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Search for universality in disordered 2D ferromagnets

B. Derrida, B. W. Southern ([1]) and D. Stauffer ([2])

Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette, France

Résumé. — Des calculs de matrice de transfert pour les modèles d'Ising et de Potts à 3 états sur le réseau carré, avec des interactions aléatoires prenant deux valeurs, donnent des résultats compatibles avec une valeur universelle du rapport $\langle M^4 \rangle / \langle M^2 \rangle^2$ indépendante de l'amplitude du désordre. Nos résultats suggèrent que la valeur de ce rapport diffère de celle du système pur pour le modèle de Potts à 3 états. Pour le modèle d'Ising, nos résultats sont compatibles avec une valeur universelle unique pour le système pur et pour le système désordonné.

Abstract. — Transfer matrix simulations of the Ising and the 3-state Potts models on the square lattice, with strong and weak interactions distributed randomly, are compatible with a universal ratio of $\langle M^4 \rangle / \langle M^2 \rangle^2$ independent of the amount of disorder. Our results suggest that this ratio differs from that of the pure system in the 3-state Potts model, but in the Ising case the data are consistent with the same universal value for both the disordered and the pure system.

1. Introduction.

Disordered ferromagnets near their Curie points have been studied for many years [1]. One of the most interesting questions to ask is whether disorder is able to change the nature of the phase transition. In the weak-disorder limit, the Harris criterion [2] gives a widely accepted answer to this question. According to Harris, if the specific heat exponent α_p of the pure system is positive, disorder is relevant, that is for any amount of disorder the critical behaviour should be changed. On the contrary, if α_p is negative, a small enough amount of disorder should not change the critical exponents. In all cases ($\alpha_p >$ or < 0), the situation of strong disorder [3, 4] is less clear, but one expects that even if $\alpha_p < 0$, the critical behaviour could be changed above a certain amount of disorder. In particular, for strong disorder it has not yet been established if the critical exponents should change continuously with the amount of disorder, if they adopt new universal values corresponding to the existence of

a random fixed point or if they remain the same as in the pure case.

Standard Monte Carlo simulations on the disordered Ising model first failed to see any difference [3] from the pure exponents ; more recent simulations indicated a continuous variation of the effective exponents with disorder, towards the values of the percolation exponents [4]. Reference [5] gives some numerical tests of the log log $(T - T_c)$ behaviour in the specific heat, as predicted theoretically [6] in two dimensions, and of the spontaneous magnetization. For dynamic relaxation, a temperature dependent exponent has been suggested (see Ref. [7] for recent literature).

In the present work, we investigate the case of the Ising model and of the 3-state Potts model in two dimensions for different strengths of disorder. Since disorder is marginally ($\alpha_p = 0$) irrelevant for the Ising case [6] and is relevant ($\alpha_p = 1/3$) for the Potts case [8] one expects that a small amount of disorder should not change the critical behaviour in the Ising case but should change it in the Potts case. The quantity that we measure is related to the ration [9, 10]

$$R = \langle M^4 \rangle / \langle M^2 \rangle^2 \qquad (1)$$

of the fourth and second moments of the total magnetization M. This ratio is expected to be universal [9, 10]

([1]) *Permanently at* Department of Physics, University of Manitoba, Winnipeg, Manitoba, R3T 2N2, Canada.
([2]) LHMP, ESPCI, 10, rue Vauquelin, F-75231 Paris Cedex 05 ; *permanently at* Inst. Theor. Phys., University, 5000 Köln 41, F.R.G.

at the critical point and has already been determined for both the pure Ising [10] and the percolation [11] model. The number R is universal but depends on the shape of the system. For example, if one measures R on rectangles of size $N \times L$, in the limit $N \to \infty$ and $L \to \infty$, the ratio R depends only on the anisotropy ratio N/L. In the present work, we consider the strip geometry [10], i.e. the case $N \to \infty$, $L \to \infty$ and $N/L \to 0$. We measure the following quantity :

$$A_N = \lim_{L \to \infty} L(1 - \langle M^4 \rangle_{L,N}/3 \langle M^2 \rangle^2_{L,N})/N \quad (2)$$

for several different strip widths N and for several strengths of disorder. Since at the critical point the limit of A_N as $N \to \infty$ is expected to be universal, the existence of a random fixed point would mean that the limit is independent of the strength of disorder. On the other hand, if the system has continuously varying exponents, then we would expect that the limit of A_N varies continuously with the amount of disorder.

