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DISCRETE STATE COUPLED TO A CONTINUUM. CONTINUOUS TRANSITION BETWEEN THE WEISSKOPF-WIGNER EXPONENTIAL DECAY AND THE RABI OSCILLATION.

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1. Introduction

WEISSKOPF and WIGNER have shown a long time ago that when a discrete state $|\phi_i>$ is weakly coupled to a broad continuum, the probability that the system remains in $|\phi_i>$ decreases exponentially, in an irreversible way. One can then ask the following question : How does this behaviour change when the width W_o of the continuum is decreased, or when, W_o remaining constant, the coupling V between the discrete state and the continuum is increased ? One knows of course another extreme case, the one where the width of the continuum is so small that it can be considered as a discrete state $|\phi_j>$. Then, the coupling V between $|\phi_i>$ and $|\phi_j>$ induces reversible oscillations between $|\phi_i>$ and $|\phi_j>$, with a frequency proportional to $<\phi_i|V|\phi_j>$ and which is the well known Rabi nutation frequency.

In this paper, we show how it is possible, with a very simple model and with elementary graphic constructions, to understand the continuous transition between the Weisskopf-Wigner exponential decay and the Rabi oscillation.

So many publications have been devoted to the problem of the coupling between discrete states and continuums that it seems extremely difficult to try to present an exhaustive review on this subject. We therefore apologize for not giving any bibliography at the end of this paper.

2. Presentation of a simple model

2.1. Notations

We consider an unperturbed hamiltonian H_o having only one non-degenerate discrete state $|\phi_i>$, with an energy E_i, and one continuum of states $|\beta, E>$, labelled by their energy E, which varies from 0 to $+\infty$, and some other quantum numbers β. The density of states in the continuum will be noted $\rho(\beta, E)$.

$$\begin{cases} H_o\,|\phi_i> = E_i\,|\phi_i> & \\ H_o\,|\beta, E> = E\,|\beta, E> & 0 \leqslant E < \infty \end{cases} \qquad (1)$$

One adds to H_o a coupling λV proportional to a dimensionless parameter λ. When $\lambda >> 1$, the coupling is strong, when $\lambda << 1$ it is weak. The operator V is assumed to have non zero matrix elements only between the discrete state and the continuum, and they are noted $v(\beta, E)$,

$$< \phi_i|V|\beta, E > = v(\beta, E) \tag{2}$$

All other matrix elements of V are equal to zero

$$< \phi_i|V|\phi_i > = < \beta, E|V|\beta', E' > = 0 \tag{3}$$

2.2. Decay amplitude $U_i(t)$ and Fourier transform $b_i(E)$ of the decay amplitude

What we have to calculate is the matrix element of the evolution operator between $|\phi_i >$ and $< \phi_i|$,

$$U_i(t) = < \phi_i|e^{-i(H_0+\lambda V)t}|\phi_i > , \tag{4}$$

which represents the decay amplitude, i.e. the probability amplitude that the system, starting at t = 0 in $|\phi_i >$, remains in this state after a time t. One can easily show from Schrödinger equation that $U_i(t)$ satisfies an integro-differential equation which is not easy to solve.

It is much simpler to take a different approach and to calculate the Fourier transform $b_i(E)$ of $U_i(t)$, rather than $U_i(t)$ itself

$$U_i(t) = \int_{-\infty}^{+\infty} dE\ e^{-iEt}\ b_i(E) \tag{5}$$

The first step in this direction is to compute the matrix element of the resolvent operator $G(Z) = [\ Z - H\ ]^{-1}$ between $|\phi_i >$ and $< \phi_i|$

$$G_i(Z) = < \phi_i\ |\ \frac{1}{Z - H}\ |\ \phi_i > \tag{6}$$

where Z is the complex variable and $H = H_o + \lambda V$ the total hamiltonian. One can easily show that $G_i(Z)$ satisfies an algebraic equation (much simpler than an integro-differential equation) which can be exactly solved for the simple model considered here. The corresponding calculations are sketched in the appendix. In the same appendix, we give the connection between $G_i(Z)$ and $b_i(E)$, displayed by the equation

$$b_i(E) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left[G_i(E-i\varepsilon) - G_i(E+i\varepsilon) \right] \tag{7}$$

and from which it is possible to derive an explicit expression for $b_i(E)$. Before giving this expression, it will be useful to introduce and to discuss 2 functions of E which play an important role in the problem.

