# Fluctuations in Radiative Processes

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## Abstract

The purpose of these two lectures is to present a brief review of theoretical works done at Ecole Normale Superieure in collaboration with Jean Dalibard, Jacques Dupont-Roc and Claude Fabre. The motivation of these works is to try to understand the dynamics of an atomic system coupled to the radiation field and to get some physical insight in radiative processes in terms of fluctuations of the two interacting systems. By radiative processes, we mean spontaneous emission of radiation by an excited atom and radiative corrections such as the Lamb-shift or $g - 2$. We will focus here on "spontaneous" effects which are not induced by an incident field, so that the two interacting systems are the atom and the vacuum field.

The calculations presented in these lectures are based on the quantum theory of radiation and are limited to order 1 in the fine structure constant $\alpha = q^2/4\pi\varepsilon_0\hbar c$, and to the non relativistic domain (electron velocity $\ll c$). Our motivation is *not* to present a new method for calculating radiative processes (the covariant Q.E.D. formalism is well established), but rather to try to understand the physical mechanisms. We would like also to establish some connections with the quantum theory of damping and to discuss radiative processes in terms of master equations, Heisenberg–Langevin equations and linear response functions.

There are usually two extreme points of view for interpreting radiative processes. The first one considers the interaction of the electron with "vacuum fluctuations" as the basic process which, for example, triggers the spontaneous emission of radiation by an excited atom or produces a vibration of the atomic electron which is responsible for an averaging of the Coulomb potential of the nucleus (Welton's picture for the Lamb-shift) [1]. One must not forget however that the picture of the electron spin vibrating in vacuum fluctuations leads to the wrong sign for $g - 2$ (see Section 5.2 below). The second point of view tries to understand all radiative processes in terms of interaction of the electron with its self field, which gives rise to the well known "radiation reaction" [2–5]. It must be kept in mind however that the vacuum field and the field commutation relations cannot be discarded [6]. This raises the following question. Are these two pictures "two sides of the same quantum mechanical coin", as mentioned

by Senitzky [7], or is it possible to identify their respective contributions?

Actually, the same question can be asked for a small system $\mathscr{S}$ (generalizing the atom) coupled to a large reservoir $\mathscr{R}$ (generalizing the field). Is the evolution of $\mathscr{S}$ due to the "reservoir fluctuations" of $\mathscr{R}$ acting upon $\mathscr{S}$, or should we invoke a "self-reaction", $\mathscr{S}$ perturbing $\mathscr{R}$ which reacts back on $\mathscr{S}$? It turns out that such a generalization of the problem is useful since it leads to theoretical expressions, with a more transparent structure, which is not overlooked by simplifications specific to a particular choice of $\mathscr{S}$ and $\mathscr{R}$. This is why lecture I will deal with the dynamics of a small systems $\mathscr{S}$ coupled to a large reservoir $\mathscr{R}$. The energy shifts and damping rates of $\mathscr{S}$ will be calculated to order 2 in the coupling constant and interpreted in terms of two important statistical functions of the two interacting systems, which are the symmetric correlation functions and the linear susceptibilities [8]. These general results will then be applied in lecture II to the particular case where $\mathscr{R}$ is the vacuum field and $\mathscr{S}$ an atomic electron [9].

## LECTURE I — DYNAMICS OF A SMALL SYSTEM COUPLED TO A LARGE RESERVOIR

### 1. Hamiltonian — Assumptions concerning the reservoir

The Hamiltonian of the total system can be written:

$$H = H_R + H_S + V \tag{1}$$

where $H_R$ ($H_S$) is the Hamiltonian of $\mathscr{R}$ ($\mathscr{S}$), and:

$$V = -gRS \tag{2}$$

is the coupling between both systems. In eq. (2), $g$ is a coupling constant, $R$ and $S$ are *Hermitian* reservoir and system operators. The following calculations could be easily generalized for more complicated forms of $V$, such as $-g \sum_i R_i S_i$.

We make two assumptions concerning $\mathscr{R}$. (i) $\mathscr{R}$ is a reservoir, which means that the modification of the state of $\mathscr{R}$ due to the coupling $V$ is negligible. If $\varrho(t)$ is the density operator of the total system, we have for the reduced density operator of $\mathscr{R}$:

$$\sigma_R(t) = \mathrm{Tr}_R \, \varrho(t) \simeq \sigma_R(0) = \sigma_R. \tag{3}$$

In writing eq. (3), we have implicitly supposed that $\sigma_R(0)$ is a stationary state with respect to $H_R$,

$$[H_R, \sigma_R(0)] = 0 \tag{4}$$

which does not evolve under the effect of $H_R$. (ii) The fluctuations of $\mathscr{R}$ are fast. More precisely, we suppose:

$$v\tau_c/\hbar \ll 1 \tag{5}$$

where $v$ gives the order of magnitude of $V$ and $\tau_c$ is the correlation time of the observable $R$ of $\mathcal{R}$ appearing in eq. (2). Condition (5) means that, during the correlation time of $R$, the effect of the coupling between $\mathcal{S}$ and $\mathcal{R}$ is negligible. Such a condition reminds the one appearing in the theory of brownian motion and expressing that, during a collision time $\tau_c$, the velocity change of the heavy particle is very small.

A first consequence of these assumptions is the existence of two time scales, the correlation time $\tau_c$ and the relaxation time $T_R$ such that:

$$\tau_c \ll T_R. \tag{6}$$

We will see later on that the damping rate, $1/T_R$, of $\mathcal{S}$, due to the coupling with $\mathcal{R}$, is of the order of:

$$\frac{1}{T_R} \sim \frac{v^2 \tau_c}{\hbar^2} = \frac{1}{\tau_c} \frac{v^2 \tau_c^2}{\hbar^2}. \tag{7}$$

Equation (6) is a consequence of eqs. (5) and (7).

The existence of two time scales gives the possibility to compute "coarse-grained" rates of variation for $\mathcal{S}$, by averaging the instantaneous rates over an interval $[t, t + \Delta t]$ such that:

$$\tau_c \ll \Delta t \ll T_R. \tag{8}$$

Condition (8) allows two simplifications. First, since $\Delta t \ll T_R$, a perturbative calculation of the evolution of $\mathcal{S}$ between $t$ and $t + \Delta t$ is possible. Then, since $\Delta t \gg \tau_c$, one can neglect the correlations between $\mathcal{S}$ and $\mathcal{R}$ which exist at the initial time $t$ of the interval $[t, t + \Delta t]$, and which last for a time $\tau_c \ll \Delta t$. These correlations are described by the difference between the density operator $\varrho(t)$ of the total $\mathcal{R} + \mathcal{S}$ system and the tensor product of the reduced operators of $\mathcal{R}$ and $\mathcal{S}$:

$$\varrho_{\text{correl}}(t) = \varrho(t) - [\text{Tr}_R \, \varrho(t)] \otimes [\text{Tr}_S \, \varrho(t)]. \tag{9}$$

A detailed discussion of the conditions of validity of this second approximation is outside the scope of this paper. Neglecting $\varrho_{\text{correl}}(t)$ transforms a reversible equation of motion for $\varrho(t)$ into a master equation for $\text{Tr}_R \, \varrho(t)$, which, as we will see later on, is irreversible. We have to suppose that the initial state of the $\mathcal{R} + \mathcal{S}$ system is such that the correlations which appear between $\mathcal{S}$ and $\mathcal{R}$ are not "pathological" and can be neglected in the coarse grained rate of variation. Such an assumption is implicit in most quantum theories of damping. Sometimes, one explicitly supposes that the initial state of the total $\mathcal{R} + \mathcal{S}$ system, at $t = t_0$, is factorized. By introducing in this way a privileged time $t_0$ in the past of $t$, one breaks the symmetry between the two arrows of time starting from $t$.

