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Statistical mechanics of point particles with a gravitational interaction

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Abstract. – We study the dynamics of N point particles with a gravitational interaction. The divergence of the microcanonical partition function prevents this system from reaching equilibrium. Assuming a random diffusion in phase space we deduce a scaling law involving time, which is numerically checked for 3 interacting masses in a quadratic nonsymmetrical potential. This random walk on the potential energy scale is studied in some detail and the results agree with the numerics.

The dynamics of N point masses with a gravitational interaction is of great interest in understanding many structures observed in astrophysics [1]. Some studies [2] restrict themselves to collisionless systems; this is a mean-field approach which is not based upon a systematic expansion in a small parameter, as in plasma physics, and its applicability to systems of point masses is hard to assess. Other studies [3, 4] assume that a temperature T can be defined for a gas of point masses, that is that it can reach thermal equilibrium. But the existence of such a statistical equilibrium requires to cut off the interaction at short distances: without such a cut-off, the partition function heavily diverges. Therefore one cannot define any finite invariant Lebesgue continuous measure of the phase space. Hence equilibrium statistical mechanics cannot be applied here as based upon the ergodic assumption: the system cannot fill uniformly all the accessible phase space in the course of time. Nevertheless, as we shall see, the study of this divergence can give some hints on the main features of the long-term dynamics. This divergence already occurs with $N = 3$ particles, suggesting that many features of the N particles dynamics are present for 3 particles in a confining potential. This divergence was mentioned by Padmanabhan [5], but he ignored the problem by introducing an arbitrary cut-off on the distances, therefore making the equilibrium problem well defined so.

We have considered a quadratic and nonsymmetrical confining potential $V = x^2 + 3.1y^2 + 2z^2$: with a symmetrical potential, because of the restriction of phase space due to conservation

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of angular momentum that makes the microcanonical partition function converge, the system would reach true equilibrium. In this case, by looking at the dynamics as a kind of diffusion in a phase space of infinite volume, one gets a scaling law for the long-time behavior of the potential energy of a pair: $\langle E_{\rm p}\rangle \propto t^{1/3}$. Numerical simulations confirm those predictions.

We propose to unify two points of view that are usually not related to each other: the "statistical mechanical" one, with all the machinery of Gibbs-Boltzmann ensemble, and the observations by Heggie [6] that collisions between close pairs ("hard binaries") and a third slow mass tend (on average) to bring the pair to a lower energy state. This letter presents itself as an attempt to understand a fundamental problem in statistical physics, not as a way of describing real astrophysical objects, a nontrivial task anyway. Nevertheless, it is certainly of interest to understand what happens exactly in a model system to build on solid ground statistical mechanics of large assemblies of interacting point masses.

Let us consider N particles of unit mass in a box, attracting each other with a $1/r^2$ central force in a d-dimensional universe. The gravitational constant G is set to 1. The energy being fixed by the initial conditions, a microcanonical approach should be used. The corresponding partition function is formally

$$Z = \int_{\rm box} \delta\left(E - \frac{1}{2}\sum p_i^2 + \sum_{i\neq j}\frac{1}{|\vec{r_i}-\vec{r_j}|}\right) {\rm d}\vec{r_i}{\rm d}\vec{p_i}. \tag{1}$$

The change of variables ${\rm d}\vec{p_i} = (S_{dN}/2)(p^2)^{(dN-2)/2} d(p^2)$, where S_n is the unit sphere surface in $\mathbb{R}^n$, yields

$$Z = \frac{S_{dN}}{2^{(4-dN)/2}}\int {\rm d}\vec{r_i}\left(E + \sum_{i\neq j}\frac{1}{|\vec{r_i}-\vec{r_j}|}\right)^{(dN-2)/2}. \tag{2}$$

