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RICCI FLOWS AND INFINITE DIMENSIONAL ALGEBRAS

STRINGS 2004

RG (Ricci) flow

$$\frac{\partial G_{\mu\nu}}{\partial t} = -R_{\mu\nu} + \ldots$$

β-function eqs of 2-D σ-models

$G_{\mu\nu}(X)$: target space metric

$t = \log \Lambda^{-1}$ RG time : ws length

More generally allow reparametrizations along the flow

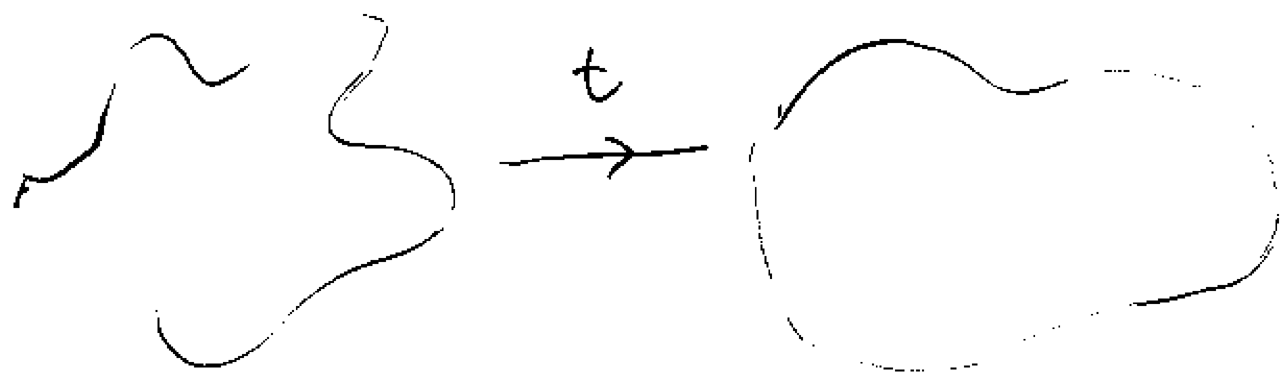
$$\frac{\partial G_{\mu\nu}}{\partial t} = -R_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

generated by vector field ξ_μ.

Also important tool in mathematics in dimensions 2, 3, ...

$$\frac{\partial}{\partial t}(\text{Volume}) = -\frac{1}{2}\int_M d^n X \sqrt{\det G}\, R[G]$$

Geometric deformations



as dynamical system in superspace

Non-linear generalization of heat flow eq

since for $G_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \longrightarrow$ weak

$R_{\mu\nu} \sim -\nabla^2 h_{\mu\nu}$

$\frac{\partial h_{\mu\nu}}{\partial t} = \nabla^2 h_{\mu\nu}$ dissipative properties

Tachyon condensation (gravity regime)

- present characteristic examples / features
- cast into zero-curvature form using novel ∞-dim algebra (2D)