## 2. Master equation for the small system

### 2.1. *Structure of the master equation* [10]

In interaction representation with respect to $H_S + H_R$ (where the operators are noted $\tilde{\varrho}(t)$, $\tilde{V}(t)$, ...), the coarse grained rate of variation of $\tilde{\varrho}(t)$ is given by:

$$\frac{\Delta \tilde{\varrho}(t)}{\Delta t} = \frac{\tilde{\varrho}(t + \Delta t) - \tilde{\varrho}(t)}{\Delta t} = \frac{1}{i\hbar} \frac{1}{\Delta t} \int_t^{t+\Delta t} dt' \, [\tilde{V}(t'), \tilde{\varrho}(t)]$$
$$+ \frac{1}{(i\hbar)^2} \frac{1}{\Delta t} \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \, [\tilde{V}(t'), [\tilde{V}(t''), \tilde{\varrho}(t'')]]. \tag{10}$$

Equation (10) is exact. We now introduce two approximations:

(i) In the last term of eq. (10), we replace $\tilde{\varrho}(t'')$ by $\tilde{\varrho}(t)$. This means that we limit the calculation of $\Delta \tilde{\varrho}/\Delta t$ to order 2 in $V$ (such a perturbative calculation is valid since $\Delta t \ll T_R$).

(ii) In both terms of eq. (10), we neglect the initial correlations between $\mathcal{S}$ and $\mathcal{R}$ at time $t$, and we take:

$$\tilde{\varrho}(t) \simeq [\text{Tr}_R \, \tilde{\varrho}(t)] \otimes [\text{Tr}_S \, \tilde{\varrho}(t)] \simeq \tilde{\sigma}_S(t) \sigma_R \tag{11}$$

where we have used eq. (3) and where:

$$\tilde{\sigma}_S(t) = \text{Tr}_R \, \tilde{\varrho}(t) \tag{12}$$

is the reduced density operator of $\mathcal{S}$.

If, in addition we assume that:

$$\text{Tr}_R \, \sigma_R \tilde{V} = 0 \tag{13}$$

which expresses that the "mean field" of $\mathcal{R}$ "seen" by $\mathcal{S}$ vanishes (if this was not the case, it could be reincluded in $H_R$), we get, by taking the trace of eq. (10) with respect to $\mathcal{R}$:

$$\frac{\Delta \tilde{\sigma}_S}{\Delta t} = \frac{1}{(i\hbar)^2} \frac{1}{\Delta t} \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \, \text{Tr}_R \, [\tilde{V}(t'), [\tilde{V}(t''), \tilde{\sigma}_S(t) \sigma_R]] \tag{14}$$

with:

$$\tilde{V}(t') = -g\tilde{R}(t')\tilde{S}(t') \tag{15}$$

and a similar equation for $\tilde{V}(t'')$.

The double commutator of eq. (14) is a product of reservoir and system operators, in interaction representation, i.e., evolving freely, under the effect of $H_R$ or $H_S$. When the trace is taken over $R$, one sees that the reservoir appears in eq. (10), only through two time averages such as $\text{Tr}_R \, [\sigma_R \tilde{R}(t') \tilde{R}(t'')]$ which depend only on $t' - t''$, because of the stationary character of $\sigma_R$ (see eq. (4)), and which vanish if $t' - t'' \gg \tau_c$, by definition of the correlation time $\tau_c$. This shows that the domain of integration of eq. (14), in the $t'$, $t''$ plane, reduces to a narrow strip, along the diagonal $t' = t''$, with a width $\tau_c$ and an area of the order of $\tau_c \Delta t$. The right side of eq. (14) is consequently of the order of:

$$\frac{1}{\hbar^2} \frac{1}{\Delta t} \tau_c \Delta t \, v^2 \tilde{\sigma} \sim \frac{v^2 \tau_c}{\hbar^2} \tilde{\sigma}. \tag{16}$$

This shows that the master equation is a linear differential equation coupling $\Delta \tilde{\sigma}(t)/\Delta t$ to $\tilde{\sigma}(t)$, with coefficients of the order of the damping rate $1/T_R$ introduced above in eq. (7). We will not give here the explicit expression of the coefficients of the master equation and refer the reader to [10] for more details. We prefer to focus here on the physical meaning of the two time averages of $\mathcal{R}$ appearing in eq. (14).

### 2.2. *Statistical functions of $\mathcal{R}$ and $\mathcal{S}$ appearing in the master equation*

The real and imaginary parts of the two time average $\text{Tr}_R \, [\sigma_R \tilde{R}(t) \tilde{R}(t - \tau)]$ have a clear physical meaning [8, 11].

The real part,

$$C_R(\tau) = \frac{1}{2} \text{Tr}_R \, \{\sigma_R[\tilde{R}(t)\tilde{R}(t - \tau) + \tilde{R}(t - \tau)\tilde{R}(t)]\} \tag{17}$$

is a *symmetric correlation function* which describes the dynamics of the fluctuations of $R$ in the state $\sigma_R$.

The imaginary part,

$$\chi_R(\tau) = \frac{i}{\hbar} \text{Tr}_R \, \{\sigma_R[\tilde{R}(t), \tilde{R}(t - \tau)]\} \theta(\tau), \tag{18}$$

which we have multiplied by the Heaviside functions $\theta(\tau)$, equal to 1 for $\tau > 0$ and to 0 for $\tau < 0$, is a *linear susceptibility* which describes the linear response of $\langle R \rangle$ to a perturbation, $-\lambda(t)R$ proportional to $R$, the initial state of $\mathscr{R}$ being $\sigma_R$

$$\langle R(t) \rangle = \int_{-\infty}^{+\infty} \chi_R(t - t')\lambda(t') \, \mathrm{d}t' \tag{19}$$

Our procedure for extracting the physical content of the master equation is to express every result in terms of $C_R(\tau)$ and $\chi_R(\tau)$ (or of their Fourier transforms $\hat{C}_R(\omega)$ and $\hat{\chi}_R(\omega)$), and also in terms of similar functions introduced for the observable $S$ of $\mathscr{S}$ appearing in the expression (2) of $V$

$$C_S^a(\tau) = \tfrac{1}{2}\langle a|\tilde{S}(t)\tilde{S}(t - \tau) + \tilde{S}(t - \tau)\tilde{S}(t)|a\rangle \tag{20a}$$

$$\chi_S^a(\tau) = \frac{1}{i\hbar}\langle a|[\tilde{S}(t), \tilde{S}(t - \tau)]|a\rangle\theta(\tau). \tag{20b}$$

The averages in (20) are taken in an energy level $|a\rangle$ of $H_S$, which is a stationary state of $\mathscr{S}$.

