Rapidly rotating quantum gases. Lecture 2

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Outline

- Density-phse representation
- Excitation modes of the vortex lattice in the LLL
- One-body density matrix
- Melting of the lattice
- Strongly correlated states

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Density-phase representation

$$\hat{H} = \int d^2\mathbf{r}\left[-\hat{\psi}^\dagger\frac{\hbar^2}{2m}\Delta_\mathbf{r}\hat{\psi} + \frac{g}{2}\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} + V(\mathbf{r})\hat{\psi}^\dagger\hat{\psi} - \Omega\hat{\psi}^\dagger\hat{L}\hat{\psi}\right]$$

$$i\hbar\frac{\partial\hat{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\Delta_\mathbf{r}\hat{\psi} + g\hat{\psi}^\dagger\hat{\psi}\hat{\psi} + V(\mathbf{r})\hat{\psi} - \Omega\hat{L}\hat{\psi}$$

$$\hat{\psi} = \exp i\hat{\Phi}\sqrt{\hat{n}};\quad \hat{\psi}^\dagger = \sqrt{\hat{n}}\exp -i\hat{\Phi};\quad [\hat{n}(\mathbf{r}), \hat{\Phi}(\mathbf{r}')] = i\delta(\mathbf{r}-\mathbf{r}')$$

$n = n_0(\mathbf{r}) + \delta\hat{n};\ \ \hat{\Phi} = \Phi_0(\mathbf{r}) + \delta\hat{\Phi}$ Small fluctuations of the density

Take zero and linear orders of NLSE with respect to $\delta\hat{n}$ and $\nabla\delta\hat{\Phi}$

Zero order $\Rightarrow$ GP equation for $\Psi_0(\mathbf{r}) = \sqrt{n_0(\mathbf{r})}\exp[i\Phi_0(\mathbf{r})]$

$$-\frac{\hbar^2}{2m}\Delta_\mathbf{r}\Psi_0 + g|\Psi_0|^2\Psi_0 + V(\mathbf{r})\Psi_0 - \Omega\hat{L}\Psi_0 = \mu\Psi_0$$

LLL $\Rightarrow$ $\Psi_0 = \sqrt{n_0}f_0(z)\exp(-|z|^2/2)$

Zero order euation

Projected GP equation

$$\hat{P}F(z,\bar{z}) = \frac{1}{\pi}\int dw d\bar{w} \exp[-|w|^2 + z\bar{w}]F(w,\bar{w}) \;\Rightarrow \text{LLL}$$

$\Omega = \omega \;\Rightarrow$ geometry of an infinite plane

$$\frac{Ng}{\pi}\int dw d\bar{w} \mathrm{e}^{-2w\bar{w}+z\bar{w}}|f_0(w)|^2 f_0(w) = \tilde{\mu} f_0(z);\; \tilde{\mu} = \mu - \hbar\Omega$$

Triangular vortex lattice $f_0(z) = (2v)^{1/4}\vartheta_1(\sqrt{\pi v}z, q)\,\mathrm{e}^{z^2/2}$

$$q = \exp(i\pi\tau),\ \tau = u + iv,\ v = \sqrt{3}/2,\ u = -1/2$$

$$\tilde{\mu} = \alpha n g; \quad \alpha = 0.1596$$

First order

Linear order $\Rightarrow$ equations for $\delta\hat{n}$ and $\delta\Phi$ $\Rightarrow$

solution in terms of elementary excitations $(u_{\mathbf{k}}, \tilde{v}_{\mathbf{k}})$

$$\delta\hat{n} = \sqrt{n_0}\mathrm{e}^{-|z|^2/2}\sum_{\mathbf{k}}[u_{\mathbf{k}}\exp[-i\Phi_0] - \tilde{v}^*_{\mathbf{k}}\exp[i\Phi_0]]\exp[-i\epsilon_{\mathbf{k}}t]\hat{a}_{\mathbf{k}} + \text{h.c.}$$

$$\delta\hat{\Phi} = \frac{-i\mathrm{e}^{-|z|^2/2}}{2\sqrt{n_0}}\sum_{\mathbf{k}}[u_{\mathbf{k}}\exp[-i\Phi_0] + \tilde{v}^*_{\mathbf{k}}\exp[i\Phi_0]]\exp[-i\epsilon_{\mathbf{k}}t]\hat{a}_{\mathbf{k}} + \text{h.c.}$$