Sausage model

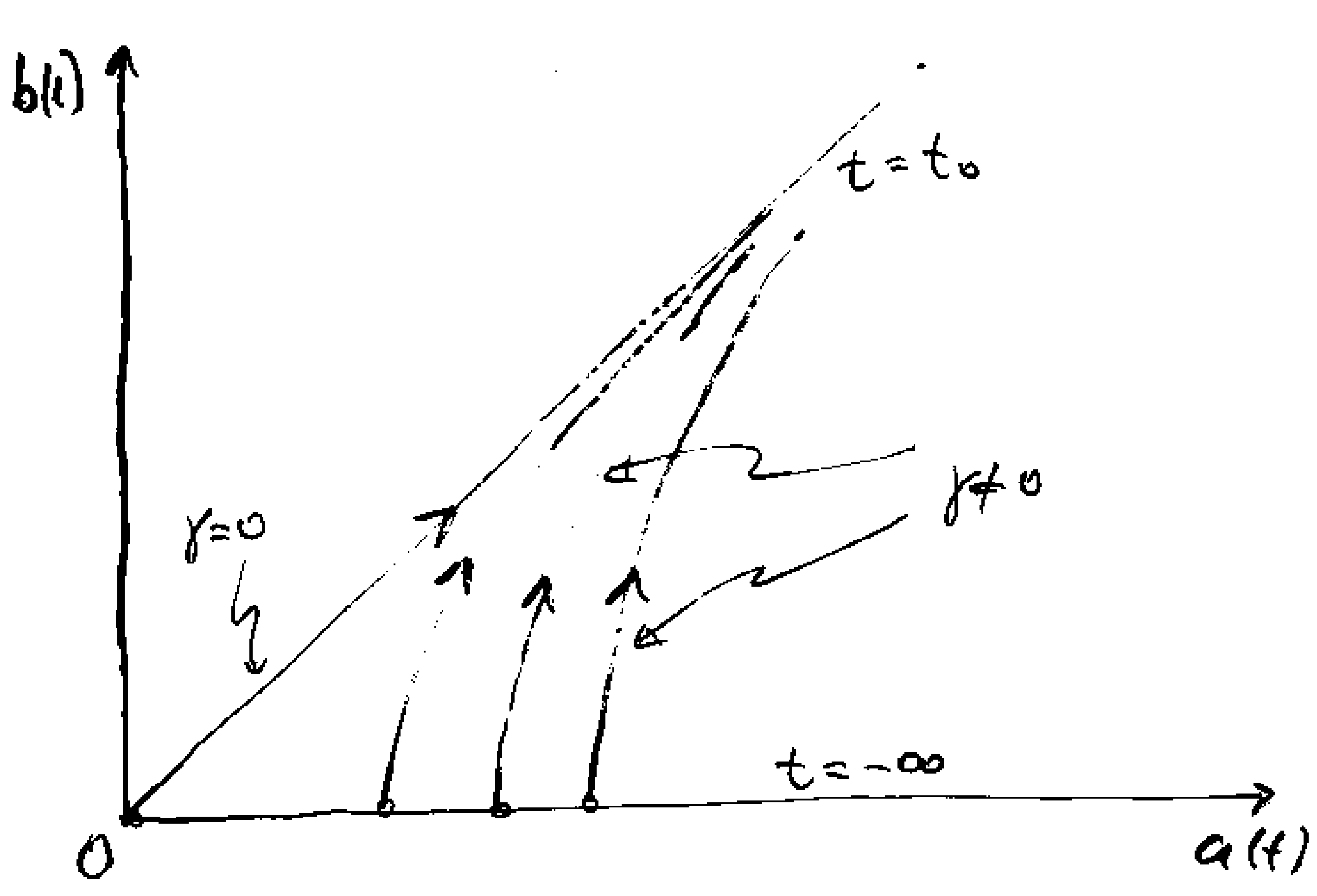
Axially symmetric deformations of S^2

$$S_t = \int e^{\Phi(Y;t)} \left((\partial_\mu X)^2 + (\partial_\mu Y)^2 \right)$$

