Statistical Field Theory and Applications: An Introduction for (and by) Amateurs.

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<u>The XY model</u>

The XY model is a statistical spin model with spin variables $\vec{S}_i$, on each site i of the lattice Λ, which are two component unit vectors, $\vec{S}_i^2 = 1$. The energy of a configuration $[\vec{S}]$ is defined as $E[\vec{S}] = -\sum_{[ij]} \vec{S}_i \cdot \vec{S}_j$ where the sum runs over neighboor points on Λ. Parametrising the unit spin vectors $\vec{S}_i$ by an angle Θ_i defined modulo 2π, we write the configuration energy as

$$E[\vec{S}] = -\sum_{[ij]} \cos(\Theta_i - \Theta_j).$$

The partition function is $Z = \int [\prod_i \frac{d\Theta_i}{2\pi}] \exp\Big(\beta \sum_{[i,j]} \cos(\Theta_i - \Theta_j)\Big)$ with $\beta = 1/k_B T$ the inverse temperature.

Here is the solution of the problem on the XY model given in Section 9.1.

• *<u>IA- The XY model on a lattice: High temperature expansion</u>*

The aim of this section is to study the high temperature ($\beta \ll 1$) behavior of the XY model. It is based on rewriting the Boltzmann sums in terms of dual flow variables.

IA-1 Explain why we can expand $e^{\beta\cos\Theta}$ in series as $e^{\beta\cos\Theta} = I(\beta)\,\big(1 + \sum_{n\neq 0} t_n(\beta) e^{in\Theta}\big)$, where $I(\beta)$ and $t_n(\beta)$ are some real β-dependent coefficients. We set $t_0(\beta) = 1$.

IA-2 By inserting this series in the defining expression of the partition function and by introducing integer variables $u_{[ij]}$ on each edge $[ij]$ of the lattice Λ, show that the partition function can be written as $Z = I(\beta)^{N_\mathrm{e}} \cdot \hat{Z}$ with N_e the number of edges and

$$\hat{Z} = \sum_{[u],\ [\partial u = 0]} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the partition sum is over all configurations $[u]$ of integer edge variables $u_{[ij]}$ such that, for any vertex $i \in \Lambda$, the sum of these variables arriving at i vanishes, i.e. $\sum_j u_{[ij]} = 0$.
<u>Remark:</u> The variables u are attached to the edge of the lattice and may be thought of as 'flow variables'. The condition that their sum vanishes at any given vertex is a divergence free condition. The divergence at a vertex i of a configuration $[u]$ is defined as $(\partial u)_i := \sum_j u_{[ij]}$.

IA-3 Let i_1 and i_2 be two points of Λ and $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ be the two-point spin correlation function. Explain why $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle = \mathrm{Re}\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle$.
Show that,

$$\langle e^{-i(\Theta_{i_1} - \Theta_{i_2})} \rangle = \frac{1}{\hat{Z}} \cdot \sum_{\substack{[u] \\ [\partial u = \delta_{\cdot;i_1} - \delta_{\cdot;i_2}]}} \prod_{[ij]} t_{u_{[ij]}}(\beta),$$

where the sum is over all integer flow configurations such that their divergence is equal to $+1$ at point i_1, to -1 at point i_2, and vanishes at any other vertex.

IA-4 Show that $t_n(\beta) = t_{-n}(\beta) \simeq \frac{\beta^n}{2^n n!}$ as $\beta \to 0$.
Argue, using this asymptotic expression for the $t_n(\beta)$'s, that the leading contribution to the spin correlation functions at high temperature comes from flow configurations with $u = 0$ or $u = \pm 1$ on each edge of the lattice.

IA-5 Deduce that, at high temperature, the correlation function $\langle \vec{S}_{i_1} \cdot \vec{S}_{i_2} \rangle$ decreases exponentially with the distance between the two points i_1 and i_2.
Show that the correlation length behaves as $\xi \simeq a/\log(2/\beta)$ at high temperature.

• *IB- Low temperature expansion*

The aim of this section is to study the low temperature ($\beta \gg 1$) behavior of the XY model. It consists in expanding the interaction energy $\cos(\Theta_i - \Theta_j)$ to lowest order in the angle variables so that we write the configuration energy as (up to an irrelevant additive constant)

$$E[\vec{S}] = \text{const.} + \frac{1}{2} \sum_{[i,j]} (\Theta_i - \Theta_j)^2 + \cdots .$$

This approximation neglects the 2π-periodicity of the angle variables.

IB-1 Argue that the higher order terms in this expansion, say the terms proportional to $\sum_{[i,j]} (\Theta_i - \Theta_j)^4$, are expected to be irrelevant and can be neglected.

