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Statistical properties of randomly broken objects and of multivalley structures in disordered systems

B Derrida† and H Flyvbjerg‡

† Service de Physique Théorique, CEN Saclay, F-91191 Gif-sur-Yvette, France
‡ The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

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Abstract. The statistical properties of the multivalley structure of disordered systems and of randomly broken objects have many features in common. For all these problems, if W_s denotes the weight of the sth piece, we show that the probability distributions $P_1(W_1)$ of the largest piece W_1, $P_2(W_2)$ of the second largest piece, and $\Pi(Y)$ of $Y=\Sigma_s W_s^2$ always have singularities at $W_1=1/n$, $W_2=1/n$ and $Y=1/n$, $n=1,2,3,\ldots$.

1. Introduction

Most of the interesting properties of disordered systems are due to the fact that phase space is broken into many valleys. This many-valley structure is responsible for very slow relaxations, dependence on initial conditions and remanent effects. In several cases which have been studied recently (mean field theory of spin glasses (Mézard *et al* 1984a, b, 1985, Derrida and Toulouse 1985), random networks of automata (Derrida and Flyvbjerg 1986, 1987)) it has been shown that phase space is composed of several valleys. These different valleys have different weights. There are usually an infinite number of them but only a few have a large weight. Moreover, the weights of the valleys are not self-averaging, i.e. they fluctuate from sample to sample even in the thermodynamic limit.

In the present paper, we study the statistical properties of quantities which describe the multivalley structure of disordered systems and compare them with those of randomly broken objects.

We will consider here three different examples.

(i) The random map model (Derrida and Flyvbjerg 1986, 1987) which is a disordered system with deterministic dynamics. The model is just defined as a random map of a system of N points into themselves. For this problem, phase space is broken into several basins of attraction (or valleys). One can associate to the sth basin of attraction a weight W_s which is just its normalised size (i.e. W_s is the probability that a randomly chosen configuration belongs to the sth valley).

(ii) The mean field theory of spin glasses (Mézard *et al* 1984a, b, 1985, Derrida and Toulouse 1985) at thermal equilibrium. In the thermodynamic limit, the picture which comes out of the replica approach is that phase space is again composed of many valleys, the weight W_s of the sth valley being given by its free energy f_s as

$$W_s=\exp(-f_s/T). \tag{1}$$

(iii) The problem of a randomly broken interval. We break the unit interval into an infinite number of pieces by a random process. We call the length of the sth piece W_s.

These three examples share the property that the breaking is sample dependent. Several quantities have already been studied for these problems: for example, the average number of pieces at weight W or the correlation functions of weights. In the present work we try to go further in the study of the statistical properties of such broken objects and show that many behaviours are very similar in these problems. In particular, we will see that the probability distributions of several quantities have singularities which are always located at the same place.

The paper is organised as follows. In § 2 we define the basic quantities that we are going to study or use: the functions $g(W)$, $g(W, W'), \ldots$, the functions $f(W)$, $f(W, W'), \ldots$, the probability distributions $P_1(W)$ of the largest weight, $P_2(W)$ of the second largest weight and $\Pi(Y)$ of Y where Y is defined by

$$Y = \sum_s W_s^2 \tag{2}$$

where in (2) the sum runs over all the weights W_s. In § 3 we define a model of random breaking of an interval and we compute for it the functions g defined in § 2.

In § 4 we describe how the probability distributions $P_1(W)$, $P_2(W)$ and $\Pi(Y)$ can be obtained numerically for our three examples. In § 5 we show how the singularities at $W = \frac{1}{2}, \frac{1}{3}, \ldots$ in $P_1(W)$, $P_2(W)$ can be understood. Lastly in § 6 we compute the singular behaviours of $\Pi(Y)$ at $Y = \frac{1}{2}, \frac{1}{3}, \ldots$.

2. Definitions

Let us consider an object of normalised size (for example, phase space of a disordered system or more generally any broken object). Once the system is broken into many pieces, the sth piece having a weight W_s, one has of course

$$\sum_s W_s = 1 \tag{3}$$

because of the normalisation.

For each sample, i.e. for each way of breaking the object, one can introduce a function $H(W)$ defined by

$$H(W) = \sum_s W_s \delta(W - W_s). \tag{4}$$

$H(W)$ depends of course on the breaking, so there is a probability distribution of $H(W)$. One can then define $g(W)$, $g(W, W')$ and $g(W, W', W'')$ as

$$g(W) = \overline{H(W)} \tag{5}$$

$$g(W, W') = \overline{H(W)H(W')} \tag{6}$$

$$g(W, W', W'') = \overline{H(W)H(W')H(W'')} \tag{7}$$

where in (5)-(7) the bar denotes the average over all the possible breakings of the object.

In, for example, the language of the random map $g(W)$, $g(W, W')$ and $g(W, W', W'')$ have very simple interpretations: $g(W)$ is the average number of times that a randomly chosen configuration belongs to a basin of weight W and $g(W, W')$

is the average number of times that two randomly chosen configurations belong to basins of weights W and W' respectively, and so on.

These functions $g(W)$, $g(W, W')$, ..., are related to the average number $f(W, W')$ of pairs of different pieces having weights W and W':

$$f(W) = \sum_s \overline{\delta(W - W_s)} = \frac{g(W)}{W} \tag{8}$$

$$\begin{aligned} f(W, W') &= \sum_s \sum_{s' \neq s} \overline{\delta(W - W_s)\delta(W' - W_{s'})} \\ &= \frac{g(W, W')}{WW'} - \frac{g(W)\delta(W - W')}{W}. \end{aligned} \tag{9}$$

The functions f or g have already been obtained for the random map (RM) model (Derrida and Flyvbjerg 1986, 1987)

$$f_{\mathrm{RM}}(W) = \tfrac{1}{2} W^{-1}(1 - W)^{-1/2} \tag{10a}$$

$$f_{\mathrm{RM}}(W, W') = \tfrac{1}{4}(WW')^{-1}(1 - W - W')^{-1/2} \tag{10b}$$

and for the infinite-range spin glass (SG) (using the replica approach of Mézard *et al* (1984a, b))

$$f_{\mathrm{SG}}(W) = \frac{W^{y-2}(1 - W)^{-y}}{\Gamma(y)\Gamma(1-y)} \tag{11a}$$

$$f_{\mathrm{SG}}(W, W') = \frac{(1-y)(WW')^{y-2}(1 - W - W')^{1-2y}}{\Gamma(y)\Gamma(y)\Gamma(2-2y)} \tag{11b}$$

$$\begin{aligned} &f_{\mathrm{SG}}(W_1, W_2, \ldots, W_K) \\ &\quad = \frac{(1-y)^{K-1}\Gamma(\mathrm{K})}{\Gamma^K(y)\Gamma(K - Ky)} \left(\prod_{i=1}^{K} W_i\right)^{y-2} \left(1 - \sum_{i=1}^{K} W_i\right)^{K-Ky-1} \end{aligned} \tag{11c}$$

where y $(0 < y < 1)$ is a parameter which contains all the physical parameters (temperature, magnetic field, etc).