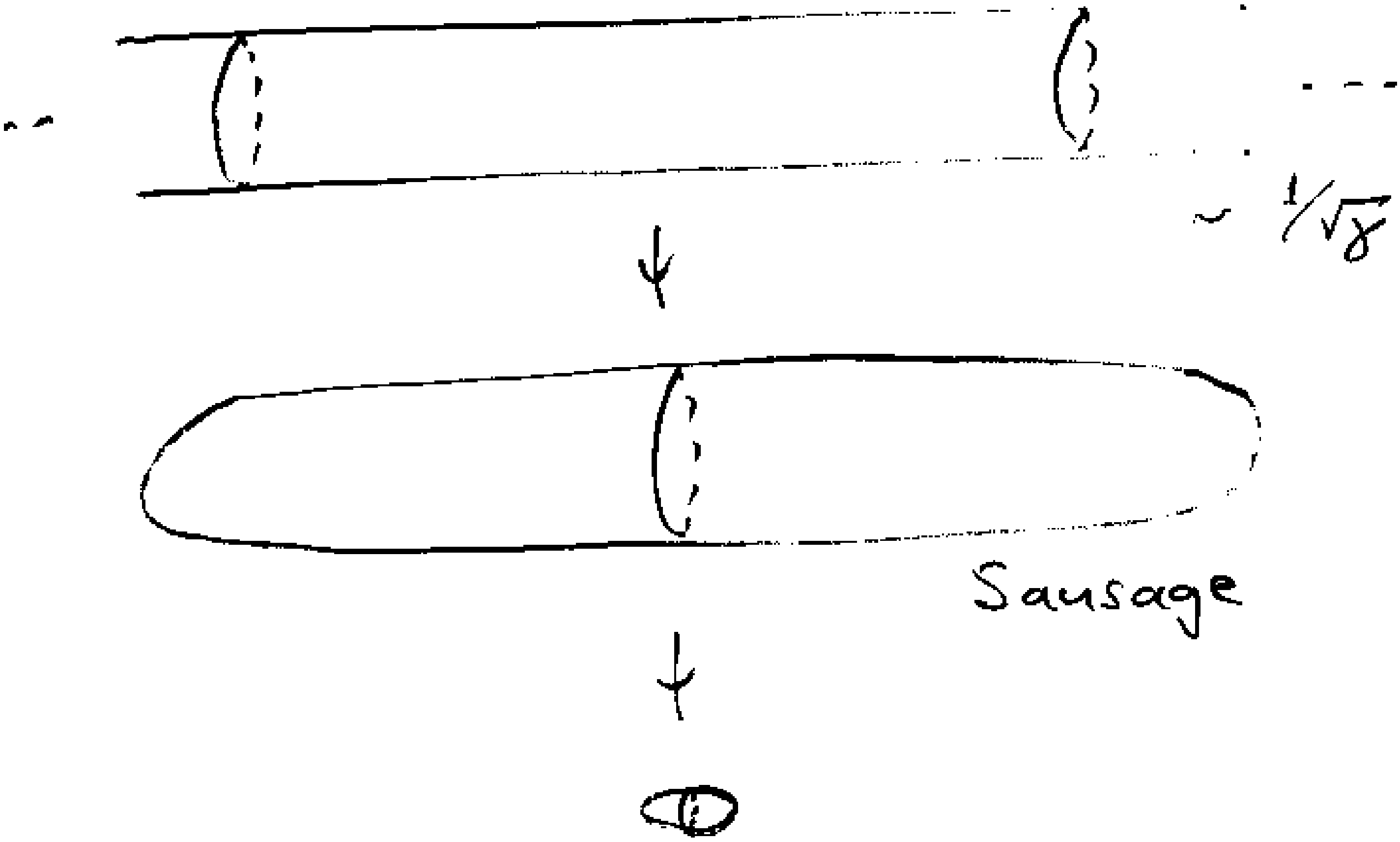
$$X : 0 \ldots 2\pi \,, \qquad Y : -\infty, \ldots +\infty$$

$$e^{\Phi(Y;t)} = \frac{2}{a(t) + b(t)\cosh 2Y}$$

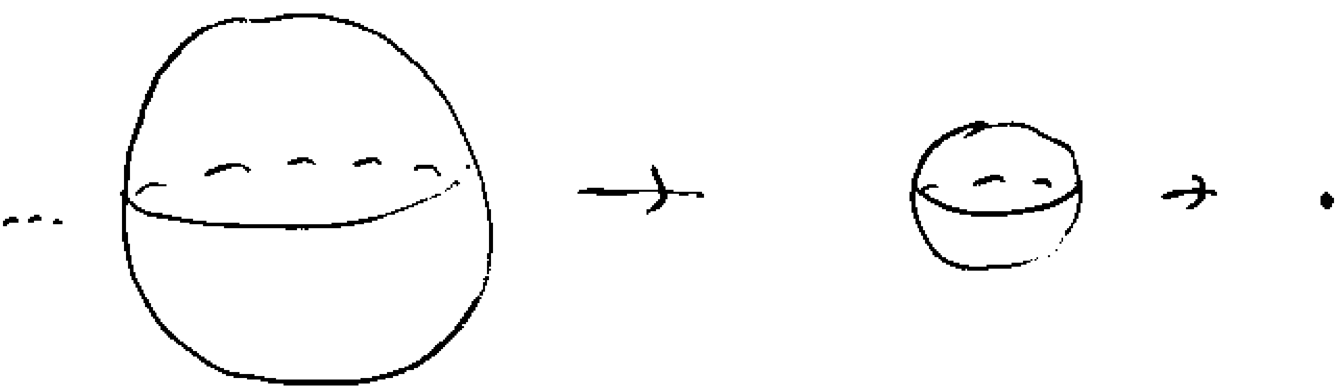
$$a(t) = \gamma \coth\left(2\gamma (t_0 - t)\right) \,, \quad b(t) = \frac{\gamma}{\sinh\left(2\gamma (t_0 - t)\right)}$$



Generic trajectory ($\gamma \neq 0$)