### 2.3. *Structure of the results concerning the energy shifts of $\mathscr{S}$* [8]

A first category of terms appearing in the master equation (14) correspond to a reversible evolution of an hamiltonian type. They describe shifts of the energy levels $|a\rangle$ of $\mathscr{S}$ produced by the coupling with $\mathscr{R}$. When expressed in terms of $C_R$, $\chi_R$, $C_S^a$, $\chi_S^a$, the energy shift $\delta E_a$ appears as

$$\begin{aligned}
\delta E_a = &-\frac{g^2}{2}\int_0^\infty \mathrm{d}\tau \, C_R(\tau)\chi_S^a(\tau) \\
&-\frac{g^2}{2}\int_0^\infty \mathrm{d}\tau \, C_S^a(\tau)\chi_R(\tau).
\end{aligned} \tag{21}$$

The important point in (21) is that $C_R$ is associated with $\chi_S^a$, and $C_S^a$ with $\chi_R$. This leads to very simple and clear physical pictures. The first term of eq. (21) described processes in which $\mathscr{R}$ fluctuates (in a way characterized by $C_R$), polarizes $\mathscr{S}$ (which corresponds with its linear susceptibility $\chi_S^a$), and interacts with this induced polarization. This is the usual picture one would expect for the effect of reservoir fluctuations. The second term of eq. (21) is associated with a different type of process. The small system in level $|a\rangle$ fluctuates (in a way characterized by $C_S^a$), polarizes $\mathscr{R}$, i.e., produces a "field" (proportional to the linear response $\chi_R$ of $\mathscr{R}$) which reacts back on $\mathscr{S}$. This second effect is a kind of "self reaction".

### 2.4. *Structure of the results for the energy damping rates of $\mathscr{S}$* [8]

The non hamiltonian terms of the master equation (14) describe an irreversible damping due to the $\mathscr{R}$–$\mathscr{S}$ coupling. For example, $\langle a| \, \mathrm{d}H_S/\mathrm{d}t|a\rangle$ is the rate of energy loss (or gain) by $\mathscr{S}$ in level $|a\rangle$. We get for such a term a structure similar to the one of eq. (21)

$$\begin{aligned}
\langle a| \, &\mathrm{d}H_S/\mathrm{d}t|a\rangle = \\
&-g^2\pi\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \hat{C}_R(\omega) \, i\omega[\hat{\chi}_S^{a*}(\omega) - \chi_S^a(\omega)] \\
&+g^2\pi\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \hat{C}_S^a(\omega) \, i\omega[\hat{\chi}_R^*(\omega) - \hat{\chi}_R(\omega)]
\end{aligned} \tag{22}$$

and, consequently, a similar physical interpretation.

It must be noted that we get in eq. (22) the *dissipative* parts ($\hat{\chi}''$) of $\hat{\chi}_R$ and $\hat{\chi}_S$, whereas it can be shown that the energy

shifts $\delta E_a$ given in (21) only depend on the *reactive* parts ($\hat{\chi}'$). This is in agreement with the reversible or irreversible nature of the processes described by eqs. (21) and (22).

Finally, we can note that, if $\mathscr{R}$ in thermal equilibrium, $\hat{C}_R$ and $\hat{\chi}_R''$ are proportional. This is the well known fluctuation dissipation theorem [11]. It is clear however that the previous considerations are still valid in more general situations where $\mathscr{R}$ is in a stationary non thermal state.

### 2.5. *Quasiclassical limit*

When $\mathscr{S}$ is a quasiclassical system, with closely spaced, locally equidistant energy levels, the set of populations of the various energy levels of $\mathscr{S}$ can be approximated by an energy distribution function $\mathscr{P}(E, t)$. We have shown in Ref. [8] that $\mathscr{P}(E, t)$ obeys a Fokker–Planck equation and that the drift and diffusion terms are respectively associated with self reaction and reservoir fluctuations. The drift term only depends on $\chi_R$ (and not on $C_R$) and describes the mean energy loss of $\mathscr{S}$ per unit time due to self reaction. On the other hand, the diffusion term, which only depends on $C_R$, describes a random walk in the ladder of energy levels of $\mathscr{S}$ induced by reservoir fluctuations. These general results can be applied to the problem of the spontaneous emission of a large angular momentum and give some physical insight in Dicke's superradiance [12].

## 3. Heisenberg–Langevin equations for the observables of $\mathscr{S}$

In Section 2, we have used the Schrödinger picture and studied the evolution of the density operator of $\mathscr{S}$. We switch now to the Heisenberg picture and consider the rate of variation of the observables $G_S$ of $\mathscr{S}$.

### 3.1. *Principle of the calculation*

We start from the Heisenberg equation for the observable $R$ of $\mathscr{R}$ appearing in the expression (2) of $V$. The solution of this equation can be written

$$R(t) = R_f(t) + R_s(t) \tag{23}$$

where

$$R_f(t) = R_{\text{free}}(t) = e^{iH_R(t - t_0)/\hbar}R(t_0) \, e^{-iH_R(t - t_0)/\hbar} \tag{24}$$

is the "free reservoir field" calculated to order 0 in $V$, and where $R_S(t) = R_{\text{source}}(t)$ represents the terms of order 1 and higher in $V$ in eq. (23), which describe the "source field" due to the coupling with $\mathscr{S}$.

We then insert eq. (23) into the equation of motion of a general observable $G_S$ of $\mathscr{S}$, more precisely in the term describing the effect of the coupling $V$ on $\mathrm{d}G_S/\mathrm{d}t$:

$$\left(\frac{\mathrm{d}G_S(t)}{\mathrm{d}t}\right)_{\text{coupling}} = -\frac{g}{i\hbar}[G_S(t), R(t)S(t)] = gN(t)R(t) \tag{25}$$

where

$$N(t) = \frac{i}{\hbar}[G_S(t), S(t)] \tag{26}$$

is another observable of $\mathscr{S}$.

It is tempting now to consider that the contribution of $R_f(t)$ in eq. (25) represents the effect of reservoir fluctuations, whereas the one of $R_S$ is associated with self reaction. But we are immediately faced with a problem of order between commuting observables [7, 13–15].

### 3.2. *Indetermination in the order of commuting observables*

The observables $N(t)$ and $R(t)$ appearing in eq. (25) commute at any time since they are respectively associated with $\mathscr{S}$ and $\mathscr{R}$. Equation (25) can therefore be written

$$\left(\frac{\mathrm{d}G_\mathrm{S}(t)}{\mathrm{d}t}\right)_{\mathrm{coupl}} = \lambda g N(t) R(t) + (1 - \lambda) g R(t) N(t) \qquad (27)$$

with $\lambda$ arbitrary. The rate (27) does not depend on $\lambda$.

When we insert eq. (23) into eq. (27) we get

$$\begin{cases} \left(\dfrac{\mathrm{d}G_\mathrm{S}}{\mathrm{d}t}\right)_{\mathrm{res.\,fluct.}} = \lambda g N(t) R_\mathrm{f}(t) + (1 - \lambda) g R_\mathrm{f}(t) N(t) & (28a) \\[2mm] \left(\dfrac{\mathrm{d}G_\mathrm{S}}{\mathrm{d}t}\right)_{\mathrm{self\,react.}} = \lambda g N(t) R_\mathrm{S}(t) + (1 - \lambda) g R_\mathrm{S}(t) N(t). & (28b) \end{cases}$$

The problem is that $R_\mathrm{f}(t)$ and $R_\mathrm{S}(t)$ do not commute separately with $N(t)$ as their sum (23) does. It seems therefore that the respective contributions of reservoir fluctuations and self reaction depend on $\lambda$ and can be arbitrarily changed.

