Rapidly rotating quantum gases. Lecture 1

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Outline

- Introduction. Vortices in supetrfluids
- Gross-Pitaevskii equation
- Rapid rotation. Single particle problem. Landau levels
- BEC in the lowest Landau level. Projected GP equation
- Quantum transitions in the trapped case
- Vortex lattice of trapped BEC

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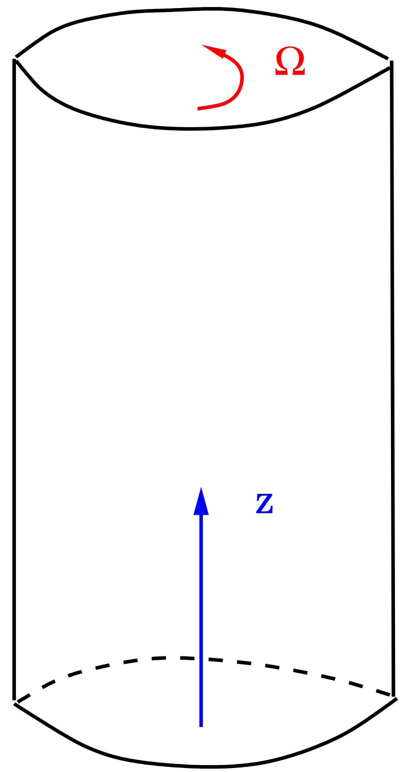
Vortices in superfluids

Rotating Bose superfluid at $T = 0$. No motion



$E_{\rm rot} = E - L_z\Omega$ is minimum. Superfluid motion around microscopically narrow lines at which the condition $\mathrm{curl}\,\mathbf{v}_s = 0$ is violated

Circulation of the superfluid velocity around a vortex line

$$\oint \mathbf{v}_s d\mathbf{l} = \oint \frac{\hbar}{m}\nabla S d\mathbf{l} = \frac{2\pi\hbar}{m}s; \quad S = \frac{1}{\hbar}[m\mathbf{v}_s\mathbf{r} - (mv_s^2/2 + \mu)t]$$

Vortices in superfluids

In the reference frame where the liquid moves with velocity $\mathbf{v}_s$:

$$\Psi_0 = \sqrt{n_0}\exp(iS) \Rightarrow \quad \mathbf{v}_s = \frac{\hbar}{m}\nabla S$$

Single valued $\Psi_0] \Rightarrow$ integer s. The circulation is quantized in units of $\hbar/m$.

Vortex line along z. Streamlines of $\mathbf{v}_z$ are circles perpendicular to z:

$$v_s = s\frac{\hbar}{mr}$$

Angular momentum $L_z = \int n_s m v_s r d^3r = \pi s \mathcal{R}^2 \mathcal{L} \hbar n_s$

Energy associated with the vortex $E_v = \frac{1}{2}\int n_s m v_s^2 d^3r = \pi n_s m s^2 \left(\frac{\hbar}{m}\right)^2 \ln\left(\frac{\mathcal{R}}{r_c}\right)$

Critical rotation frequency $\Rightarrow E_{\mathrm{rot}} = E_v - \Omega_c L_z = 0 \Rightarrow$

$$\Omega_c = \frac{E_v}{L_z} = \frac{\hbar}{m\mathcal{R}^2}\ln\left(\frac{\mathcal{R}}{r_c}\right)$$

This is for $|s| = 1$. The states with the charge $|s| > 1$ are unstable

Vortices in superfluids

For Ω greatly exceeding $\Omega_c \Rightarrow$ many vortices

$$\oint \mathbf{v}_s d\mathbf{l} = 2\pi N_v \frac{\hbar}{m}$$

$N_v \gg 1 \Rightarrow$ relations for a rotating rigid body

$v_s = \Omega r$ and $|\mathrm{curl}\mathbf{v}_s| = 2\Omega$, so that $\oint \mathbf{v}_s d\mathbf{l} = 2\Omega A$