Here E is assumed positive, a noncrucial point. The integrand diverges when two particles are too close to each other. Let us assume $|\vec{r_i}-\vec{r_j}| \geq \epsilon$. The integrand in (2) diverges as $\epsilon^{(2-dN)/2}$ when ϵ tends to zero, and the change of variables gives a term ϵ^{d-1}. Hence the partition function diverges if $N \geq 2 + 2/d$. For $d = 3$, Z diverges as soon as $N \geq 3$. If $d = 2$, the divergence is only marginal for $N = 3$: if the available phase space is reduced, by the conservation of angular momentum for instance, Z does not diverge any more. Hence the $d = 2$ case with $N = 3$ in a symmetric confining potential is a nontrivial convergent case: we will use it as a reference for an equilibrium situation. Bringing $2 + M$ particles close to each other can also cause a divergence, but it will obviously concern a smaller part of the phase space: after changes of variables, Z diverges like $\int {\rm d}\epsilon\epsilon^{d(M+1)-1+1-dN/2}$. Hence the divergence only occurs if N is bigger than $2 + 2/d + 2M$.

The dominating divergence thus comes from pairs. For the same reason, the divergence due to two different close pairs is also less strong than for one close pair only. Therefore, we can focus on the case with only one close pair: as this dominates the divergence of Z, it is more likely to occur. This makes the 3-body problem of great interest: it bears the same diverging microcanonical partition function as the few-body case, and is far more open to numerical investigations than the many-body case.

In order to examine the effect of the divergence on the long-term dynamics, we will compare what happens for $d = 2$, $N = 3$ in a confining $V = x^2 + y^2 + z^2$ potential, with $d = 3$, $N = 3$, and $V = x^2 + 3.1y^2 + 2z^2$. As long as V is nonsymmetrical, the confining potential changes nothing to the divergence of the partition function. It has been introduced because, as suggested by the preceding remark, the N-body problem can be seen in a loose sense as a collection of three-body problems. Hence the idea of examining what happens when

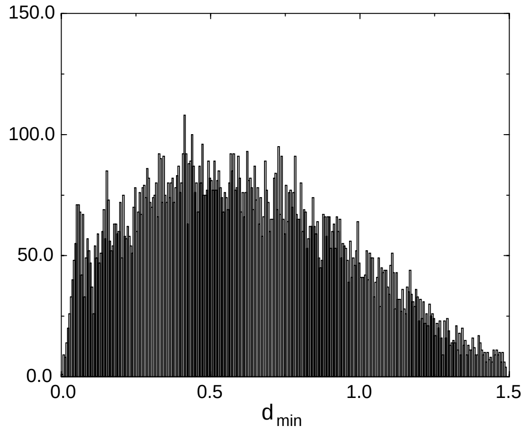

Fig. 1 – Histogram of the distance between the closest particles for $N = 3$ and $d = 2$: particles are most of the time far from each other.

perturbing two particles with a third one: if N is large, there will always be nearby a third particle to interfere. We first planned to use a box with reflecting boundary conditions, but non-soft potentials turned out to make simulations much harder. We will see that the quadratic potential changes the expected exponent, but since the divergence is still there, the basic conclusions are the same.

We used the Bulirsch-Stoer extrapolation method for solving differential equations [7]. When two particles are very close to each other, a change of variables concentrates the divergent part of the dynamics in a term of the form: $\ddot{u} = 1/u^2 +$ smaller terms. The quantity $u(t)$ is then written as $u_{\mathrm{ell}} + \delta u$, where u_{ell} is solution of $\ddot{u} = 1/u^2$. With this trick, neither forces nor accelerations get too important, allowing an efficient code, free of any arbitrary cut-off.

The above remarks led to the conclusion that the dynamics is governed by close pairs: the system spends most of the time in the region of phase space contributing the most to the divergence of the partition function. Hence one expects that in the divergent case a close pair will almost always be present in the system [8], whereas in the convergent case particles will mostly be far from each other. This is not *a priori* the same as the gravothermal catastrophe [9]: this divergence occurs whatever the size of the system, and there is no real core defined. The consequences for a big system are still not clear. In dense areas, pairs are certainly going to be formed (in agreement with Heggie's observations [8]) but it is not clear at all that the size of this dense area will tend to decrease. At least this is not an obvious consequence of our considerations. Histograms for the minimum pair distance of 3 particles (representing the time spent with particles at a given distance) are shown in figs. 1 and 2, for $d = 2$ and $d = 3$, respectively, in agreement with this predicted behavior. We have also done simulations with different forces (in $r^{-1.01}$ and $r^{-1.6}$) such that the partition function in 3D with 3 particles converges: there also the histogram of distances is not sharply peaked near $r = 0$.

