Higher-Loop Integrability in $\mathcal{N} = 4$ Gauge Theory

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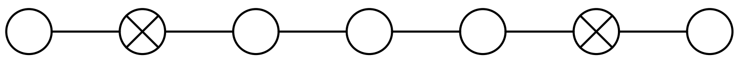


based on work with V. Dippel, C. Kristjansen, M. Staudacher
hep-th/0303060, 0307015, 0307042, 0310252, 0405001.

Large Spin Limits of AdS/CFT

AdS/CFT correspondence predicts agreement of spectra of

- Energies E in IIB string theory on $AdS_5 \times S^5$ and
- Scaling dimensions D in $\mathcal{N} = 4$ gauge theory.

Tests prevented by strong/weak nature of the duality.

Proposal: Consider states with large spin J on S^5

- BMN limit; non-planar and near $\mathcal{O}(1/J)$ extensions. [Berenstein Maldacena Nastase]
- Semiclassical Spinning Strings. [Frolov Tseytlin]

Effective coupling constant

$$\lambda' = \frac{\lambda}{J^2}.$$

- String theory: Expansion in λ' and $1/J \sim 1/\sqrt{\lambda}$,
- Gauge theory: ℓ-loop contribution suppressed by (at least) $1/J^{2\ell}$.

Expansion in λ' apparently equivalent to expansion in λ. Compare!

Three-Loop Discrepancies

BMN state with 2 excitations

$$\mathcal{O}_n \approx \sum_{p=0}^{J} \exp \frac{2\pi i n p}{J} \operatorname{Tr} \mathcal{Z}^p \phi \mathcal{Z}^{J-p} \phi \qquad D - J \approx 2\sqrt{1 + \frac{\lambda n^2}{J^2}}$$

Gauge theory dimension in near BMN limit $\mathcal{O}(1/J)$ [NB Kristjansen Staudacher]

$$D - J = 2 + \frac{\lambda n^2}{J^2}\left(1 - \frac{2}{J}\right) - \frac{\lambda^2 n^4}{J^4}\left(\frac{1}{4} + \frac{0}{J}\right) + \frac{\lambda^3 n^6}{J^6}\left(\frac{1}{8} + \frac{1}{2J}\right) + \ldots$$

Energy of near plane-wave string [Callan, Lee, McLoughlin Schwarz, Swanson, Wu]

$$E - J = 2 + \lambda' n^2\left(1 - \frac{2}{J}\right) - \lambda'^2 n^4\left(\frac{1}{4} + \frac{0}{J}\right) + \lambda'^3 n^6\left(\frac{1}{8} + \frac{0}{J}\right) + \ldots$$

Three-loop mismatch also for 3 excitations. [Callan McLoughlin Swanson]

Similar disagreement for spinning strings. [Serban Staudacher]

Outline

- How to obtain these results in gauge theory?
- What is higher-loop integrability?
- How to make use of it?
- What about the discrepancy?

Focus On

- Scaling dimensions D in $\mathrm{U}(N)$ $\mathcal{N}=4$ SYM

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle \sim \frac{1}{|x-y|^{2D}}.$$

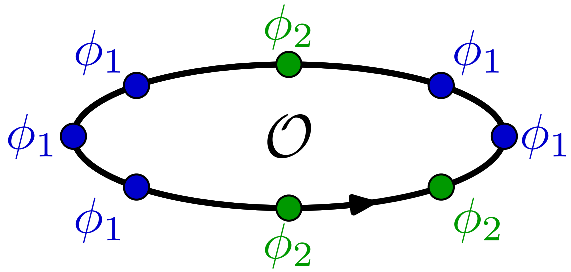
- Strict planar limit $N=\infty$.

