value corresponds to a mean number of particles very small, thus to a system well described by classical Boltzmann statistics:

$$\mu \longrightarrow -\infty \qquad : \qquad N_\lambda \simeq z e^{-\beta\epsilon_\lambda} \ . \tag{17}$$

On the contrary, a positive value which is big with respect to $k_B T$ corresponds to a very large number of particles, and thus to a highly degenerate Fermi gas. The occupation numbers N_λ are almost equal to 1 if $\epsilon_\lambda < \mu$, and to 0 otherwise.

The Bosonic Case: Bose-Einstein Statistics

For Bosons, the computation of (13) leads to the sum of a geometrical series, or:

$$\zeta_\lambda^{(B)} = \frac{1}{1-e^{-\alpha-\beta\epsilon_\lambda}} = \frac{1}{1-z e^{-\beta\epsilon_\lambda}} \ . \tag{18}$$

The number of particles is then given by:

$$N = \sum_{\lambda} N_{\lambda} \qquad \text{with} \qquad N_{\lambda} = \frac{1}{e^{\beta(\epsilon_{\lambda}-\mu)} - 1} \, . \tag{19}$$

In this case the chemical potential can take all values from $-\infty$ up to $\epsilon_{\min}$, which represents the energy of the fundamental level of $\hat{h}$'s. For a chemical potential beyond this value, the population of this fundamental level would become negative, which of course doesn't make any sense. As for the Fermi gas, the big negative values of μ correspond to a gas well described by classical physics (Boltzmann distribution):

$$\mu \longrightarrow -\infty \qquad : \qquad N_{\lambda} \simeq z e^{-\beta\epsilon_{\lambda}} \, . \tag{20}$$

2 Bose-Einstein Condensation in a Harmonic Trap

2.1 The saturation of the excited levels

Let us consider Bose-Einstein statistics given by (19). When μ goes to $\epsilon_{\min}$, at a fixed temperature, the number of particles N_0 in the fundamental level of $\hat{h}$ becomes infinite:

$$\mu \longrightarrow \epsilon_{\min} \qquad : \qquad N_0 \simeq \frac{k_B T}{\epsilon_{\min} - \mu} \tag{21}$$

If the gas is restricted to a finite box or trapped in a harmonic pit, the spectrum of $\hat{h}$'is discrete. The number of particles N' in the excited levels of $\hat{h}$ is bounded above:

$$N' = {\sum_{\lambda}}' \frac{1}{e^{\beta(\epsilon_{\lambda}-\mu)} - 1} \; < \; N'_{\max} = {\sum_{\lambda}}' \frac{1}{e^{\beta(\epsilon_{\lambda}-\epsilon_{\min})} - 1} \, , \tag{22}$$

where $\sum'$ represents the sum over all the eigenstates λ of $\hat{h}$ except the fundamental state.

In what follows, we will call *saturation number* the value of $N'_{\max}$. The existence of that number, which represents an upper bound for the number of particles which can be put in states other than the fundamental, may be considered as a signature of Bose-Einstein condensation: if, at fixed temperature, we put in the trap a number of particles N greater than $N'_{\max}$, we are sure that at least $N - N'_{\max}$ particles must belong to the fundamental state. This effect is sufficient to explain the phenomena observed in a harmonic trap.

The saturation of the excited levels of $\hat{h}$' should not be considered a phase transition. To introduce this notion, one should perform the thermodynamic limit of the system considered, and observe if the density corresponding to N' atoms placed on the excited levels also remains bounded. The answer to this question will highly depend on the system's dimensionality. Before, we are going to study the application of (22) to the case of a harmonic trap, for which this notion of thermodynamical limit is not necessary [6, 7, 8, 9].

