THE WEAKLY INTERACTING BEC

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4. Superfluid hydrodynamics

Summary of the previous lectures

- The properties of dilute Bose-Einstein condensates can be described within the mean-field approximation.
- All particles possess the same wave-function Φ solution of the Gross-Pitaevskii Equation (GPE)

$$i\hbar\partial_t\Phi = -\frac{\hbar^2}{2m}\nabla^2\Phi + U(r)\Phi + Ng\,|\,\Phi\,|^2\,\Phi$$

- In the Thomas-Fermi approximation, $Na/a_{ho}>>1$, the GPE reduces to the LDA.

Low-lying excitations

Bose-Einstein condensate close to equilibrium

$$\Phi(\mathbf{r},t) = e^{-i\mu_0 t/\hbar}\left(\varphi(\mathbf{r}) + \delta\Phi(\mathbf{r},t)\right)$$

For a weak perturbation: $\delta\Phi << \varphi$

Expansion of the time-dependent EGP to first order:

$$i\hbar\partial_t \begin{pmatrix} \delta\Phi \\ \delta\Phi^* \end{pmatrix} = \mathcal{L}_{\mathrm{GP}} \begin{pmatrix} \delta\Phi \\ \delta\Phi^* \end{pmatrix}$$

with

$$\mathcal{L}_{\mathrm{GP}} = \begin{pmatrix} h_0 + 2gN|\varphi|^2 & gN\varphi^2 \\ -gN\varphi^{*2} & -h_0 - 2gN|\varphi|^2 \end{pmatrix} \qquad h_0 = \frac{-\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) - \mu_0$$



$\mathcal{L}_{\mathrm{GP}}$ is *not* a hermitian operator!

The Bogoliubov spectrum

Find the spectrum of $\mathcal{L}_{GP}$ for **a uniform** system: $n_0=N|\varphi|^2$; $\mu_0=gn_0$.

$$\hbar\omega\begin{pmatrix}u\\v\end{pmatrix}=\mathcal{L}_{GP}\begin{pmatrix}u\\v\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}u(\mathbf{r})\\v(\mathbf{r})\end{pmatrix}=\begin{pmatrix}u_k\\v_k\end{pmatrix}\frac{e^{i\mathbf{k}.\mathbf{r}}}{\sqrt{V}}$$

$\hbar\omega$ is eigenvalue of

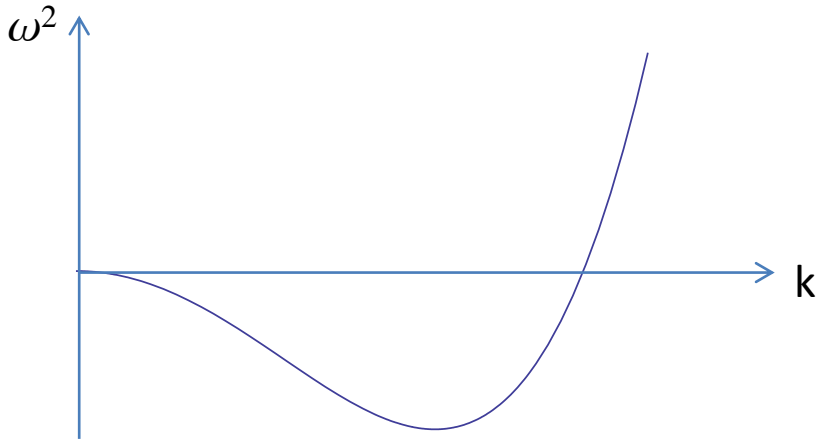
$$\mathcal{L}_{GP}=\begin{pmatrix}\hbar^2k^2/2m+gn_0 & gn_0\\ -gn_0 & -\hbar^2k^2/2m-gn_0\end{pmatrix}$$