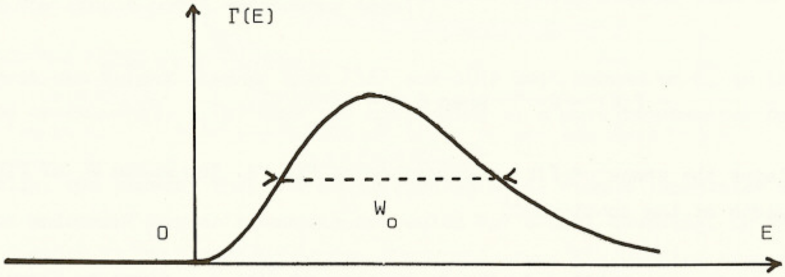

Figure 1 : Γ(E) function

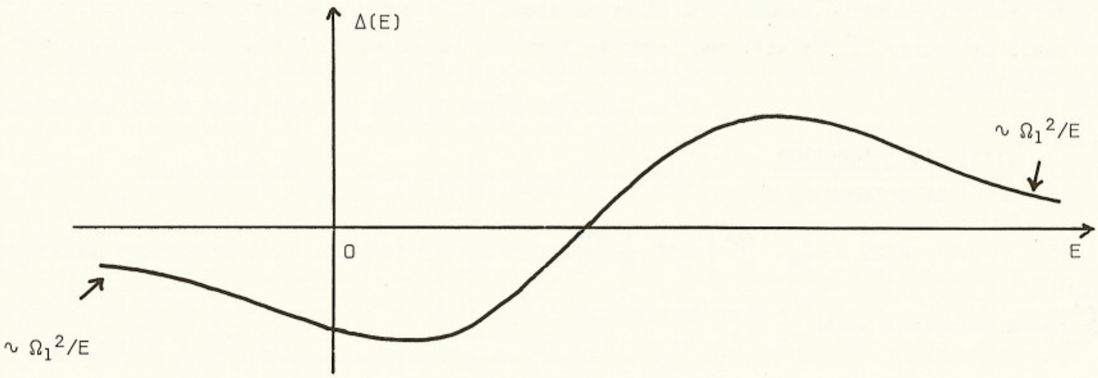

Figure 2 : Δ(E) function

2.3. Energy dependence of the coupling between the discrete state and the continuum

(i) Γ(E) function

The first function, Γ(E), is defined by :

$$\Gamma(E) = 2\pi \int d\beta \; \rho(\beta, E)\,|v(\beta, E)|^2 \tag{8}$$

Physically, Γ(E) represents the strength of the coupling between the discrete state $|\phi_i>$ and the shell of states in the continuum having the energy E. The variations with E of Γ(E) are represented on Fig. 1.

From the definition (8), it follows that :

$$\Gamma(E) \geqslant 0 \tag{9}$$

Since, the continuum is supposed to start from E = 0, ρ(β, E) = 0 for E < 0, and, consequently :

$$\Gamma(E) = 0 \quad \text{for} \quad E < 0 \tag{10}$$

Finally, $\rho(\beta, E)$ is generally an increasing function of E whereas $|v(\beta, E)|^2$ tends to zero when $E \to \infty$. We will suppose here that $|v(\beta, E)|^2$ tends to zero sufficiently rapidly so that :

$$\Gamma(E) \to 0 \qquad \text{when } E \to \infty \tag{11}$$

This explains the shape of $\Gamma(E)$ represented on Fig. 1. The width W_o of $\Gamma(E)$ can be considered as the "width of the continuum".

Let's also note that $\lambda^2\Gamma(E_i)$ is the decay rate of $|\phi_i>$, given by Fermi's golden rule.

(ii) Parameter Ω_1

From $\Gamma(E)$, it is possible to define the parameter Ω_1 by :

$$\Omega_1{}^2 = \frac{1}{2\pi}\int dE\ \Gamma(E) = \iint d\beta\ dE\ \rho(\beta, E)\ |v(\beta, E)|^2 \tag{12}$$

Ω_1 characterizes the coupling between the discrete state $|\phi_i>$ and the whole continuum (rather than the shell of energy E). We will see later on that $\lambda\Omega_1$ coincides with the Rabi frequency for very high couplings.

(iii) $\Delta(E)$ function

$\Delta(E)$ is defined by :

$$\Delta(E) = \frac{1}{2\pi}\ \mathcal{P}\int dE'\ \frac{\Gamma(E')}{E - E'} \tag{13}$$

where $\mathcal{P}$ means principal part.

It is easy to see that the variations with E of $\Delta(E)$, represented on Fig. 2, are those of a dispersion like curve. For the following discussion , it will be useful to determine the asymptotic behaviour of $\Delta(E)$ for $|E|$ very large. For $|E| \gg W_o$, one can replace in (13) E-E' by E so that, using (12), one gets :

$$\text{for } |E| \gg W_o : \qquad \Delta(E) \simeq \frac{1}{2\pi E}\int dE'\ \Gamma(E') = \frac{\Omega_1{}^2}{E} \tag{14}$$

Finally, it can be noted that $\lambda^2\Delta(E_i)$ is the Weisskopf-Wigner's result for the shift of the discrete state $|\phi_i>$ due to its coupling with the continuum.

2.4. Explicit expression for $b_i(E)$

We can now give the explicit expression of $b_i(E)$ (see the appendix for the details of the calculations) which only depends on $\Gamma(E)$ and $\Delta(E)$:

$$b_i(E) = \frac{1}{\pi}\lim_{\varepsilon \to 0_+} \frac{\varepsilon + \lambda^2\ \frac{\Gamma(E)}{2}}{\left[E - E_i - \lambda^2\Delta(E)\right]^2 + \left[\varepsilon + \lambda^2\ \frac{\Gamma(E)}{2}\right]^2} \tag{15}$$

It must be emphasized that this expression is exact and does not involve any approximation (within the simple model considered here).

One must not forget however that $\Gamma(E)$ and $\Delta(E)$ both depend on E, so that $b_i(E)$ is not a lorentzian, and consequently $U_i(t)$ does not correspond to a pure exponential decay.