$u_{\mathbf{k}}, \tilde{v}_{\mathbf{k}} \rightarrow$ solutions of projected BdG equations

$$2g\hat{P}(|\Psi_0|^2 u_{\mathbf{k}}) - g\hat{P}(\Psi_0^2\tilde{v}^*_{\mathbf{k}}) = (\tilde{\mu} + \epsilon_{\mathbf{k}})u_{\mathbf{k}}$$
$$2g\hat{P}(|\Psi_0|^2 \tilde{v}_{\mathbf{k}}) - g\hat{P}(\Psi_0^2 u^*_{\mathbf{k}}) = (\tilde{\mu} - \epsilon_{\mathbf{k}})\tilde{v}_{\mathbf{k}}$$

Solution of projected BdG equations

$$u_{\mathbf{k}} = \frac{c_{1\mathbf{k}}}{\sqrt{S}} f_0 \left(z + \frac{ik_+}{2} \right) \mathrm{e}^{ik_- z/2} \mathrm{e}^{-k^2/4} = c_{1\mathbf{k}} P(f_0 \mathrm{e}^{i\mathbf{kr}})$$

$$\tilde{v}_{\mathbf{k}} = \frac{c_{2\mathbf{k}}}{\sqrt{S}} f_0 \left(z - \frac{ik_+}{2} \right) \mathrm{e}^{-ik_- z/2} \mathrm{e}^{-k^2/4} = c_{2\mathbf{k}} P(f_0 \mathrm{e}^{-i\mathbf{kr}})$$

$$k_\pm = k_x \pm k_y; \quad c_{1\mathbf{k}} = \left[\frac{\tilde{K}(\mathbf{k}) + \epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}} \right]^{1/2} \mathrm{e}^{k^2/8}; \quad c_{2\mathbf{k}} = \left[\frac{\tilde{K}(\mathbf{k}) - \epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}} \right]^{1/2} \frac{|K_2(\mathbf{k})|}{K_2(\mathbf{k})} \mathrm{e}^{k^2/8}$$

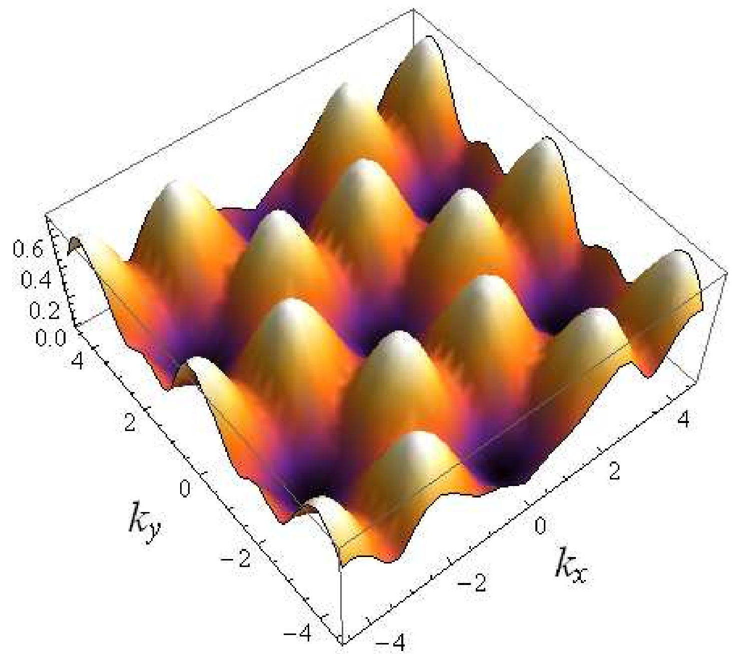
$$\tilde{K}(\mathbf{k}) = 2K_1(\mathbf{k}) - K_1(0; \quad K_1(0) = K_2(0) = \tilde{K}(0) = \alpha \simeq 1.1596$$

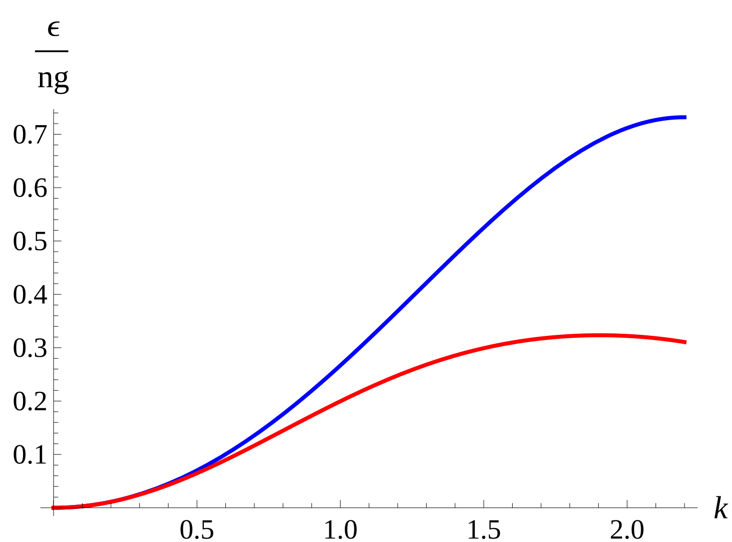
$$K_1(\mathbf{k}) = \sqrt{v} \sum_{n,m=-\infty}^{\infty} (-1)^{nm} \mathrm{e}^{-\pi v (n^2+m^2)} \mathrm{e}^{-\sqrt{\pi v} k_x n + i\sqrt{\pi v} k_y m} \mathrm{e}^{-k_x^2/4}$$