In the next section we will give explicit expressions of f for the problem of breaking the interval (BI).

There are many quantitites which characterise the multivalley structure: for a given sample, let W_1 denote the largest weight, W_2 the next largest weight, and so on. Of course $W_1, W_2, \ldots$, are sample dependent and we will let $P_1(W_1)$ denote the probability distribution of W_1, $P_2(W_2)$ the probability distribution of W_2, and so on.

One can also consider the quantity Y defined by (2) or generalisations Y_P of it:

$$Y_P = \sum_s W_s^P \tag{12}$$

$(Y = Y_2)$. Again Y and Y_P are sample dependent and are characterised by probability distributions $\Pi(Y)$ of Y and $\Pi_P(Y_P)$ for Y_P.

In § 4 we show the shapes of $P_1(W)$, $P_2(W)$ and $\Pi(Y)$ for our three examples. Let us describe first an example of interval breaking and compute the functions g corresponding to this example.

3. An example of random breaking of an interval

There are several ways of defining a random process which can break an interval into several pieces. In this section we choose one which has the advantage of showing interesting properties with simple solutions.

We start by breaking an interval into two pieces, one of weight W_1 and the other of weight $W = 1 - W_1$. We keep the piece of weight W_1 as the first pieces of our sample. At the second step we break the piece W into two pieces, one of weight W_2 which we keep and another piece of weight $W' = W - W_2$ that we are going to break at the third step and so on. After n steps, we have n pieces of our final sample: $W_1, W_2, \ldots, W_n$ and a piece W which will be broken at the next step. To make the model simple, we consider only processes which are self-similar in the following sense: the probability that W is broken into W_{n+1} and $W - W_{n+1}$ depends only on the ratio W_{n+1}/W. Therefore after an infinite number of steps the system consists of an infinite number of pieces which can be described in the following way:

$$\begin{aligned} W_1 &= x_1 \\ W_2 &= (1-x_1)x_2 \\ W_3 &= (1-x_1)(1-x_2)x_3 \\ &\vdots \\ W_n &= (1-x_1)\ldots(1-x_{n-1})x_n \\ &\vdots \end{aligned} \tag{13}$$

where all the numbers $x_1, x_2, \ldots, x_n$ are randomly distributed according to the same probability distribution $\rho(x)$. It is then clear that if $\tilde{H}(W)$ and $H(W)$ are defined by

$$\tilde{H}(W) = \sum_{s=2}^{\infty} \frac{W_s}{1-W_1} \delta\left(W - \frac{W_s}{1-W_1}\right) \tag{14a}$$

$$\begin{aligned} H(W) &= \sum_{s=1}^{\infty} W_s \delta(W - W_s) \\ &= W\delta(W - W_1) + \tilde{H}\left(\frac{W}{1-W_1}\right) \end{aligned} \tag{14b}$$

then $H(W)$ and $\tilde{H}(W)$ have the same probability distribution. Therefore

$$g(W) = \overline{H(W)} = \overline{\tilde{H}(W)} \tag{15a}$$

$$g(W, W') = \overline{H(W)H(W')} = \overline{\tilde{H}(W)\tilde{H}(W')} \tag{15b}$$

etc.

It is then easy to see that $g(W), g(W, W'), \ldots,$ must satisfy the following integral equations:

$$g(W) = W\rho(W) + \int \rho(x) g\left(\frac{W}{1-x}\right) \mathrm{d}x \tag{16a}$$

$$\begin{aligned} g(W, W') = {} & W^2 \rho(W) \delta(W - W') \\ & + W\rho(W) g\left(\frac{W'}{1-W}\right) + W'\rho(W') g\left(\frac{W}{1-W'}\right) \\ & + \int \rho(x) g\left(\frac{W}{1-x}, \frac{W'}{1-x}\right) \mathrm{d}x \end{aligned} \tag{16b}$$

etc. When solving these equations, one should remember that $g(W)=0$ for $W>1$, and $g(W, W')=0$ for $W+W'>1$.

For a general distribution $\rho(W)$, equations (16) are not simple to solve. However, there are some choices of $\rho(W)$ for which $g(W), g(W, W'), \ldots,$ have simple expressions, e.g., when one chooses

$$\rho(W)=(\alpha+1)(1-W)^{\alpha} \tag{17}$$

($\alpha=0$ corresponding to a uniform distribution), one finds for the broken interval (BI)

$$g_{\mathrm{BI}}(W)=\rho(W)=(\alpha+1)(1-W)^{\alpha} \tag{18a}$$

$$g_{\mathrm{BI}}(W, W')=(\alpha+1)W(1-W)^{\alpha}\delta(W-W')+(\alpha+1)^2(1-W-W')^{\alpha} \tag{18b}$$

etc. For the choice (17), one finds $g_{\mathrm{BI}}(W)=\rho(W)$. This simple relation between g and ρ is particular to the example (17). There is no reason why it should hold for other choices of ρ.

4. Shapes of distributions $P_1(W)$, $P_2(W)$ and $\Pi(Y)$

In this section, we show how the probability distributions $P_1(W_1)$ of the largest weight W_1, $P_2(W_2)$ of the second largest weight W_2, and $\Pi(Y)$ can be obtained easily by Monte Carlo methods.

Let us start with the random map. In our previous work (Derrida and Flyvbjerg 1986, 1987) we have already given a way of constructing numerically $P_1(W)$ and $\Pi(Y)$, as follows. Take a random map T of a set of M points into themselves. Any point has a probability $g_{\mathrm{RM}}(W)$ of belonging to a basin of weight W. Remove this basin. There remains a random map $\tilde{T}$ of a set of $M(1-W)$ points into themselves. In other words, if $\tilde{T}$ is a random map which has basins of weights $W_1, W_2, \ldots, W_n, \ldots,$ then the map T has basins of weights $W, W_1(1-W), W_2(1-W), \ldots, W_n(1-W), \ldots$. It is of course easy to relate the properties of the map T to those of the map $\tilde{T}$. If $\tilde{W}_{\max}$ and $\tilde{W}'_{\max}$ are the two largest weights of $\tilde{T}$ and $W_{\max}$ and $W'_{\max}$ are those of T, then one has

$$\begin{aligned} W_{\max} &= \max(W, (1-W)\tilde{W}_{\max}) \\ W'_{\max} &= \max[\min(W, (1-W)\tilde{W}_{\max}), (1-W)\tilde{W}'_{\max}]. \end{aligned} \tag{19}$$