2. Method.

We consider here the Ising model and the 3-state Potts model on the square lattice with nearest neighbour interaction :

Ising $\mathcal{H} = -\sum J_{ij} S_i S_j$ $S_i = \pm 1$ (3a)

Potts $\mathcal{H} = -\sum J_{ij} \delta_{\sigma_i \sigma_j}$ $\sigma_i = 1, 2, 3$.   (3b)

The total magnetization M is defined by :

Ising $M = \sum_i S_i$ (3c)

Potts $M = \sum_i \left(\delta_{\sigma_i, 1} - \frac{1}{3} \right)$.   (3d)

The interactions J_{ij} are randomly distributed according to a probability distribution ρ which consists of two delta functions :

$$\rho (J_{ij}) = (\delta (J_{ij} - K_1) + \delta (J_{ij} - K_2))/2 . \quad (4a)$$

Here we set the temperature $T = 1/k_B$. For any choice of K_1 we take for K_2 the dual value of K_1, i.e. K_1 and K_2 satisfy the following relationships :

Ising $\tanh (K_1) = \exp(- 2 K_2)$ (4b)

Potts $(\exp(K_1) - 1)(\exp(K_2) - 1) = q$

(here : $q = 3$).   (4c)

The model then is selfdual and we assume that, as in the pure case, the selfdual point is the critical point. Here we set the temperature $T = 1/k_B$. For any choice at criticality, and by changing K_1 we vary the amount of disorder. The limit $K_1 = K_2$ corresponds to the pure system whereas the limit $K_1 \to \infty$ and $K_2 \to 0$ gives the percolation limit. For $K_1 = K_2$ one should recover the

results of reference [10] : $A_N \to 2.46$ in the Ising case as $N \to \infty$.

In the limit $K_1 \to \infty$ a little work is needed to relate our results to those of reference [11]. In this limit, the strip is composed of clusters ; the sites within each cluster are connected by bonds of strength K_1. In the Ising case, the magnetization is given by

$$M = \sum_k \varepsilon_k M_k \quad (5)$$

where M_k is the number of sites in cluster k and $\varepsilon_k = \pm 1$ since the clusters are free to orient themselves. This gives $\langle M^2 \rangle = \sum_k M_k^2$ and $\langle M^4 \rangle = \sum_k M_k^4 + 6 \sum_{k < m} M_k^2 M_m^2$. If one calls n_s the number of clusters of size s per unit length on a strip of given width N and length L, then one has

$$\langle M^2 \rangle = L \sum_s s^2 n_s \quad (6a)$$

$$\langle M^4 \rangle = 3 \langle M^2 \rangle^2 - 2 L \sum_s s^4 n_s . \quad (6b)$$

Therefore the ratio A_N becomes in the percolation limit $K_1 \to \infty$, from equation (2) :

$$A_N = (2/3 N) \sum_s s^4 n_s / \left(\sum_s s^2 n_s \right)^2 . \quad (7)$$

This ratio has been measured (see definitions (4) and (5) and result (33) of Ref. [11]) at the percolation threshold $A_N \to (2/3) \times 9.9 = 6.6$ as $N \to \infty$.