Single Trace Operators and Spin Chains

Single trace operator, two complex scalars ϕ_1, ϕ_2 (a.k.a. $\mathcal{Z}, \phi$ or Z, X)

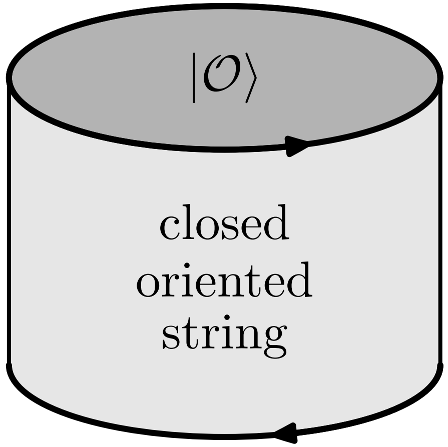
$$\mathcal{O} = \operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2$$

Length L: # of fields

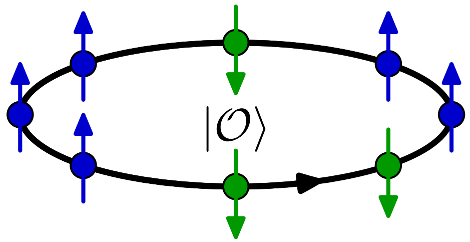


Identify $\phi_1 = |\uparrow\rangle$, $\phi_2 = |\downarrow\rangle$

$$|\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$



Length L: # of sites

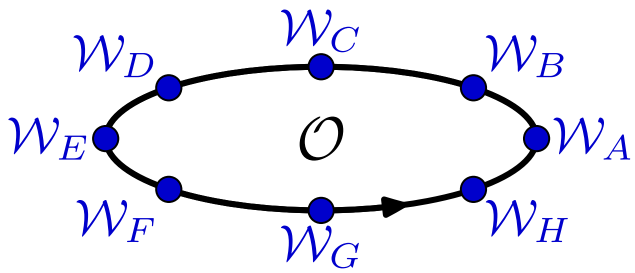


Operator mixing, quantum superposition: $|\mathcal{O}\rangle = *|\ldots\rangle + *|\ldots\rangle + \ldots$

Full $\mathcal{N} = 4$ SYM and Subsectors

Generic single trace state

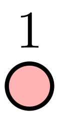
$$\mathcal{O} = \operatorname{Tr} \mathcal{W}_A \mathcal{W}_B \mathcal{W}_C \mathcal{W}_D \mathcal{W}_E \mathcal{W}_F \mathcal{W}_G \mathcal{W}_H$$



$\mathfrak{su}(2)$ Subsector

$$\mathcal{W} \in \{\phi_1, \phi_2\}$$

(fundamental $\mathbf{2}$)

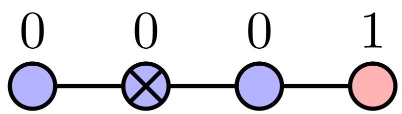


$\mathfrak{su}(2)$

$\mathfrak{su}(2|3)$ Subsector

$$\mathcal{W} \in \{\phi_{1,2,3}, \psi_{1,2}\}$$

(fundamental $\mathbf{3}|\mathbf{2}$)

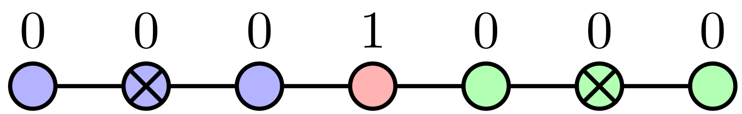


$\mathfrak{su}(2|3)$

Full $\mathcal{N} = 4$ SYM

$$\mathcal{W} \in \{\mathcal{D}^k \Phi, \mathcal{D}^k \Psi, \mathcal{D}^k \mathcal{F}\}$$

(non-compact rep.)



$\mathfrak{psu}(2,2|4)$

and many more . . .