Similarly Y and $\tilde{Y}$ are related by

$$Y=W^2+(1-W)^2\tilde{Y}.$$

One can thus construct in this way a sequence of N samples very easily by an iterative procedure. Suppose that we have built the nth sample and we know its properties $W_{\max}(n)$, $W'_{\max}(n)$, $Y(n)$, etc, then one chooses a weight W_n at random according to the probability distribution $g_{\mathrm{RM}}(W)=\frac{1}{2}(1-W)^{-1/2}$ and then one has

$$\begin{aligned} W_{\max}(n+1) &= \max[(1-W_n)W_{\max}(n), W_n] \\ W'_{\max}(n+1) &= \max[\min((1-W_n)W_{\max}(n), W_n), (1-W_n)W'_{\max}(n)] \\ Y(n+1) &= W_n^2+(1-W_n)^2Y(n). \end{aligned} \tag{20}$$

By iterating this procedure sufficiently many times, one can get $P_1(W)$, $P_2(W)$ and $\Pi(Y)$ as the histograms of $W_{\max}(n)$, $W'_{\max}(n)$ and $Y(n)$. The results obtained after $N=10^7$ iterations are shown in figure 1.

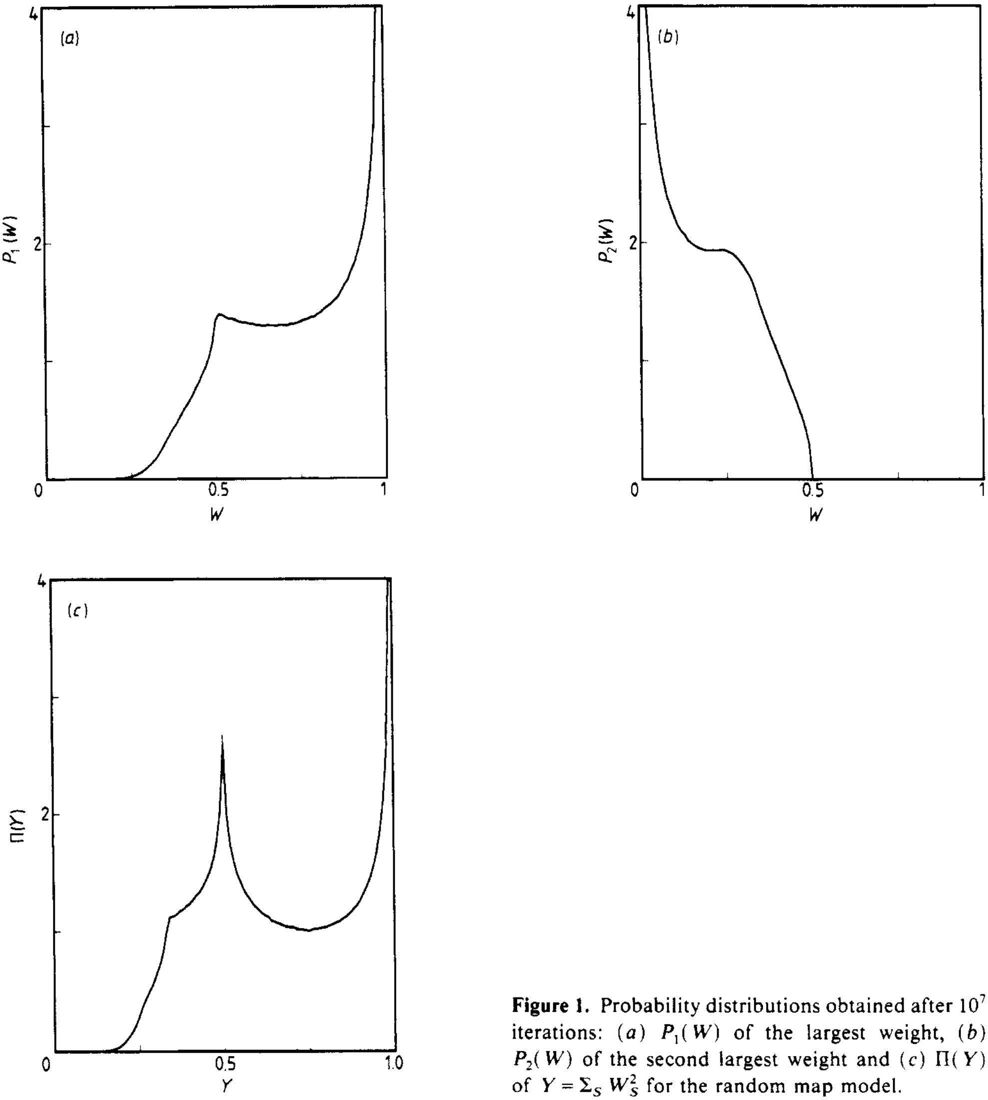

Figure 1. Probability distributions obtained after 10^7 iterations: (a) $P_1(W)$ of the largest weight, (b) $P_2(W)$ of the second largest weight and (c) $\Pi(Y)$ of $Y = \Sigma_S W_S^2$ for the random map model.

The case of the breaking of the interval is described by similar equations. Again, to go from one sample to the next, one needs only to iterate (20) (this is easily seen by looking at (13)). The only difference from the random map is that in (20) W_n is chosen according to the probability $\rho(W)$. In figure 2 we show $P_1(W)$, $P_2(W)$ and $\Pi(Y)$ obtained after $N = 10^7$ in the case of $\rho(x) = 1$.

The case of the mean field theory of spin glasses is a little more difficult because we did not find a way of constructing a new sample from an old one by just adding one weight. However, there is an easy way of constructing samples, from the knowledge (11c) of $f_{SG}(W_1, W_2, \ldots, W_K)$ for all K, by the following procedure. Choose a sequence of random numbers $x_1, x_2, \ldots, x_n$ where x_n is chosen according to the probability distribution $\rho_n(x)$ given by

$$\rho_n(x_n) = \frac{\Gamma(n - ny + y)}{\Gamma(n - ny)\Gamma(y)} x_n^{y-1}(1 - x_n)^{n(1-y)-1}. \tag{21}$$