IB-2 Write the expression of the partition function Z of the model within this approximation. Explain why, in this approximation, the theory may be viewed as a Gaussian theory.

IB-3 Let $G_\beta(x)$ be the two-point function of this Gaussian theory. Show that $G_\beta(x) = \beta^{-1} G(x)$ with

$$G(x) = \int_{-\pi/a}^{+\pi/a} \frac{d^2p}{(2\pi/a)^2} \frac{e^{ip\cdot x}}{4 - 2(\cos ap_1 + \cos ap_2)},$$

with p_1, p_2 the two components of the momentum p and a the lattice mesh.

IB-4 Let i_1 and i_2 be two points on Λ and x_1 and x_2 be their respective Euclidean positions. Let $C_\alpha(x_1, x_2) = \langle e^{i\alpha(\Theta_{i_1} - \Theta_{i_2})} \rangle$ with α integer. Show that

$$C_\alpha(x_1, x_2) = e^{-\frac{\alpha^2}{\beta}\left(G(0) - G(x_1 - x_2)\right)}.$$

IB-5 Explain why $G(x)$ is actually IR divergent[1] and what is the origin of this divergence, but that $G(0) - G(x)$ is finite for all x. Show that

$$G(0) - G(x) = \frac{1}{2\pi} \log(|x|/a) + \text{const.} + O(1/|x|).$$

IB-6 Deduce that the correlation functions C_α decrease algebraically at large distance according to

$$C_\alpha(x_1, x_2) \simeq \text{const.}\, (a/|x_1 - x_2|)^{\alpha^2/2\pi\beta}.$$

Compare with the high temperature expansion.

• *II- The role of vortices in the XY field theory*

[1] So that, when defining $G(x)$, we implicitly assumed the existence of an IR cut-off, say $|p| > 2\pi/L$ with L the linear size of the box on which the model is considered.

The previous computations show that the model is disordered at high temperature but critical at low temperature with temperature dependent exponents. The aim of this section is to explain the role of topological configurations, called vortices, in this transition.

We shall now study the model in continuous space, the Euclidean plane $\mathbb{R}^2$, but with an explicit short distance cut-off a. We shall consider the XY system in a disc of radius L.

In the continuous formulation, the spin configurations are then maps Θ from $\mathbb{R}^2$ to $[0, 2\pi]$ modulo 2π. The above Gaussian energy is mapped into the action

$$S_0[\Theta] = \frac{\kappa}{2} \int d^2x (\nabla\Theta)^2,$$

with a coefficient κ proportional to β.

II-1 Argue that the coefficient κ cannot be absorbed into a rescaling of the field variable Θ?

II-2 A vortex, centred at the origin, is a configuration such that $\Theta^{\pm}_{\rm v}(z) = \pm\mathrm{Arg}(z)$, with z the complex coordinate on $\mathbb{R}^2$, or in polar coordinates[2], $\Theta^{\pm}_{\rm v}(r, \phi) = \pm\phi$.
Show that $\Theta^{\pm}_{\rm v}$ is an extremum of S_0 in the sense that $\nabla^2\Theta^{\pm}_{\rm v} = 0$ away from the origin.
Show that $\oint_{C_0} d\Theta^{\pm}_{\rm v} = \pm 2\pi$ for C_0 a small contour around the origin.

II-3 Let a_0 be a small short distance cut-off and let $\mathbb{D}(a_0)$ be the complex plane with small discs of radius a_0 around the vortex positions cut out. Prove that, evaluated on $\Theta^{\pm}_{\rm v}$, the action S_0 integrated over $\mathbb{D}(a_0)$ (with an IR cut-off L) is

$$S^{(1)}_{\rm vortex} = \frac{\kappa}{2} \int_{\mathbb{D}(a_0)} d^2x\, (\nabla\Theta^{\pm}_{\rm v})^2 = \pi\kappa \log\big[L/a_0\big].$$

Give an interpretation of the divergence as $a_0 \to 0$.

II-4 What is the entropy of single vortex configurations? Show that the contribution of single vortex configurations to the free energy is

$$e^{-F^{(1)}_{\rm vortex}} \simeq \text{const.} \left(\frac{L}{a_0}\right)^2 e^{-\pi\kappa \log[L/a_0]}$$

Conclude that vortex configurations are irrelevant for $\pi\kappa > 2$ but relevant for $\pi\kappa < 2$.

- *III- The XY field theory and the sine-Gordon model*

The aim of this section is to analyse this phase transition using renormalization group arguments via a mapping to the so-called sine-Gordon field theory.