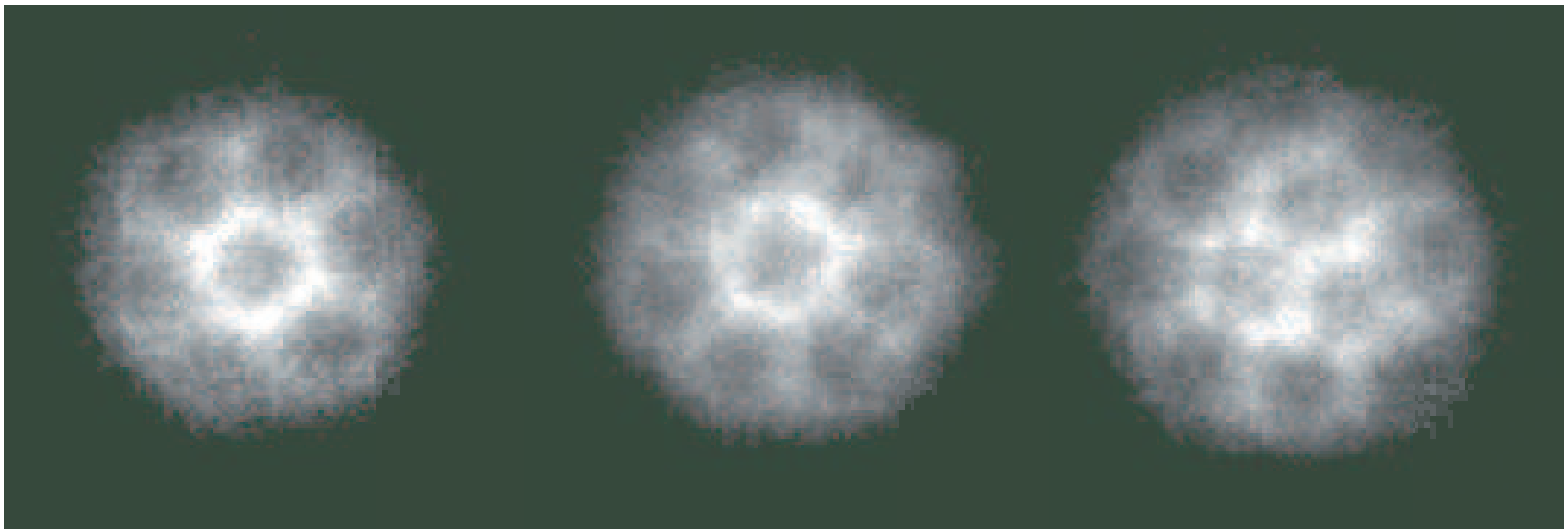
$$n_v = \frac{N_v}{A} = \frac{m\Omega}{\pi\hbar}$$

Experiments

Superfluid ^{4}He W.F. Vinen (1961), R.E. Packard/T.M. Sanders (1972)

Bose-condensed ultracold atomic gases

JILA (E.A. Cornell), ENS (J. Dalibard), MIT (W. Ketterle)



Gross-Pitaevskii equation for the vortex state

Straight vortex line along the z-axis and $|s| = 1$ L.P. Pitaevskii (1963)

$$\psi_0 = \sqrt{n_0} f(r) \exp(i\phi) \quad r = \sqrt{x^2 + y^2}$$

Laplacian $\Rightarrow \quad \Delta_{\mathbf{r}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$

Gross-Pitaevskii equation $-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) f + n_0 g |f|^2 f - \mu f = 0$