Actually, one expects that the deviations of $b_i(E)$ from a lorentzian are very small for $\lambda << 1$, since Weisskopf-Wigner's results are valid for a weak coupling. On the other hand, for $\lambda >> 1$, one expects to find 2 sharp maxima for $b_i(E)$ since the Fourier transform of the Rabi sinusoïd, which must appear at very strong coupling, is formed by 2 delta functions.

It is precisely for understanding the deformations of $b_i(E)$ when λ increases that we introduce now some simple graphic constructions.

3. Some simple graphic constructions

3.1. Construction of $b_i(E)$

In Fig. 3, we have represented 3 functions of E : $\lambda^2\Gamma(E)$, $\lambda^2\Delta(E)$, $E-E_i$ (straight line of slope 1 intersecting the E axis at E_i). Let's consider now, for each value of E, a vertical line of abscissa E, and let's call A, B, C, D the intersections of this vertical line with respectively the E axis, $\lambda^2\Gamma(E)$, $\lambda^2\Delta(E)$, $E-E_i$. We have :

$$AB = \lambda^2\Gamma(E) \qquad\qquad CD = E - E_i - \lambda^2\Delta(E) \tag{16}$$

so that the expression (15) of $b_i(E)$ can be rewritten as :

$$b_i(E) = \frac{1}{\pi} \lim_{\varepsilon \to 0_+} \frac{\varepsilon + \frac{AB}{2}}{(CD)^2 + \left(\varepsilon + \frac{AB}{2}\right)^2} \tag{17}$$

which gives the possibility of determining $b_i(E)$ graphically for each value of E.

Since everything is positive in the denominator of (17), one expects to find a maximum of $b_i(E)$ when E is such that $CD = E - E_i - \lambda^2\Delta(E) = 0$. Thus, the abscissas, E_m, of the maximums of $b_i(E)$ are given by :

$$E_m - E_i - \lambda^2\Delta(E_m) = 0 \, . \tag{18}$$

3.2. Positions of the maximums of $b_i(E)$

According to (18), the positions of the maximums of $b_i(E)$ can be obtained by studying the intersections of $(E-E_i) / \lambda^2$ with $\Delta(E)$. The corresponding graphic construction is represented on Fig. 4.

For a weak coupling ($\lambda << 1$), $(E-E_i) / \lambda^2$ is practically a vertical line, so that there is only one solution to equation (18), $E_m \simeq E_i$. A better approximation of E_m is obtained by replacing in the small term $\lambda^2\Delta(E_m)$ of (18) E_m by E_i, which gives :