We noticed that the system cannot explore uniformly the entire phase space. It will explore new parts of it as time goes by, corresponding to closer and closer pairs. Contrary to what happens at equilibrium, the system stays forever in a transitory regime: the probability distribution for the distance r between the closest particles will depend on time. It will never average itself: the average of r on some large time interval Δt at different times never reaches an asymptotic value. Let us now focus on those pairs. Once a pair exists, its energy will

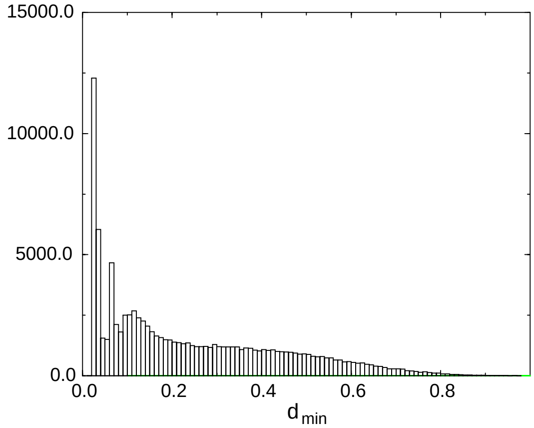

Fig. 2 – Histogram of the distance between closest particles for $N = 3$ and $d = 3$, with a non-symmetrical potential: there is most of the time a binary in the system.

be nearly constant, except during short-range collisions with the third particle. Therefore it makes sense to explore the evolution of this energy.

Let us consider a pair with an internal distance of order r. The potential energy of the pair is of order $1/r$. From the virial theorem applied to the Keplerian orbit, its kinetic and negative total energy are of order $1/r$ (this theorem does not apply here because the pair is interacting with the third particle, but it can be used to find orders of magnitude). To ensure conservation of the total energy, the kinetic energy of the center of mass of the pair has to be of order $1/r$ as well. This is contrary to the equipartition of energy, but equipartition has no reason to hold true in this system without well-defined equilibrium.

Now what will the time $t_{\rm col}$ be before our pair encounters the third particle? The pair scans a volume $v_G r^{d-1}{\rm d}t$ during ${\rm d}t$. Hence if n denotes the concentration of third particles, one gets:

$$t_{\rm col} \simeq (n v_G r^{d-1})^{-1}.$$

After such a close collision, there will be some energy transfer. Hence r will change; because of the infinite volume in phase space corresponding to r small, it will decrease on average. Heggie [6] has found that a pair whose potential energy is smaller (in absolute value) than the kinetic energy of the incoming third particle will most of the time not survive an encounter. On the contrary, hard pairs (whose potential energy is greater than the kinetic energy of the third mass) get harder on average at each collision: particles get closer. What we also claim here is that this also happens if the third particle is not picked out from an infinite set of slow, "cold" particles.

But the fact that the kinetic energy of the center of mass increases also increases the effective size of our system: with a harmonic potential, this size is of order v_G, thus n is of order $r^{d/2}$. Then, by assuming that the time scale of the system is of the same order as $t_{\rm col}$, one gets the scaling

$$r \simeq t^{2/(3-3d)}.$$

In the absence of a quadratic potential, for example in the case of a box of constant volume, there is still scale invariance when the energy of a pair is changed, but the exponent is not the same: one gets instead $r \simeq t^{2/(3-2d)}$.

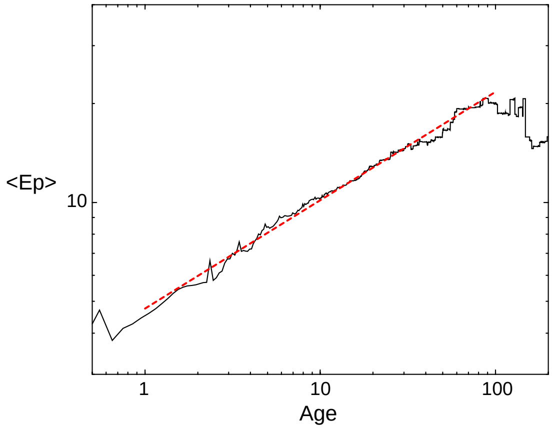

Fig. 3 – Mean energy of a pair as a function of its age for 3 particles in 3 dimensions. The dashed line corresponds to $x^{1/3}$.