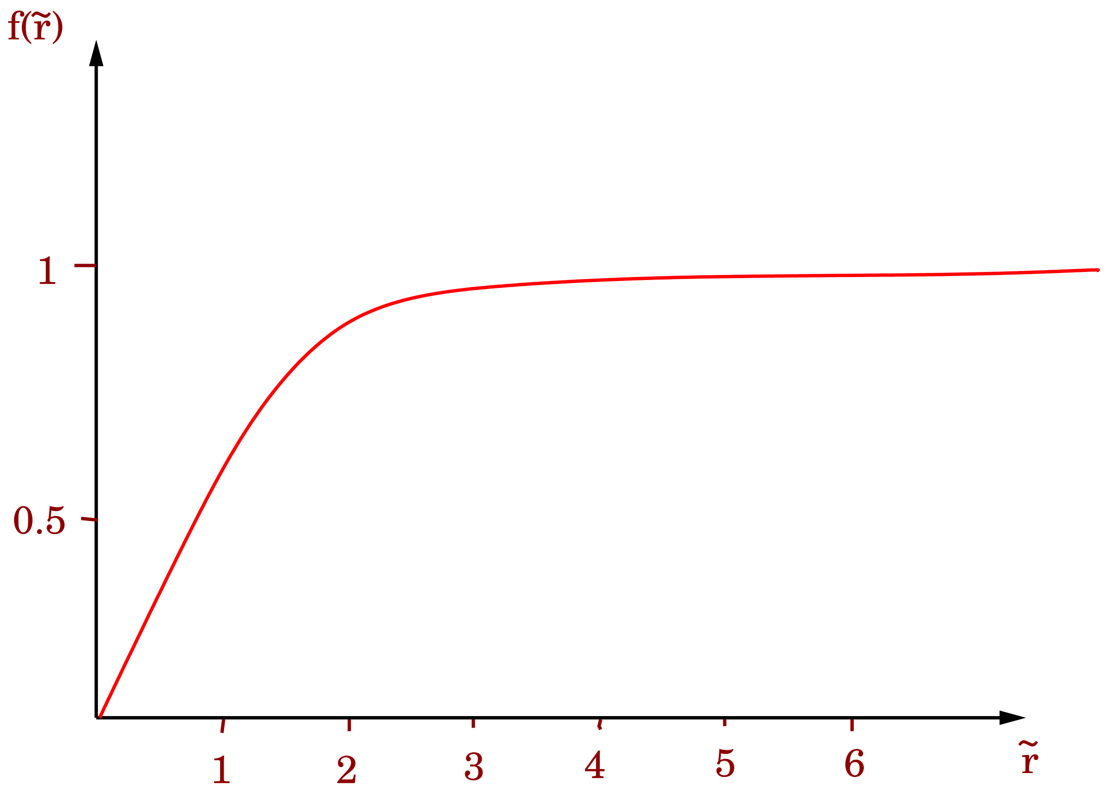
Energy scale $\mu = n_0 g$ and length scale $\xi = \hbar/\sqrt{m\mu}$

$$\frac{d^2 f}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{df}{d\tilde{r}} - \frac{f}{\tilde{r}^2} - f^3 + f = 0; \quad \tilde{r} = \sqrt{2} r/\xi$$

$$f \propto \tilde{r}, \qquad \tilde{r} \to 0$$

$$f \propto \left(1 - \frac{1}{2\tilde{r}^2}\right), \qquad \tilde{r} \to \infty$$

Gross-Pitaevskii equation for the vortex state

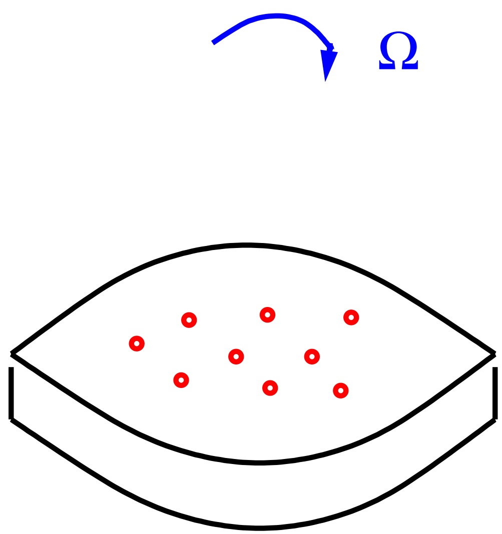


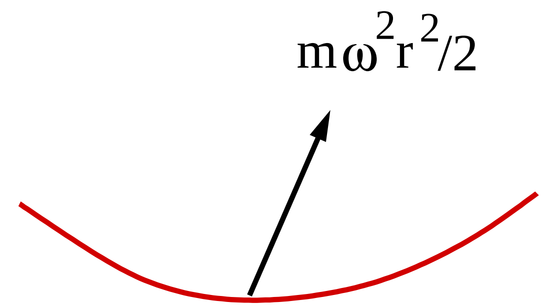
$r_c \sim \xi$ and using $\mu = n_0 g$ and $g = 4\pi\hbar^2 a/m \Rightarrow \;\; r_c \sim 1/\sqrt{n_0 a} \gg n_0^{-1/3}$ $(n_0 a^3 \ll 1)$

Topological quantum number, circulation. The vortex can only decay when going to the border of the system

Rapid rotation. Single particle problem

Non-interacting particles in the x, y plane rotating with frequency Ω around the z-axis in the harmonic potential $V(r) = m\omega^2 r^2/2;\ \ r = \sqrt{x^2 + y^2}$