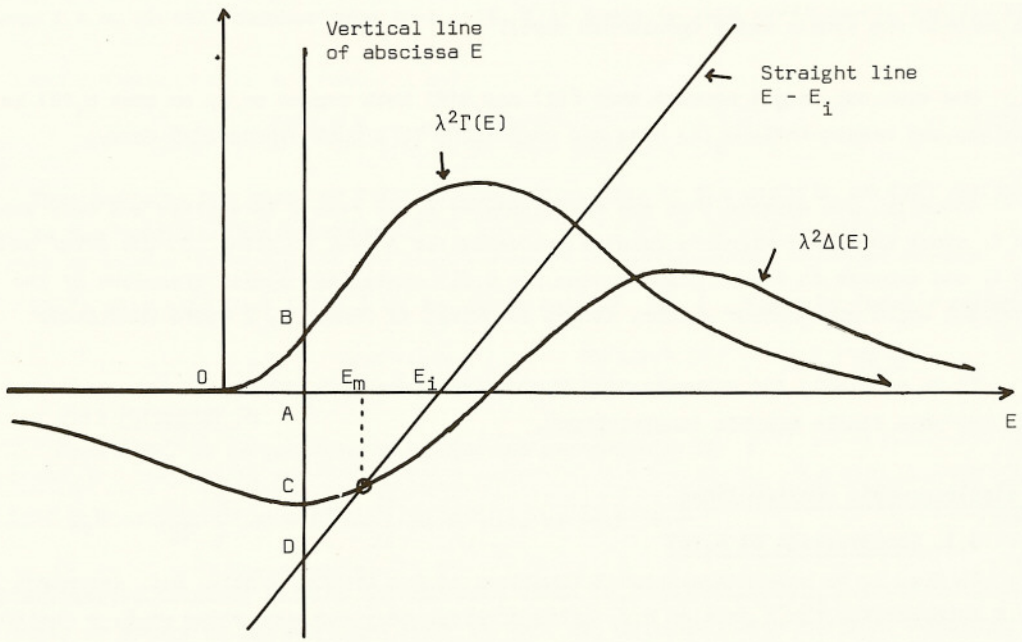

Fig. 3 : *Graphic construction of $b_i(E)$*

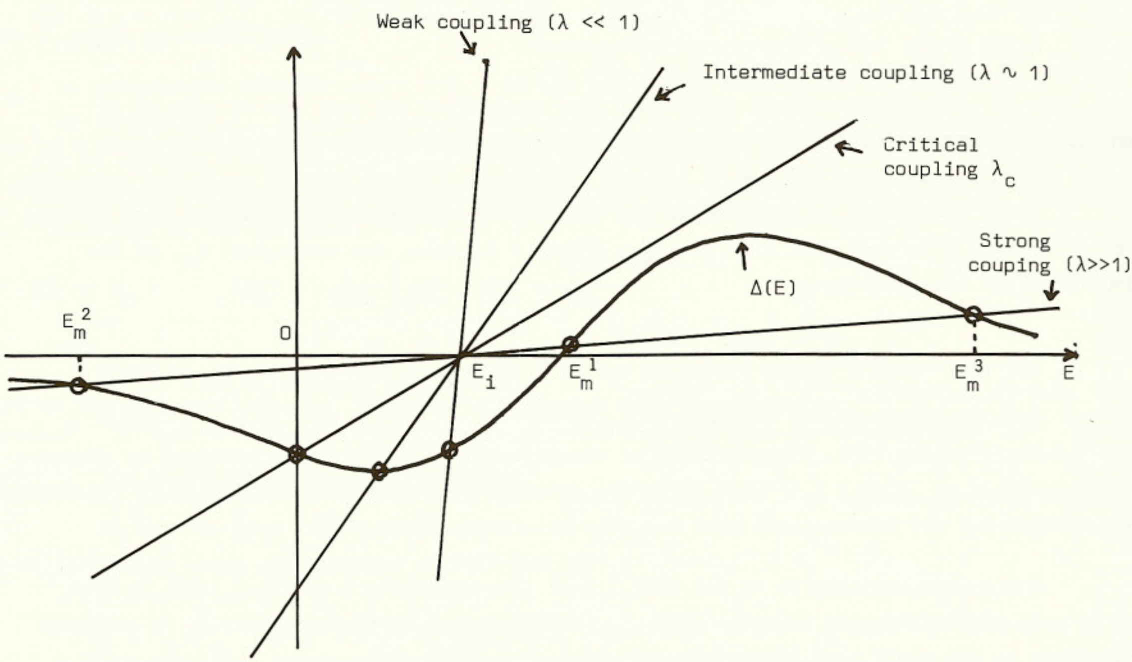

Fig. 4 : *Graphic determination of the positions of the maximums of $b_i(E)$*

$$E_m \simeq E_i + \lambda^2 \Delta(E_i) \quad . \tag{19}$$

When the coupling λ increases, one sees on Fig. 4 that the abscissa E_m of the intersection of $\Delta(E)$ with $(E-E_i) / \lambda^2$ decreases until λ reaches a critical value λ_c above which E_m takes a negative value. (New intersections of the 2 curves can also appear.) . The value of λ_c is obtained by putting $E_m = 0$ in (18) :

$$\lambda_c^2 = - \frac{E_i}{\Delta(0)} = \frac{2\pi \, E_i}{\displaystyle\int_0^\infty dE' \, \frac{\Gamma(E')}{E'}} \tag{20}$$

We will come back in § 4 on the physical meaning of this critical coupling.

At very strong couplings ($\lambda >> 1$), the slope of $(E-E_i) / \lambda^2$ becomes very small and one sees on Fig. 4 that $\Delta(E)$ and $(E-E_i) / \lambda^2$ intersect in general in 3 points with abscissas E_m^1, E_m^2, E_m^3. E_m^1 is approximately equal to the abscissa of the zero of $\Delta(E)$. E_m^2 and E_m^3 correspond to the points far in the wings of $\Delta(E)$, where the asymptotic expression (14) can be used. It is therefore possible, for evaluating E_m^2 and E_m^3 to transform (18) into :

$$E_m - E_i - \frac{\lambda^2 \, \Omega_1^2}{E_m} = 0 \quad . \tag{21}$$

Neglecting E_i in comparison to E_m, one gets :

$$(E_m)^2 - \lambda^2 \, \Omega_1^2 = 0 \tag{22}$$

which finally gives :

$$E_m^2 \simeq - \lambda\Omega_1 \; , \qquad\qquad E_m^3 \simeq + \lambda \, \Omega_1 \tag{23}$$

4. Discussion of the various regimes

4.1. Weak coupling limit. Corrections to the Weisskopf-Wigner's result

When $\lambda << 1$, the $\left[E-E_i-\lambda^2\Delta(E) \right]^2$ term in the denominator of (15) is much larger than all other ones, except around $E = E_i$ where it vanishes. It is therefore a good approximation to replace, in the small terms $\lambda^2\Gamma(E)$ and $\lambda^2\Delta(E)$, E by E_i, since it is only around $E = E_i$ that these small terms are not negligible compared to $\left[E-E_i-\lambda^2\Delta(E) \right]^2$. One gets in this way :

$$b_i(E) = \frac{1}{\pi} \; \frac{\lambda^2 \, \dfrac{\Gamma_i}{2}}{\left[E-E_i-\lambda^2\Delta_i \right]^2 + \left(\lambda^2 \, \dfrac{\Gamma_i}{2} \right)^2} \tag{24}$$

where : $\qquad \Gamma_i = \Gamma(E_i) \qquad\qquad \Delta_i = \Delta(E_i)$. (25)

The Fourier transform of (24) is :