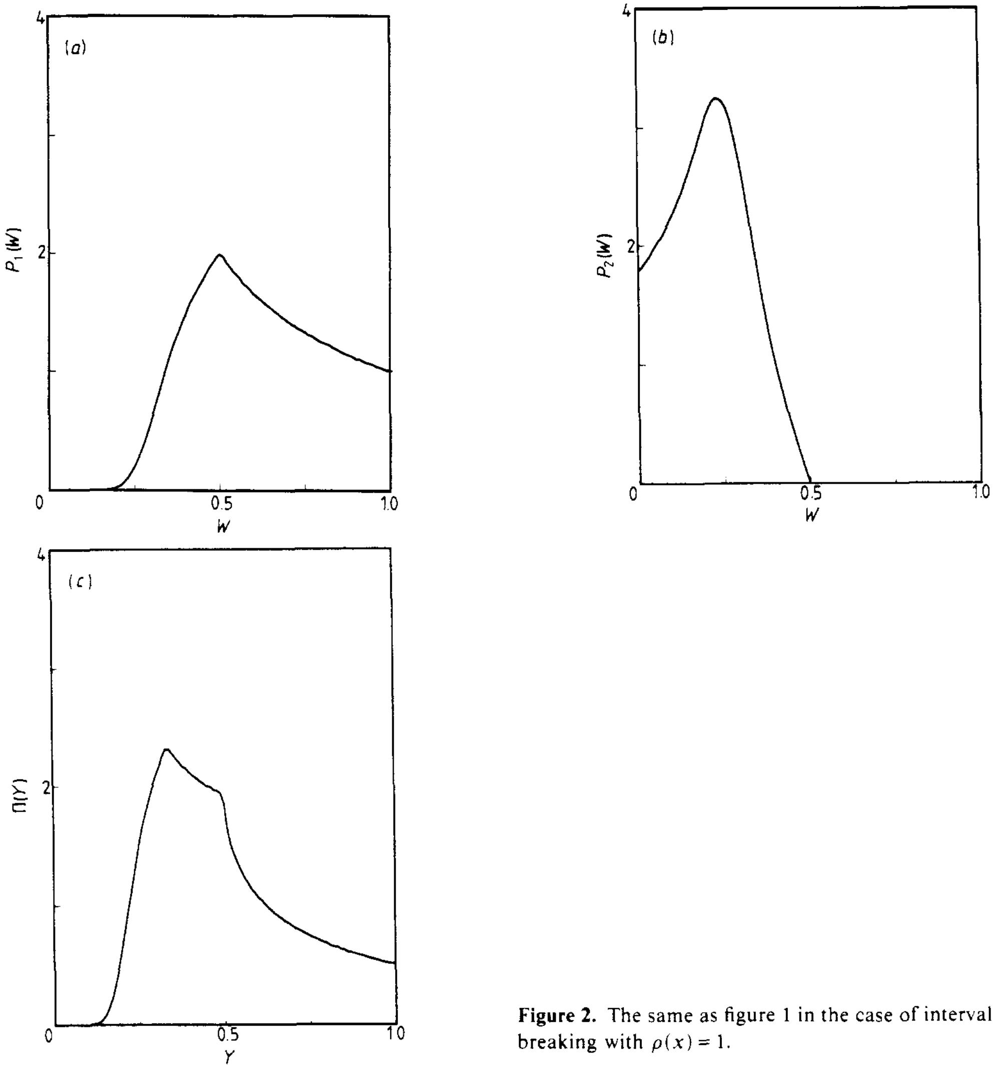

Figure 2. The same as figure 1 in the case of interval breaking with $\rho(x) = 1$.

Then build the sample for the spin glass problem by calculating the weights $W_1, W_2, \ldots, W_n$ by

$$\begin{aligned} W_1 &= x_1 \\ W_2 &= (1 - W_1)x_2 \\ &\vdots \\ W_n &= (1 - W_1 - W_2 - \ldots - W_{n-1})x_n. \end{aligned} \tag{22}$$

The way in which (21) and (22) can be related to the expression (11*c*) of $f_{SG}(W_1, W_2, \ldots, W_K)$ is quite easy to understand, as follows. Once the first $K-1$ weights have been constructed, using $f_{SG}(W_1, W_2, \ldots, W_K)$, then one knows the probability distribution of W_K (which depends on K and $W_1, W_2, \ldots, W_{K-1}$). More precisely, from $f(W)$ one knows that the probability $R_1(W_1)$ that the physical system is in a valley of weight W_1 is

$$R_1(W_1) = W_1 f(W_1).$$

Now assume that we know that there are valleys of weights $W_1, W_2, \ldots, W_{K-1}$ and we want to know the probability $R_K(W_K)$ that the system is in the Kth valley of weight W_K. Then one has

$$R_K(W_K) = \frac{W_K f_K(W_1, W_2, \ldots, W_{K-1}, W_K)}{\int \mathrm{d}W\, W f_K(W_1, W_2, \ldots, W_{K-1}, W)}$$

$$= \frac{\Gamma(K - Ky + y)}{\Gamma(y)\Gamma(K - Ky)} \left(\frac{W_K}{1-S}\right)^{y-1} \left(1 - \frac{W_K}{1-S}\right)^{K-Ky-1} \left(\frac{1}{1-S}\right) \tag{23}$$

where

$$S = 1 - W_1 - W_2 - \ldots - W_{K-1}. \tag{24}$$

This means that to choose W_K one needs only to choose a number x_K at random distributed according to $\rho_K(x)$ given by (21) and then $W_K = (1 - W_1 - W_2 - \ldots - W_{K-1})x_K$.

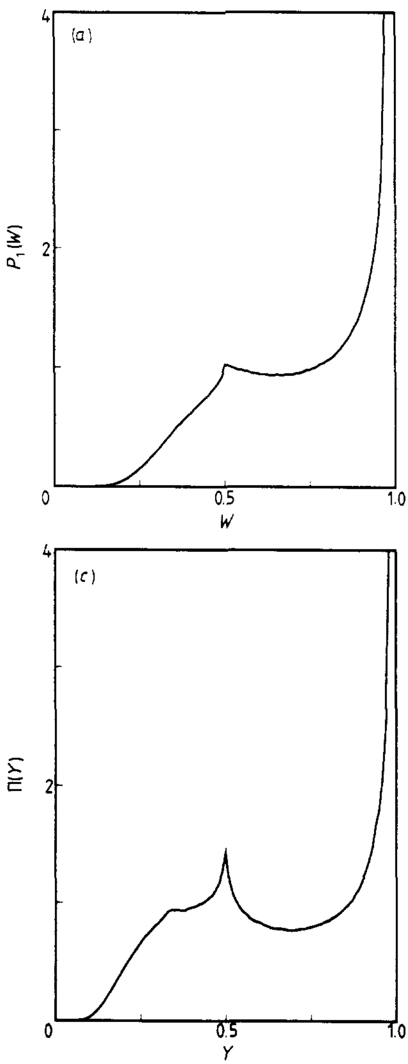

Figure 3. The same as figure 1 for the infinite-range spin glass with $y = 0.7$.