Bogoliubov spectrum: $\hbar\omega=\pm\sqrt{\hbar^2k^2/2m(\hbar^2k^2/2m+2gn_0)}$

$$u_k=\cosh(\theta_k)$$
$$v_k=\sinh(\theta_k)$$
$$\tanh(2\theta_k)=\frac{gn}{\hbar^2k^2/2m+gn}$$

Bogoliubov Spectrum (II)

g<0: collapse instablity



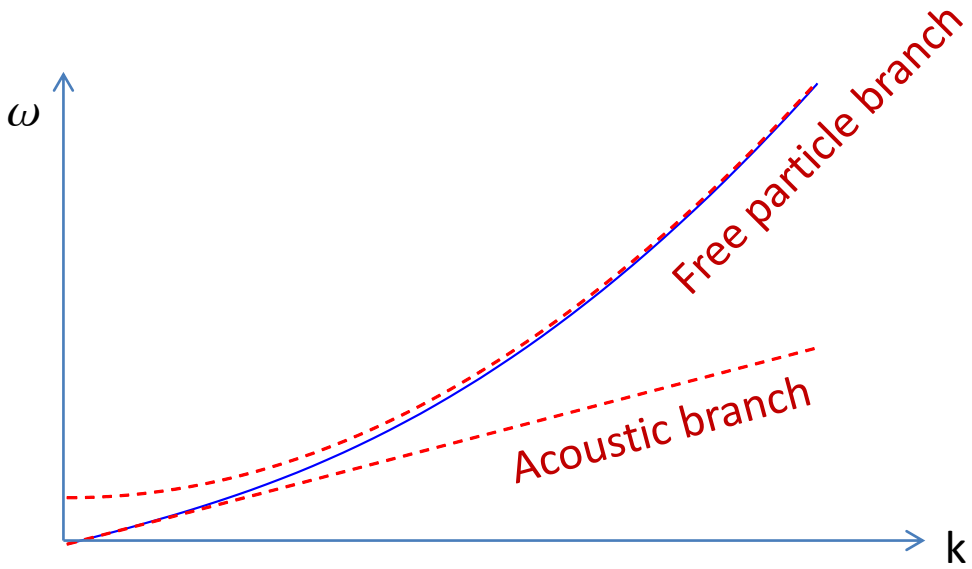
g>0:Bogoliubov spectrum

Acoustic branch

$$\omega = ck$$

Sound velocity $\mu_0 = mc^2$

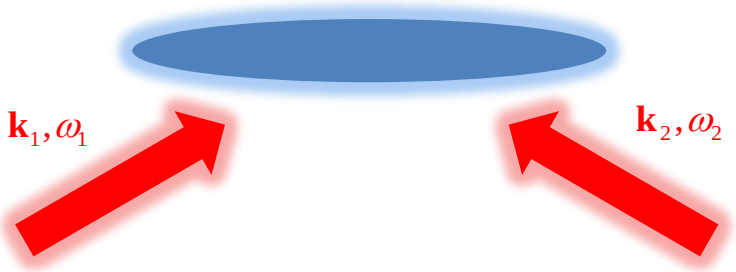
Crossover for $\frac{\hbar^2 k^2}{2m} \simeq \mu_0$