$$K_2(\mathbf{k}) = \sqrt{v} \sum_{n,m=-\infty}^{\infty} (-1)^{nm} \mathrm{e}^{-\pi v (n^2+m^2)} \mathrm{e}^{-\sqrt{\pi v}(k_x - ik_y)(n+m)} \mathrm{e}^{-k_x^2/2 + ik_x k_y/2}$$

Excitation spectrum

$$\epsilon_{\mathbf{k}}^2 = |2K_1(\mathbf{k}) - K_0|^2 - |K_2(\mathbf{k})|^2$$





Low-energy excitations

$$kl \ll 1 \Rightarrow \quad K_1 = \alpha\left[1 - \frac{k^2}{8} + \frac{(\eta+1)k^4}{64}\right];\ K_2 = \alpha\left(1 - \frac{k^2}{4} + \frac{k^4}{32}\right);\ \eta = 0.8219$$

$$\epsilon = \frac{\alpha\sqrt{\eta}}{4}\, ng(kl)^2 \simeq 0.2628\, ng\,(kl)^2$$

Exactly coincides with Sonin (2005)

Tight confinement in one direction $\Rightarrow g = 2\sqrt{2\pi}\hbar^2 a/ml_0;\ l_0 = \sqrt{\hbar/m\omega_0}$

Rb^{87} $\Omega = 100$ Hz $\omega_0 = 300$ Hz $\Rightarrow ng/\hbar\Omega \simeq 0.1$ at $n \simeq 3 \times 10^8$ cm^{-2} (LLL!)

Low-energy excitations $\Rightarrow \epsilon < 1$ Hz

$\nu = \pi n l^2 \gg 1 \ \rightarrow$ mean-field regime

Brief historical overview

Elastic oscillations of a vortex lattice in incompressible superfluids

$\Rightarrow \quad \epsilon(k) \propto k$ Tkachenko (1966)

Finite compressibility $\Rightarrow \quad \epsilon(k) \propto k^2$

Volovik/Dotsenko (1979); Baym/Chandler (1983); Sonin (1987)

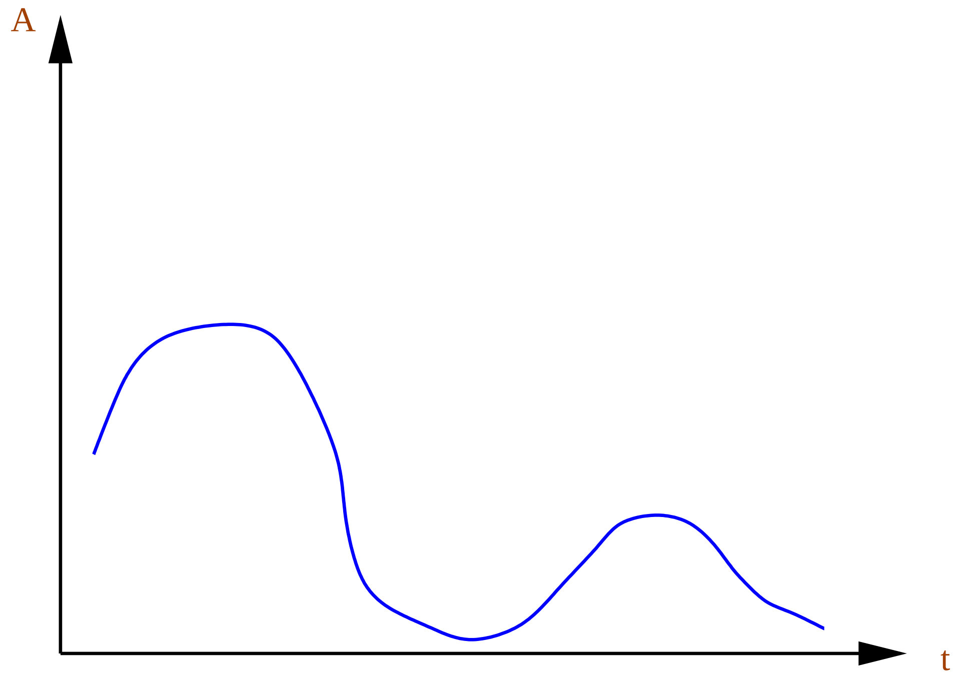
Hydrodynamic approach Baym (2004); Sonin (2005); Fetter