### 3.3. *Physical argument removing this indetermination*

This argument, introduced in references [8, 9], is the following. All orders are of course *mathematically* equivalent in eq. (27), but there is only *one* order having a physical meaning. We first note that, if $G_\mathrm{S}$ represents a physical quantity, it must be *hermitian*, as well as its rate $\mathrm{d}G_\mathrm{S}/\mathrm{d}t$. Now, if the total rate $(\mathrm{d}G_\mathrm{S}/\mathrm{d}t)$ is split into two rates, eqs. (28a) and (28b) representing two distinct physical processes, these two rates must be *separately hermitian* if we want them to have a physical meaning. This imposes $\lambda = 1 - \lambda$, i.e., $\lambda = 1/2$, which corresponds to be *completely symmetrical order*.

### 3.4. *Structure of the results obtained with the completely symmetrical order*

We just give here the results of the calculations presented in Ref. [8].

Starting from eq. (28), with $\lambda = 1/2$, we integrate these equations (in interaction representation) between $t$ and $t + \Delta t$, and, as in the previous Section 2, we calculate to order 2 in $V$ coarse grained rates of variation for $G_\mathrm{S}(t)$. The equation obtained in this way for $\Delta G_\mathrm{S}(t)/\Delta t$ has the structure of a Langevin equation, with a Langevin force and a "friction force" describing not only damping processes but also energy shifts. The Langevin force has a zero reservoir average, is of first order in $V$, and comes from reservoir fluctuations. The "friction force" has a non zero reservoir average, is of second order in $V$ and comes from both reservoir fluctuations and self-reaction.

The important point is that, when one takes the reservoir average of these rates

$$(\Delta G_\mathrm{S}/\Delta t)_\mathrm{R} = \mathrm{Tr}_\mathrm{R}\,(\sigma_\mathrm{R} \Delta G_\mathrm{S}/\Delta t) \qquad (29)$$

one finds that the terms coming from reservoir fluctuations (self reaction) have the structure of integrals of products of $C_\mathrm{R}\chi_\mathrm{S}$ ($C_\mathrm{S}\chi_\mathrm{R}$), as in eqs. (21) and (22). This shows that the symmetrical order, imposed by physical arguments, leads, in the Heisenberg picture, to results in complete agreement with those derived in the previous section from the Schrödinger picture.

Finally, to conclude this lecture I, we can say that, by two different methods, we have identified, in the equations of motion of $\mathscr{S}$, two types of terms corresponding to two different physical processes:

(i) $\mathscr{R}$ fluctuates and polarizes $\mathscr{S}$.
(ii) $\mathscr{S}$ fluctuates and polarizes $\mathscr{R}$.

We have obtained for the energy shifts and damping rates of $\mathscr{S}$ simple and general expressions in terms of correlation functions and linear susceptibilities of the two interacting systems. We can apply now these general results to the discussion of spontaneous emission and radiative corrections.

## LECTURE II – APPLICATION TO SPONTANEOUS EMISSION AND RADIATIVE CORRECTIONS

We consider now the particular case where $\mathscr{S}$ is an atomic electron and $\mathscr{R}$ is the vacuum field. It is clear that the energy shifts of $\mathscr{S}$ represent in this case radiative corrections (such as the Lamb-shift or $g - 2$), whereas expressions such as eq. (22) describe damping rates due to spontaneous emission. We must note also that, because of the continuum nature of the frequency spectrum of the field modes, the correlation time $\tau_\mathrm{c}$ of vacuum fluctuations is very short so that condition (5) is fulfilled.

Section 4 is devoted to a spinless electron coupled to the vacuum field. Spin and magnetic effects will then be discussed in Section 5.

### 4. Spinless electron coupled to the vacuum field

#### 4.1. *Nonrelativistic hamiltonian in Coulomb gauge and long wavelength approximation*

We consider an electron with charge $q$, mass $m$, position $r$, momentum $p$. The hamiltonian of the electron-field system can be written (in Coulomb gauge)

$$H = \frac{1}{2m}[p - qA(r)]^2 + V(r) + \varepsilon_{\mathrm{Coul}} + H_\mathrm{R}, \qquad (30)$$

where $V(r)$ is a static potential, binding the electron near the origin, $A(r)$ is the vector potential of the quantized radiation field, with the following usual mode expansion.

$$A(r) = \sum_{k\varepsilon} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega L^3}}\,[a_{k\varepsilon}\varepsilon\,\mathrm{e}^{\mathrm{i}k\cdot r} + a_{k\varepsilon}^+\varepsilon\,\mathrm{e}^{-\mathrm{i}k\cdot r}]. \qquad (31)$$

$H_\mathrm{R}$ is the energy of the free radiation field

$$H_\mathrm{R} = \sum_{k\varepsilon} \hbar\omega[a_{k\varepsilon}^+ a_{k\varepsilon} + \tfrac{1}{2}] \qquad (32)$$

$a_{k\varepsilon}^+$ and $a_{k\varepsilon}$ being the creation and annihilation operators for a photon of wave vector $k$, polarization $\varepsilon$, energy $\hbar\omega$ ($L^3$ is the quantization volume). Finally, $\varepsilon_{\mathrm{Coul}}$ is the energy of the Coulomb field of the electron.

The long wavelength approximation consists in replacing $A(r)$ by $A(0)$ in eq. (30) and is valid if the variations of the field are negligible over the spatial extension of the electron wave function (of the order of the Bohr radius $a_0$). We will introduce a cut off in the mode expansion of the fields at $k_\mathrm{M} = \omega_\mathrm{M}/c$ with

$$\begin{cases} \text{Rydberg} \ll \hbar\omega_\mathrm{M} \ll mc^2 & (33a) \\ k_\mathrm{M}a_0 \ll 1 & (33b) \end{cases}$$

so that we can make the non relativistic and long wavelength

approximations (the effect of relativistic modes will be discussed at the end of Section 5). Condition Rydberg $\ll \hbar\omega_M$ means that we keep a frequency spectrum much wider than the characteristic electron frequency. With the same cut off for the longitudinal field, $\varepsilon_{Coul}$ becomes finite and equal to

$$\varepsilon_{Coul} = \frac{q^2 k_M}{4\pi\varepsilon_0} = \delta m_1 c^2 \tag{34}$$

where $\delta m_1$ is a mass correction associated with the Coulomb field.

Finally, the Hamiltonian (30) can be rewritten

$$H = H_e + \delta m_1 c^2 + q^2 \frac{A^2(0)}{2m} + H_R - q \sum_{i=x,y,z} \frac{p_i}{m} A_i(0) \tag{35}$$

where

$$H_e = \frac{p^2}{2m} + V(r) \tag{36}$$

is a pure electronic Hamiltonian. The third term of eq. (36) is a pure field operator which will be interpreted later on (see Section 4.3.3). The last term of eq. (35) is a coupling between the electron and the field which can be written, as in eq. (2)

$$V = -q \sum_i R_i S_i \tag{37}$$

with

$$\begin{cases} R_i = A_i(0) & \text{(38a)} \\ S_i = p_i/m & \text{(38b)} \end{cases}$$

### 4.2. *Correlation functions and linear susceptibilities for the vacuum field and for the electron* [9]

In order to apply the general results of Lecture I (Section 2), we need the correlation functions and linear susceptibilities for the operators $A_i(0)$ and $p_i/m$ appearing in the electron field coupling (37).