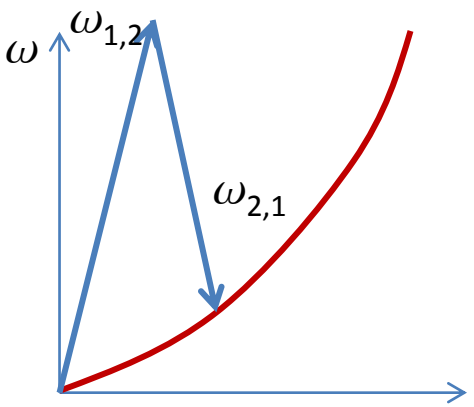


Measurement of the Bogoliubov spectrum

Steinhauer *et al.* Phys. Phys. Lett. **88**, *120407, (2002).*

Bragg spectroscopy of elementary excitations

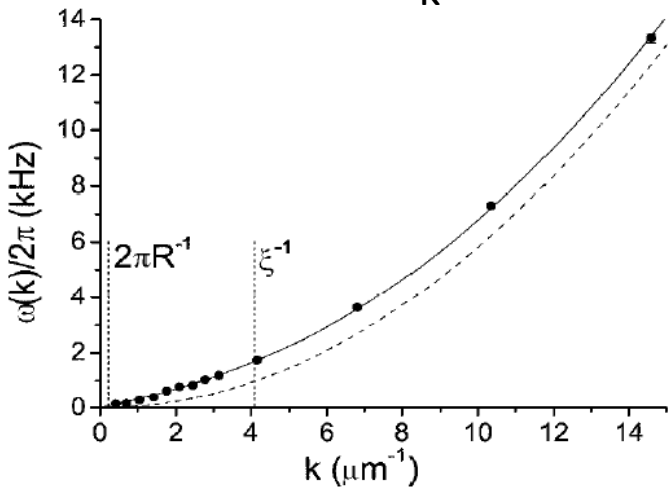




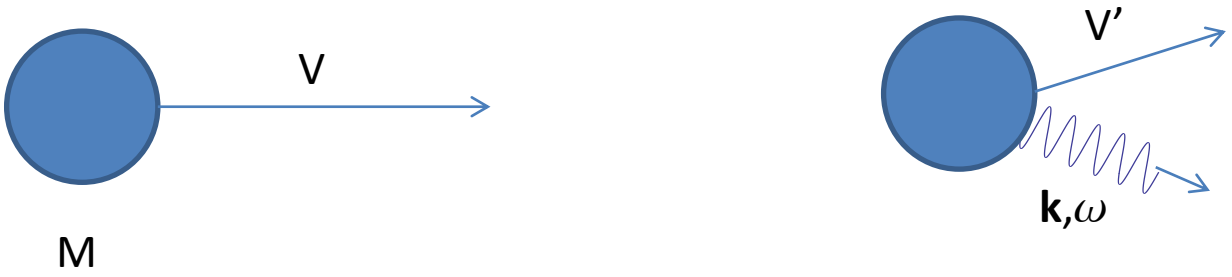
Energy momentum conservation

Resonant transfer of photons for

$$\omega_{\text{Bogo}}(\mathbf{k}_1 - \mathbf{k}_2) = |\omega_1 - \omega_2|$$

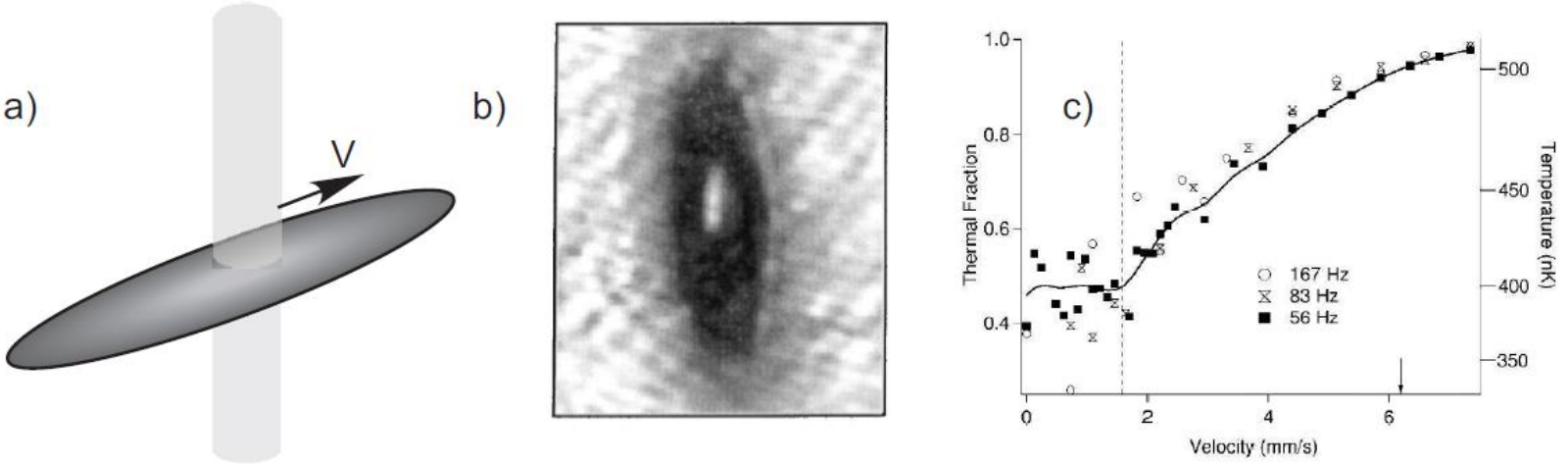


Landau's criterion for superfluidity



Energy-momentum conservation: the motion of the impurity is damped by emission of elementary excitations if

$$V \geq \min \frac{\omega_k}{k} = c$$



Raman *et al.* PRL **83**, 2502 (1999).

Superfluid Hydrodynamics

Alternative approach to the dynamical properties of a Bose-Einstein condensate

Madelung's Transform: $\varphi(\mathbf{r},t) = \sqrt{\frac{n(\mathbf{r},t)}{N}} e^{i\chi(\mathbf{r},t)}$

Using Gross-Pitaevskii's Equation

$$\mathbf{v} = \frac{\hbar}{m}\nabla\chi$$ Potential flow

$$\partial_t n + \nabla(n\mathbf{v}) = 0$$ Conservation equation

$$m\partial_t v = -\nabla\left(gn + U + mv^2/2 - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2 n \right)$$ Euler's Equation

Long wave-length/Thomas-Fermi/large cloud limit

Hydrodynamic equations for a viscousless fluid.