[NB, PhD to appear]

Dilatation Generator

Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathfrak{D}(g)$

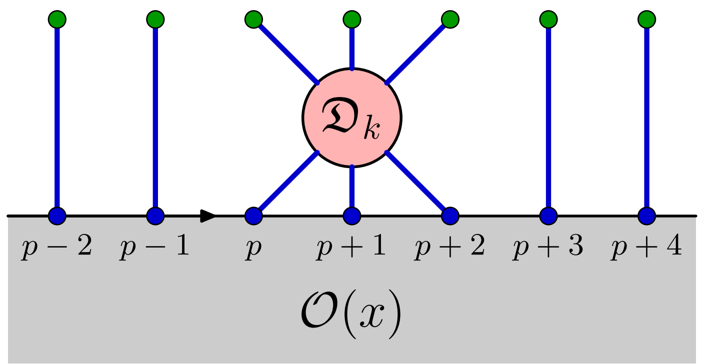
$$\mathfrak{D}(g)\,\mathcal{O} = D_{\mathcal{O}}(g)\,\mathcal{O}.$$

Quantum corrections in perturbation theory: $g \sim \sqrt{\lambda}$

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2\mathfrak{D}_2 + g^3\mathfrak{D}_3 + g^4\mathfrak{D}_4 + \ldots.$$

Local action along spin chain (homogeneous)

$$\mathfrak{D}_k = \sum_{p=1}^{L}$$



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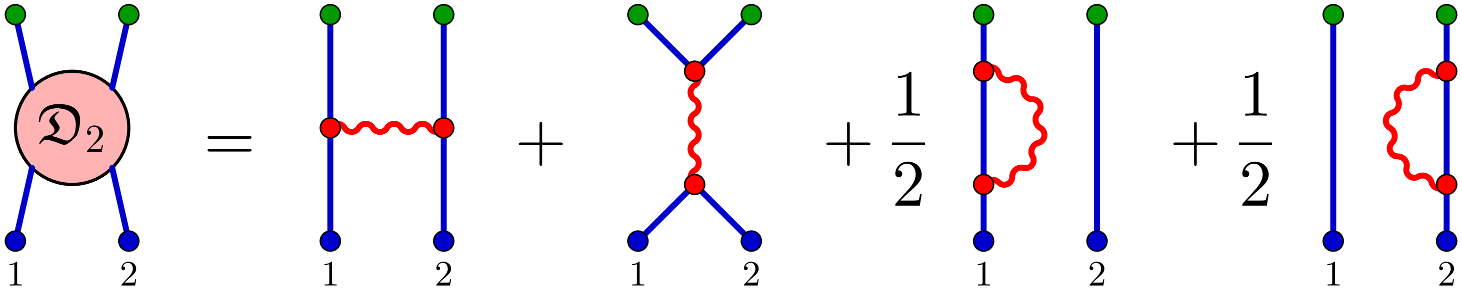
One-Loop

One-loop $\mathcal{O}(g^2)$ dilatation operator $\mathfrak{D}_2$:

$\mathfrak{D}_{2(12)} =$



Extract logarithmic piece of Feynman diagrams.

$\mathfrak{su}(2)$ Subsector

$$\mathfrak{D}_{2(12)} = 1 - P_{(12)}.$$

[Minahan Zarembo]

$1 = I_{(12)}$: identity

$P_{(12)}$: permutation

Heisenberg chain

$\mathfrak{su}(2|3)$ Subsector

$$\mathfrak{D}_{2(12)} = 1 - SP_{(12)}.$$

[NB hep-th/0310252]

SP_{12}: graded perm.