Replacing in eqs. (17) and (18), $\tilde{R}(t)$ by $\tilde{A}_i(0, t)$ and $\tilde{R}(t - \tau)$ by $\tilde{A}_j(0, t - \tau)$, using the expansions of $\tilde{A}_i(0, t)$ and $\tilde{A}_j(0, t - \tau)$ in $a_{k\varepsilon}$ and $a_{k\varepsilon}^+$ and the well known commutation relations and matrix elements of $a_{k\varepsilon}$ and $a_{k\varepsilon}^+$, we get

$$\begin{cases} C_{ij}^R(\tau) = \frac{\hbar\delta_{ij}}{12\pi^2\varepsilon_0 c^3} \int_{-\omega_M}^{+\omega_M} |\omega| \, e^{i\omega\tau} \, d\omega & \text{(39a)} \\ \chi_{ij}^R(t) = \frac{\delta_{ij}}{3\pi^2\varepsilon_0 c^3} \left[ \omega_M \delta(\tau) - \frac{\Gamma}{2}\delta'(\tau) \right]\theta(\tau) & \text{(39b)} \end{cases}$$

and the corresponding expressions for the Fourier transforms

$$\begin{cases} \hat{C}_{ij}^R(\omega) = \frac{\hbar\delta_{ij}}{12\pi^2\varepsilon_0 c^3} |\omega| & \text{for } -\omega_M \leqslant \omega \leqslant \omega_M \quad \text{(40a)} \\ = 0 & \text{elsewhere} \\ \hat{\chi}_{ij}^R(\omega) = \frac{\delta_{ij}}{6\pi^3\varepsilon_0 c^3} \left[ \omega_M - i\frac{\Gamma}{2}\omega \right]. & \text{(40b)} \end{cases}$$

We study now the $\hbar$ dependence of $C_R$ and $\chi_S$. This allows a simple discussion of classical versus quantum effects as far as the field is concerned. Radiation reaction effects are proportional to $C_S \chi_R$. It appears on eqs. (39b) and (40b) that $\chi_R$ is independent of $\hbar$ and has the same value as in classical field theory. Such a result is easy to understand. It comes from the fact that the field is a set of harmonic oscillators, and

it is well known that the linear susceptibility of an harmonic oscillator is independent of $\hbar$ and independent of the state of the oscillator. We expect therefore that all radiation reaction effects, coming from $C_S \chi_R$, will have the same form in quantum and classical radiation theories. On the other hand, $C_R$ is proportional to $\hbar$, as it appears on eqs. (39a) and (40a). This means that quantum theory of radiation is essential for explaining the correlation function of vacuum fluctuations. It follows that all vacuum fluctuation effects, coming from $C_R \chi_S$, have a quantum nature. It must be noted however that, to second order in $q$, the contribution of the vacuum field to these effects appears only through $C_R$. Once $C_R$ is given by the quantum theory of radiation, the effect of vacuum fluctuations can be calculated semiclassically. It is the same as the effect of a classical random field having the same correlation function, i.e., a spectral power density equal to $\hbar\omega/2$ per mode.

For the sake of completeness, we give also the Fourier transforms of the correlation functions and linear susceptibilities, in the electronic energy level $|a\rangle$ (eigenstate of the hamiltonian $H_e$ given in eq. (36)), of the electronic observables $\tilde{p}_i(t)/m$ and $\tilde{p}_j(t - \tau)/m$

$$\hat{C}_{S_{ij}}^a(\omega) = \frac{1}{2m^2} \sum_b \{(p_i)_{ab}(p_j)_{ba}\delta(\omega - \omega_{ab})$$
$$+ (p_j)_{ab}(p_i)_{ba}\delta(\omega + \omega_{ab})\} \tag{41}$$

$$\hat{\chi}_{S_{ij}}^a(\omega) = \frac{1}{2\pi\hbar m^2} \sum_b \left\{ (p_i)_{ab}(p_j)_{ba}\mathscr{P}\frac{1}{\omega - \omega_{ab}} \right.$$
$$\left. - (p_j)_{ab}(p_i)_{ba}\mathscr{P}\frac{1}{\omega + \omega_{ab}} \right\}$$
$$+ \frac{i}{2\hbar m^2} \sum_b \{(p_i)_{ab}(p_j)_{ba}\delta(\omega - \omega_{ab})$$
$$- (p_j)_{ab}(p_i)_{ba}\delta(\omega + \omega_{ab})\}, \tag{42}$$

where $\Sigma_b$ means of sum over a complete set of eigenstates of $H_e$, $\hbar\omega_{ab} = E_a - E_b$, and $\mathscr{P}$ means principal part. The first two lines of eq. (42) give the reactive part of $\hat{\chi}_S$, the last two the dissipative one.

### 4.3. *Energy shifts of electronic energy levels*

4.3.1. *Contribution of radiation reaction.* From the expressions of $C_S$ and $\chi_R$ given above, we get for the second term of eq. (21), which describes the contribution of radiation reaction to $\delta E_a$

$$(\delta E_a)_{\text{rad. react.}} = -\frac{4}{3}\frac{\delta m_1}{m}\langle a|\frac{p^2}{2m}|a\rangle \tag{43}$$

where $\delta m_1 c^2$ is the Coulomb self energy given in eq. (34).

Combining eq. (43) with the unperturbed kinetic energy $\langle a|p^2/2m|a\rangle$ gives

$$\left(1 - \frac{4}{3}\frac{\delta m_1}{m}\right)\langle a|\frac{p^2}{2m}|a\rangle \simeq \langle a|\frac{p^2}{2(m + \frac{4}{3}\delta m_1)}|a\rangle. \tag{44}$$

It appears therefore that radiation reaction changes the mass appearing in the kinetic energy from $m$ to $m + \frac{4}{3}\delta m_1$

$$m \rightarrow m + \frac{4}{3}\delta m_1. \tag{45}$$

The mass corrections appearing in the kinetic energy term (44), and in eq. (34), which can be interpreted as a correction to the rest mass energy, differ by a factor 4/3. This

discrepancy is due to the non-covariant cut off (33) and also exists in classical theory.

4.3.2. *Contribution of vacuum fluctuations.* From the expressions of $C_R$ and $\chi_S$ given above, we get for the first term of eq. (21), which describes the contribution of vacuum fluctuations to $\delta E_a$

$$(\delta E_a)_{\text{vac. fluct.}} = \frac{-q^2}{6\pi^2 \varepsilon_0 m^2 c^3} \sum_b \omega_{ab} |(\boldsymbol{p})_{ab}|^2 \log \frac{\omega_M}{|\omega_{ab}|}. \quad (46)$$

Introducing an average atomic frequency $\bar{\omega}$ defined by

$$\sum_b \omega_{ab} |(\boldsymbol{p})_{ab}|^2 \log \frac{\mu_M}{|\omega_{ab}|} = \log \frac{\omega_M}{\bar{\omega}} \sum_b \omega_{ab} |(\boldsymbol{p})_{ab}|^2 \quad (47)$$

and using

$$\sum_b \omega_{ab} |(\boldsymbol{p})_{ab}|^2 = \frac{1}{2\hbar} \langle a|[[H_e, \boldsymbol{p}], \boldsymbol{p}]|a\rangle$$

$$= -\frac{\hbar}{2} \langle a|\Delta V(\boldsymbol{r})|a\rangle \quad (48)$$

we get

$$(\delta E_a)_{\text{vac. fluct.}} = \langle a|\delta V(\boldsymbol{r})|a\rangle \quad (49a)$$

where

$$\delta V(\boldsymbol{r}) = \frac{q^2 \hbar}{12\pi^2 \varepsilon_0 m^2 c^3} \log \frac{\omega_M}{\bar{\omega}} \Delta V(\boldsymbol{r}) \quad (49b)$$

is a correction to the potential energy $V(\boldsymbol{r})$ appearing in eq. (36), proportional to $\Delta V(\boldsymbol{r})$, and associated with vacuum fluctuations.