2.2 Saturation in a Harmonic Isotropic Trap

For an isotropic trap of frequency $\nu = \omega / 2\pi$, the energy levels of $\hat{h}$' are characterized by the three quantum numbers $(n_x, n_y, n_z) \equiv \boldsymbol{n}$, which characterize the oscillator's state of vibration upon the three axes. The corresponding energy is:

$$\epsilon_{\boldsymbol{n}} = \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right) \qquad \epsilon_{\min} = \frac{3}{2}\hbar\omega \; . \tag{23}$$

Moreover, each energy level has a degeneracy g_n given by:

$$g_n = \frac{(n+1)(n+2)}{2} \qquad n = n_x + n_y + n_z \; . \tag{24}$$

The saturation number can be simply written:

$$N'_{\max} = \sum_{(n_x,n_y,n_z)\neq(0,0,0)} \frac{1}{e^{(n_x+n_y+n_z)\xi} - 1} = \sum_{n=1}^{\infty} \frac{g_n}{e^{n\xi} - 1} \qquad \text{with} \qquad \xi = \frac{\hbar\omega}{k_B T} \; . \tag{25}$$

In the limit where the vibration quantum $\hbar\omega$ is very small compared to the thermal energy $k_B T$, that is when $\xi \ll 1$, this discrete sum can be approximated by replacing it with an integral (see appendix). We obtain:

$$N'_{\max} \simeq 1.202 \left(\frac{k_B T}{\hbar\omega} \right)^3 \; . \tag{26}$$

The quality of this approximation can be evaluated on figure 1, which gives the variation of the discrete sum (25) and of the approximated result (26) as a function of $k_B T / \hbar\omega$. When $k_B T$ becomes larger than $25\,\hbar\omega$, the two results differ by less than 5%. In practice a typical harmonic trap frequency is of magnitude 100 Hz, or $h\nu/k_B \sim 5$ nK. For a gas cooled to 200 nK, the maximal number of atoms out of the fundamental state is then about 80 000.

2.3 The Equilibrium Distribution in a Harmonic Trap

Given the saturation number for a harmonic trap, we can describe the equilibrium state of a N bosons system in this trap, when its temperature changes [10]. We will limit ourselves, in the following discussion, to the case of a number of atoms large in front of 1. This case corresponds to the solid line of figure 2. The dotted line of this Figure 2 gives some indication on the necessary modifications for lower numbers of atoms ($N = 100$ respectively). To simplify notations, we shall shift the origin of energies by $3/2\,\hbar\omega$ to bring the energy of the fundamental level to 0.

The critical temperature for which $N'_{\max} = N$ can be deduced from (26). It is given by:

$$k_B T_c = 0.94\,\hbar\omega\, N^{1/3} \tag{27}$$

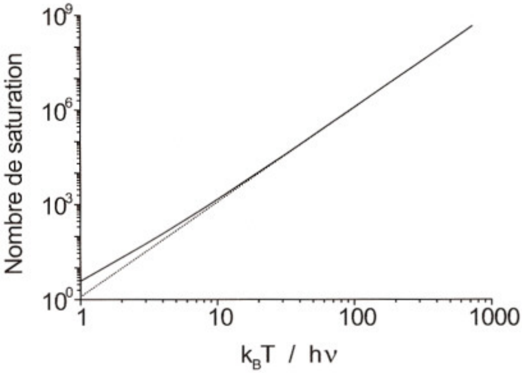

Figure 1: *Saturation number in a three-dimensional isotropic harmonic trap. The solid line gives the exact result (25) and the dotted line represents the approximated result (26).*

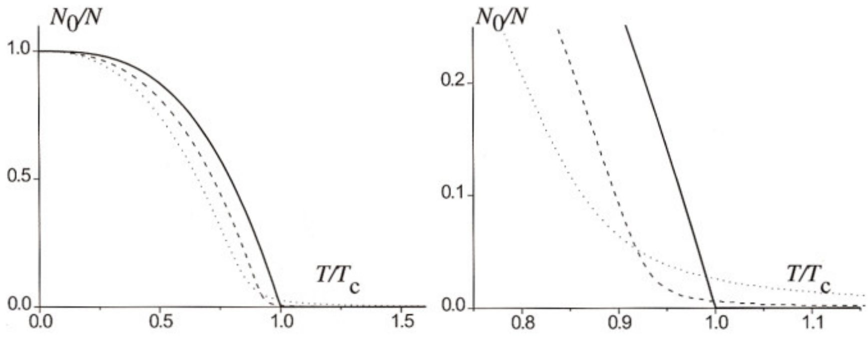

Figure 2: *Fraction of condensed atoms N_0/N as a function of the reduced temperature T/T_c. The solid curve corresponds to the limit of a large number of atoms (eq. 32). The dotted and dashed curves are the exact solutions of (19) for $N = 100$ and $N = 1000$ respectively. The right figure is an enlargement near the condensation transition.*

and is therefore large with respect to $\hbar\omega/k_B$: the approximate expressions computed in the previous paragraph are valid.