Simple solutions of the hydrodynamic equations

Stationnary solution:

$$v = 0$$

$$gn(\mathbf{r}) + U(\mathbf{r}) = \text{cte}$$ Thomas-Fermi profile!

Homogeneous system close to equilibrium:

$$\chi = \chi_1 e^{i(\mathbf{k}.\mathbf{r}-\omega t)}$$

$$n = n_0 + n_1 e^{i(\mathbf{k}.\mathbf{r}-\omega t)}$$

$$\Rightarrow \omega = ck \quad \text{with} \quad c = \sqrt{gn_0 / m}$$

In a harmonic trap: scaling solutions

$$\omega_1 = \sqrt{2}\omega_\perp$$

Cylindrical trap:

$$\omega_\pm^2 = \frac{1}{2}\left[3\omega_z^2 + 4\omega_\perp^2 \pm \sqrt{9\omega_z^4 + 16\omega_\perp^4 - 16\omega_z^2\omega_\perp^2} \right]$$

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5. Strongly interacting gases

The Bogoliubov Hamiltonian

Second quantized Hamiltonian:

$$H = \int d^3\mathbf{r}\hat{\Psi}(\mathbf{r})^\dagger h_0 \hat{\Psi}(\mathbf{r}) + \frac{g}{2}\hat{\Psi}(\mathbf{r})^\dagger \hat{\Psi}(\mathbf{r})^\dagger \hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r})$$

$$\hat{\Psi}(r) = \sqrt{N}\varphi(r) + \delta\hat{\Psi}$$

Expand H with $\delta\Psi$

$$H = E_0 + \frac{1}{2}\int d^3\mathbf{r} \begin{pmatrix} \delta\hat{\Psi} \\ -\delta\hat{\Psi}^\dagger \end{pmatrix}^\dagger \mathcal{L}_{GP} \begin{pmatrix} \delta\hat{\Psi} \\ \delta\hat{\Psi}^\dagger \end{pmatrix} + \ldots$$

