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Transfer-matrix approach to percolation and phenomenological renormalization

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Résumé. — Une méthode de matrice de transfert est utilisée pour calculer la longueur de corrélation de rubans de largeur finie dans le problème de percolation par sites et par liaisons. A partir de la connaissance de ces longueurs de corrélation, nous calculons les seuils et l'exposant critique ν par la méthode de renormalisation phénoménologique.

Abstract. — A transfer-matrix method is used to calculate the correlation length for strips of finite width in the bond and site percolation problem. From the knowledge of these correlation lengths we compute the thresholds and the critical exponent ν by the phenomenological renormalization method.

1. **Introduction.** — We present a new approach to percolation, based on an adaptation of the transfer-matrix method to this type of problems. The transfer-matrix method is an old tool of statistical mechanics, which is generally useful to compute the physical properties of one-dimensional systems with a finite number of states. It is shown here how to generalize it to various situations in percolation theory and how to extract information on two-dimensional systems from the results on strips of different widths.

The standard transfer-matrix method amounts to breaking the partition function of a system with N columns into a sum of terms, where each term corresponds to a configuration of the n spins in the last column (Fig. 1). For instance, for a strip of Ising spins of width $n = 2$, there are four terms $Z_N(++)$, $Z_N(+-)$, $Z_N(-+)$ and $Z_N(--)$, corresponding to the four possibilities for the spins. For a system with $(N+1)$ columns, the $\{ Z_{N+1} \}$ are related to the $\{ Z_N \}$ by linear relations which may be written in matrix form. The largest eigenvalue of this transfer-matrix yields the free energy and from the ratio of the two largest eigenvalues one obtains the correlation length along the strip.

Though Onsager was able to use this method to solve the two-dimensional Ising model, such an extension is not possible in general. To study two-dimensional systems, one possibility is to consider strips of finite width n, which have a finite number of states per column, and to extrapolate the results for large n, using for instance finite-size scaling arguments. Another possibility consists in using the phenomenological renormalization-group equations developed by Nightingale [1] and extended by Sneddon [2], which are particularly well-suited when the correlation length is known.

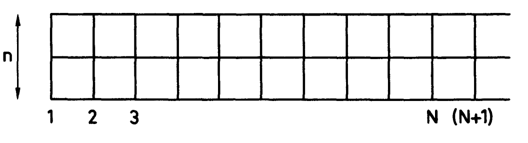

Fig. 1. — Strip of n rows and N columns.

The aim of the present letter is to demonstrate the feasibility and interest of the approach rather than to obtain very accurate numbers. The central part consists in the analytical calculation of the correlation length for various strips. The phenomenological renormalization-group theory is then used to obtain estimates for the percolation threshold p_c and the critical exponent ν associated with the divergence of the correlation length, for both bond and site percolation. These estimates are in satisfactory agreement with the known or accepted values [3], in spite of the limited amount of work involved.

There are *a priori* several meaningful ways to define a correlation length ξ for percolation problems [4].

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The definition used here is that the probability of the first and Nth columns to be connected decays as $\exp(-N/\xi)$ for large N. It is the direct counterpart of the usual definition for Ising strips based on the decay of the spin correlation function. This correlation length may be obtained as the largest eigenvalue of a matrix, the elements of which are explicit functions of the probability p of available sites or bonds. This analytical character is an advantage over Monte-Carlo methods, since it eliminates the need to perform averages, and it compensates for the limitation on the size of the matrices one can eventually handle.

Another approach based on a real-space renormalization-group theory has been recently proposed by Reynolds *et al.* [5], with very good quantitative results. Briefly, these authors consider a square cell of the original lattice and map it onto a site or a cell of the renormalized lattice. They study different prescriptions to decide whether the new site is occupied or empty. The renormalization-group equations thus involve directly the occupation probability, and the results are systematically improved by going to larger cells. Our method is rather different, since an infinite object (the strip) is considered at the outset. It also treats on the same level the site and bond problems, and does not introduce a choice between different prescriptions in the renormalization. More extensive work is needed to see whether the transfer-matrix method may be pushed far enough to provide results of comparable accuracy to existing theories.