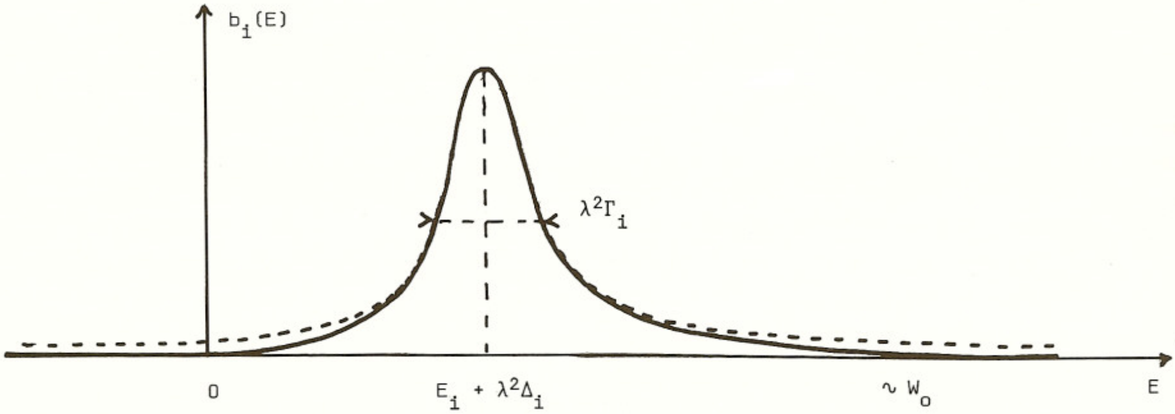

Fig. 5 : *Shape of $b_i(E)$ for a weak coupling. In dotted lines, lorentzian corresponding to the Weisskopf-Wigner's result. In full lines, better approximation for $b_i(E)$*

$$U_i(t) = e^{-i\left[E_i+\lambda^2\Delta_i\right]t}\ e^{-\lambda^2\Gamma_i t/2} \qquad (26)$$

This is the well known Weisskopf-Wigner's result : the energy of the discrete state is shifted by an amount $\lambda^2\Delta_i$ and the population of this state decays with a rate $\lambda^2\Gamma_i$.

A better approximation would be to replace $\Gamma(E)$ and $\Delta(E)$ by Γ_i and Δ_i in the denominator of (15) where the E dependence is essentially determined by the $\left[E-E_i-\lambda^2\Delta(E)\right]^2$ term, but to keep $\Gamma(E)$ in the numerator :

$$b_i(E) = \frac{1}{\pi}\ \frac{\lambda^2\ \frac{\Gamma(E)}{2}}{\left[E-E_i-\lambda^2\Delta_i\right]^2 + \left[\frac{\lambda^2\ \Gamma_i}{2}\right]^2} \qquad (27)$$

One gets in this way corrections to the Weisskopf-Wigner's result due to the E dependence of the coupling with the continuum.

We have represented on Fig. 5, in dotted lines, the lorentzian associated with equation (24), in full lines, the better approximation (27). Since $\Gamma(E)$ vanishes for $E < 0$ and tends to zero for $E >> W_o$, one sees first that the wings of the curve in full lines tend to zero more rapidly than for a lorentzian and that the domain of existence of this curve is an interval of width W_o. It follows that, at very short times ($t << \frac{1}{W_o}$), the decay amplitude $U_i(t)$ may be shown to vary not linearly in t, as $1-\lambda^2\left(\frac{\Gamma_i}{2} + i\Delta_i\right)t$, but quadratically as $1 - \frac{\lambda^2\Omega_1{}^2\ t^2}{2}$. Another correction comes from the fact that $\Gamma(E) \equiv 0$ for $E < 0$ and consequently $b_i(E) \equiv 0$ for $E < 0$. The long time behaviour of the decay amplitude is determined by the E dependence of $b_i(E)$ around $E = 0$. If one assumes for $\Gamma(E)$ a power law dependence, $\Gamma(E) \simeq E^n$ for E small, one finds that, at very long times ($t >> 1/\Gamma_i$), the decay amplitude does not decrease exponentially but as $1/t^{n+1}$.

When λ increases, more important deviations from the exponential decay law occur, due to the appearance of broad structures in the wings of $b_i(E)$. New zeros of $E-E_i-\lambda^2\Delta(E)$ can appear, giving rise to new maximums for $b_i(E)$. It is therefore useful to understand the shape of $b_i(E)$ near these maximums.

4.2. Expansion of $b_i(E)$ near a maximum

Around a zero E_m of equation (18), one can write :

$$\begin{cases} \Gamma(E) \simeq \Gamma(E_m) = \Gamma_m \\ \Delta(E) \simeq \Delta(E_m) + (E-E_m)\,\Delta'(E_m) = \Delta_m + (E-E_m)\,\Delta'_m \end{cases} \tag{28}$$

$$E-E_i-\lambda^2\Delta(E) = \underbrace{E_m-E_i-\lambda^2\Delta(E_m)}_{=0} + E-E_m-\lambda^2\left[\Delta(E)-\Delta(E_m)\right]$$

$$\simeq (E-E_m)\,(1-\lambda^2\Delta'_m) \tag{29}$$

so that we get :

$$b_i(E) \simeq \frac{1}{1-\lambda^2\Delta'_m}\;\frac{1}{\pi}\lim_{\varepsilon\to 0_+}\;\frac{\frac{\gamma_m}{2}}{(E-E_m)^2+\left(\frac{\gamma_m}{2}\right)^2} \tag{30}$$

with :

$$\gamma_m = \frac{\varepsilon+\lambda^2\Gamma_m}{1-\lambda^2\Delta'_m}\quad . \tag{31}$$

We therefore conclude that, around $E = E_m$, $b_i(E)$ has the shape of a lorentzian, centered at $E = E_m$, having a width γ_m and a weight $1\,/\,(1-\lambda^2\Delta'_m)$.