The physical interpretation of the energy correction (49), which coincides with the standard non relativistic expression for the Lamb-shift [16] is well known [1]. The electron, vibrating in vacuum fluctuations, averages the binding static potential $V(\boldsymbol{r})$ over a finite spherical volume, and this explains the correction to $V(\boldsymbol{r})$, proportional to $\Delta V(\boldsymbol{r})$. For a Coulomb potential, $V(\boldsymbol{r})$ is proportional to $1/r$, and $\Delta V(\boldsymbol{r})$ to $\delta(\boldsymbol{r})$. It follows that only $s$-states are shifted. Since the two states $2s_{1/2}$ and $2p_{1/2}$ in hydrogen have the same average kinetic energy, the energy correction (43) due to radiation reaction cannot remove the degeneracy between the two states, and we conclude that the Lamb shift is essentially due to the correction (49) to the potential energy, i.e. to vacuum fluctuations.

4.3.3. *Interpretation of the $q^2 A^2(0)/2m$ term.* We interpret now the third term of eq. (35), which is a pure field operator. Taking the vacuum average value of this operator gives, to order 2 in $q$

$$\langle 0| \frac{q^2}{2m} A^2(0)|0\rangle = \sum_{\omega \leqslant \omega_M} \frac{q^2 \mathscr{E}_\omega^2}{2m\omega^2} \quad (50)$$

where $\mathscr{E}_\omega$ is the vacuum field in mode $\omega$. Physically, such an energy represents the mean kinetic of vibration, $\mathscr{E}_{\text{vibr}}$, of the electron in vacuum fluctuations. It can be written

$$\mathscr{E}_{\text{vibr}} = \delta m_2 c^2 \quad (51)$$

and appears as a correction to the rest mass energy of the electron, associated with vacuum fluctuations, in the same way as $\varepsilon_{\text{Coul}}$, in eq. (34), appears as a correction to the rest mass energy due to self reaction, since it comes from the Coulomb self field of the electron.

It may appear surprising that our calculation doesn't give any correction to the kinetic energy associated with the mass correction $\delta m_2$ due to vacuum fluctuations. One would expect to find, as in Section 4.3.1, a term of the order of

$$-\frac{\delta m_2}{m} \langle a| \frac{\boldsymbol{p}^2}{2m} |a\rangle. \quad (52)$$

Actually, this comes from the fact that $\delta m_2$ is of a higher order in $1/c$ than $\delta m_1$, as it appears when we express eqs. (34) and (50) in terms of the fine structure constant $\alpha$ and of $\hbar\omega_M/mc^2$

$$\frac{\delta m_1}{m} = \frac{\alpha}{\pi} \frac{\hbar\omega_M}{mc^2} \quad (53a)$$

$$\frac{\delta m_2}{m} = \frac{\alpha}{2\pi} \left(\frac{\hbar\omega_M}{mc^2}\right)^2. \quad (53b)$$

Pushing our non relativistic calculation to higher orders in $1/c$ actually gives extra terms such as eq. (52) [17].

To conclude our discussion of energy shifts, we can say that, to lowest order in $1/c$, radiation reaction changes the kinetic energy by increasing the mass of the electron by $4\delta m_1/3$, whereas vacuum fluctuations change the potential energy by producing a vibration of the electron.

4.4. *Rate of electronic energy loss*

As in the previous section, we insert the values of $C_R$, $C_S$, $\chi_R$, $\chi_S$, given in Section 4.2, in the two terms of eq. (22) which represent respectively the contributions of vacuum fluctuations and radiation reaction to the rate of variation of the electron energy in level $a$.

4.4.1. *Contribution of radiation reaction.* The last term of eq. (22) becomes here

$$\left(\langle 0, a| \frac{dH_e}{dt} |0, a\rangle\right)_{\text{rad. react.}} = -\frac{q^2}{6\pi m^2 \varepsilon_0 c^3} \sum_b \sum_i |(p_i)_{ab}|^2 \omega_{ba}^2. \quad (54)$$

Using

$$\frac{1}{m}(p_i)_{ab} \omega_{ab} = \frac{1}{m\hbar} \langle a|[H_e, p_i]|a\rangle$$

$$= -\frac{i}{m}(\dot{p}_i)_{ab} = -i(\ddot{r}_i)_{ab} \quad (55)$$

we can transform eq. (54) into

$$\left(\langle 0, a| \frac{dH_e}{dt} |0, a\rangle\right)_{\text{rad. react.}} = -\frac{2}{3} \frac{q^2}{4\pi\varepsilon_0 c^3} \langle a|\ddot{\boldsymbol{r}}^2|a\rangle. \quad (56)$$

We find that the rate of energy loss is proportional to the square of the acceleration of the radiating charge, which is a purely classical result. We note also that, if radiation reaction was alone, the ground state would be unstable since the average value of $\ddot{\boldsymbol{r}}^2$ does not vanish in the ground state.

4.4.2. *Contribution of vacuum fluctuations.* We find for the first term of eq. (22)

$$\left(\langle 0, a| \frac{dH_e}{dt} |0, a\rangle\right)_{\text{vac. fluct.}} = -\frac{q^2}{6\pi m^2 \varepsilon_0 c^3}$$

$$\times \sum_b \sum_i |(p_i)_{ab}|^2 \omega_{ab} |\omega_{ab}|. \quad (57)$$

Distinguishing the terms $\omega_{ab} > 0$ ($E_a > E_b$) and the terms

$\omega_{ab} < 0$ $(E_a < E_b)$, we can transform eq. (57) into

$$\left(\langle 0, a| \frac{\mathrm{d}H_e}{\mathrm{d}t} |0, a\rangle\right)_{\text{vac. fluct.}} = + \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0 c^3}$$

$$\times \left\{ \sum_{\substack{b \\ E_b > E_a}} (\ddot{\boldsymbol{r}})_{ab} \cdot (\ddot{\boldsymbol{r}})_{ba} - \sum_{\substack{b \\ E_b < E_a}} (\ddot{\boldsymbol{r}})_{ab} \cdot (\ddot{\boldsymbol{r}})_{ba} \right\}. \quad (58)$$

We have mentioned above (in Section 4.2) that the effect of vacuum fluctuations is equivalent to the one of a fluctuating field having a spectral power density $\hbar\omega/2$ per mode. Such a fluctuating field can induce transitions from level $a$ to higher levels $b$, which corresponds to an energy gain for the atom (first term of the bracket of eq. (58)), as well as transitions from level $a$ to lower levels $b$, which corresponds to an energy loss for the atoms (second term of the bracket of eq. (58)).

If we write $\langle a|\ddot{\boldsymbol{r}}^2|a\rangle$ in eq. (56) as $\langle a|\ddot{\boldsymbol{r}}^2|a\rangle = \Sigma_b(\ddot{\boldsymbol{r}})_{ab} \cdot (\ddot{\boldsymbol{r}})_{ba}$ and if we add eqs. (56) and (58), we find of course that spontaneous transitions can occur only from level $a$ to lower levels, under the combined effect of vacuum fluctuations and radiation reaction (the first term of eq. (58) is cancelled by a similar term of eq. (56) and the second term of eq. (58) is doubled).