For the 3-state Potts model, the ratio $\langle M^4 \rangle / \langle M^2 \rangle^2$ has not yet been measured to our knowledge in the pure case. However in a way similar to the Ising case one can show that the limit $K_1 \to \infty$.

$$A_N = (1/2 N) \sum_s s^4 n_s / \left(\sum_s s^2 n_s \right)^2 . \quad (8)$$

Let us now describe how we computed A_N for random strips. The interactions J_{ij} are randomly determined by Monte Carlo sampling. As is usual with transfer matrix methods, at each step in the calculation the system is represented by a vector $\mathbf{V}$ of 2^N components in the Ising case (3^N for the 3-state Potts model). We multiply this vector by the transfer matrix T_i, which is random since it contains the random interactions J_{ij}.

Then the free energy f of the system is given by

$$f = \lim_{L \to \infty} \frac{1}{L} \ln \| \Pi_i T_i \mathbf{V} \| / \| \mathbf{V} \| \quad (9)$$

where the product Π runs from $i = 1$ to L.

$\langle M^2 \rangle$ and $\langle M^4 \rangle$ can be obtained by introducing a magnetic field h and by computing the field derivatives of f at $h = 0$. To avoid the difficulty of computing derivatives numerically, we have followed the expan-

sion described in reference [11]. We represent the vector **V**, which depends on h, by 5 vectors **W** :

$$\mathbf{V} = \mathbf{W}^{(0)} + h\mathbf{W}^{(1)} + h^2\mathbf{W}^{(2)} +$$
$$+ h^3\mathbf{W}^{(3)} + h^4\mathbf{W}^{(4)} + O(h^5), \quad (10a)$$

the transfer matrix T_i by 5 matrices U :

$$T_i = U_i^{(0)} + hU_i^{(1)} + h^2 U_i^{(2)} +$$
$$+ h^3 U_i^{(3)} + h^4 U_i^{(4)} + O(h^5), \quad (10b)$$

and we perform the multiplication T_i **V** order by order in h. In this way we get the four derivatives of f with respect to h with the same accuracy. Then we get A_N by the following formula :

$$A_N = - (1/3 N) \times$$
$$\times d^4 f(h)/dh^4 \big|_{h=0} \big/ \{d^2 f(h)/dh^2 \big|_{h=0}\}^2. \quad (11)$$

3. Results.

In figure 1 we show the variation of A_N for fixed $N = 7$ with disorder in the two-dimensional Ising model. We see that from this type of plot a continuous variation of A_N with disorder would be concluded. However, a better analysis based on accurate data should take into account the systematic N dependence, and this is what the next figures try to show.
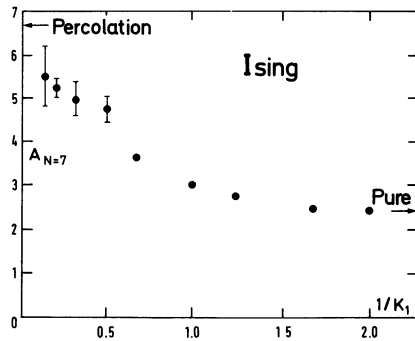

Fig. 1. — Variation of A_N for fixed $N = 7$ with $1/K_1$, i.e. with the amount of disorder, for the Ising case in (1, 0) direction. The pure (2.46) and the percolation (6.6) limit are denoted by arrows. Similar trends of the effective ratio are observed for the Potts case.

We shall first describe our results for the Ising model. We have computed A_N with $K_1 = \ln(1 + \sqrt{2})/2 = 0.44$ (pure case), $K_1 = 0.8$, $K_1 = 1.0$ and $K_1 = 1.5$ for strips of widths N up to 10 in the (1, 0) direction, and up to 6 in the (1, 1) direction. The strip lengths L varied from 25 000 to 500 000. The pure results are those of table I in reference [10]. We see in figure 2 that as N increases, the curves corresponding to the pure case and the case $K_1 = 0.8$ appear roughly parallel. This would mean that the $N \to \infty$ limit of A_N for $K_1 = 0.8$ is different from the pure case. However, for
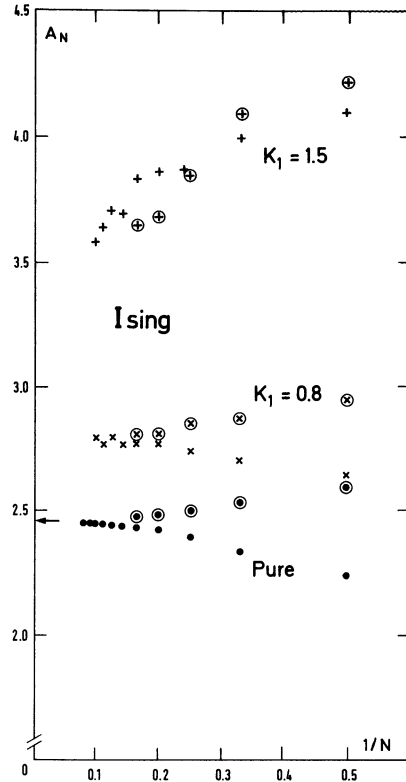