Of course, these results are only valid if $\Gamma(E)$ and $\Delta(E)$ do not vary rapidly with E in an interval γ_m around $E = E_m$.

4.3. Physical meaning of the critical coupling

When $\lambda > \lambda_c$, one zero E_m of equation (18) becomes negative. Since $\Gamma(E) = 0$ for $E < 0$, it follows that $\Gamma(E_m) = \Gamma_m = 0$, and consequently, according to (31) :

$$\gamma_m = \frac{\varepsilon}{1-\lambda^2\Delta'_m} = \varepsilon' \to 0 \tag{32}$$

The expansion (30) of $b_i(E)$ around $E = E_m$ becomes (this expansion is certainly valid since $\gamma_m = 0$)

$$b_i(E) = \frac{1}{1-\lambda^2\Delta'_m}\;\frac{1}{\pi}\lim_{\varepsilon'\to 0}\;\frac{\varepsilon'/2}{(E-E_m)^2+(\varepsilon'/2)^2} \tag{33}$$

$$= \frac{1}{1-\lambda^2\Delta'_m}\,\delta(E-E_m)$$

We therefore conclude that, above the critical coupling, a δ function appears in $b_i(E)$ in the negative region of the E axis, i.e. below the continuum, giving rise to an undamped oscillation

$$\frac{1}{1-\lambda^2\Delta'_m}\; e^{-iE_m t} \qquad (34)$$

in the decay amplitude. This means that, above the critical coupling, a new discrete state appears below the continuum.

It will be useful for the following to determine the position E_m and the weight $1/(1-\lambda^2\Delta'_m)$ of this discrete state when λ still increases and becomes verylarge. We have already seen in § 3.2 that E_m tends to $E_m^2 = -\lambda\Omega_1$ (see equation 23). Replacing $\Delta(E)$ by its asymptotic expression (14), one easily finds $\Delta'(E) = -\Omega_1{}^2/E^2$ and $\Delta'_m = \Delta'(-\lambda\Omega_1) = -1/\lambda^2$ so that :

$$\frac{1}{1-\lambda^2\Delta'_m} \;\rightarrow\; \frac{1}{2} \quad \text{if} \quad \lambda \rightarrow \infty \qquad (35)$$

It follows that, for very strong couplings, the new discrete state has an energy $-\lambda\Omega_1$ and a weight 1/2.

4.4. Strong coupling limit. Corrections to the Rabi oscillation

We have already mentioned in § 3.2 that, when $\lambda \gg 1$, $b_i(E)$ exhibits 3 maximums located at E_m^1, $E_m^2 \simeq -\lambda\Omega_1$, $E_m^3 \simeq \lambda\Omega_1$

From the results of the previous section, a delta function with a weight 1/2, $\frac{1}{2}\,\delta(E+\lambda\Omega_1)$, is associated with E_m^2 .

The expansion (30) of $b_i(E)$ shows that, around $E_m^3 = \lambda\Omega_1$, $b_i(E)$ has the shape of a lorentzian, with a weight $1/(1-\lambda^2\Delta'_m)$ which can be shown, as in § 4.3, to tend to 1/2 when $\lambda \rightarrow \infty$, and with a width which, according to (31), is equal to :

$$\gamma_m \simeq \frac{\lambda^2\,\Gamma(\lambda\Omega_1)}{1-\lambda^2\,\Delta'(\lambda\Omega_1)} \simeq \frac{1}{2}\,\lambda^2\,\Gamma\,(\lambda\Omega_1)\ . \qquad (36)$$

If $\Gamma(E)$ decreases asymptotically more rapidly than $1/E^2$, it follows from (36) that γ_m tends to zero when $\lambda \rightarrow \infty$ (γ_m tends to a constant value if $\Gamma(E)$ behaves asymptotically as $1/E^2$ and diverges if $\Gamma(E)$ decreases more slowly than $1/E^2$).

It remains to understand the contribution of E_m^1 which is close to the zero of $\Delta(E)$ (see Fig. 4). Coming back to the expression (15) of $b_i(E)$, one sees that, in the interval $0 \leqslant E \leqslant W_o$, $E-E_i$ can be neglected in comparison to $\lambda^2\Gamma(E)$ and $\lambda^2\Delta(E)$ since $\lambda \gg 1$, so that :