Single particle Hamiltonian $\hat{H}^{(1)} = -\frac{\hbar^2}{2m}\Delta_{\mathbf{r}} + \frac{m\omega^2 r^2}{2} - \hbar\Omega\hat{L}_z$

$$\hat{L}_z = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right); \quad \Omega < \omega$$

$$\hat{H}^{(1)} = -\frac{\hbar^2}{2m}(\nabla_{\mathbf{r}} - i\mathbf{A})^2 + (\omega^2 - \Omega^2)\frac{mr^2}{2}$$

$$\mathbf{A} = m\Omega(\hat{\mathbf{z}} \times \mathbf{r})/\hbar$$

Landau levels

Common eigenbasis of $\hat{L}_z$ and $\hat{H}^{(1)}$:

$$\Phi_{jk}(\mathbf{r}) = \exp\left(\frac{r^2}{2l^2}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^k \exp\left(-\frac{r^2}{l^2}\right); \;\; l = (\hbar/m\omega)^{1/2}$$

$$E_{jk} = \hbar\omega + \hbar(\omega - \Omega)j + \hbar(\omega + \Omega)k$$

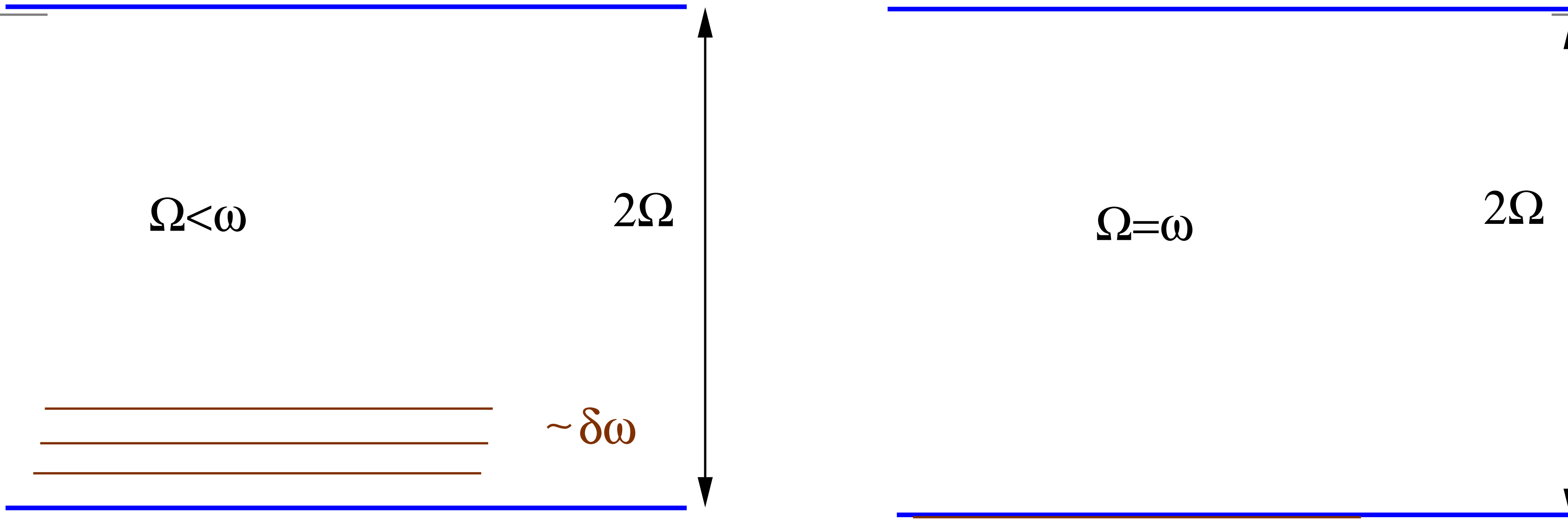
Rapid rotation $\Rightarrow \quad \delta\omega = (\omega - \Omega) \ll \Omega$. Landau levels

$\delta\omega = 0 \Rightarrow$ infinite plane geometry and $E_k = \hbar\Omega(2k+1)$

Level spacing $2\hbar\Omega$. Lowest Landau level (LLL, $k = 0$) $\Rightarrow$

$$\psi(\mathbf{r}) = f(z)\exp\left(-\frac{r^2}{2l^2}\right); \;\; z = x + iy$$

Landau levels



$\omega > \Omega \Rightarrow$ sublevels ($\delta\omega \ll \Omega$)

$$E_{jk} = \hbar\omega + \hbar(\Omega + \omega)k + \hbar\delta\omega j$$

LLL is not degenerate $f_j(z) = \dfrac{z^j}{l^{j+1}\sqrt{\pi j!}}$

$z^j = r^j \exp(ij\phi) \rightarrow$ j is the orbital angular momentum

Rapidly rotating BEC. GP equation

$$T = 0; \quad \delta\omega \ll \Omega; \quad ng \ll \hbar\Omega$$

BEC in the lowest Landau level

$$\psi_0 = \sqrt{n} f(z) \exp\left(-\frac{r^2}{2l^2}\right)$$

GP equation $\Rightarrow$ $$\left[-\frac{\hbar^2}{2m}\Delta_{\mathbf{r}} + \frac{m\omega^2 r^2}{2} - i\hbar\Omega\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + ng|\psi_0^2 - \mu\right]\psi_0 = 0$$