Full $\mathcal{N} = 4$ SYM

$$\mathfrak{D}_{2(12)} = 2h(J_{(12)}).$$

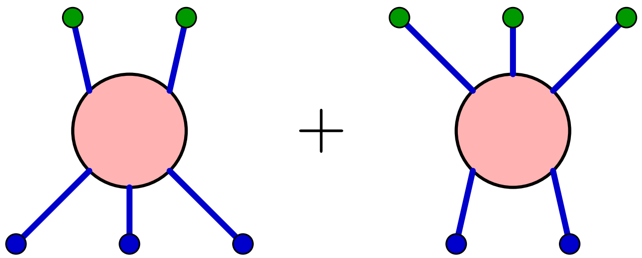
[NB hep-th/0307015]

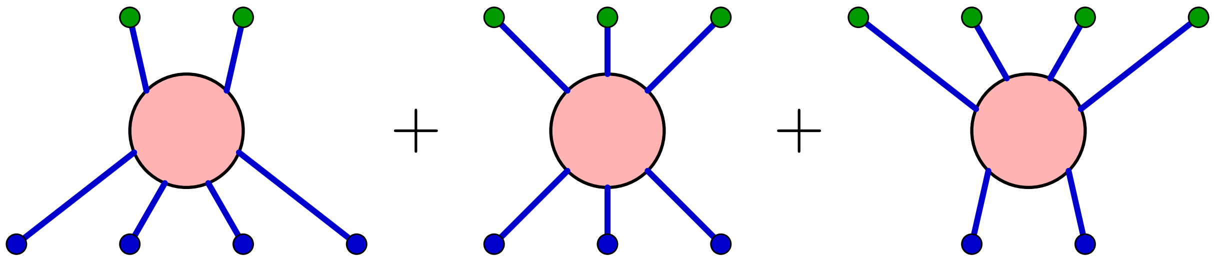
$J_{(12)}$: "total spin" op.

harmonic n. $h(s) = \sum_{k=1}^{s} \frac{1}{k}$

Higher-Loops

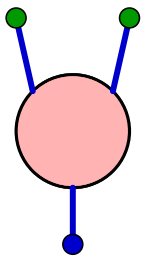
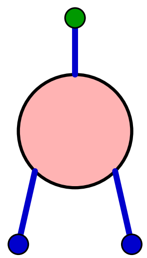
Contribution to dilatation generator at $\mathcal{O}(g^k)$ has (up to) $k+2$ legs

$\mathfrak{D}_3 =$ 

$\mathfrak{D}_4 =$  $+ \ldots$

At higher-loops: Length L fluctuates, dynamic spin chain. [NB hep-th/0310252]

Also (super)momenta $\mathfrak{Q}, \mathfrak{P}$ & (super)boosts $\mathfrak{S}, \mathfrak{K}$ are corrected, e.g.

$\mathfrak{Q}_1, \mathfrak{P}_1 =$  $\mathfrak{S}_1, \mathfrak{K}_1 =$ 

Algebraic Construction

Direct computation of higher-loops is extremely laborious.

Proposal: Try to reconstruct $\mathfrak{D}(g)$ from known properties.

(Example: $\mathfrak{D}_2$ for $\mathfrak{su}(2)$ sector)

- Consider all possible structures (# of legs) — ($I_{(12)}$ and $P_{(12)}$)
- Assume most general form — ($\mathfrak{D}_{2(12)} = \alpha I_{(12)} + \beta P_{(12)}$)
- Demand closure of symmetry algebra — (no constraint here)

$$[\mathfrak{D}(g), \mathfrak{Q}(g)] = +\tfrac{1}{2}\mathfrak{Q}(g), \quad [\mathfrak{D}(g), \mathfrak{S}(g)] = -\tfrac{1}{2}\mathfrak{S}(g), \quad \ldots$$

To fix remaining coefficients, may use

- BMN scaling behavior — ($\beta = -\alpha$)
- Known results — (from Konishi: $\alpha = 1$)
- Integrability (later . . .) — (no constraint here)

(find $\mathfrak{D}_{2(12)} = I_{(12)} - P_{(12)}$)

Algebraic Construction: Results

Full $\mathcal{N}=4$ SYM: at one-loop

- Fixed by algebra up to one overall constant (g). [NB, PhD to appear]

$\mathfrak{su}(2|3)$ **Subsector**: at three-loops (using BMN scaling) [NB hep-th/0310252]

- Dimension of Konishi

$$D_{\mathcal{K}} = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \ldots$$

  Confirmed by explicit computation. [Moch Vermaseren Vogt][Kotikov, Lipatov Onishchenko Velizhanin]
- Agrees with BMN matrix model. [Klose Plefka]
- Yields near-BMN result from beginning of talk.