*4.4.3. Application to a 2-level atom.* When applied to a 2-level atom (ground state g and excited state e), the previous results take a very simple form.

First, we find that vacuum fluctuations stabilize the ground state, as already mentioned by Fain [18], since the energy gain due to vacuum fluctuations exactly compensates the energy loss due to radiation reaction.

Then, we find that vacuum fluctuations and radiation reaction contribute equally to the spontaneous emission rate from the upper level e. The two rates are equal. This explains the factor 2 missing in elementary calculations of the spontaneous emission rate from e and considering only one physical process, emission by an accelerated charge or transition induced by a fluctuating field with a spectral density equal to $\hbar\omega/2$ per mode.

# 5. Spin and magnetic effects

In this last section, we try to understand the electron spin anomaly, $g - 2$, and in particular its positive sign. So, we introduce the spin degrees of freedom and the magnetic couplings.

## 5.1. *New terms in the Hamiltonian*

To keep the calculations as simple as possible we consider a single electron in a uniform magnetic field $\boldsymbol{B}_0$ parallel to the 0z axis.

The electronic hamiltonian $H_e$ (which replaces eq. (36)) is now

$$H_e = \frac{\pi_0^2}{2m} - \frac{q}{m} \boldsymbol{S} \cdot \boldsymbol{B}_0 \quad (59)$$

where

$$\boldsymbol{\pi}_0 = \boldsymbol{p} - q\boldsymbol{A}_0(\boldsymbol{r}) \quad (60)$$

$\boldsymbol{A}_0$ being the static vector potential associated with $\boldsymbol{B}_0$. The first term of eq. (60) is the kinetic energy of the electron, since $\boldsymbol{\pi}_0/m$ is the electron velocity. The second term of eq. (59) is the coupling of the spin magnetic moment of the electron, $2(q/2m)\boldsymbol{S}$, where $\boldsymbol{S}$ is the spin operator, with the magnetic

field $\boldsymbol{B}_0$ (in absence of radiative corrections, the g factor of the electron is equal to 2).

The interaction of the electron with the quantized radiation field is now

$$V = -\frac{q}{m} \boldsymbol{\pi}_0 \cdot \boldsymbol{A}(\boldsymbol{r}) - \frac{q}{m} \boldsymbol{S} \cdot \boldsymbol{B}(\boldsymbol{r}) + \frac{q^2}{2m} \boldsymbol{A}^2(\boldsymbol{r}). \quad (61)$$

In the long wavelength approximation, the first and last terms of eq. (60) are the same as for the spinless electron (see eq. (35)), except that $\boldsymbol{p}$ is replaced by $\boldsymbol{\pi}_0$, given in eq. (60). The second term of eq. (61) is new and represents the coupling of the spin with the quantized radiation field $\boldsymbol{B}$.

## 5.2. *Failure of Welton's picture applied to the spin magnetic moment*

It has been known for a long time [1, 19] that the picture of an electron spin oscillating in vacuum fluctuations leads to the wrong sign for $g - 2$. The fact that such a picture leads to a decrease of $g$, and, consequently, to a negative sign for $g - 2$, is easy to understand. The angular oscillation of the spin driven by the vacuum fluctuations of $\boldsymbol{B}$ produces an angular spreading of the spin, and, consequently, a *decrease* of the effective coupling of the spin with the static field $\boldsymbol{B}_0$.

What is missing in the previous description is the coupling of the electron velocity with the vector potential of the quantized radiation field. More precisely, we must consider the whole dynamics of the electron coupled to the vacuum field and study how the energy levels of the electron in the static field $\boldsymbol{B}_0$ are shifted by radiation reaction and vacuum fluctuations [17].

## 5.3. *Corrections to cyclotron and Larmor frequencies. Why is g — 2 positive*

Before considering the energy shifts produced by radiation reaction and vacuum fluctuations, we have first to give the unperturbed eigenvalues of $H_e$, which can be written

$$(n + \tfrac{1}{2})\hbar\omega_c + m_S \hbar\omega_L. \quad (62)$$

In the first term of eq. (62), $n$ is an integer ($n = 0, 1, 2, \ldots$) and $\omega_c$ is the *cyclotron frequency of the charge* given by

$$\omega_c = -\frac{qB_0}{m}. \quad (63)$$

The corresponding energy levels are the well known Landau levels of a charged particle in a uniform static field (we have supposed here that the electron velocity along the direction 0z of $\boldsymbol{B}_0$ is zero). In the second term of (62), $m_S = +1/2$ or $-1/2$ labels the eigenvalue $m_S\hbar$ of $S_z$ and $\omega_L$ is the *Larmor frequency of the spin* which can be written

$$\omega_L = -g \frac{q}{2m} B_0 \quad (64)$$

with $g = 2$.

When we introduce the coupling (61) with the quantized radiation field, the energy levels given in eq. (62) are shifted, and the cyclotron and Larmor frequencies are changed to $\tilde{\omega}_c$ and $\tilde{\omega}_L$

$$\omega_c \to \tilde{\omega}_c, \qquad \omega_L \to \tilde{\omega}_L. \quad (65)$$

The g factor of the electron in the presence of radiative

corrections is defined by

$$\frac{\tilde{g}}{2} = \frac{\tilde{\omega}_L}{\tilde{\omega}_c}. \tag{66}$$

To understand eq. (66), we note that, if $\tilde{m}$ is the renormalized mass appearing in the perturbed cyclotron frequency

$$\tilde{\omega}_c = -q \frac{B_0}{\tilde{m}}, \tag{67}$$

then, the Larmor frequency $\tilde{\omega}_L$ is expressed by eq. (66) as

$$\tilde{\omega}_L = -\tilde{g} \frac{q}{2m} B_0, \tag{68}$$

i.e., in terms of $\tilde{g}$ and of the renormalized Bohr magneton. Furthermore, we must emphasize that the more precise determinations of $\tilde{g}$ are given by a ratio of two measured frequencies, as in eq. (66) [20].

In absence of radiative corrections, $\omega_L = \omega_c$ and $g = 2$. To understand why $\tilde{g} - 2$ is positive, we have to understand why $\tilde{\omega}_L$ is larger than $\tilde{\omega}_c$ in eq. (66). We give now the conclusions of a non relativistic calculation of $\tilde{\omega}_L$ and $\tilde{\omega}_c$ [9, 17].

Consider first the contributions of radiation reaction. We find that, to lowest order in $1/c$, radiation reaction *slows down* $\omega_c$, but *not* $\omega_L$. The interpretation of this result is that, in the non relativistic domain, a charge is more coupled to its self field than a magnetic moment. The cyclotron precession of the charge is more perturbed than the Larmor precession of the spin.

The contribution of vacuum fluctuations appears only to the next order in $1/c$ and describes relativistic and magnetic effects (including the angular oscillation of the spin driven by vacuum fluctuations) which, as a whole, reduce both $\omega_c$ and $\omega_L$ (at this order in $1/c$, new terms such as spin orbit couplings must be added to $H_e$ and $V$, see [17]).

The conclusion of this calculation is that, in the non relativistic domain, the main effect (to lowest order in $1/c$) is *a slowing down of $\tilde{\omega}_c$ by radiation reaction*, and this explains why $\tilde{g} = 2\tilde{\omega}_L/\tilde{\omega}_c$ becomes larger than 2. Similar conclusions have been obtained by Grotch and Kazes [21].