Special trajectory ($\gamma = 0$)



until it shrinks to zero size volume

$$V \sim t_0 - t$$

Change variables

$$\tanh Y = \mathrm{sn}(\tilde{Y};k) \;;\quad k = \tanh\left(\gamma(t_0 - t)\right)$$

with modulus $k : 1 \ldots\ldots 0$

as $t : -\infty \ldots\ldots t_0$

Metric

$$ds_t^2 = \frac{k}{\gamma}\left(d\tilde{Y}^2 + \mathrm{sn}^2\left(\tilde{Y} + K(k)\right) dX^2\right)$$

with $\tilde{Y} : -K(k) \ldots\ldots K(k)$ two tips

Also need to compensate with

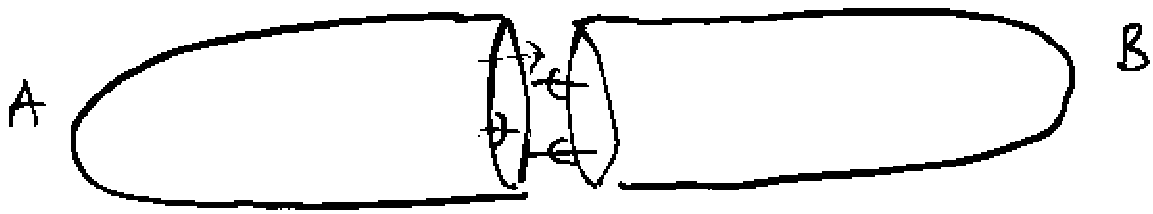
$\xi_\mu = \nabla_\mu \tilde{\Phi}$,

$$\tilde{\Phi}(\tilde{Y}) = \log \Theta\left(\tilde{Y} + K(k)\right) + \frac{1}{2}\left(\frac{E(k)}{K(k)} - \frac{1}{2}k'^2\left(1 + \frac{1}{\gamma}\right)\right)\tilde{Y}^2$$

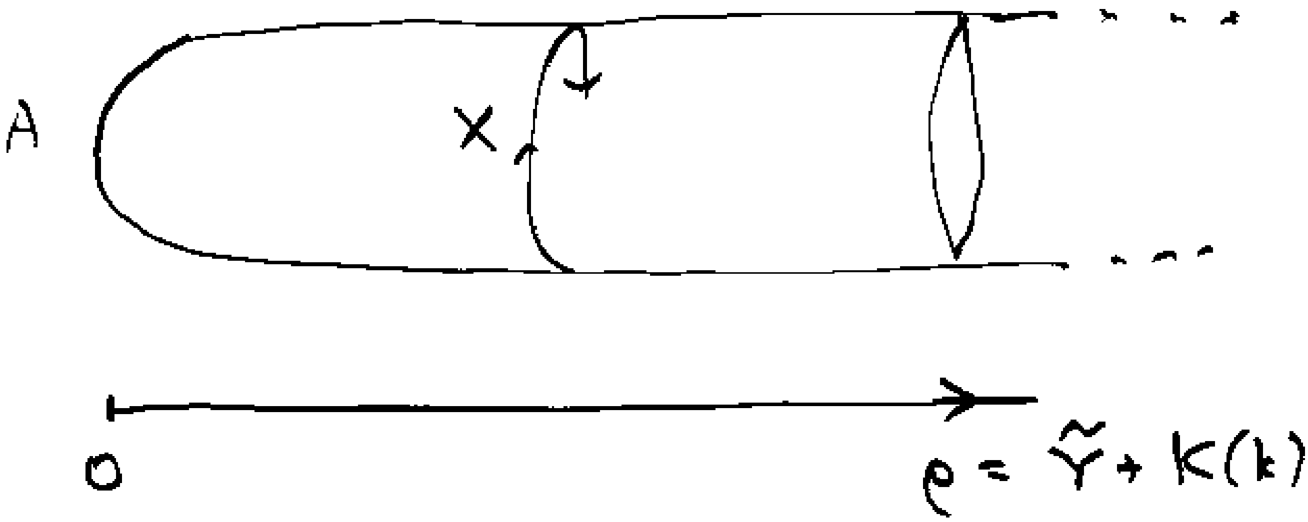
so that RG flow reads

$$\frac{\partial}{\partial t} G_{\mu\nu} = -R_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