In figures 3 and 4 we show $P_1(W)$, $P_2(W)$ and $\Pi(Y)$ obtained as the histograms of the $W_{\max}$, $W'_{\max}$ and Y of $N=10^7$ samples ($y=0.7$ for figure 3 and $y=0.9$ for figure 4). For each sample, one needs in principle to build an infinite number of weights W_s. In practice, we construct only a finite number S of weights W_s and we stop our calculations when

$$\sum_{s=1}^{S} W_s > 1-\varepsilon \qquad \text{with } \varepsilon = 10^{-2}.$$

Then the error on Y does not exceed ε^2 and the error on $W_{\max}$ and $W'_{\max}$ is usually zero because all the weights W_s for $s>S$ would not exceed ε.

From a practical point of view, there is a technical difficulty in generating the random numbers x_n distributed according to the distribution $\rho_n(x_n)$ given by (21) because $\int_0^y \rho_n(x)\,dx$ is not a simple function of y. To do so, we have used a rejection method which is described by Knuth (1981).

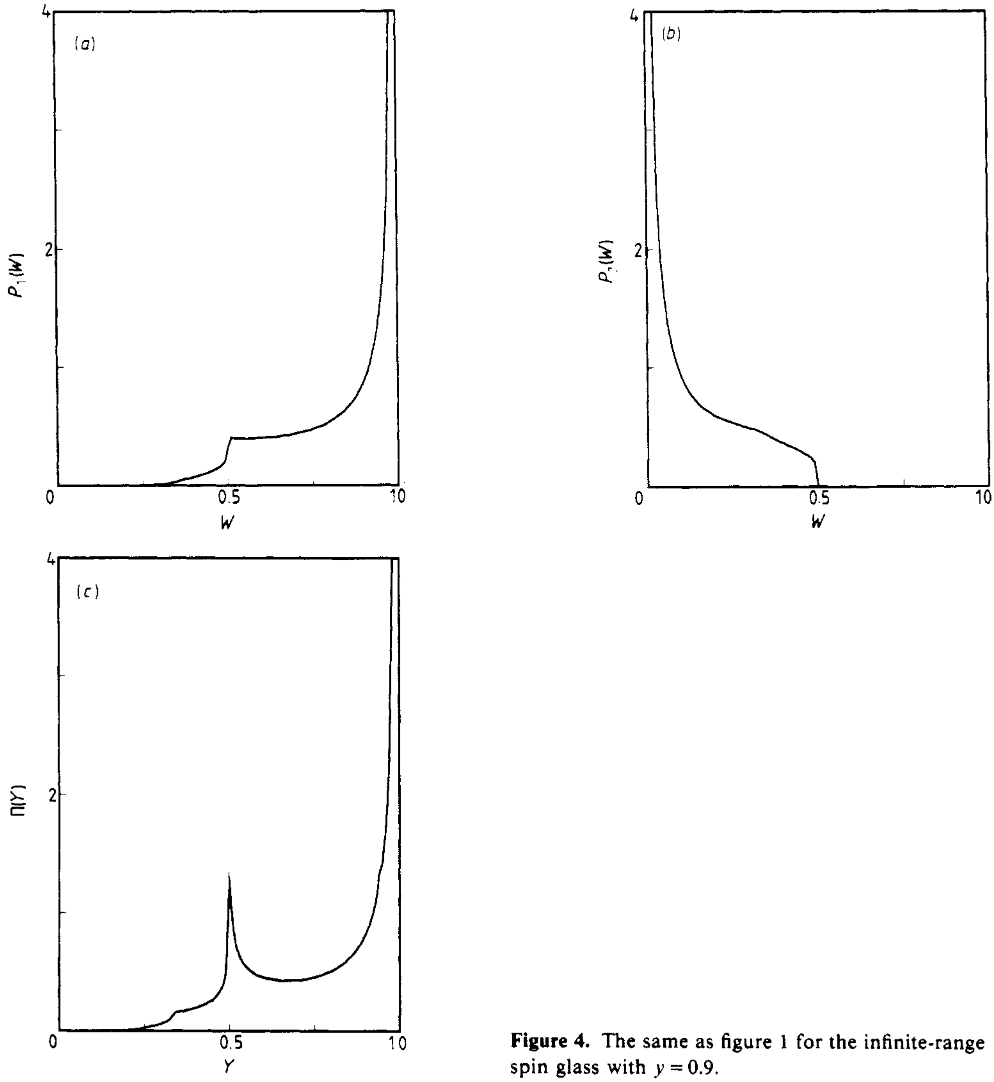

Figure 4. The same as figure 1 for the infinite-range spin glass with $y=0.9$.

The shapes of $\Pi(Y)$ for $y=0.7$ and 0.9 have already been given by Mézard *et al* (1984a, b, 1985). The shapes they found are rather similar to those of our figures 3 and 4. However, they did not find the singularities at $Y=\frac{1}{2}$ and $Y=\frac{1}{3}$ which can be seen on our figures 3 and 4. This is because they computed an approximate $\Pi(Y)$ from the knowledge of the first moments of Y. Such approximate methods are not well adapted to observe singularities in $\Pi(Y)$.

By comparing figures 1, 2 and 3 we see that our three examples give very similar shapes for $P_1(W)$, $P_2(W)$ and $\Pi(Y)$. All these distributions seem to have singular behaviours at $\frac{1}{2}$ and $\frac{1}{3}$. The purpose of the next sections is to explain these singularities and to discuss how general they are.

5. Singular behaviour in the distributions $P_1(W)$ and $P_2(W)$ of the largest and second largest weights

In this section we see that the distributions $P_1(W)$, $P_2(W)$, $P_3(W)$, ..., can be computed analytically from the knowledge of $f(W)$, $f(W, W')$ and $f(W, W', W'')$ and that these distributions have in general singularities at $W=1, \frac{1}{2}, \frac{1}{3}, \ldots, 1/n, \ldots$.

If there is a piece with weight $W>\frac{1}{2}$, this weight must be the largest one. Therefore, one always has

$$P_1(W)=f(W) \qquad \text{for } W>\tfrac{1}{2}. \tag{25}$$

If there is a piece with weight W, for $\frac{1}{3}<W<\frac{1}{2}$, this piece can either be the largest one or the second largest one. Therefore

$$P_1(W)+P_2(W)=f(W) \qquad \text{for } \tfrac{1}{3}<W<\tfrac{1}{2} \tag{26}$$

etc, and

$$P_1(W)+P_2(W)+\ldots+P_n(W)=f(W) \qquad \text{for } 1/(n+1)<W<1/n \tag{27}$$

where $P_n(W)$ is the probability distribution that the nth largest weight is W.