We have numerically checked this law by looking at the average behaviour of the energy of a pair. For this we considered that a pair is born in the system when the pair energy of the two closest particles in the system becomes negative. The pair will then exist until this energy becomes positive. During its lifetime, this pair can encounter the third particle (we have defined such an encounter by the fact that the distance to the third particle is less than 1), and particles inside the pair can change. Thus we can define the age of a pair. We have represented in fig. 3 the mean energy of a pair as a function of its age. We find $\langle E_\mathrm{p} \rangle \simeq (\mathrm{age})^{1/3}$ on two decades in time, as expected from the preceding scaling law.

One thus gets the following picture of the dynamics: a pair appears, lives for some time during which its representation point walks at random in phase space until the energy of the pair becomes positive again (although after a collision a pair is usually closer than before, there will be with probability 1 a collision destroying the pair after a long time); another pair comes up almost immediately, and the process repeats itself, independent of what happened before. This picture does not contradict Heggie's findings: they are valid for the average collision only. We have tried to write a diffusion equation for this process. One problem is that because of the finite size of the system, there is a minimum energy for the pair, which is difficult to include in this framework.

Let us consider a set of different systems, and let us look for the distribution of the energy of pairs of this set. When a pair is destroyed in a system, another one will appear with an energy close to the minimum one, which would break the scale invariance. Let us thus forget this recreation process, and focus for the moment on the evolution of a pair until its destruction: we will consider systems containing initially one pair, and forget a system when its pair is destroyed.

The dynamical equation would look like

$$\frac{\partial p(E,t)}{\partial t} = \int (n_\mathrm{col}(E')F(E',E)p(E',t) - n_\mathrm{col}(E)F(E,E')p(E,t))\mathrm{d}E'. \qquad (3)$$

Here $p(E,t)$ denotes the pair energy distribution, n_col the collision rate and $F(E,E')$ the probability that a collision will make the energy of a pair jump from E to E'. Assuming scale invariance, F must only depend on the ratio E'/E. It is then natural to use as a new variable $x = \ln(E)$. The important feature of this diffusion equation is the behaviour of $n_\mathrm{col} \propto E^{-3}$.

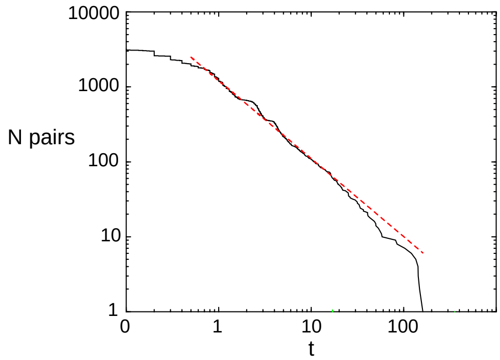

Fig. 4 – Number of pairs remaining at time t, for a big number of systems containing one pair at time $t = 0$. The dashed line corresponds to t^{-1}.

Assuming F peaked around 1, one replaces the right side of eq. (3) by a sum of diffusion, drift and loss terms:

$$\partial_t p \ = \ (\exp[-3x]p)_{xx} + \alpha(\exp[-3x]p)_x + \delta \exp[-3x]p(x).$$

Denoting $u(x,t) = \exp[-3x]p(x,t)$, this equation becomes

$$\partial_t u = \exp[-3x](u_{xx} + \alpha u_x + \delta u).$$

The case $\alpha = 0$, $\delta = 0$ (keeping only the diffusion term) gives a solution with a moving front: $u(x,t) \propto \exp[-e^{3x}/9t]/t$. The general case has a stationary solution $u(x) = \exp[kx]$ where k is solution of $k^2 + \alpha k + \delta = 0$. Hence one can look for solution of the form $u(x,t) = \exp[kx]v(3x - \ln(t))/t$. Finally, the general solution is $u(x,t) \propto \exp[kx + (2k+\alpha)(\ln t - 3x)/3 - e^{3x - \ln t}/9]/t$. p can thus be written: $p(x,t) = \tilde{p}(3x - \ln t)/t^{-k/3}$.