Think sausage as



Around tip A



have $\quad sn(\rho ; k) \underset{k \to 1}{\simeq} \tanh \rho$

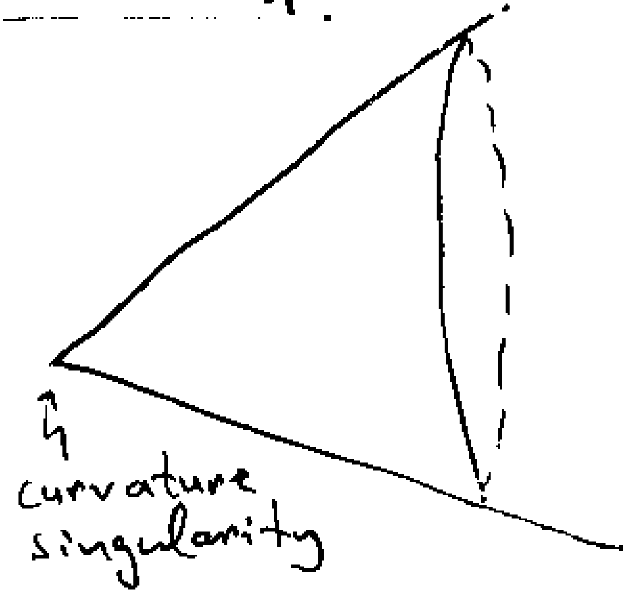
and $$ds^2 \simeq \frac{1}{\gamma}\left(d\rho^2 + \tanh^2\rho \, dX^2\right)$$

Two black-holes "eat" each other

Decay of cone C/Z_n.

$$ds^2 = dr^2 + r^2 d\phi^2$$

$$\phi : 0 \dots \frac{2\pi}{n}$$



RG flow

$$\frac{\partial}{\partial t} G_{\mu\nu} = -R_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

Consider

$$ds^2 = t\left(f^2(r)\,dr^2 + r^2 d\phi^2\right)$$

$$\xi_r = \frac{1}{2} r f(r) \quad ; \quad \xi_\phi = 0$$

Find

$$\left(\frac{1}{f(r)} - 1\right) \exp\left(\frac{1}{f(r)} - 1\right) = C e^{-\frac{1}{2}r^2}$$

$C \geq 0$ integration constant

Parametrize $G = (n-1)\exp(n-1)$

Absorb t-dependence

$$\rho := r\sqrt{t}$$

and keep ρ fixed.

$\underline{t \to 0 \ (\Leftrightarrow r \to \infty)}$: $\quad f \to 1$

$$ds^2 \simeq d\rho^2 + \rho^2 d\phi^2$$

$$\phi : 0 \ldots 2\pi$$

Cone of opening angle $2\pi/n$ $\quad (C/Z_n)$

$\underline{t \to \infty \ (\Leftrightarrow r \to 0)}$: $\quad f \to 1/n$

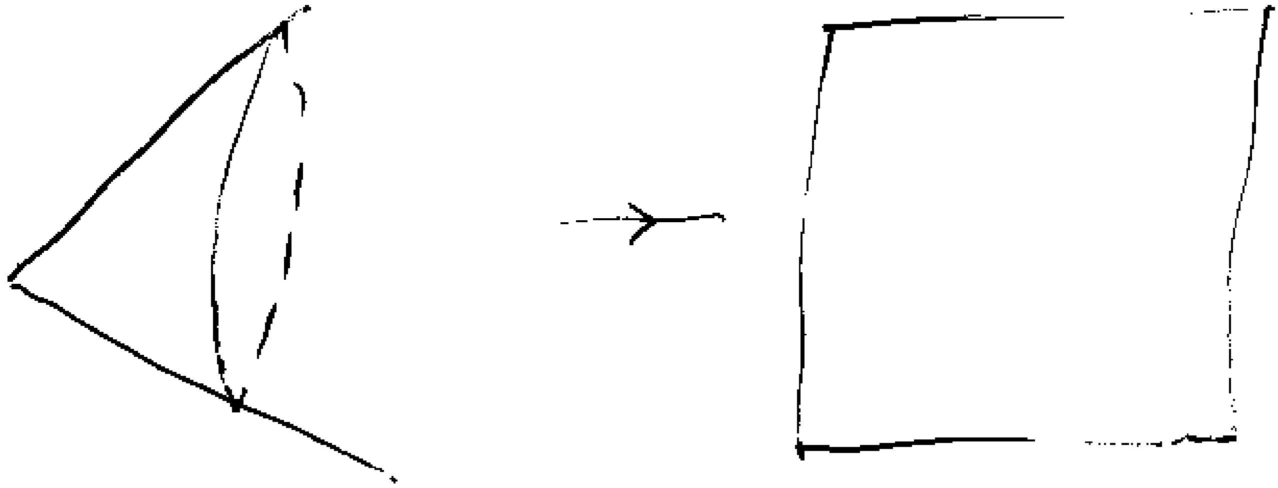
$$ds^2 \simeq \frac{1}{n^2} dr^2 + r^2 d\phi^2$$