For similar reasons, if one defines $Q_{n,m}(W, W')$ as the probability distribution that the nth largest piece has weight W and the mth largest piece has weight W', one has

$$Q_{1,2}(W, W')=f(W, W') \qquad \text{for } W>W'>\tfrac{1}{3} \tag{28}$$

$$Q_{1,2}(W, W')+Q_{1,3}(W, W')+Q_{2,3}(W, W')=f(W, W') \qquad \text{for } W>W'>\tfrac{1}{4} \tag{29}$$

and so on

$$\sum_{1\leq n<m\leq p} Q_{n,m}(W, W')=f(W, W') \qquad \text{for } W>W'>1/(p+1). \tag{30}$$

More generally, if one denotes by $R_{n_1,n_2,\ldots,n_K}(W_1, W_2, \ldots, W_K)$ the probability that the n_1th largest piece has weight W_1, the n_2th largest piece has weight W_2, etc, then one has

$$\sum_{1\leq n_1<n_2<\ldots<n_K\leq p} R_{n_1,n_2,\ldots,n_K}(W_1, W_2, \ldots, W_K)$$
$$=f(W_1, W_2, \ldots, W_K) \qquad \text{for } W_1>W_2>\ldots>W_K>1/(p+1). \tag{31}$$

From these relations it is possible to obtain $P_1(W)$, $P_2(W)$, $P_3(W)$, ..., in the various intervals. For example, using (28) one has

$$P_2(W)=\int_W^1 \mathrm{d}x\, f(x, W) \qquad \text{for } \tfrac{1}{3}<W \tag{32}$$

which expresses that there is a piece whose weight is larger than W and therefore from (26) one obtains

$$P_1(W) = f(W) - \int_W^1 \mathrm{d}x\, f(x, W) \qquad \text{for } \tfrac{1}{3} < W. \tag{33}$$

Using (31) for $\frac{1}{4} < W < W'$ one obtains in the same way

$$\begin{aligned} P_3(W) &= \int_W^1 \mathrm{d}x \int_W^x \mathrm{d}y\, f(x, , y, W) \\ Q_{2,3}(W', W) &= \int_{W'}^1 \mathrm{d}x\, f(x, W', W) \\ Q_{1,3}(W', W) &= \int_W^{W'} \mathrm{d}x\, f(W', x, W) \\ Q_{1,2}(W', W) &= f(W', W) - Q_{1,3}(W', W) - Q_{2,3}(W', W) \end{aligned} \tag{34}$$

and therefore for $\frac{1}{4} < W < W'$ one obtains

$$\begin{aligned} P_2(W) &= \int_W^1 \mathrm{d}x\, Q_{1,2}(x, W) \\ &= \int_W^1 \mathrm{d}x\, f(x, W) - 2\int_W^1 \mathrm{d}x \int_W^x \mathrm{d}y\, f(x, y, W) \end{aligned} \tag{35}$$

and

$$P_1(W) = f(W) - \int_W^1 \mathrm{d}x\, f(x, W) + \int_W^1 \mathrm{d}x \int_W^x \mathrm{d}y\, f(x, y, W). \tag{36}$$

These results can be generalised to any interval

$$1/(n+1) < W < 1/n. \tag{37}$$

After some calculations one finds that on this interval the probability $P_j(W)$ that the jth largest piece has a weight W is given by

$$P_j(W) = \sum_{i=j}^{n} (-1)^{i-j} \frac{(i-1)!}{(i-j)!(j-1)!} I_i(W) \tag{38}$$

where

$$I_i(W) = \int_W^1 \mathrm{d}x_1 \int_W^{x_1} \mathrm{d}x_2 \ldots \int_W^{x_{i-2}} \mathrm{d}x_{i-1}\, f(x_1, x_2, \ldots, x_{i-1}, W). \tag{39}$$

We will not give the derivation of (38) here: it is a direct generalisation of the derivation which led to (33), (35) and (36). Let us just mention that (38) can be obtained as a consequence of

$$I_i(W) = \sum_{j=i}^{n} \frac{(j-1)!}{(i-1)!(j-i)!} P_j(W). \tag{40}$$

We see from formulae (25), (32), (36) and (38) that the expression for $P_1(W)$ is different on each interval $1/(n+1) < W < 1/n$. The same is true for P_2, P_3 and all the

P_j. Since the analytic expressions of the P_j are different in the different intervals $1/(n+1) < W < 1/n$, one expects to see singularities of $P_j(W)$ at all the values $W = 1/n$. These singularities are, however, weaker and weaker as n increases: they are due to the fact that $I_{n+1}(W) = 0$ for $W > 1/n$ and $I_{n+1}(W) \neq 0$ for $W < 1/n$, so when n increases the singularity due to $I_n(W)$ is weaker and weaker because $I_n(W)$ is an integral over more and more variables.

As an example let us give the expressions for $P_1(W)$ and $P_2(W)$ for our three examples. For $W > \frac{1}{2}$

$$P_1^{\mathrm{RM}}(W) = \frac{1}{2W} \frac{1}{(1-W)^{1/2}} \tag{41a}$$

$$P_1^{\mathrm{SG}}(W) = \frac{W^{y-2}(1-W)^{-y}}{\Gamma(y)\Gamma(1-y)} \tag{41b}$$

$$P_1^{\mathrm{BI}}(W) = (\alpha+1)(1-W)^{\alpha}/W \tag{41c}$$

and for $\frac{1}{3} < W < \frac{1}{2}$

$$P_2^{\mathrm{RM}}(W) = \frac{1}{4W} \frac{1}{(1-W)^{1/2}} \left\{ \log\left[1 + \left(1 - \frac{W}{1-W}\right)^{1/2}\right] - \log\left[1 - \left(1 - \frac{W}{1-W}\right)^{1/2}\right] \right\} \tag{42a}$$

$$P_2^{\mathrm{SG}}(W) = \int_W^1 f^{\mathrm{SG}}(W', W)\, \mathrm{d}W' \tag{42b}$$

$$P_2^{\mathrm{BI}}(W) = \frac{1}{W} \log\left(\frac{1-W}{W}\right) \tag{42c}$$

$$P_1(W) = f(W) - P_2(W) \tag{42d}$$

where (42c) for the breaking of the interval is the expression in the case $\rho(W) = 1$.

These expressions agree with the numerical results given in § 4.

One should notice that there could be other singularities in $P_1(W), P_2(W), \ldots$, or the singularities at $W = 1/n$ could disappear if the functions $f(W_1, \ldots, W_K)$ had some non-analyticities in the domain $W_1 \geq 0, \ldots, W_K \geq 0$, $1 - W_1 - \ldots - W_K \geq 0$. For the three examples we have described here the functions f are analytic and therefore one does not expect other singularities.

6. Singularities in $\Pi(Y)$

In the previous section we have seen that, knowing the first K functions $f(W_1), f(W_1, W_2), \ldots, f(W_1, W_2, \ldots, W_K)$, it is possible to calculate the distributions $P_1(W), \ldots, P_K(W)$ for $W > 1/(K+1)$.