$\mathfrak{su}(2)$ **Subsector**: at five-loops (using BMN scaling & integrability)

- Reproduces BMN energy formula $D-J=\sum_k\sqrt{1+\lambda' n_k^2}$. [NB, Dippel Staudacher]

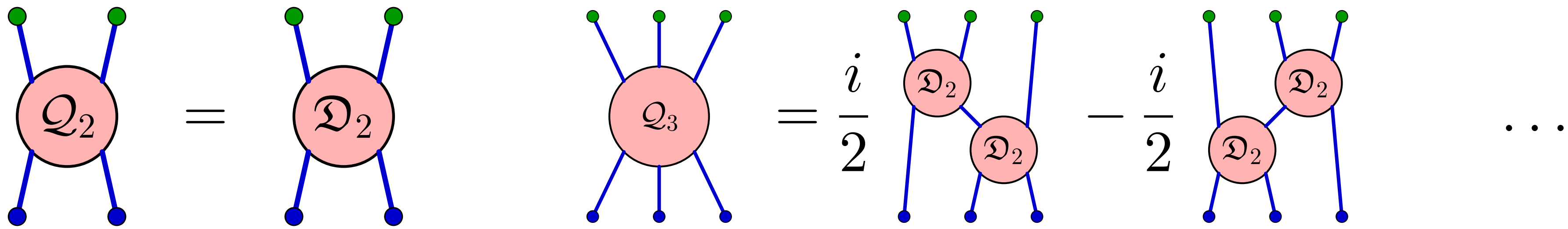
One-Loop Integrability

Only in planar limit!

Existence of higher charges $\mathcal{Q}_{2,3,4,\ldots}$ (scalar, commuting),

$$[\mathfrak{J}_0, \mathcal{Q}_r] = [\mathcal{Q}_r, \mathcal{Q}_s] = 0, \qquad \mathfrak{D}_2 = \mathcal{Q}_2.$$

Structure of charges



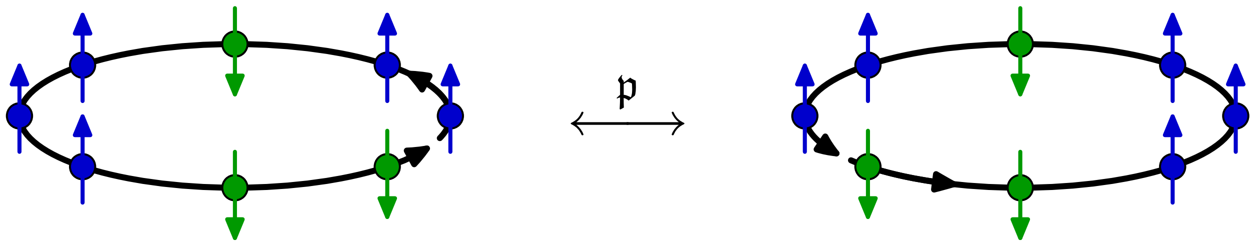
One-loop integrability found for

- $\mathfrak{so}(6)$ **Subsector** of scalars Φ_m [Minahan Zarembo]
- **Complete** $\mathcal{N} = 4$ **SYM:** $\mathfrak{su}(2,2|4)$ super spin chain. [NB Staudacher]
- Some subsectors of large N_c QCD.

Test for Integrability

Consider parity $\mathfrak{p}$ (charge conjugation/spin chain/world sheet):

$$\operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2 \overset{\mathfrak{p}}{\longleftrightarrow} \operatorname{Tr} \phi_2\phi_2\phi_1\phi_1\phi_1\phi_2\phi_1\phi_1$$



Even/odd charges have even/odd parity

$$\mathfrak{p}\,\mathcal{Q}_r\mathfrak{p}^{-1} = (-1)^r\,\mathcal{Q}_r.$$

Implies degenerate pairs of opposite parity: (only in planar limit!)

$$D_+ = D_-.$$

Test for integrability.