The Bogoliubov Hamiltonian: Homogeneous system

Expand the field operators in the plan-wave basis $\delta\hat{\Psi} = \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}.\mathbf{r}}}{\sqrt{V}} \hat{a}_{\mathbf{k}}$

$$H = E_0 + \frac{1}{2}\sum_k \begin{pmatrix} \hat{a}_k \\ -\hat{a}_{-k}^{\dagger} \end{pmatrix}^{\dagger} \begin{pmatrix} \hbar^2 k^2 / 2m + gn_0 & gn_0 \\ -gn_0 & -\hbar^2 k^2 / 2m - gn_0 \end{pmatrix} \begin{pmatrix} \hat{a}_k \\ \hat{a}_{-k}^{\dagger} \end{pmatrix}$$

Bogoliubov transform:

$$\hat{b}_k = u_k \hat{a}_k + v_k \hat{a}_{-k}^{\dagger}$$
$$\hat{b}_{-k}^{\dagger} = u_k \hat{a}_{-k}^{\dagger} + v_k \hat{a}_k$$

For $u_k^2 - v_k^2 = 1$, the b_k's are bosonic annihilation operators.

$$\hat{H} = E_0 + \sum_k \hbar\omega_k \hat{b}_k^{\dagger} \hat{b}_k$$

Quantum depletion and Lee-Huang Yang corrections

The ground state corresponds to the vacuum of the Bogoliubov excitations.

Even at T=0, states of finite momentum are populated

$$N_{k\neq 0} = \sum_{k\neq 0} \langle a_k^\dagger a_k \rangle = \sum_k v_k{}^2$$

Quantum depletion of the ground state:

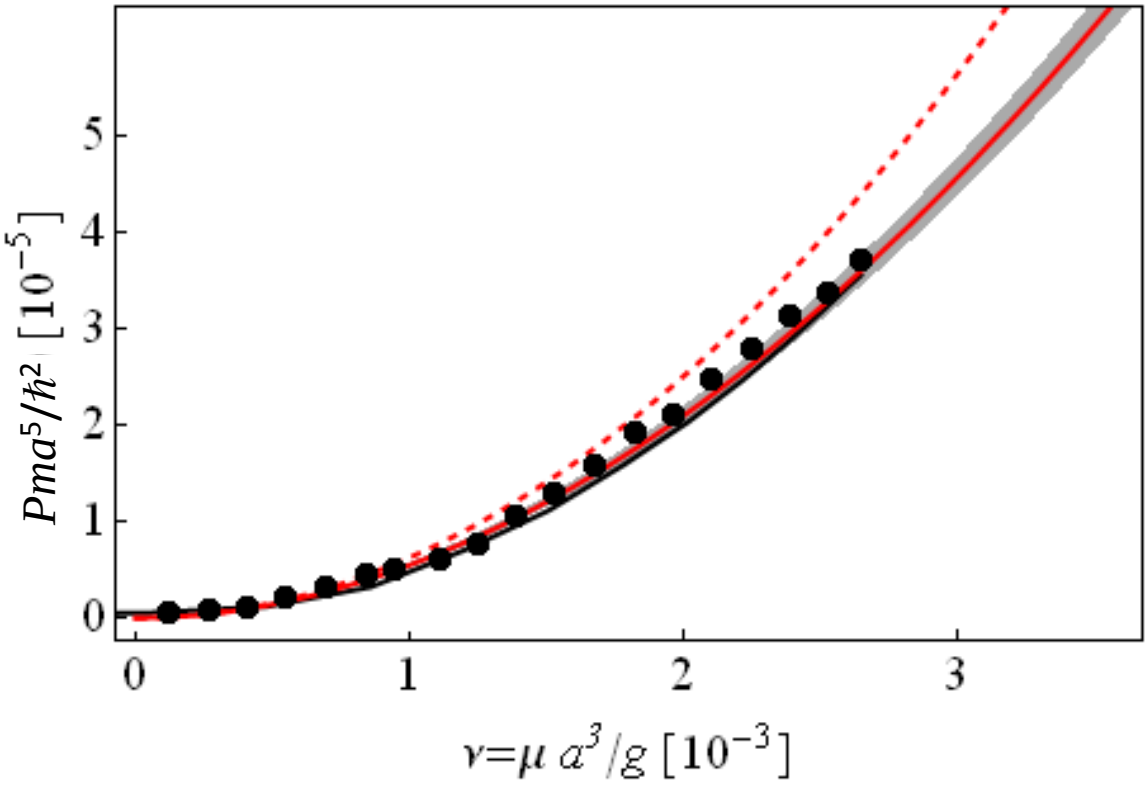
$$\frac{\delta N}{N} = \frac{8}{3\sqrt{\pi}} \sqrt{na^3}$$