We shall consider a gas of vortices. The field configuration $\Theta^{(M)}_{\rm v}$ for a collection of M vortices of charges q_a centred at positions x_a is given by the sum of single vortex configuration:

$$\Theta^{(M)}_{\rm v} = \sum_{a=1}^{M} q_a\, \mathrm{Arg}(z - z_a).$$

We shall admit that the action of such configuration is

$$S^{(M)}_{\rm vortex} = -2\pi\left(\frac{\kappa}{2}\right) \sum_{a\neq b} q_a q_b \log\left(\frac{|x_a - x_b|}{a_0}\right) + 2\pi\left(\frac{\kappa}{2}\right)\left(\sum_b q_b\right)^2 \log\left(\frac{L}{a_0}\right) + \sum_a \beta\epsilon_c,$$

[2] We recall the expression of the gradient in polar coordinates: $\nabla\Theta = (\partial_r\Theta, \frac{1}{r}\partial_\phi\Theta)$. The Laplacian is $\nabla^2 F = \frac{1}{r}\partial_r(r\partial_r)F + \frac{1}{r^2}\partial^2_\phi F$.

where ϵ_c is a 'core' energy (which is not taken into account by the previous continuous description).

- *IIIA- The XY field theory: Mapping to the sine-Gordon theory*

The aim of this section is to analyse this phase transition using renormalization group arguments via a mapping to the so-called sine-Gordon field theory.

We shall consider a gas of vortices. The field configuration $\Theta_{\mathrm{v}}^{(M)}$ for a collection of M vortices of charges q_a centred at positions x_a is given by the sum of single vortex configuration:

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We shall admit that the action of such configuration is

$$S_{\mathrm{vortex}}^{(M)} = -2\pi(\frac{\kappa}{2}) \sum_{a \neq b} q_a q_b \, \log\big(\frac{|x_a - x_b|}{a_0}\big) + 2\pi(\frac{\kappa}{2})(\sum_a q_b)^2 \log\big(\frac{L}{a_0}\big) + \sum_a \beta\epsilon_c,$$

where ϵ_c is a 'core' energy (which is not taken into account by the previous continuous description).

This mapping comes about when considering a gas of pairs of vortices of opposite charges $\pm$, so that the vortex system is neutral ($\sum_a q_a = 0$). We denote x_j^+ (resp. x_j^-) the positions of the vortices of charge $+$ (resp. $-$).

The vortex gas is defined by considering all possible vortex pair configurations (with arbitrary number of pairs) and fluctuations around those configurations. We set $\Theta = \Theta_{\mathrm{v}}^{(2n)} + \theta_{\mathrm{sw}}$ and associate to each such configuration a statistical weights e^{-S} with action given by

$$S = S_{\mathrm{vortex}}^{(2n)}[x_j^+, x_j^-] + S_0[\theta_{\mathrm{sw}}],$$

with $S_0[\theta_{\mathrm{sw}}]$ the Gaussian action $\frac{\kappa}{2}\int d^2x (\nabla\theta_{\mathrm{sw}})^2$. We still assume a short-distance cut-off a.

IIIA-1 Write the expression of the action $S_{\mathrm{vortex}}^{(2n)}[x_j^+, x_j^-]$ for a collection of n pairs of vortices at positions $x_j^\pm$, $j = 1, \cdots, n$.

IIIA-2 Argue that the partition function of the gas of vortex pairs is given by the product $Z = Z_{\mathrm{sw}} \times Z_{\mathrm{vortex}}$ with Z_{sw} the partition function for the Gaussian free field θ_{sw} and

$$Z_{\mathrm{vortex}} = \sum_{n \geq 0} \frac{\mu^{2n}}{n! \cdot n!} \times \int \big(\prod_{j=1}^{n} d^2x_j^+ \prod_{j=1}^{n} d^2x_j^-\big) \, \frac{\prod_{i<j}(|x_i^+ - x_j^+|/a)^{2\pi\kappa}(|x_i^- - x_j^-|/a)^{2\pi\kappa}}{\prod_{i,j}(|x_i^+ - x_j^-|/a)^{2\pi\kappa}},$$

with $\mu = \big(\frac{a_0}{a}\big)^{\pi\kappa} e^{-\beta\epsilon_c}$.

IIIA-3 The aim of the following questions is to express Z_{vortex} as a path integral over an auxiliary bosonic field φ. Let $\tilde{S}_\kappa[\varphi] = \frac{1}{2\kappa}\int d^2x (\nabla\varphi)^2$ be a Gaussian action. Show that, computed with this Gaussian action,

$$\langle e^{i2\pi\varphi(x)} e^{-i2\pi\varphi(y)} \rangle_{\tilde{S}_\kappa} = \frac{1}{|x-y|^{2\pi\kappa}}.$$

<u>*Hint:*</u> The Green function associated to the action $\tilde{S}_\kappa[\varphi]$ is $G(x,y) = -\frac{\kappa}{2\pi}\log\big(|x-y|/a\big)$.