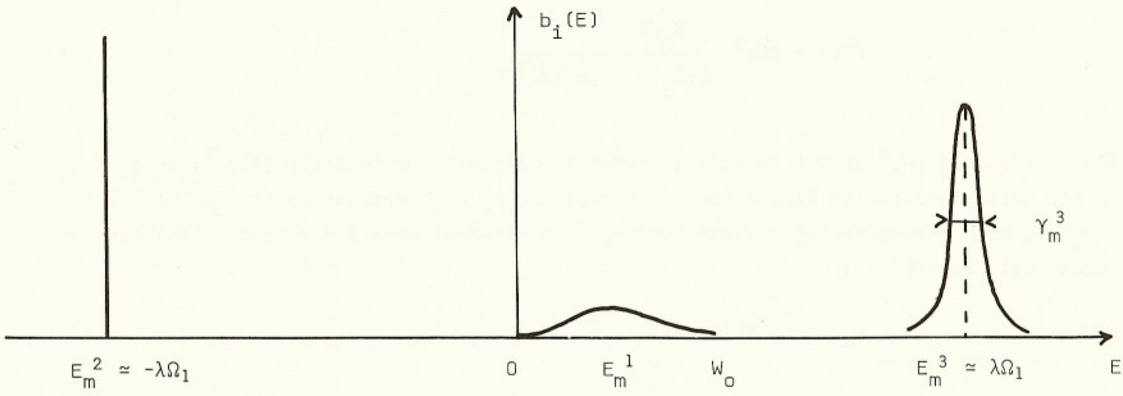

Fig. 6 : *Shape of $b_i(E)$ for a strong coupling*

$$b_i(E) \simeq \frac{1}{\pi}\frac{1}{\lambda^2}\frac{\Gamma(E)/2}{\left(\Delta(E)\right)^2 + \left(\Gamma(E)/2\right)^2} \tag{37}$$

In the interval $0 \lesssim E \lesssim W_o$, $b_i(E)$ behaves as a curve having a width of the order of W_o and a weight tending to zero as $1/\lambda^2$ when $\lambda \to \infty$ (we don't use the expansion (30) since $\Gamma(E)$ and $\Delta(E)$ vary rapidly in the interval $0 \lesssim E \lesssim W_o$).

All these results are summarized in Fig. 6. One deduces the following conclusions for the Fourier transform $U_i(t)$ of $b_i(E)$.

Since the weight of the central curve of Fig. 6 tends to zero when $\lambda \to \infty$, whereas the weights of the two other narrow curves tend to 1/2, we have essentially for $U_i(t)$ an oscillation of the form $\frac{1}{2}\left[e^{i\lambda\Omega_1 t} + e^{-i\lambda\Omega_1 t}\right] = \cos\lambda\Omega_1 t$ due to the beat between the Fourier transforms of the two narrow curves. This is precisely the Rabi oscillation. There are however corrections to this oscillation :

(i) At very short times ($t \ll 1/W_o$), small corrections in $1/\lambda^2$ appear, associated with the central curve of Fig. 6 and damped with a time constant of the order of $1/W_o$.

(ii) The contribution to $U_i(t)$ of the narrow curve located at $E_m^3 = \lambda\Omega$ is <u>damped</u> with a time constant of the order of $1/\gamma_m$, so that, at very long times ($t \gg 1/\gamma_m$), only survives the contribution $e^{i\lambda\Omega_1 t}/2$ of the delta function.

This last point clearly shows that we can never get an undamped Rabi oscillation. The coupling with the continuum introduces a fundamental irreversibility in the problem, which cannot completely disappear, even at very strong couplings.

<u>Remark</u>

So far, we have considered a true continuum, with a density of states starting at $E = 0$, and equal to zero below this value. In some simple models, one can also consider a continuum extending over the whole E axis, and giving rise to a $\Gamma(E)$ function having a lorentzian shape with a width W_o, and centered at $E = E_o$:

$$\Gamma(E) = 2\Omega_1^2 \frac{W_o/2}{(E-E_o)^2 + (W_o/2)^2} \tag{38}$$

[The coefficient $2\Omega_1^2$ is put in (38) in order to maintain the relation (12)]. In such a case, the delta function of Fig. 6 has to be replaced by a lorentzian as the one centered at $E = E_m^3$, both curves having a width tending to a constant when $\lambda \to \infty$ (since $\Gamma(E)$ behaves asymptotically as $1/E^2$) :

$$\gamma_m = \frac{1}{2} \lambda^2 \Gamma(\pm\lambda\Omega_1) \to \frac{W_o}{2} \tag{39}$$

One therefore finds that, in the limit of strong coupling, the Rabi oscillation is completely damped to zero with a time constant $4/W_o$.

This simple result can be obtained more easily by considering the continuum as an unstable state with a complex energy $E_o - iW_o/2$ and by describing the coupling between the discrete state and the continuum by a "non hermitian hamiltonian",

$$\begin{pmatrix} E_i & \lambda\Omega_1 \\ \lambda\Omega_1 & E_o - i\frac{W_o}{2} \end{pmatrix} \tag{40}$$

5. Conclusion

In this paper, we have presented a model of a discrete state coupled to a continuum, sufficiently simple for allowing an exact solution. Using graphic constructions we have shown how the Weisskopf-Wigner's exponential decay is progressively changed into a Rabi type damped oscillation when the coupling constant increases from very small to very high values.

It would be interesting to investigate possible applications of this model. Suppose that one excites with a monochromatic light a transition connecting a discrete bound state to a narrow autoionizing level near the ionization limit. The atom in the bound state in presence of N photons can be associated with the discrete state $|\phi_i\rangle$ of this model whereas the autoionizing continuum with N-1 photons can be associated with the states $|\beta,E\rangle$. Varying N, i.e. the light intensity, amounts to vary the coupling which is proportional to $\sqrt{N}$. At very low intensities, one can of course define a probability of ionization per unit time. At very high intensities, one expects to find some ringings in the photocurrent associated with a Rabi type oscillation. Of course, other atomic states exist and a simple $\Gamma(E)$ function of the type of Fig. 1 is not realistic so that it would be necessary to complicate the model. Another possible application would be to extend this formalism to Liouville space in order to study the transition between Markovian and non-Markovian regimes in quantum statistical problems.