### 5.4. Outline of a completely relativistic calculation

Because of the cut off introduced at $\omega_M$ (see eq. (33)), the previous calculation considers only the coupling of the electron with non relativistic modes of the radiation field.
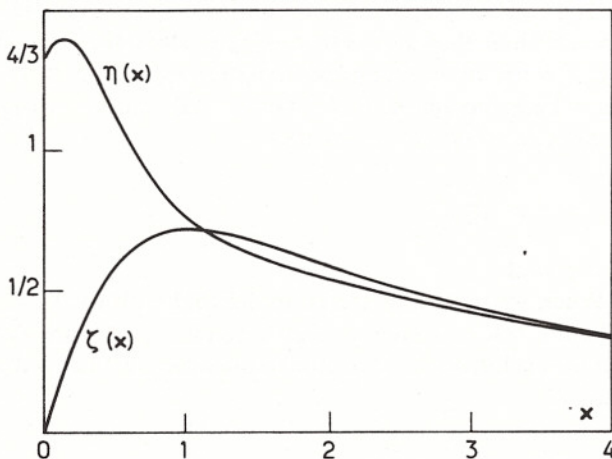


*Fig. 1.* Graphs of the functions $\eta(x)$ and $\zeta(x)$.

We have tried to evaluate the contribution of relativistic modes, for which $\hbar\omega_M$ can be of the order or larger than $mc^2$. Starting from the full relativistic Hamiltonian, for coupled quantized Dirac and Maxwell fields, we have derived an effective Hamiltonian giving the energy levels of a non relativistic electron (i.e. in a frame where this electron is moving slowly), and including the contribution of virtual emissions and reabsorptions of photons of any frequency $\omega$. (In the virtual intermediate state, the electron can be relativistic and electron–positron pairs can be created). We just give here the main results of such a calculation which is presented in Ref. [22].

To order 1 in $\alpha$, we find that the effective Hamiltonian $H_{eff}$ has the following form

$$H_{eff} = \frac{\pi_0^2}{2m} \left[ 1 - \frac{\alpha}{\pi} \int_0^{x_M} \eta(x) \, dx \right]$$
$$- \frac{q}{m} S \cdot B_0 \left[ 1 - \frac{\alpha}{\pi} \int_0^{x_M} \zeta(x) \, dx \right] \tag{69}$$

where $x = \hbar\omega/mc^2$ is related to the frequency $\omega$ of the photon which is virtually emitted and reabsorbed, $x_M$ is cut off, which is much stronger than 1 (so that the effect of relativistic modes is taken into account), $\pi_0$ is the same as in eq. (60).

The two functions $\eta(x)$ and $\zeta(x)$, which describe how the various modes of the radiation field contribute to the modification of the cyclotron frequency (first term of eq. (69)), and of the Larmor frequency (second term of eq. (69)), are given by

$$\eta(x) = x \left[ 1 - \frac{x}{(1 + x^2)^{1/2}} + \frac{4}{3x(1 + x^2)^{1/2}} \right.$$
$$\left. - \frac{x}{3(1 + x^2)^{3/2}} - \frac{2x}{(1 + x^2)^{5/2}} \right] \tag{70b}$$

$$\zeta(x) = x \left[ \frac{5}{3} \left( 1 - \frac{x}{(1 + x^2)^{1/2}} \right) + \frac{2}{3} \frac{x}{(1 + x^2)^{3/2}} \right] \tag{70a}$$

and are represented on Fig. 1. For $x \ll 1$ (non-relativistic domain), the main correction comes from $\eta(x)$ and produces a decrease of the cyclotron frequency (because of the minus sign in the first term of eq. (69)). This confirms the previous calculation discussed in Section 5.3. For $x \gg 1$, $\eta(x)$ and $\zeta(x)$ are both equivalent to $3/2x$, leading to the same logarithmic divergence for the coefficients of $\pi_0^2/2m$ and $-(q/m)S \cdot B_0$.

To see how this divergence can be reabsorbed in the mass normalization, we come back now to the definition (66) of $\tilde{g}$. Since the correction factors for $\tilde{\omega}_c$ and $\tilde{\omega}_L$ are the two brackets of eq. (69), we get

$$\frac{\tilde{g}}{2} = \frac{\tilde{\omega}_L}{\tilde{\omega}_c} = \frac{1 - \frac{\alpha}{\pi} \int_0^{x_M} \zeta(x) \, dx}{1 - \frac{\alpha}{\pi} \int_0^{x_M} \eta(x) \, dx} \tag{71}$$

The two integrals of eq. (71) are of the order of $\log x_M$. Since $e^{\pi/\alpha} \gg 1$, we can choose $1 \ll x_M \ll e^{\pi/\alpha}$, i.e. take a very high cut off and simultaneously have the two terms in $\alpha$ very small compared to 1, so that eq. (71) can be written

$$\frac{\tilde{g}}{2} \simeq 1 + \frac{\alpha}{\pi} \int_0^{x_M} [\eta(x) - \zeta(x)] \, dx. \tag{72}$$

The function $\eta(x) - \zeta(x)$ is represented in Fig. 2 and the integral of eq. (72), which is no longer divergent, is equal to
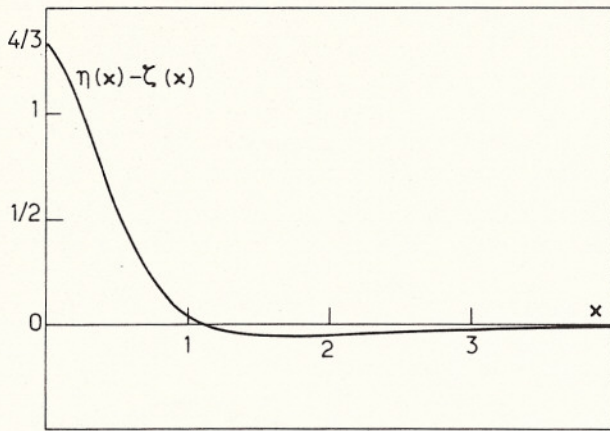
*Fig. 2.* Graph of the function $\eta(x) - \zeta(x)$.

$1/2$, if $x_M$ is large enough. We get therefore the correct result (to order 1 in $\alpha$) for the spin anomaly

$$\frac{\tilde{g}}{2} = 1 + \frac{\alpha}{2\pi}. \tag{73}$$

The curve of Fig. 2 shows how the various modes of the field contribute to $\tilde{g} - 2$. It clearly appears on this curve that the main contribution comes from $x < 1$ and that it is not necessary to invoke ultra high relativistic modes for explaining the sign of $g - 2$ (actually the contribution of the domain $x > 1$ to the integral is negative!). The physical interpretation derived from the non relativistic calculation is therefore confirmed.

To conclude this Lecture II, we can summarize the main results which have been obtained:

(i) It is possible to make a clear separation between the effects of radiation reaction and those of vacuum fluctuations (to order 1 in $\alpha$).

(ii) Radiation reaction effects are exactly the same in classical and quantum theories of radiation. Vacuum fluctuation effects appear as induced by a fluctuating field having a spectral power density equal to $\hbar\omega/2$ per mode.

(iii) Radiative corrections can be interpreted in terms of simple physical pictures: vibration of the charge and spin in vacuum fluctuations, electromagnetic inertia slowing down the motion of the charge.

(iv) The Lamb shift is mainly due to vacuum fluctuations. The spin anomaly $g - 2$ is mainly due to radiation reaction.

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