[Grabowski Mathieu]

Higher-Loop Integrability

No formalism yet (R-matrix, Yang-Baxter equation, ...).

Existence of higher charges $\mathcal{Q}_r(g)$, [NB Kristjansen Staudacher]

$$[\mathfrak{J}(g), \mathcal{Q}_r(g)] = [\mathcal{Q}_r(g), \mathcal{Q}_s(g)] = 0, \qquad \mathfrak{D}(g) = \mathfrak{D}_0 + g^2 \mathcal{Q}_2(g).$$

Test: Planar parity pairs preserved at higher-loops

$$D_+(g) = D_-(g).$$

Higher-loop integrability for

- $\mathfrak{so}(6)$ **Subsector**: Sector not closed at higher-loops due to mixing.
- $\mathfrak{su}(2|3)$ **Subsector**: Observed at three-loops (pairs). [NB hep-th/0310252]
  Even through length fluctuates.
- $\mathfrak{su}(2)$ **Subsector**: Construct five-loops dilop. via integrability. [NB, Dippel Staudacher]

Long-Range Bethe Ansatz

$\mathfrak{su}(2)$ subsector isomorphic to Inozemtsev chain up to three-loops. [Serban Staudacher]
Generalization of Bethe equations to all-loops (asymptotically). [NB, Dippel Staudacher]
Algebraic equations for the Bethe roots u_k:

$$\frac{x(u_k - \frac{i}{2})^L}{x(u_k + \frac{i}{2})^L} = \prod_{j=1}^{K} \frac{u_k - u_j - i}{u_k - u_j + i}, \qquad x(u) = \frac{u}{2} + \frac{u}{2}\sqrt{1 - \frac{2g^2}{u^2}}$$

with charge eigenvalues $\mathcal{Q}_r$ and resulting dimension D

$$\mathcal{Q}_r = \sum_{k=1}^{K} \frac{i}{r-1}\left(\frac{1}{x(u_k - \frac{i}{2})^{r-1}} - \frac{1}{x(u_k + \frac{i}{2})^{r-1}}\right), \qquad D = L + g^2 \mathcal{Q}_2.$$

- Asymptotic: Realiable only when loop order ℓ less than length L.
- Spectrum ($L \leq 10$) agrees with five-loop spin chain model.

Spinning Strings at Higher-Loops

Thermodynamic limit $L, K \to \infty$, effective coupling $\tilde{g} = g/L$.
Roots $\tilde{x}_k = x(u_k)/L$ condense on cuts $\mathcal{C}$ with density $\rho(\tilde{x})$

$$\frac{1}{1 - \frac{\tilde{g}^2}{2\tilde{x}^2}} \frac{1}{\tilde{x}} + 2\pi n_{\tilde{x}} = \int_{\mathcal{C}} \frac{d\tilde{x}' \rho(\tilde{x}')}{1 - \frac{\tilde{g}^2}{2\tilde{x}'^2}} \left(\frac{2}{\tilde{x} - \tilde{x}'} - \frac{\tilde{g}^2}{\tilde{x}\tilde{x}'^2} \frac{1}{1 - \frac{\tilde{g}^2}{2\tilde{x}\tilde{x}'}} \right).$$