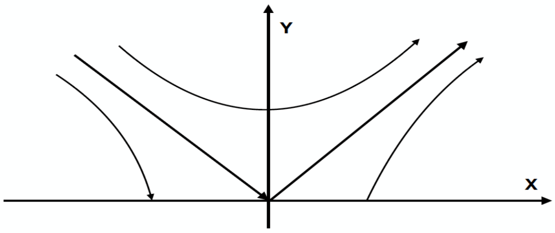

Figure 1: The XY RG flow.

IIIA-4 What is the scaling dimension (computed with the Gaussian action $\tilde{S}_\kappa[\varphi]$) of the operators $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$?
Deduce that the perturbation $\cos(2\pi\varphi)$ is relevant for $\pi\kappa < 2$ and irrelevant for $\pi\kappa > 2$.
Is the the perturbation $(\nabla\varphi)^2$ relevant or irrelevant?

IIIA-5 Show that Z_{vortex} can be written as the partition function of Gaussian bosonic field with action $S_{sG}[\varphi]$,

$$Z_{\text{vortex}} = \int [D\varphi]\, e^{-S_{sG}[\varphi]},$$

where the action S_{sG} is defined as

$$S_{sG}[\varphi] = \int d^2x \Big[\frac{1}{2\kappa}(\nabla\varphi)^2 - 2\mu\cos(2\pi\varphi)\Big].$$

This is called the sine-Gordon action.
Hint: Compute perturbatively the above partition function as a series in μ while paying attention to combinatorial factors.

- *The XY field theory: The renormalization group analysis*

IIIB-1 We now study the renormalization group flow of the action S_{sG} for κ close to the critical value $\kappa_c = 2/\pi$. We let $\kappa^{-1} = \kappa_c^{-1} - \delta\kappa$ and write

$$S_{sG}[\varphi] = \tilde{S}_{\kappa_c}[\varphi] - \int d^2x \Big[\frac{1}{2}(\delta\kappa)(\nabla\varphi)^2 + 2\mu\cos(2\pi\varphi)\Big]$$

Show that, to lowest order, the renormalization group equations for the coupling constants $\delta\kappa$ and μ are of the following form:

$$\begin{aligned} \dot{(\delta\kappa)} = \ell\partial_\ell\,(\delta\kappa) &= b\,\mu^2 + \cdots \\ \dot{\mu} = \ell\partial_\ell\,\mu &= a\,(\delta\kappa)\,\mu + \cdots \end{aligned}$$

with a and b some positive numerical constants.
Hint: It may be useful to first evaluate the OPE of the fields $(\nabla\varphi)^2$ and $\cos(2\pi\varphi)$.

IIIB-2 We redefine the coupling constants and set $X = a\,(\delta\kappa)$ and $Y = \sqrt{ab}\,\mu$ such that the RG equations now reads $\dot{X} = Y^2$ and $\dot{Y} = XY$.
Show that $Y^2 - X^2$ is an invariant of this RG flow.
Draw the RG flow lines in the upper half plane $Y > 0$ near the origin.

IIIB-3 We look at the flow with initial condition $X_I < 0$ and Y_I.
Show that if $Y_I^2 - X_I^2 < 0$ and $X_I < 0$, then the flow converges toward a point on the line $Y = 0$. Deduce that for such initial condition the long distance theory is critical. Compare with section I-B.

IIIB-4 Show that if $Y_I^2 - X_I^2 > 0$ and $X_I < 0$, the flow drives X and Y to large values.
Let $Y_0^2 = Y_I^2 - X_I^2$ with $Y_0 > 0$. Show that the solution of the RG equations are

$$\log\big(\frac{\ell}{a}\big) = \frac{1}{Y_0}\Big[\arctan\big(\frac{X(\ell)}{Y_0}\big) - \arctan\big(\frac{X_I}{Y_0}\big)\Big].$$

IIIB-5 The initial condition X_I and Y_I are smooth functions of the temperature T of the XY model. The critical temperature T_c is such that $X_I + Y_I = 0$. We take the initial condition to be near the critical line $X_I + Y_I = 0$ with $X_I < 0$. We let $X_I = -Y_I(1+\tau)$ in which $\tau \ll 1$ is interpreted at the distance from the critical temperature: $\tau \propto (T - T_c)$.
For $\tau > 0$, we define the correlation length as the length ξ at which $X(\ell)$ is of order 1.
Why is this a good definition?
Show that

$$\xi/a \simeq \text{const.}\ e^{\text{const.}/\sqrt{\tau}}.$$