2. **The transfer-matrix.** — An approach using the transfer-matrix method would be to consider the low-temperature limit of the dilute Ising system. A similar scheme has been carried out on some problems involving frustration [6] but it involves products of random matrices and the simplicity of the method is lost.

We use here another formulation, which is simpler and looks more promising. For percolation, the quantity of interest is the probability $P(N)$ that at least one site of the Nth column is connected to the 1st column by a path of present bonds (or sites). To calculate $P(N+1)$, the knowledge of $P(N)$ is clearly not sufficient. One needs more detailed information on column N. $P(N)$ is the sum of the probabilities $p_i(N)$ of different configurations for the sites of column N. These configurations depend on the number and the position of the sites which are connected to the 1st column and on the connections among the other sites *via* the parts of the clusters included between columns 1 and N.

Knowing the $p_i(N)$, the $p_i(N+1)$ are given by

$$p_i(N+1) = \sum_j q_{ij}\, p_j(N)$$

where q_{ij} is the probability that the vertical bonds of column N and the horizontal bonds between columns N and $N+1$ are such that the configuration j at column N gives rise to configuration i at column $N+1$. The matrix $\{ q_{ij} \}$ is the transfer-matrix. This procedure of breaking $P(N)$ into more detailed probabilities is identical to the procedure mentioned above for Z_N in the Ising model.

In order to illustrate the method, we give now an explicit example on a strip of width $n = 3$ with periodic boundary conditions. Due to the symmetries of the strip, one has only to consider the probabilities of the 4 configurations described by figure 2 :

— $p_1(N)$: which is the probability that one site of column N is connected to column 1 and the other two sites are not connected together (through a path entirely between column 1 and N),

— $p'_1(N)$: one site is connected to column 1 and the other two sites $\underline{\text{are}}$ connected together,

— $p_2(N)$: two sites are connected to column 1,

— $p_3(N)$: all three sites are connected to column 1.

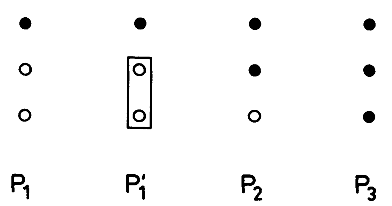

Fig. 2. — The states of a strip of width $n = 3$ with periodic boundary conditions, for bond percolation : the black points are connected to the 1st column, the white points are not. In p'_1 the two white points are linked together.

It is essential to distinguish p'_1 from p_1, because such configurations account for the possibility that a connecting path goes backwards for a while. For larger n, one would have to consider all the possible connections between the sites which are not connected to the 1st column. The number of these configurations increases rapidly with n.

To write explicitly the transfer matrix, it is convenient to express it as a product $M = M_{\mathrm{H}} M_{\mathrm{V}}$ where the contribution of vertical bonds M_{V} and of horizontal bonds M_{H} are :

$$M_{\mathrm{V}} = \begin{pmatrix} (1-p)^3 & 0 & 0 & 0 \\ p(1-p)^2 & (1-p)^2 & 0 & 0 \\ 2\,p(1-p)^2 & 0 & (1-p)^2 & 0 \\ p^2(3-2\,p) & (2\,p-p^2) & (2\,p-p^2) & 1 \end{pmatrix}$$

$$M_{\mathrm{H}} = \begin{pmatrix} p & p(1-p^2) & 2\,p(1-p) & 3\,p(1-p)^2 \\ 0 & p^3 & 0 & 0 \\ 0 & 0 & p^2 & 3\,p^2(1-p) \\ 0 & 0 & 0 & p^3 \end{pmatrix}.$$

For large N all probabilities $p_i(N)$ behave as λ^N, where λ is the largest eigenvalue of the transfer-matrix. The global probability that columns 1 and N are connected has the same behaviour, so the correlation length is

$$\xi = -1/\ln \lambda . \quad (1)$$

The case where the Nth column is not connected to the first one at all may be omitted, since it just insures conservation of total probability. If it is included, the matrix has 1 as largest eigenvalue and the second largest eigenvalue yields the correlation length, in exact correspondence with the transfer-matrix formulation for spin systems.