APPENDIX

Calculation of $G_i(Z)$

Let $G(Z) = (Z - H_o - \lambda V)^{-1}$ and $G_o(Z) = (Z - H_o)^{-1}$ be the perturbed and unperturbed resolvent operators. Using the identity :

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{B}(B-A)\frac{1}{A} \tag{A-1}$$

one gets the equation :

$$G(Z) = G_o(Z) + G_o(Z)\,\lambda V\, G(Z) \tag{A-2}$$

which can be iterated to give :

$$G(Z) = G_o(Z) + \lambda\, G_o(Z)\, V\, G_o(Z) + \lambda^2\, G_o(Z)\, V\, G_o(Z)\, V\, G(Z) \tag{A-3}$$

Taking the matrix element of (A-3) between $|\phi_i>$ and $<\phi_i|$ and using the properties (2) and (3) of V, one gets :

$$G_i(Z) = \frac{1}{Z-E_i} + 0 + \frac{\lambda}{Z-E_i}\iint d\beta'\, dE'\, \frac{\rho(\beta',E')\,|v(\beta',E')|^2}{Z - E'}\, G_i(Z) \tag{A-4}$$

from which one deduces, using (8) :

$$G_i(Z) = \cfrac{1}{Z-E_i - \cfrac{\lambda^2}{2\pi}\displaystyle\int dE'\, \frac{\Gamma(E')}{Z-E'}} \tag{A-5}$$

If one is only interested in the values $G_i(E \pm i\varepsilon)$ of $G_i(Z)$ near the real axis, one gets from (A-5) :

$$G_i(E\pm i\varepsilon) = \cfrac{1}{E-E_i \pm i\varepsilon - \cfrac{\lambda^2}{2\pi}\displaystyle\int dE'\, \frac{\Gamma(E')}{E-E'\pm i\varepsilon}} \tag{A-6}$$

Since

$$\lim_{\varepsilon \to 0_+} \frac{1}{E-E'\pm i\varepsilon} = \mathcal{P}\,\frac{1}{E-E'} \mp i\pi\,\delta(E-E') \tag{A-7}$$

we finally have, according to (13) :

$$\lim_{\varepsilon \to 0_+} G_i(E\pm i\varepsilon) = \lim_{\varepsilon \to 0_+} \frac{1}{\left[E-E_i-\lambda^2\Delta(E)\right] \pm i\left[\varepsilon + \lambda^2\,\frac{\Gamma(E)}{2}\right]} \tag{A-8}$$

This shows that $G_i(Z)$ has a cut along the real axis (The limits of $G_i(Z)$ are not the same according as Z tends to the real axis from below and from above).

Connection between $b_i(E)$ and $G_i(E\pm i\varepsilon)$

From the evolution operator $U(t) = e^{-iHt}$, one can introduce the 2 operators :

$$K_\pm (t) = \pm \theta (\pm t)\, U(t) \tag{A-9}$$

where $\theta(x)$ is the Heaviside function $\left[\theta(x) = 1 \text{ for } x > 0,\ \ \theta(x) = 0 \text{ for } x < 0\right]$.

Let us now define the Fourier transforms of $K_\pm(t)$ by :

$$K_\pm(t) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} dE\, e^{-iEt}\, G_\pm(E) \tag{A-10}$$

Inverting (A-10), one gets :

$$\begin{aligned} G_+(E) &= \frac{1}{i} \int_{-\infty}^{+\infty} dE\, e^{iEt}\, K_+(t) = \frac{1}{i} \int_0^{\infty} dE\, e^{iEt}\, U(t) \\ &= \frac{1}{i} \lim_{\varepsilon \to 0_+} \int_0^{\infty} dE\, e^{i(E-H+i\varepsilon)t} = \lim_{\varepsilon \to 0_+} \frac{1}{E+i\varepsilon-H} \end{aligned} \tag{A-11}$$

and similarly :

$$G_-(E) = \lim_{\varepsilon \to 0_+} \frac{1}{E-i\varepsilon-H} \tag{A-12}$$

Now, since $\theta(x) + \theta(-x) = 1$, we have from (A-9) :

$$U(t) = K_+(t) - K_-(t) \tag{A-13}$$

Inserting (A-11) and (A-12) into (A-10) and then into (A-13), we finally get :

$$U_i(t) = \int_{-\infty}^{+\infty} dE\, e^{-iEt}\, b_i(E) \tag{A-14}$$

where :

$$\begin{aligned} b_i(E) &= \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left\{ \langle \phi_i \,|\, \frac{1}{E-i\varepsilon-H} \,|\phi_i \rangle - \langle \phi_i |\, \frac{1}{E+i\varepsilon-H} \,|\, \phi_i \rangle \right\} \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left[G_i(E-i\varepsilon) - G_i(E+i\varepsilon) \right] \end{aligned} \tag{A-15}$$

which proves equation (7). Inserting (A-8) into (A-15) gives equation (15).