$$\exp\left(-\frac{r^2}{2l^2}\right)\left[\hbar\delta\omega z\frac{d}{dz} + ngf(z)f^*(\bar{z})\exp\left(-\frac{r^2}{l^2}\right) - \tilde{\mu}\right]f(z) = 0$$

$$\bar{z} = x - iy; \quad \tilde{\mu} =' \mu - \hbar\omega$$

Projection on the LLL $\Rightarrow$ $\hat{P} = \dfrac{1}{2\pi}\exp(z\bar{z}' - z'\bar{z}'/2)$

$$\hat{P}\psi(z,\bar{z}) = \frac{1}{2\pi}\int \psi(z',\bar{z}')\exp(z\bar{z}' - z'\bar{z}'/2)\,dz'd\bar{z}'$$

$$\hbar\delta\omega z\frac{df(z)}{dz} + \frac{ng}{2\pi}\int f^2(z')f^*(\bar{z}')\exp(z\bar{z}' - 2z'\bar{z}')\,dz'd\bar{z}' - \tilde{\mu}f(z) = 0$$

Vortex lattice

Infinite plane geometry ($\delta\omega = 0$) $\Rightarrow \;\; \psi_0(z) = \theta(z)\exp(z^2/2 - z\bar{z}/2)$

Triangular lattice $\theta(z) = (2v)^{1/4}\nu_1(\sqrt{\pi v}z, \tau)$

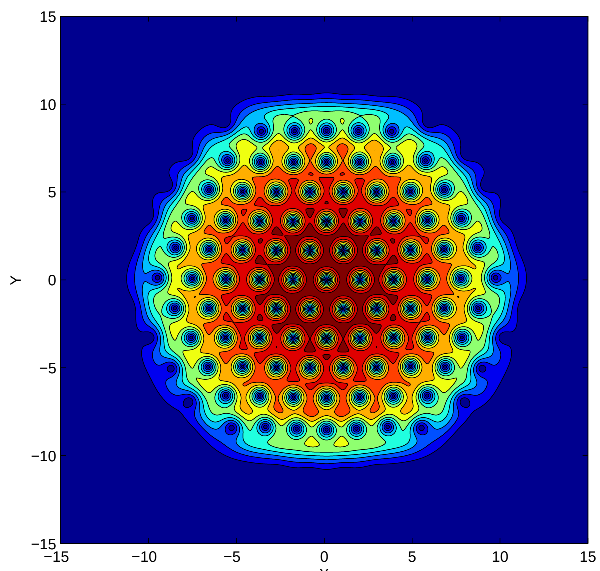
$$\nu_1(\zeta, \tau) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n \exp\{i\pi\tau(n+1/2)^2 + 2i\zeta(n+1/2)\}$$

$$\tau = u + iv; \quad v = \sqrt{3}/2; \;\; u = -1/2; \quad \tilde{\mu} = \alpha n g; \;\; \alpha = 1.1596$$

$\sim l \Rightarrow$ period of the lattice and the size of the vortex core

Numbar of vortices $N_v \sim A/l^2$. Number of particles $N \sim nA$

Criterion of the mean-field regime $N_v \ll N \Rightarrow \;\; nl^2 \gg 1$



Trapped BEC in the LLL

$\Omega < \omega$ ($\Omega \gg \delta\omega > 0$) Trapped BEC in the LLL

Vortex lattice. Mean-field Quantum Hall regime.

Number of vortices is much smaller than the number of particles

JILA experiment (E. Cornell group) and ENS experiment (J. Dalibard group)

Theoretical studies based on the GP equation and hydrodynamic approach

Last 10 years Ho, Baym, Fetter, Cooper, Aftalion, Dalibard, Sonin, etc.

Is it possible to have rapidly rotating BEC without vortices?

If yes, then how the vortices start to appear?