$$\omega(k) \propto \sqrt{4\Omega^2 + (s^2 + A)k^2} \Rightarrow \text{ inertial}$$

$$\epsilon(k) \propto \frac{sk^2}{\sqrt{4\Omega^2 + (s^2 + A)k^2}}$$

Brief historical overview

JILA experiment (E. Cornell group)



Typical picture from the JILA experiment

Calculation of the observed frequencies: Anglin, Baym

Mizushima et al, Bigelow group, Stringari group

Problem

Non-condensed density $n' = \langle \hat{\psi}'^{\dagger} \hat{\psi}' \rangle = \exp(-|z|^2) \sum_{\mathbf{k}} |v_{\mathbf{k}}|^2$

$$n' \sim \int \frac{d^2k}{\epsilon_{\mathbf{k}}} \sim \int \frac{dk}{k}$$

Low-momentum divergence. No true BEC.

This is not a problem!

One-body density matrix

$$g_1(\mathbf{r}) = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(0)\rangle = \Psi_0^*(\mathbf{r})\Psi_0(0)\exp\left\{-\frac{1}{2}\langle(\delta\hat{\Phi}(\mathbf{r}) - \delta\hat{\Phi}(0))^2\rangle\right\}$$

$$\delta\Phi(\mathbf{r}) = -\frac{i}{2}\sum_{\mathbf{k}}\frac{(c_{1\mathbf{k}} + c_{2\mathbf{k}})}{\sqrt{N}}\exp(i\mathbf{k}\mathbf{r}/2)\hat{a}_{\mathbf{k}} + h.c.$$

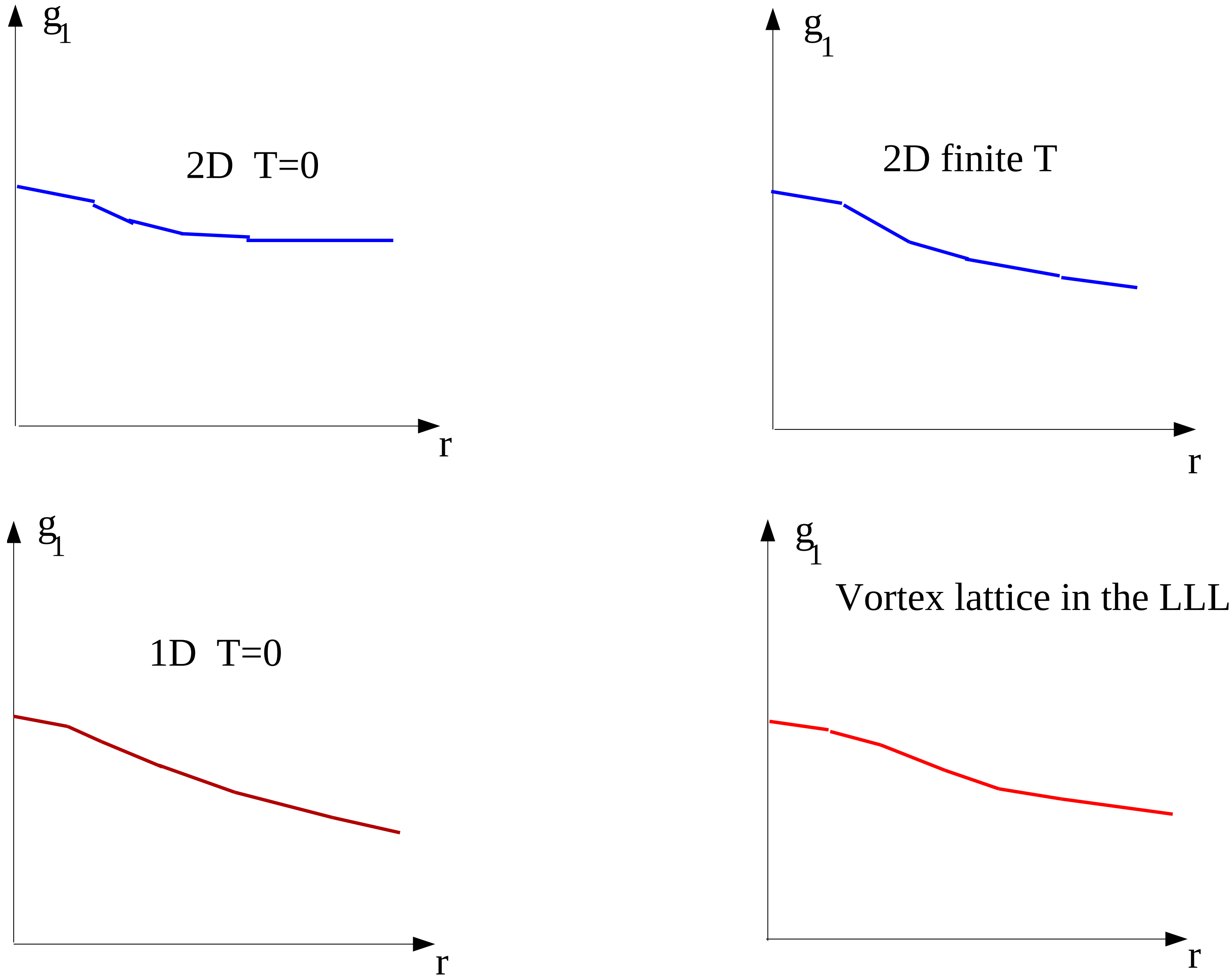
$$\langle(\delta\hat{\Phi}(\mathbf{r}) - \delta\hat{\Phi}(0))^2\rangle = \alpha g\int\frac{d^2k}{(2\pi)^2}\frac{(1+2N_k)}{\epsilon_k}[1 - J_0(kr/2)]$$