In the preceding, it is not stated if the sites of column N are connected to a given site of column 1, or if they are connected to anyone of them. This distinction just amounts to consider the action of the transfer-matrix on different vectors, and it makes no difference for the correlation length.

3. **Phenomenological renormalization.** — 3.1 PRINCIPLE OF THE METHOD. — Suppose the correlation length is known, as a function of the probability p, for two strips of different widths n and m ($n > m$). A correspondence between them may be established through a contraction by a factor n/m; for the correlation length to be contracted in the same ratio as the width, the probability p' for the strip of width m must be such that :

$$\xi_m(p') = (m/n)\, \xi_n(p) . \quad (2)$$

Following Nightingale [1], this equation may be regarded as a renormalization equation for the parameter p. Its content has been recently discussed by dos Santos and Sneddon [7].

The relation may be written as :

$$p' = R_{n,m}(p) . \quad (3)$$

Suppose the function $R_{n,m}(p)$ depends only on the ratio n/m, then it defines an exact renormalization scheme since it holds for n and m infinite and n/m fixed. The fixed point of eq. (2) or (3) is then the exact threshold and is given by :

$$\xi_m(p_c) = (m/n)\, \xi_n(p_c) . \quad (4)$$

The value of the critical exponent ν is :

$$\nu = \frac{\ln (n/m)}{\ln (\mathrm{d}p'/\mathrm{d}p \mid p_c)} . \quad (5)$$

In fact, $R_{n,m}(p)$ does not depend only on the ratio n/m and any choice of n and m yields an approximation. The best results are expected to be obtained for n large and $m = n - 1$ [1, 2].

3.2 RESULTS. — We have carried out the computation of the correlation length for the first few values of n, with two kinds of boundary conditions (periodic or free b.c.), for both bond and site percolation. The results of the phenomenological renormalization are given in table I for the bond problem and in table II for the site problem, and compared to the known or presently accepted values. The size of the matrix $M_n(S_n \times S_n)$ involved is also indicated, as a measure of the amount of work involved.

The numerical values so obtained are quite satisfactory, and several remarks may be made on the general trends :

— the matrices are smaller for the site problem. It might be feasible to study the convergence law of p_c

Table I. — *Results of the phenomenological renormalization for bond percolation, using strips of width n and m.* * *The value given for ν is a compromise between current conjectures* (refs. [5, 8]).

	Periodic b.c.			Free b.c.		
n-m	S_n	p_c	ν	S_n	p_c	ν
2-1	2	0.502 6	1.24	2	0.595 5	1.24
3-2	4	0.485 6	1.20	6	0.541 9	1.29
4-3	9	0.491 3	1.24			
Expected value		0.5	1.34 ± 0.02 *			

Table II. — *Results for site percolation.*

	Periodic b.c.			Free b.c.		
n-m	S_n	p_c	ν	S_n	p_c	ν
2-1	2	0.733 9	1.56	2	0.733 9	1.56
3-2	3	0.582 1	1.49	5	0.671 1	1.51
4-3	5	0.591 0	1.47	9	0.644 2	1.48
5-4	7	0.588 7	1.41			
Expected value [5]		0.593 1 ± 0.000 6	1.34 ± 0.02			

(bond), then use this knowledge to improve estimates of p_c (site) ;

— p_c converges faster for periodic boundary conditions, but its convergence is not monotonic while for free conditions the convergence is more regular, though slower. Larger sizes are needed to see which case is most suitable for extrapolation ;

— the critical exponent seems in general little affected by the choice of conditions.

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