Quantum transition from zero to one vortex state

State without vortices $\Rightarrow f_0(z) = 1/\sqrt{\pi}$

$$\tilde{\mu}_0 = \frac{ng}{2\pi}$$

,

$$E_0 = \frac{N^2 g}{4\pi}$$

State with one vortex $\Rightarrow f_1(z) = z/\sqrt{\pi}$

$$\tilde{\mu}_1 = \hbar\delta\omega + \frac{ng}{4\pi}$$

$$E_1 = \hbar\delta\omega N + \frac{N^2 g}{8\pi}$$

The state without vortices remains the ground state when $E_0 < E_1 \Rightarrow$

$$\hbar\delta\omega > \frac{ng}{8\pi}$$

Quantum transition from zero to one vortes state

At the point where $\frac{ng}{8\pi} = \hbar\delta\omega$

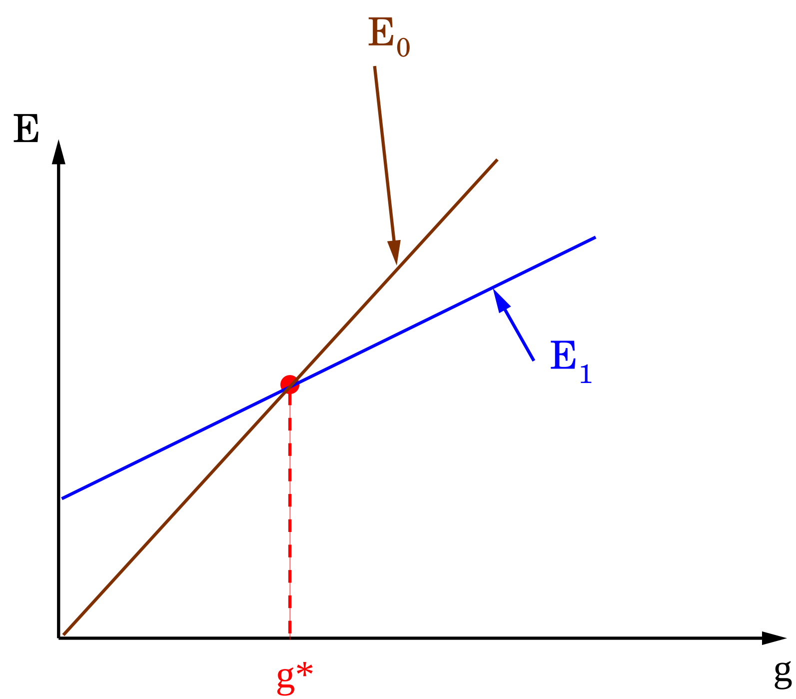
there is a quantum phase transition to the state with one vortex

The chemical potential undergoes a jump $\Rightarrow \tilde{\mu}_1 - \tilde{\mu}_0 = -\hbar\delta\omega$