$$T = 0 \Rightarrow \quad \langle(\delta\hat{\Phi}(\mathbf{r}) - \delta\hat{\Phi}(0))^2\rangle_0 \simeq \frac{2}{\sqrt{\eta}}\frac{1}{\nu}\ln\left(\frac{e^C r}{2l}\right)$$

$$g_1(r) \propto \left(\frac{l}{r}\right)^{1/\sqrt{\eta}\pi n l^2}, \quad r \gg l \quad \text{Baym (2004)}$$

Phase coherence length $l_\phi \sim l\exp(nl^2) >>> l$ is extremely large

One-body density matrix



Nature of the vortex state in the LLL

No long-range order. No true BEC in the thermodynamic limit

Algebraic order. QuasiBEC with an extremely large phase coherence length

$\epsilon_{\mathbf{k}} \propto k^2$. No superfluidity

Landau criterion is not satisfied

Analogy with ”flux flow resistivity” in superconductors

Melting of the vortex lattice

One cannot locate the vortex point to a distance smaller than

the mean separation between particles $\Rightarrow (\delta r)^2 \sim \frac{1}{2\pi n}$

From the Lindemann melting criterion $\Rightarrow \frac{(\delta r)^2}{l^2} \approx 0.02$

we estimate the critical value of the filling factor

(ratio of the number of particles to the number of vortices) $\nu_c = \pi n_c l^2 \sim 10$

More controlled calculation involves the consideration

of collective modes of the lattice. This increases ν_c

Exact diagonalization for a small system gives $\nu_c \approx 6$

Cooper/Wilkin/Gunn (2001)

Strongly correlated states ($\nu < \nu_c$)

Poor understanding for ν just below ν_c

Good understanding for small filling factors

$$\nu = \frac{1}{2} \Rightarrow \text{ Laughlin state}$$

$$\Psi_{Ln}(\{z_i\}) \propto \prod_{i<j}^{N} (z_i - z_j)^2$$

Exact ground state for contact repulsion at $L = N(N-1)$

Ψ_{Lh} vanishes when two coordinates coincide

Zero local two-body correlation

Laughlin state

The average density is uniform. Vortices are not localized in space

Translational symmetry is not broken. Vortices are bound to the particles

Ψ_{Lh} changes phase by $2 \times 2\pi$ when any particle

encircles the position of another particle

Each particle thus experiences 2 vortices

bound to the position of every other particle

The total number of vortices is $N_v = (N-1)$, so that $\nu = N/N_v = 1/2$

Incompressible state with gapped excitations in the bulk and gapless edge modes

Exact diagonalization $\Rightarrow$ incompressibility and the gap of $0.05g/l^2$

Moore-Read state

$$\Psi_{MR} \propto \hat{S}\left[\prod_{i<j\leq N/2}(z_i - z_j)^2 \prod_{N/2<l<m}(z_l - z_m)^2\right]$$

$$\nu = 1$$

Non-abelian statistics for quasiparticle excitations