Fig. 2. — Variation of A_N with $1/N$ for the Ising case. The encircled symbols correspond to the (1, 1) direction, the others to the (1, 0) direction.

$K_1 = 1.5$ we see a downward slope and a downward curvature of A_N as a function of $1/N$. Therefore the linear extrapolation of the results for small sizes ($N = 2, 3, 4$) would give a limit value much too high.

The question that we now must ask is whether the results we have for $K_1 = 0.8$ will extrapolate, as figure 2 suggests, to A_N near 2.85, or if the curvature observed for larger disorder ($K_1 = 1.5$) will appear also in the case $K_1 = 0.8$. Looking at figure 2 does not give a clear answer to this question except that the results obtained in the (1, 0) and (1, 1) directions for the same amount of disorder, $K_1 = 0.8$, do not seem to extrapolate linearly to the same value. Therefore at least one of these two sets of data should exhibit curvature.

In order to see if our results would be compatible with a marginaly irrelevant disorder [6] in the Ising case, we present $1/(A_N(K_1) - A_N^{pure})$ versus $\log N$ in figure 3. Since disorder is marginal here, a logarithmic approach to the limit $N \to \infty$ is possible [12], i.e. $A_N(K_1) - A_N^{pure} \propto 1/\log$ (Const. N) and thus :

$$1/(A_N(K_1) - A_N^{pure}) \propto \text{Const.} + \log N. \quad (12)$$

We see in figure 3 for $K_1 = 0.8$, $K_1 = 1.0$ as well as $K_1 = 1.5$ (which is already a rather strong disorder since $K_1/K_2 \simeq 30$ in this latter case) that our data are consistent with equation (12) and thus with the idea
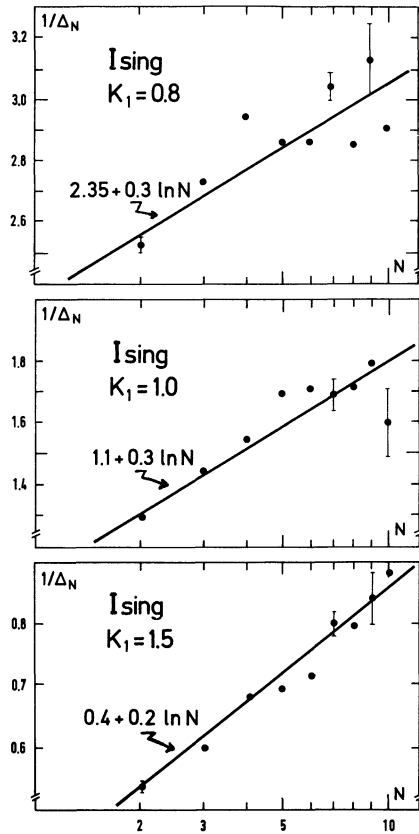
Fig. 3. — Variation of $1/\Delta_N$ with N (logarithmic scale) for the Ising case ; $\Delta_N = A_N(K_1) - A_N^{pure}$. (Tests of Eq. (12).

than $A_N(K_1)$ approaches the same limit for all K_1, including the pure case. As the extrapolated curve for $K_1 = 0.8$ in figure 2 shows, one might need extremely large widths $N \geq 10^5$ to get A_N closer to the pure value 2.46 than our last (maximum) values near 2.8.