Quantum phase transition $\Rightarrow$ the transition at $T = 0$

under a change of one of the parameters

Jump in $\mu \Rightarrow$ First order quantum transition



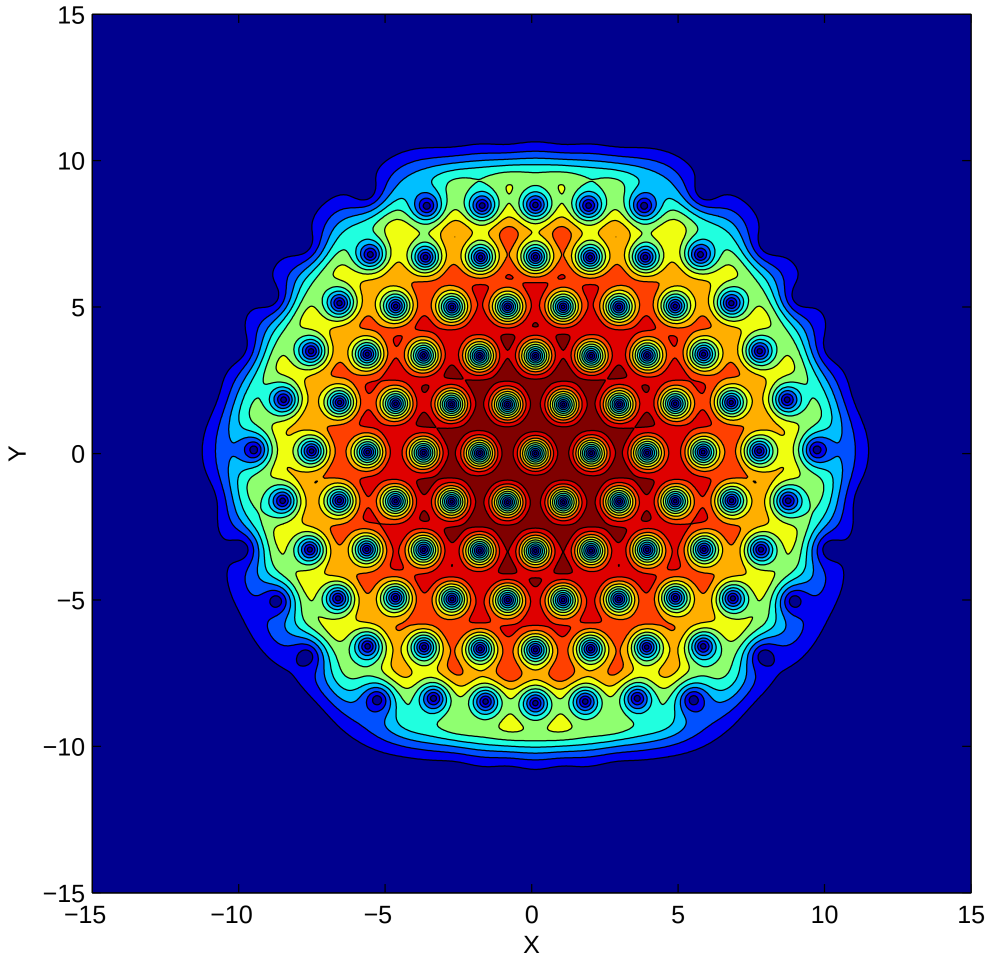
Trapped vortex lattice

How to increase the number of vortices?

Decrease $\delta\omega$ or increase g

Many vortices $\Rightarrow$ trapped vortex lattice

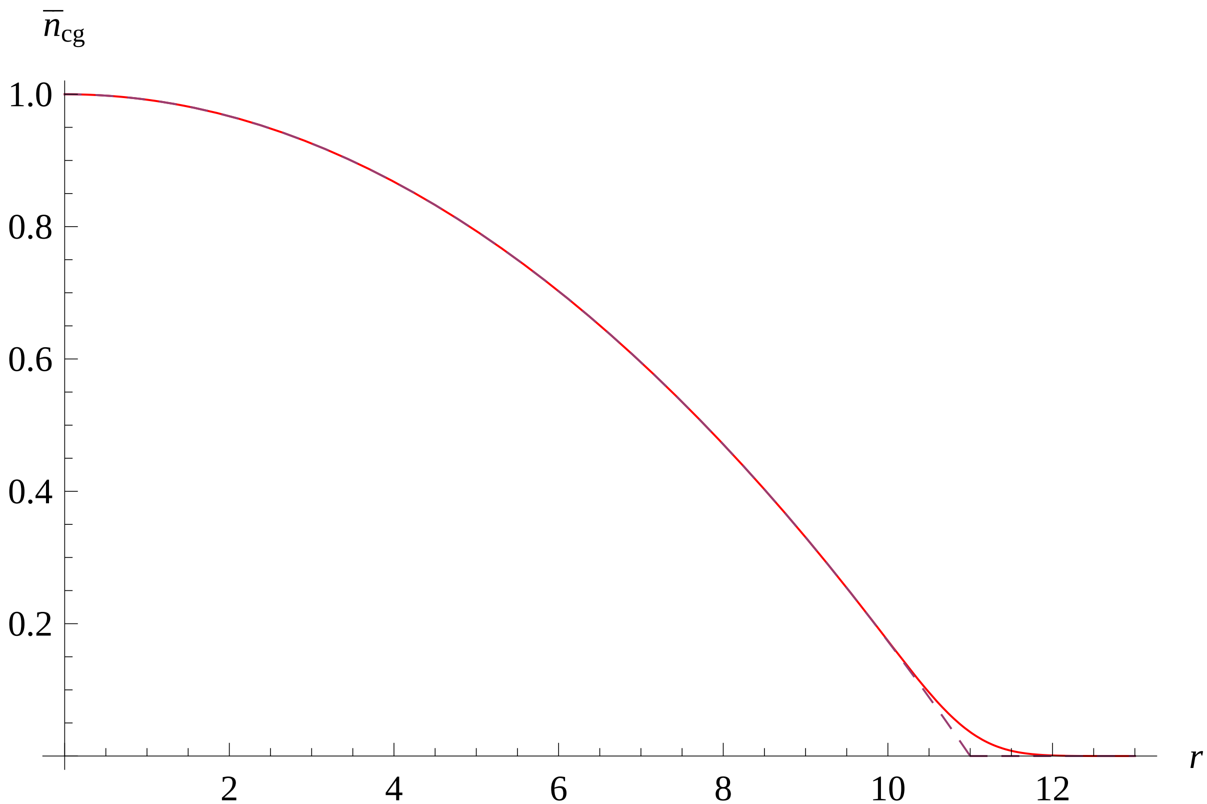
Approximately Thomas-Fermi density profile