$$= \frac{1}{n^2}\left(dr^2 + r^2 d\tilde{\phi}^2\right)$$

$$\tilde{\phi} : 0 \ldots 2\pi$$

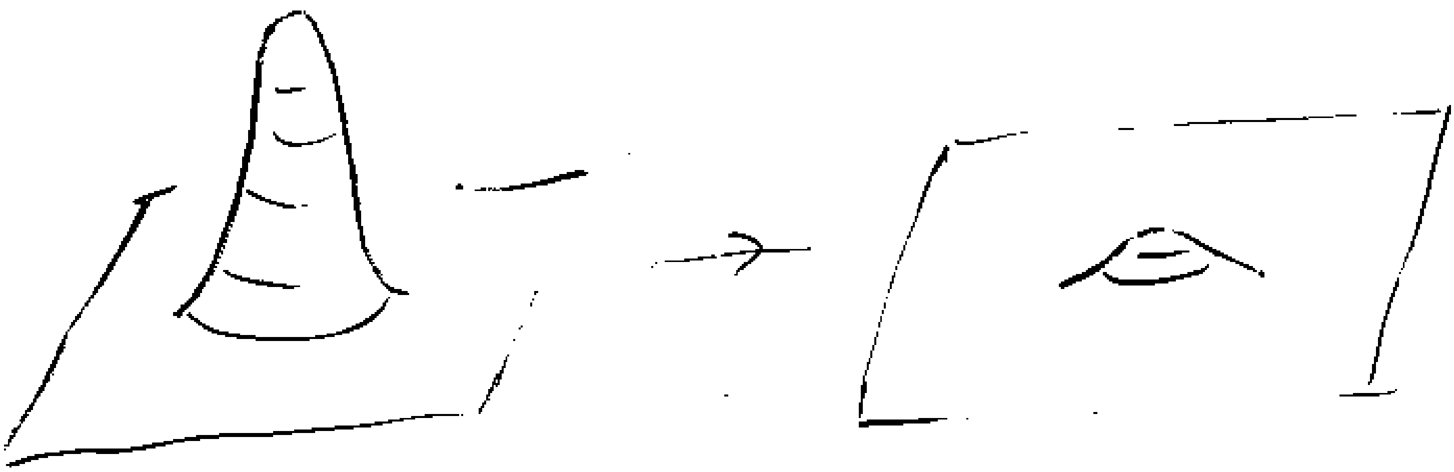
plane written in polar coordinates

Decay process



by dissipating curvature all over space

Analogous to heat equation



Gaussian spreading in time

$t=0$: delta function

- localized tachyon condensation

Ricci flows in 2-D

Write metric in conformally flat form

$$ds_t^2 = 2\, e^{\Phi(z,\bar z;t)}\, dz\, d\bar z$$

Since $R_{+-} = -\partial\bar\partial\Phi$ we have

$$\boxed{\partial\bar\partial\Phi(z,\bar z;t) = \frac{\partial}{\partial t} e^{\Phi(z,\bar z;t)}}$$

Viewed as Toda-like system

$$\partial\bar\partial\Phi(z,\bar z;t) = \int dt'\, K(t,t')\, e^{\Phi(z,\bar z;t')}$$

for $\boxed{K(t,t') = \frac{\partial}{\partial t}\delta(t-t')}$

c.f. $\partial\bar\partial\varphi_i(z,\bar z) = \sum_j K_{ij}\, e^{\varphi_j(z,\bar z)}$

$$\varphi_i(z,\bar z) \rightsquigarrow \Phi(z,\bar z;t)$$

$$K_{ij} \rightsquigarrow K(t,t')$$

continuous index, as in master field

Algebraic setting: $H(t)$, $X^\pm(t)$

$$[X^+(t), X^-(t')] = \delta(t-t')\, H(t')$$

$$[H(t), X^\pm(t')] = \pm K(t,t')\, X^\pm(t')$$

$$[H(t), H(t')] = 0$$

with Cartan kernel $K(t,t')$.