The knowledge of a finite number of functions $f(W)$ does not allow us to calculate parts of $\Pi(Y)$. There is, however, some information about $\Pi(Y)$ contained in these functions, for example, some moments of $\Pi(Y)$ and $\Pi_P(Y_P)$

$$\bar{Y} = \int f(W)\, W^2\, \mathrm{d}W = \int g(W)\, W\, \mathrm{d}W \tag{43}$$

$$\overline{Y^2} = \int g(W, W')\, WW'\, \mathrm{d}W\, \mathrm{d}W' \tag{44}$$

and more generally

$$\overline{(Y_P)^n} = \int \ldots \int g(W_1, \ldots, W_n) W_1^{P-1} \ldots W_n^{P-1} \, \mathrm{d}W_1 \ldots \mathrm{d}W_n \tag{45}$$

but, of course, the singularities of $\Pi(Y)$ or $\Pi_P(Y_P)$ cannot be extracted from the knowledge of a finite number of moments.

We believe that the singularities in $\Pi(Y)$ that we observed in § 4 are quite general and occur in all kinds of problems of random breaking. We did not, however, find a general proof of their existence, so we will discuss these singularities only in the restricted case of our three examples.

Let us first discuss the case of the random map and of the random breaking of the interval. We have seen in § 4 that the distribution $\Pi(Y)$ was nothing but the histogram of the $Y(n)$ where the sequence $Y(n)$ is constructed by the random process (20)

$$Y(n+1) = W_n^2 + (1 - W_n)^2 Y(n) \tag{46}$$

where the W_n are randomly chosen according to a given distribution $\rho(W)$ which is arbitrary for the problem of randomly breaking the interval and which is given by

$$\rho_{\mathrm{RM}}(W) = \tfrac{1}{2}(1 - W)^{-1/2} \tag{47}$$

for the random map problem.

So $Y(n)$ and $Y(n+1)$ have the same limiting distribution $\Pi(Y)$ for $n \to \infty$ and from (46) one sees that $\Pi(Y)$ obeys the integral equation

$$\begin{aligned}\Pi(Y) &= \int_0^1 \rho(W)\, \mathrm{d}W \int_0^1 \Pi(Y')\, \mathrm{d}Y' \, \delta[Y - W^2 - (1 - W)^2 Y'] \\ &= \int_0^{\prime} \frac{\rho(W)\, \mathrm{d}W}{(1 - W)^2} \Pi\left(\frac{Y - W^2}{(1 - W)^2}\right)\end{aligned} \tag{48}$$

which can be rewritten as

$$\Pi(Y) = \int_0^{Y^{1/2}} \frac{\rho(W)\, \mathrm{d}W}{(1 - W)^2} \Pi\left(\frac{Y - W^2}{(1 - W)^2}\right) \qquad \text{for } Y < \tfrac{1}{2} \tag{49a}$$

$$\Pi(Y) = \left(\int_0^{W_-} + \int_{W_+}^{Y^{1/2}}\right) \frac{\rho(W)\, \mathrm{d}W}{(1 - W)^2} \Pi\left(\frac{Y - W^2}{(1 - W)^2}\right) \qquad \text{for } Y > \tfrac{1}{2} \tag{49b}$$

where

$$W_\pm = \tfrac{1}{2}[1 \pm (2Y - 1)^{1/2}]. \tag{49c}$$

Equation (48) can be reduced to (49) because $\Pi(Y)$ and $\rho(W)$ have their support on [0, 1].

For Y close to 1, $Y = 1 - \varepsilon$, $\varepsilon \sim 0$, $W_+ = 1 - \frac{1}{2}\varepsilon - \frac{1}{4}\varepsilon^2 + \mathrm{O}(\varepsilon^3)$. Therefore, and because Π is normalised, it follows from (48) that when $\alpha \leqslant 0$

$$\Pi(1 - \varepsilon) = \frac{4}{\varepsilon^2} \rho(1 - \tfrac{1}{2}\varepsilon) \int_{1 - \varepsilon/2 - \varepsilon^2/4}^{1 - \varepsilon/2 - \varepsilon^2/8} \mathrm{d}W \, \Pi\left(\frac{1 - \varepsilon - W^2}{(1 - W)^2}\right) = \tfrac{1}{2}\rho(1 - \tfrac{1}{2}\varepsilon) \tag{50}$$

because the integral from 0 to W is negligible. For the breaking of the interval according to the distribution $\rho(W)$ given in (17) (or according to any *other* distribution $\rho(W)$ with exponent α in $W = 1$) we see that

$$\Pi_{\mathrm{BI}}(Y \sim 1) \propto (1 - Y)^\alpha + \mathrm{O}[(1 - Y)^{\alpha + 1}]. \tag{51}$$

For the random map, for which $\alpha=-\frac{1}{2}$, we have

$$\Pi_{\mathrm{RM}}(Y\sim 1)=\frac{1}{[8(1-Y)]^{1/2}}+\mathrm{O}[(1-Y)^{1/2}]. \tag{52}$$

Since (48) gives different relations for $\Pi(Y)$ according to whether Y is larger or smaller than $\frac{1}{2}$ it is not surprising that $\Pi(Y)$ is singular at $Y=\frac{1}{2}$. This singularity may be viewed as 'inherited' from the singularity at $Y=1$: for $Y=\frac{1}{2}$ the function $(Y-W^2)/(1-W)^2$ has a maximum value of 1 occurring at $W=\frac{1}{2}$. Consequently the integral in (48) picks up a contribution from $\Pi(Y\sim 1)$. With $\Pi^{(\mathrm{sing})}$ denoting the dominant part of the singularity in $\Pi(Y)$ we have

$$\Pi^{(\mathrm{sing})}(\tfrac{1}{2}\varepsilon)\propto\int \mathrm{d}W\,\Pi^{(\mathrm{sing})}\{(1+4\varepsilon)[1-8(W-\tfrac{1}{2})^2+\mathrm{O}((W-\tfrac{1}{2})^3)]\}. \tag{53}$$

So for $\Pi^{(\mathrm{sing})}(Y\sim 1)\propto|1-Y|^{\alpha}$ we have

$$\int_b \mathrm{d}W\,|4\varepsilon-8(W-\tfrac{1}{2})^2|^{\alpha}\propto\varepsilon^{\alpha+1/2} \tag{54}$$

where $b=\frac{1}{2}$ for $\varepsilon<0$ and $b=\frac{1}{2}+(\varepsilon/2)^{1/2}$ for $\varepsilon>0$. Equation (54) is obtained for any $\rho(W)$ with exponent α in $W=1$. For the random map $\alpha=-\frac{1}{2}$ and (54) signals a logarithmic singularity.