For the 3-state Potts model our data are shown in figure 4 for $K_1 = \ln (1 + \sqrt{3}) \simeq 1.005$, $K_1 = 1.5$ and $K_1 = 3.0$. Table I gives the results for the pure case. One sees that now the difference in A_N between the pure case and the case $K_1 = 1.5$ increases with increasing N. This is consistent with the relevant disorder in the 3-state Potts model. The question to be asked is

Table I. — *Ratios A_N for the pure 3-state Potts model Assuming a parabolic variation with $1/N$ for $N > 4$ we extrapolate A_N to about* 0.83 ± 0.03.

N	A_N
2	0.951182630
3	0.946850397
4	0.942262384
5	0.933858915
6	0.924933240
7	0.916833401
8	0.909816194

whether the limit value for A_N is the same for different amounts of disorder, or if it varies continuously with the strength of disorder. Looking at figure 4 we see that the small-width data $(2 \leqslant N \leqslant 5)$ indicate that the results for different values of K_1 would extrapolate to different limits for A_N. This would mean that the exponents vary with the amount of disorder. However our data for larger N show a downward trend for $K_1 = 3$, which would mean that the extrapolated values for $K_1 = 1.5$ and $K_1 = 3.0$ are closer. If we try to extrapolate them to the same value we would get from figure 4 :

$$\lim_{N \to \infty} A_N(K_1) = 1.5 \pm 0.3 \qquad (13)$$

for the disordered 3-state Potts model, compared with a limit below 0.9 in the pure case (Table I).

In figure 4 we have plotted our results *versus* $1/N$. We also tried other extrapolations. By plotting A_N *versus* $N^{-1/5}$ we found that our two sets of data could be extrapolated to the same value $\simeq 1.4$, consistent with equation (13).
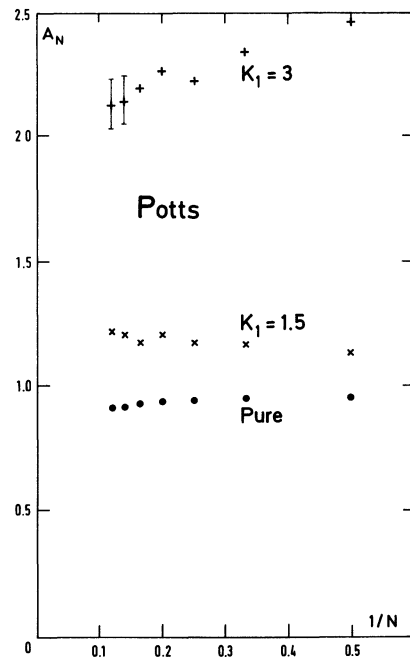

Fig. 4. — Variation of A_N with $1/N$ for the 3-state Potts model.

4. Conclusion.

We have seen that for the Ising and 3-state Potts models, the variation of the ratio A_N with disorder decreases as N increases. Thus what at first sight seems to be a continuously varying ratio becomes more constant if the quality of the data is improved. Our results are still not sufficient to ensure that the limiting value of A_N is the same for all amounts of disorder. However, they are *consistent* with the following pos-

sibilities. For the Ising case the limit for A_N is the same for all amounts of disorder as well as the pure case. For the 3-state Potts model the limiting value of A_N is about 1.5 ± 0.3 for all amounts of disorder, which differs from < 0.9 for the pure case.

It would, of course, be interesting to improve the statistics and to go to even larger widths N in order to confirm the trend we have observed here. We think that at least 50 hours of Cray computer time would be needed in order to improve significantly the results presented here from 5 hours of Cray. Finally let us just mention that we tried to use other (continuous) distributions of bonds, as well as bond dilution (K_1 finite, $K_2 = 0$). We did not find any qualitative change in the results nor that continuous distributions were giving faster converging results.

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