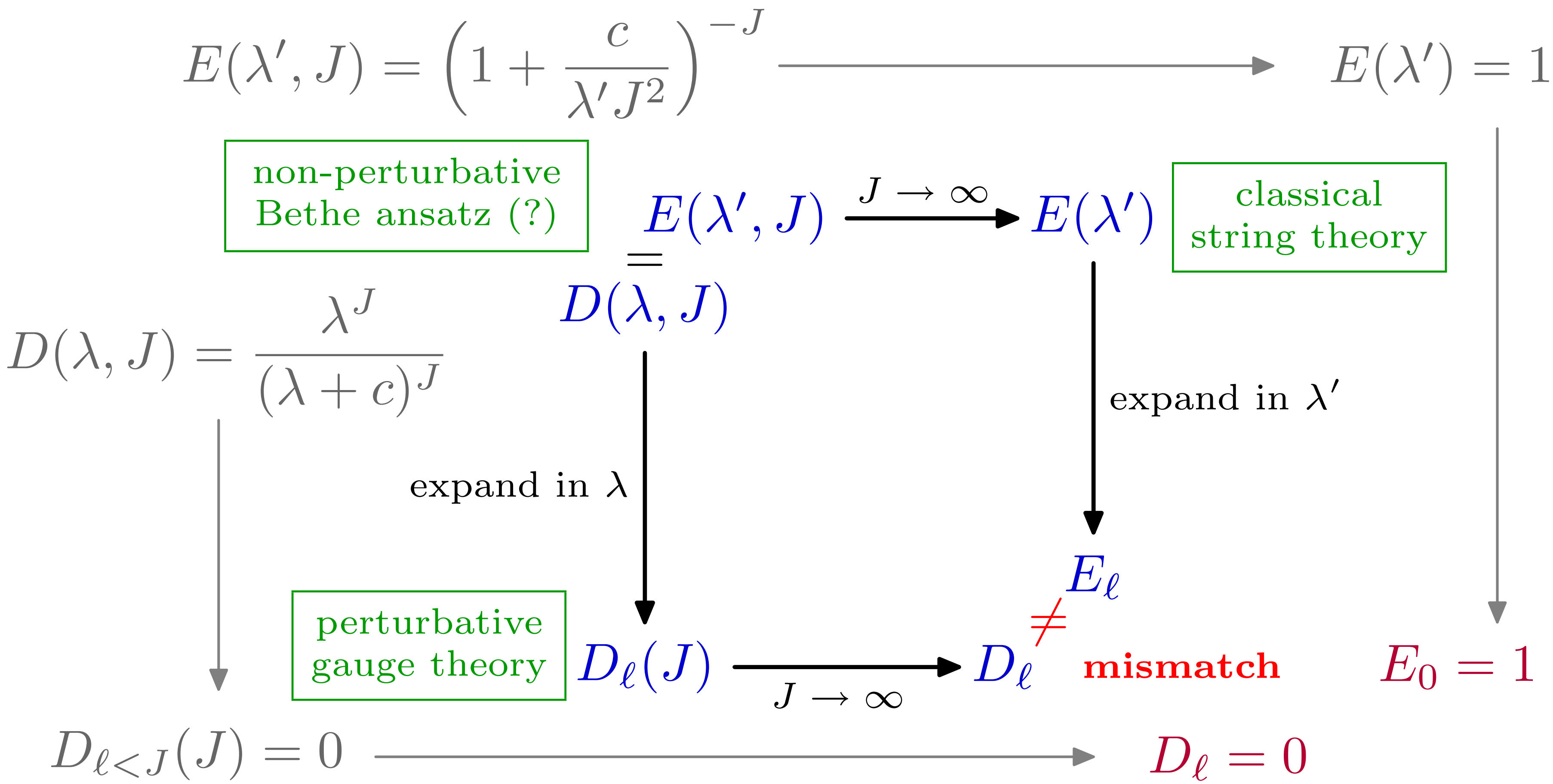
$$\mathcal{Q}_r = \frac{1}{L^{r-1}} \int_{\mathcal{C}} \frac{d\tilde{x}\, \rho(\tilde{x})}{1 - \frac{\tilde{g}^2}{2\tilde{x}^2}} \frac{1}{\tilde{x}^r}, \qquad D = L + g^2 \mathcal{Q}_2.$$