Now let us consider more carefully the function

$$Y'(W)=(Y-W^2)/(1-W)^2. \tag{55}$$

It is maximal at $W=Y$ with the value

$$Y'_{\max}=Y/(1-Y). \tag{56}$$

We have already seen that, since $Y'_{\max}=1$ for $Y=\frac{1}{2}$, the singularity of $\Pi(Y)$ at $Y=1$ gives rise to a singularity at $Y=\frac{1}{2}$. In general $Y'_{\max}=1/n$ for $Y=1/(n+1)$, giving rise to singularities at $Y=1/n$, $n=1,2,3,\ldots$:

$$\Pi^{(\mathrm{sing})}(1/n+\varepsilon)\propto\int \mathrm{d}W\,\Pi^{(\mathrm{sing})}\left(\frac{1}{n+1}\left[1+\frac{n^2}{n-1}\varepsilon\right]\left\{1-\frac{n^3}{(n-1)^2}\left(W-\frac{1}{n}\right)^2+\mathrm{O}\left[\left(W-\frac{1}{n}\right)^3\right]\right\}\right). \tag{57}$$

Thus for

$$\Pi^{(\mathrm{sing})}\left(Y\sim\frac{1}{n-1}\right)\propto\left(Y-\frac{1}{n-1}\right)^{\beta} \tag{58}$$

we have

$$\Pi^{(\mathrm{sing})}\left(\frac{1}{n}+\varepsilon\right)\propto\int_{1/n}\mathrm{d}W\left[(2n-1)\varepsilon-\frac{n^2}{n-1}\left(W-\frac{1}{n}\right)^2\right]^{\beta}\propto\varepsilon^{\beta+1/2}. \tag{59}$$

Random breaking of an interval according to a distribution $\rho(W)$ with exponent α in $W=1$ thus gives

$$\Pi^{(\mathrm{sing})}_{\mathrm{BI}}(Y-1/n)\propto(Y-1/n)^{(n-1)/2+\alpha}. \tag{60}$$

An integer exponent in (60) (as occurs, for example, for the random map for which $\alpha = -\frac{1}{2}$) signals a logarithmic singularity.

The line of arguments just presented for $\Pi(Y)$ may be applied to any of the probability distributions Π_P for the quantities Y_p defined in (12). One finds singularities at the points

$$Y_P = (1/n)^{P-1} \qquad n = 1, 2, 3, \ldots \tag{61}$$

and the same exponents as for $\Pi(Y)$

$$\Pi_P^{(\text{sing})}(Y_P \sim [1/n]^{P-1}) \propto (Y_P \sim [1/n]^{P-1})^{(n-1)/2+\alpha} \tag{62}$$

where α is the exponent of $\rho(W)$ at $W = 1$.

Finally, for the spin glass case the integral equation (48) generalises to an infinite set of integral equations coupling as many probability distributions $\Pi_{P,j}(Y_{P,j})$, $j = 1, 2, 3, \ldots$, where $\Pi_{P,1} = \Pi_P$ and

$$Y_{P,j} = \left(\sum_{s=j}^{\infty} W_s^P\right)\left(\sum_{s=j}^{\infty} W_s\right)^{-P}.$$

Our two other examples (BI and RM) are special cases for which $\Pi_{P,j} = \Pi_P$ for all j. The line of arguments presented above generalises again, and after some calculations one finds in the spin glass case that $\Pi_P(Y_P)$ are again singular at $Y_P = (1/n)^{P-1}$, $n = 1, 2, 3, \ldots$:

$$\Pi_P^{(\text{sing})}(Y_P \sim (1/n)^{P-1}) \propto (Y \sim (1/n)^{P-1})^{(n-3)/2+n(1-y)}. \tag{63}$$

Here y is a parameter between 0 and 1 containing the physical parameters of the system. We notice in (63) that the *locations* of the singular points do not depend on y, and are identical to those found for random breaking of the unit interval, cf (62). The *exponents* on the other hand do not depend on P. This is again similar to the breaking of the interval (62) though the n dependence of the exponents differs between (62) and (63). Equation (63) shows that, for any value of y between 0 and 1, $\Pi_P(Y_P)$ diverges at $Y_P = 1$ with exponent $-y$. These results agree with what was predicted by Mézard *et al* (1984a, b). For y between $\frac{3}{4}$ and 1, $\Pi_P(Y_P)$ diverges also at $Y_P = (\frac{1}{2})^{P-1}$, with exponent $-2(y-\frac{3}{4})$. For $y = \frac{3}{4}$ the divergence is logarithmic. For no value of y between 0 and 1 does $\Pi_P(Y_P)$ diverge at $Y_P = (1/n)^{P-1}$, $n = 3, 4, 5, \ldots$.

7. Conclusion

In the present paper we have shown that our three examples (the random map, the infinite-range spin glass and the problem of the randomly broken interval) have very similar behaviours as far as the shapes of $\Pi(Y)$ and of the probability distributions $P_1(W)$ and $P_2(W)$ of the largest and second largest pieces are concerned. We have seen that the values $Y = 1/n$ and $W = 1/n$ for $n = 1, 2, 3, \ldots$, are always singular points of these probability distributions and we have computed some of these singular behaviours. The presence of these singularities is certainly more general than our examples and they should be observed in a very large class of theoretical problems (Y is, for example, the inverse participation ratio in localisation) or of experimental situations (for example, breaking dishes, nuts, chalk, particles, etc). We were first surprised to see them in the random map problem and to recover them in the infinite-range spin glass and in the randomly broken interval. We think, however, that their presence could have been easily guessed: when one breaks an object into only two pieces (Bray and Moore 1985) of weights W and $1-W$, one has, of course, $P_1(W)=0$ for $W<\frac{1}{2}$, $P_2(W)=0$ for $W>\frac{1}{2}$ and $\Pi(Y)=0$ for $Y<\frac{1}{2}$ because $Y=W^2+(1-W^2)$ cannot be less than $\frac{1}{2}$. So already in this very simple case, one starts to see the singularities at $W=\frac{1}{2}$ and $Y=\frac{1}{2}$.

It would be interesting to know whether these singularities can be observed in any measurable quantity. Also it would be interesting to know what information about the random breaking process is contained in the knowledge of the distributions $P_1(W), P_2(W), \ldots, \Pi(Y)$. In other words: by looking at the pieces of many randomly broken objects, what can be said about their history? We think that an interesting model to study would be one for which at time t, a piece of weight W' which has been created at time t', has a probability $P(W, W', t, t')$ of breaking into two pieces of weight W and $W'-W$. What could be said about the function $P(W, W', t, t')$ by knowing the statistical properties of the pieces at an observation time t_0?

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