Other generators obtained by successive $[\,,\,]$

<u>Associated Toda system</u>

$$\partial\bar\partial\, \Phi(z,\bar z;t) = \int dt'\, K(t,t')\, e^{\Phi(z,\bar z;t')}$$

is cast into zero curvature form

$$[\partial + A(z,\bar z),\ \bar\partial + B(z,\bar z)] = 0$$

with
$$\boxed{\begin{aligned} A(z,\bar z) &= H(\Psi) + X^+(1) \\ B(z,\bar z) &= X^-(e^\Phi) \end{aligned}}$$

choose $\bar\partial \Psi = e^\Phi$, $\partial e^\Phi = K(\Psi, e^\Phi)$

smear $A(f) = \int dt\, f(t)\, A(t)$

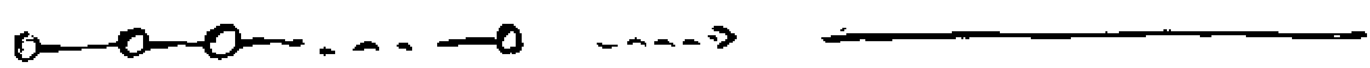
Examples

- $K(t,t') = \delta(t-t')$

  $SL(2)$ - current algebra

- $K(t,t') = -\partial_t^2\,\delta(t-t')$

  arises as (continuous) large N limit



A_n $\qquad\qquad\qquad\qquad$ A_∞

$$K_{ij} = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix} \dashrightarrow K(t,t') = -\partial_t^2\,\delta(t-t')$$

- $K(t,t') = \partial_t\,\delta(t-t')$

  Ricci flow algebra

Kac-Moody algebra with anti-symmetric Cartan kernel

General solution

Introduce (formally) highest weight state

$$X^+(t')|t\rangle = 0, \quad H(t')|t\rangle = \delta(t-t')|t\rangle$$

with $\langle t|t\rangle = 1$. Then,

$$\boxed{\Phi(z,\bar{z};t) = \Phi_0(z,\bar{z};t) - \int dt' K(t',t) \log \langle t'|M_+^{-1} M_-|t\rangle}$$

where

$$\Phi_0(z,\bar{z};t) = f^+(z;t) + f^-(\bar{z};t)$$

is 1-parameter family of 2-D free fields and

$$M_\pm(z_\pm;t) = \underline{P}\exp\left[\int^{z_\pm} dz'_\pm \int^t dt' e^{f^\pm(z'_\pm;t')} X^\pm(t')\right]$$

$z_+ = z, \quad z_- = \bar{z}$

Formal power series expansion in terms of

$$\langle t|X^+(t_1)\ldots X^+(t_m)X^-(t'_m)\ldots X^-(t'_1)|t\rangle$$

for all $m = 1, 2, \ldots \infty$

Exponential growth.

$$\cdots \oplus G_{-2} \oplus \underbrace{G_{-1} \oplus G_0 \oplus G_{+1}}_{\text{local part}} \oplus G_{+2} \oplus \cdots$$

$G_{\pm 1} : X^{\pm}$; $G_0 : H$

$G_{\pm 2} = [G_{\pm 1}, G_{\pm 1}]$

$G_{\pm 3} = [G_{\pm 1}, G_{\pm 2}]$

etc ....

For RG flow algebra have exponential growth since for $n \geqslant 2$

if $G_{\pm n}$ spanned by $X^{(1)}_{\pm n}, \ldots, X^{(d_n)}_{\pm n}$

then $G_{\pm(n+1)}$ spanned by $2d_n$ elements

- $X^{(s)}_{\pm(n+1)}(\varphi) = [X_{\pm 1}(1), X^{(s)}_{\pm n}(\varphi)] - [X_{\pm 1}(\varphi), X^{(s)}_{\pm n}(1)]$ for $1 \leq s \leq d_n$
- $X^{(s)}_{\pm(n+1)}(\varphi) = [X_{\pm 1}(1), X^{(s)}_{\pm n}(\varphi)] + [X_{\pm 1}(\varphi), X^{(s)}_{\pm n}(1)]$ for $d_n + 1 \leq s \leq 2d_n$

OUTLOOK

Integrability of Ricci flows
in 2-dim target spaces

- general solution
- new ∞-dim algebra

Need to explore more:

- algebraic structure
  (exponential growth etc..)
- application of algebraic
  techniques for more general
  dynamical problems
  (eg real time evolution ...)