Energy of the ground state (Lee-Huang-Yang formula)

$$E = E_{\text{Mean Field}} \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{na^3} \right)$$

Experimental demonstration of the Lee-Huang-Yang corrections

Averaging images of ^{7}Li between 1440 and 2150 a_0



$$E/N = \frac{gn}{2}\left(1 + \alpha\sqrt{na^3}\right)$$

We find : 4.5(7)

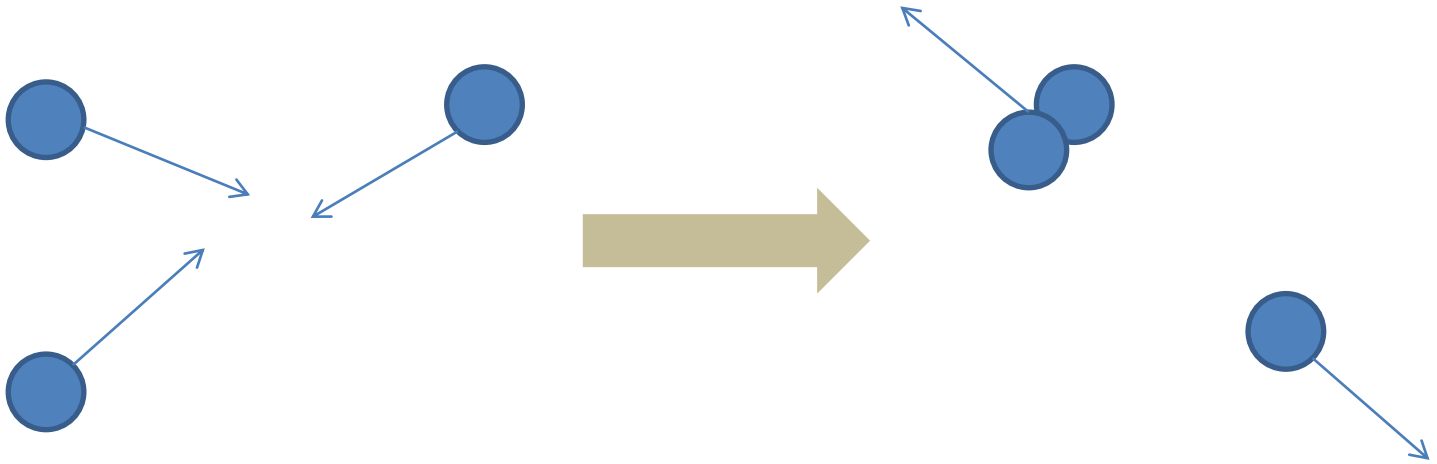
$$\frac{128}{15\sqrt{\pi}} \simeq 4.81$$

Navon *et. al.*, PRL **107**,135301 (2011)

And beyond Lee-Huang-Yang?

When getting closer to the Feshbach resonance the lifetime of the could decreases!