At high temperature, the saturation number $N'_{\max}$ (proportional to T^3) is much larger N. The gas is then only very weakly degenerate and one can apply Boltzmann's statistics (20). The fugacity is determined using $N = \sum_{\boldsymbol{n}} N_{\boldsymbol{n}} \simeq \sum_n g_n z e^{-\xi n}$, which gives after summation of a triple geometric series:

$$z = N \left(1 - e^{-\xi}\right)^3 \simeq N\,\xi^3 \simeq 1.202 \frac{N}{N'_{\max}} \ll 1 \,, \tag{28}$$

form which one deduces the occupation of each level:

$$N_{n_x,n_y,n_z} = N\,e^{-\xi(n_x+n_y+n_z)} \left(1 - e^{-\xi}\right)^3 \simeq N\,\xi^3\,e^{-\xi(n_x+n_y+n_z)} \,. \tag{29}$$

In particular, the proportion of atoms in the fundamental state N_0/N is given by $\xi^3 = \left(\hbar\omega/k_BT\right)^3$ and is very small compared to 1. The distributions in position and velocity of the atoms are Gaussian, with respective variances $k_BT/(m\omega^2)$ and k_BT/m.

When temperature goes down and nears T_c, the fugacity z increases and nears 1. One can compute it by solving the transcendental equation:

$$N = \sum_{n=0}^{\infty} \frac{g_n}{z^{-1}e^{\xi n} - 1} \simeq N_0 + \xi^{-3} g_3(z) \qquad \text{avec} \qquad N_0 = \frac{z}{1-z} \tag{30}$$

where one performed an approximation similar to (50-53). Let us remark that it is essential to exclude the contribution of the fundamental level in this transformation of a discrete sum into an integral. If this is not done, the lower bound of the integral which replaces (51) is $-1/2$ and the integral can diverge when z is close enough to 1.

For $T = T_c$, the proportion of atoms in the fundamental state is still small compared to 1, but the population of excited levels is quasi saturated. Let us remark that very many levels have a significant occupation at this point. The vibration quantum number n_v after which the occupation rate becomes smaller than 1 is of order of $k_BT/\hbar\omega$, hence $n_v \sim N^{1/3} \gg 1$. One should therefore not confuse the condensation that occurs at that point with the more trivial phenomenon that is expected in the regime of extremely low temperature, *i.e.*, $k_BT \ll \hbar\omega$, for which only the fundamental level has a significant population, no matter how many particles are present.

If the temperature comes below T_c, one observes a redistribution of particles from the excited levels towards the fundamental level, the fugacity remaining almost equal to 1. The population of excited levels decreases according to the previously determined saturation law:

$$N'(T) = 1.202 \left(\frac{k_BT}{\hbar\omega}\right)^3 = N \left(\frac{T}{T_c}\right)^3 \,, \tag{31}$$

and the population of the fundamental level is therefore:

$$N_0(T) = N\left(1 - \left(\frac{T}{T_c}\right)^3\right) . \tag{32}$$

The result (30) should therefore be understood in the following way in the regime $\xi \ll 1$:

- either $T > T_c$, and the number of atoms in the fundamental state is negligible; one has then:

$$N \simeq \xi^{-3} g_3(z) ; \tag{33}$$

- or $T < T_c$ and the distribution of excited states is saturated:

$$N \simeq N_0 + \xi^{-3} g_3(1) . \tag{34}$$

Once the temperature comes below the critical temperature, the spatial distribution and the velocity distribution of the atoms each show two well-separated components. For instance the spatial distribution is the superposition of a narrow peak, of width $\Delta x_0 = (\hbar \,/\, m\omega)^{1/2}$ corresponding to the fundamental state of the harmonic trap, and a broader peak corresponding to the fraction of non condensed atoms, of width $\Delta x' = (k_B T \,/\, m\omega^2)^{1/2}$. The ratio of these two widths is:

$$\frac{\Delta x_0}{\Delta x'} = \left(\frac{\hbar\omega}{k_B T}\right)^{1/2} \qquad \text{soit, pour} \quad T = T_c \;: \qquad \frac{\Delta x_0}{\Delta x'} \simeq N^{-1/6} \ll 1 . \tag{35}$$

Similarly, in velocity space, one finds $\Delta v_0 = (\hbar\omega/m)^{1/2}$ and $\Delta v' = (k_B T/m)^{1/2}$, which leads to a ratio $\Delta v_0/\Delta v'$ equal to the ratio $\Delta x_0/\Delta x'$.

3 Bose-Einstein Condensation in a Box

We come now to the description of Bose-Einstein condensation for a gas confined in a parallelepipedic box, which is the situation initially considered by Einstein in 1924, to discover this phenomenon. This geometry corresponds to the experiments performed on macroscopic samples confined in real containers, like the study of helium's superfluidity in a cryostat (see S. Balibar's lecture in this book), or of an exciton gas in a semi-conductor, or of a bidimensional gas of hydrogen atoms maintained in levitation above a surface of liquid helium.

3.1 Energy Levels

Let us consider a parallelepipedic box, with sides L_x, L_y, L_z. One notes $V = L_x L_y L_z$ the volume of the box. We choose here periodic boundary conditions. The eigenstates λ of the one-body Hamiltonian are plane waves $|\lambda\rangle \equiv |\boldsymbol{k}\rangle$:

$$\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}} \qquad k_i = \frac{2\pi n_i}{L_i} \qquad i = x, y, z , \tag{36}$$

where the n_i's are positive or negative integers. One has:

$$\epsilon_{\boldsymbol{k}} = \frac{2\pi^2\hbar^2}{m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right) \tag{37}$$

et $\epsilon_{\min} = 0$.

The problem that we now want to solve is the following one: when one takes the thermodynamic limit for this system, by letting the size of the box tend to infinity and keeping the particle density, the temperature, and the chemical potential fixed, how do particles fill the various energy levels? More precisely, the interesting question is to find whether the saturation of the excited levels of $\hat{h}$, found in the previous paragraph for a system of finite size, will "survive" to this limit, although the gap between $\epsilon_{\min}$ and the first excited levels of $\hat{h}$ tends to 0 as the size of the box increases towards infinity.

3.2 The Quasi One Dimensional Bose Gas

Let us start with the simplest situation, which corresponds to a strong confinement along the axes x and y: the transverse dimensions L_x and L_y of the box are assumed small and fixed. More precisely, one assumes that the energy necessary to excite the motion of a particle along these directions $\hbar^2\pi^2/(2mL_i^2)$ (with $i = x, y$) is much larger than the thermic energy k_BT. The thermodynamic limit is taken by letting L_z tend to infinity, keeping N/L_z constant, and one wants to find the behavior of the maximal linear density of the particles which are not condensed in the fundamental state of the box $N'_{\max}/L_z$.

The explicit computation of $N'_{\max}$ from (22) for the energy levels (37) is performed in a very simple way if one neglects the population of the transversally excited levels ($n_x \neq 0$ or $n_y \neq 0$). One gets:

$$N'_{\max} \simeq 2\sum_{n_z=1}^{+\infty} s\frac{1}{e^{2\pi^2\hbar^2 n_z^2/(mk_BTL_z^2)} - 1}\ . \tag{38}$$

Let us introduce the thermic wave length:

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}\ , \tag{39}$$

and define:

$$f(u) = \frac{1}{e^{\pi u^2\lambda^2/L_z^2} - 1} \tag{40}$$

Using the result (50) of the Appendix, which is valid if $\lambda \ll L_z$, one finds:

$$N'_{\max} \simeq 2\int_{1/2}^{\infty} f(u)\,du = \frac{2}{\sqrt{\pi}}\,\frac{L_z}{\lambda}\int_{\epsilon}^{\infty}\frac{dv}{e^{v^2} - 1}\ , \tag{41}$$

where the lower bound of the integral is $\epsilon = \sqrt{\pi}\,\lambda/(2L_z)$. The linear density not condensed in the fundamental level is therefore at most:

$$\Lambda'_{\max} = \frac{N'_{\max}}{L_z} = \frac{2}{\sqrt{\pi}}\,\frac{1}{\lambda}\int_{\epsilon}^{\infty}\frac{dv}{e^{v^2}-1}\,. \tag{42}$$

Let now the size L_z of the box tend to infinity. The integral which appears in (42) diverges like $1/\epsilon$, hence more precisely:

$$\Lambda'_{\max} \simeq \frac{4}{\pi}\,\frac{L_z}{\lambda^2}\,. \tag{43}$$

There is therefroe no finite limit for the saturation linear density when one takes the thermodynamic limit $L_z \to \infty$. In other words, for a fixed linear density N/L_z and a fixed temperature, there is a size L_z above which atoms will essentially fill excited levels, the fraction of atoms in the fundamental level being negligible. In this one dimensional case, the saturation of excited levels did not survive the thermodynamic limit.

3.3 The Three Dimensional Bose Gas

Let us take now $L_x = L_y = L_z = L$. The number of non condensed atoms can be written as:

$$N'_{\max} = \sum_{(n_x,n_y,n_z)}{}' \frac{1}{e^{\pi\lambda^2(n_x^2+n_y^2+n_z^2)/L^2}-1} \simeq \frac{4}{\sqrt{\pi}}\,\frac{L^3}{\lambda^3}\int_{\epsilon}^{\infty}\frac{v^2\,dv}{e^{v^2}-1} \tag{44}$$

The discrete sum is performed over all triplets which differ from zero triplet $(0,0,0)$ corresponding to the fundamental state. Similarly, the lower bound ϵ of the integral corresponds to exclude a sphere with radius of order λ/L, corresponding to the contribution of this fundamental level. In the three dimensional case here at hand, the integral that should be computed converges, even when its lower bound is replaced by 0, and its value therefore does not depend on ϵ in the limit $L \to \infty$. One finds:

$$n'_{\max}(T) = \frac{N'_{\max}}{L^3} = \frac{g_{3/2}(1)}{\lambda^3} \qquad \text{soit} \qquad n'_{\max}(T)\,\lambda^3 \simeq 2.612\,. \tag{45}$$

In this case, the saturation of excited levels "survived" passing to the thermodynamic limit. The study of this gas is done in detail in many textbooks on statistical physics and we shall only recall the major results, the road to follow in order to obtain them being similar to that followed in the case of the harmonic trap [2, 3, 4].

Consider a gas of fixed density n. In the high temperature region, characterized by $n\lambda^3 \ll 1$, the critical density $n'_{\max}(T)$, which varies as $T^{3/2}$, is much larger than n, and the gas is only very weakly degenerated. Physically, this temperature region corresponds to the case where the distance between particles, $n^{-1/3}$, is very large compared to their thermic wave length. Quantum effects are therefore in general hidden and the gas can be validly described by Boltzmann statistics. Its velocity distribution is well described by a Gaussian, of variance $k_B T/m$, and the fugacity is given by $z = n\lambda^3 \ll 1$.

When the temperature of this gas is lowered, the Gaussian approximation for the velocity distribution becomes worse and worse, and the fugacity z must be determined through the transcendental equation:

$$n = n_0 + \frac{g_{3/2}(z)}{\lambda^3} \qquad \text{avec} \qquad n_0 = \frac{1}{L^3}\,\frac{z}{1-z} \,. \tag{46}$$

Similarly to what we saw for (30), this equation must be understood in the following way:

- When $T > T_c$, where the critical temperature T_c is such that $n = n'_{\max}(T_c)$, the density in the fundamental state n_0 is negligible and one has:

$$n = \frac{g_{3/2}(z)}{\lambda^3} \,. \tag{47}$$

- When $T < T_c$, the excited levels are saturated, and one has:

$$n = n_0 + \frac{g_{3/2}(1)}{\lambda^3} \,. \tag{48}$$

  The $N - N'$ remaining atoms accumulate in the fundamental state $\boldsymbol{p} = 0$, and the condensed density is:

$$n_0(T) = n - n'_{\max}(T) = n\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right) \,. \tag{49}$$

  In contrast with the harmonic trap, this condensation happens only in the velocity space. The distribution of the atoms in position space remains uniform, as it should be taking into account the translation invariance of the system.

Appendix: Proof of Approximation (26)

When a function $f(x)$ varies slowly on an interval of length 1, one has the approximation:

$$f(1)+f(2)+f(3)+\ldots \simeq \int_{1/2}^{3/2} f(u)\,du + \int_{3/2}^{5/2} f(u)\,du + \int_{5/2}^{7/2} f(u)\,du + \ldots \tag{50}$$

For the sum (25), this hypothesis of slow variation corresponds to the case $\xi \ll 1$. One has then:

$$N'_{\max} = \frac{1}{2}\int_{1/2}^{\infty} \frac{(u+1)(u+2)\,du}{e^{u\xi}-1} \,, \tag{51}$$

hence, writing $x = u\xi$:

$$N'_{\max} \simeq \frac{1}{2\xi^3} \int_{\xi/2}^{\infty} \frac{(x+\xi)(x+2\xi)\, dx}{e^x - 1} \ . \tag{52}$$

When one expands the numerator of the integrand as $x^2 + 3x\xi + 2\xi^2$, it is easy to show that, for $\xi \ll 1$, the essential contribution comes from x^2 since the function to sum takes significant values only for $x \sim 1$. Let us keep only this term, and let us send the lower bound of the integral ($\xi/2$) to 0, which is possible since the function to sum is continuous at 0. One then finds:

$$N'_{\max} \simeq \frac{1}{2\xi^3} \int_0^{\infty} \frac{x^2\, dx}{e^x - 1} \ , \tag{53}$$

which is

$$N'_{\max} \simeq \left(\frac{k_B T}{\hbar\omega}\right)^3 g_3(1) \ . \tag{54}$$

We introduced here the two special functions:

$$g_\alpha(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^\alpha} \qquad\qquad I_\alpha(z) = \int_0^{\infty} \frac{x^{\alpha-1}\, dx}{z^{-1} e^x - 1} \ , \tag{55}$$

linked through the relation:

$$I_\alpha(z) = \Gamma(\alpha)\, g_\alpha(z) \qquad \text{avec} \qquad \Gamma(\alpha) = \int_0^{+\infty} y^{\alpha-1}\, e^{-y}\, dy \ . \tag{56}$$

One has $\Gamma(\alpha+1) = \alpha\, \Gamma(\alpha)$ with in particular:

$$\Gamma(1) = \Gamma(2) = 1 \qquad , \qquad \Gamma(3/2) = \frac{\sqrt{\pi}}{2} \qquad , \qquad \Gamma(3) = 2 \ ,$$

and

$$g_{3/2}(1) \simeq 2.612 \qquad , \qquad g_2(1) = \frac{\pi^2}{6} \qquad , \qquad g_3(1) \simeq 1.202 \ .$$

To obtain a better precision on the value of the critical temperature, one can try to evaluate the contributions of the terms $3x\xi$ and $2\xi^2$ which appear in the numerator of (52). For the term in $3x\xi$ one can still send the lower bound of the integral to 0 and one obtains a correction in ξ^{-2} to $N'_{\max}$. For the term in $2\xi^2$, the function to sum diverges like $1/x$ in 0, and one must keep the lower bound equal to $\xi/2$. The leading contribution for this last term is $\xi^{-1} \ln \xi$. Remark however that these expansions improved with respect to (54) are not very interesting in practice. If one wants to determine the saturation number with a very good precision in the case where $k_B T$ is not very large compared to $\hbar\omega$, it is faster and more secure to return to the series (25).

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