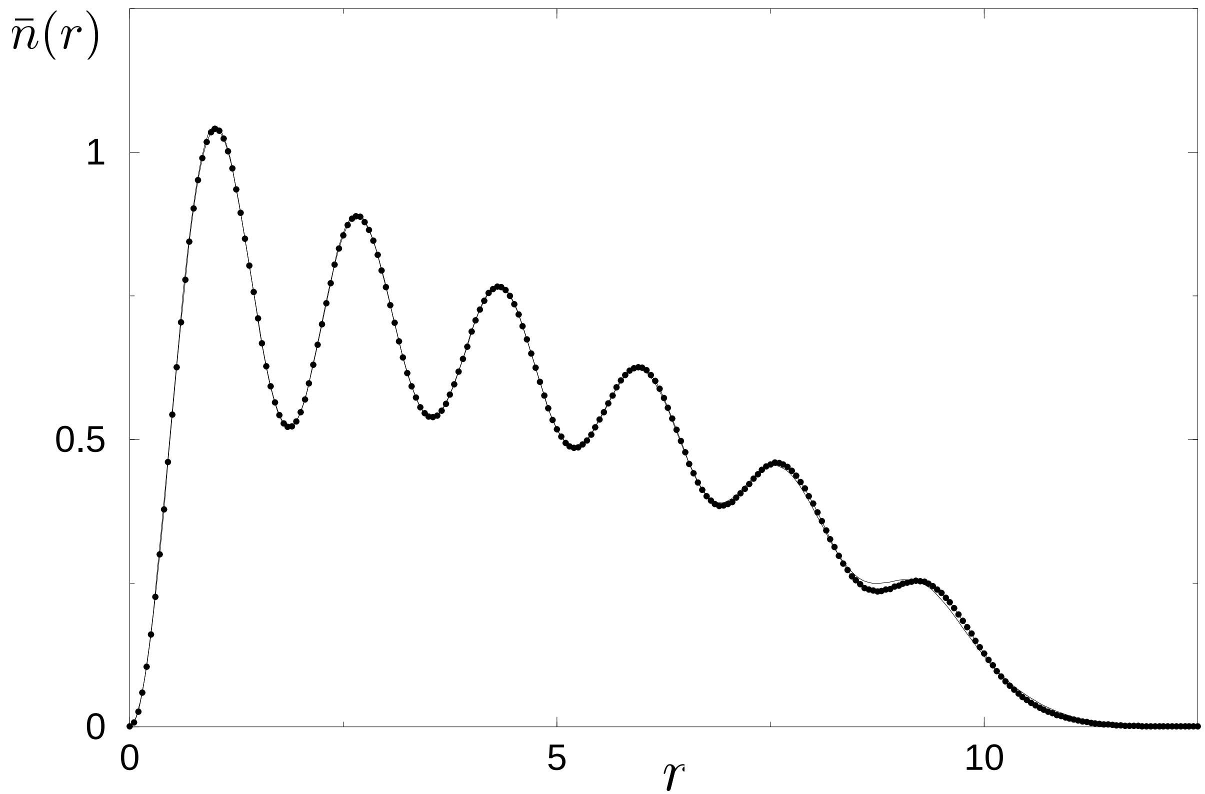
Trapped vortex lattice

Coarse grained density $\bar{n}_{cg} = \bar{n}_{2D}\left(1 - \frac{r^2}{R^2}\right)$; $\quad \bar{n}_{2D} = \frac{2N}{\pi R^2}$

BEC size $R = (2\alpha\beta/\pi)^{1/4} l; \quad \beta = \frac{Ng}{l^2 \hbar \delta\omega}$



Trapped vortex lattice



$$R \sim \left(\frac{Ng}{\hbar\delta\omega}\right)^{1/4} l$$

$$N_v \sim \frac{R^2}{l^2} \Rightarrow \quad \frac{N_v}{N} \sim \left(\frac{g}{Nl^2\hbar\delta\omega}\right)^{1/2}$$

Increase $N_v/N \Rightarrow$ melting of the vortex lattice