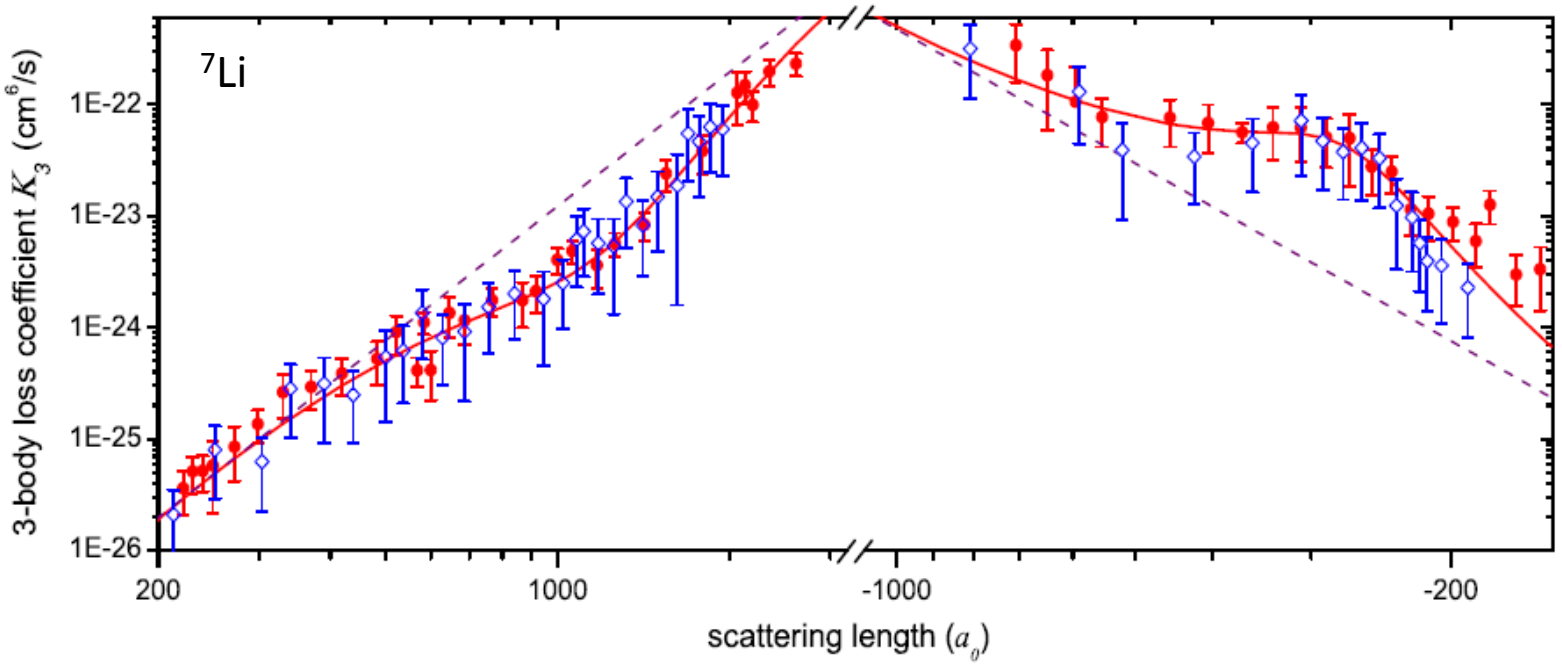
Three-body recombination:



The binding energy is large compared to the potential height: **the three atoms are lost**.

Phenomenological law : $\frac{dN}{dt} = -L_3 n^2 N$

Three-body losses in a Bose gas close to a Feshbach resonance



Gross et. al., PRL 2009 (also Pollack et. al., Science 2010), also Innsbruck, Florence, Rice…)

Dimensionally: $L_3 \propto \frac{\hbar a^4}{m}$ Quasi-equilibrium for $\mu/\hbar >> L_3 n^2$ $\Rightarrow na^3 \ll 1$

Hard to enter to strongly correlated regime with free bosons!