Bethe equations for classical string theory (charges as above) [Kazakov, Marshakov Minahan, Zarembo]

$$\frac{1}{1 - \frac{\tilde{g}^2}{2\tilde{x}^2}} \frac{1}{\tilde{x}} + 2\pi n_{\tilde{x}} = \int_{\mathcal{C}} \frac{d\tilde{x}' \rho(\tilde{x}')}{1 - \frac{\tilde{g}^2}{2\tilde{x}'^2}} \left(\frac{2}{\tilde{x} - \tilde{x}'} - \frac{\tilde{g}^2}{\tilde{x}\tilde{x}'^2} \frac{1}{1 - \frac{\tilde{g}^2}{2\tilde{x}\tilde{x}}} \right).$$

Two-loop agreement manifest. Disagreement starts at three-loops.
Guess quantum strong coupling Bethe Ansatz: near-BMN, $\sqrt[4]{\lambda}$. [Arutyunov Frolov Staudacher]

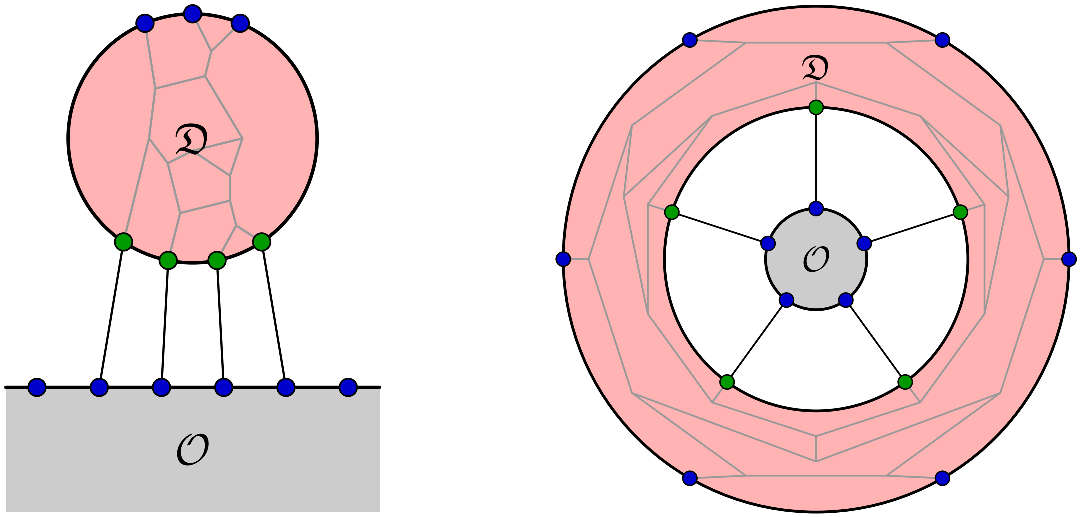
Order of Limits



- Cannot compare in perturbation theory.
- Spinning strings and near-BMN proposals do not quite work.
- Integrability might lead to the exact solution of either theory.

Wrapping Interactions

At higher loop orders there is an additional type of planar interaction



regular wrapping

- Starts contributing at $\mathcal{O}(\lambda^L)$.
  May nevertheless repair discrepancy (see example).
- Asymptotic Bethe ansatz does not incorporate wrappings.
- Algebraic construction apparently not useful here.

Conclusions & Outlook

⋆ **Higher-loop scaling dimensions in a conformal gauge theory**

⋆ **Higher-loop integrability in $\mathcal{N} = 4$ SYM**

- Bethe equations in $\mathfrak{su}(2)$ sector for all loops (asymptotically).
- Non-perturbative Bethe equations?

⋆ **Three-loop discrepancies**

- Order of limits problem. Strong/weak duality remains.
- What about non-planar BMN (at three-loops)?

⋆ **Further questions**

- Other subsectors at higher loops?
- Can one prove higher-loop integrability?
- Integrability for quantum string sigma model?
- Conformal symmetry and integrability?