Hence we get a packet propaging to high energies, with a damping term which is a power of t. $t^{-k/3}$ is thus related to the loss of pairs when there is no recreation: one expects the number $n(t)$ of pairs of a given age to decrease as $t^{k/3}$. Let us now suppose that we can neglect the width of this packet (we consider that a pair of a given age has an energy age$^{1/3}$) and take into account the recreation of pairs after their destruction.

We will now show that the fact that the system cannot reach equilibrium can be seen as the fact that this exponent $k/3$ is greater than -1 (it has been numerically found to be very close to -1: see fig. 4; the difference at long time is mainly due to poor convergence due to the small number of such events): if it is not, the integral $\int \mathrm{d}x\, x^{k/3}$ converges near infinity and one can include a constant reinjection term c in the diffusion equation to balance the destruction term. Indeed, the distribution of the energy of the pairs at time t is the sum of all the packets which started at time t', less than t, damped by $(t-t')^{k/3}$. Since this integral converges when t tends to infinity, c can be chosen so that the number of pairs is conserved. One expects thus to find equilibrium in this situation.

This is actually the case when the bounding potential is symmetrical. For $V = x^2 + y^2 + 2z^2$, the angular momentum along z is conserved, and there is true microcanonical equilibrium. That the partition function converges in this case makes it unlikely to observe the law of growth of the energy of an aging pair, since pairs have a finite lifetime. Nevertheless, one may still get scaling laws, corresponding to the occurrence of rare events (one explores the tail of a distribution), and one numerically gets $n(t) \simeq t^{-5/4}$, and $E \simeq (\text{age})^{1/2}$. The probability for a pair to have an energy between E and $E + \mathrm{d}E$ is thus the probability that it has an age between E^2 and $E^2 + 2E\mathrm{d}E$; it has then been damped by a factor age$^{-5/4}$: hence it behaves like $E \times E^{-5/2} = E^{-3/2}$. The study of the microcanonical distribution shows that, if we call Z_n the number of states with a pair of energy between 2^n and 2^{n+1}, $\sqrt{2}Z_{n+1} = Z_n$. Hence the exponent β of the probability of having a pair of energy E satisfies $1/2+n+1+\beta n+\beta = n+\beta n$: hence $\beta = 3/2$, which validates the present model.

When $k/3$ is greater than -1, the reinjection term $c(t)$ defined as the number of pairs destroyed and recreated at low energy per unit time must tend to zero when $t \to +\infty$. Expressing the conservation of the total number of pairs in terms of $c(t)$ yields

$$\int_0^t \frac{c(t-t')}{(1+t')^{-k/3}}\mathrm{d}t' = 1.$$

Considering that $c(t-t') \approx c(t)$ for $t' < t/2$ and evaluating the integral from 0 to $t/2$, we find $c(t)t^{1+k/3} = O(1)$. If we put $c(t) \propto t^{-k/3-1}$ the other half of the integral is also of order one which validates this hypothesis. The distribution $p(t', t)$ of the pair ages t' at time t is then

$$p(t', t) \propto (1 + t - t')^{-k/3-1}(1 + t')^{k/3}.$$

Equivalently, the distribution of pair energies corresponds to make the change $t' = E^3$. This shows how older and older pairs (that is, more and more energetic pairs) are found in the system in the course of time. Note that the integral of $c(t)$ diverges, which means that the pairs are nonetheless destroyed and recreated endlessly.

In conclusion, the three-body system in a trap without angular momentum conservation never reaches equilibrium. This has important consequences on various features of the dynamics. The use of a cut-off in theoretical or numerical studies seems thus unjustified, and could lead to wrong conclusions. On the other hand, the concept of diffusion in phase space seems quite relevant, since the scaling law for the energy of a pair is well checked and the solution of the diffusion equation we write coincides with the microcanonical distribution in the equilibrium case.

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