J. Phys. A: Math. Gen. 21 (1988) L509–L515. Printed in the UK

LETTER TO THE EDITOR

Statistical properties of valleys in the annealed random map model

B Derrida and D Bessis

Service de Physique Théorique, Institut de Recherche Fondamentale, CEA, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 25 February 1988

Abstract. The annealed random map model is one of the simplest models of statistical mechanics with stochastic dynamics. For this model, we define valleys by saying that two configurations submitted to the same stochastic forces belong to the same valley at time t if their trajectories have met before time t. We compute in the long-time limit the probability distribution of the number and the sizes of these valleys. We find a structure very reminiscent of the valley structure of the mean-field spin glasses with sample-to-sample fluctuations. Interpreting the annealed random map model as an aggregation model, we obtain non-self-averaging effects for the number of macroscopic clusters and for their sizes.

There are two kinds of dynamics one can consider to describe the time evolution of systems in statistical mechanics models: deterministic and stochastic dynamics.

Deterministic dynamics are defined by a map F in phase space

$$\mathscr{C}_{t+1} = F(\mathscr{C}_t) \tag{1}$$

which gives the configuration $\mathscr{C}_{t+1}$ of a system at time $t+1$ as a function of its configuration $\mathscr{C}_t$ at time t. The map $\mathscr{C}_t \to \mathscr{C}_{t+1}$ does not depend on time. For such dynamics, phase space, even when it is finite, can be decomposed into several valleys, each valley being the basin of attraction of an attractor of the map F.

Stochastic dynamics are defined by a map in phase space

$$\mathscr{C}_{t+1} = G(\mathscr{C}_t, \text{noise}_t) \tag{2}$$

which depends on some stochastic variables that we will call the noise, noise_t. So the map $\mathscr{C}_t \to \mathscr{C}_{t+1}$ changes with time because it depends on noise_t and it is only the statistical properties of this map (averages over noise_t) which do not depend on time. In most cases, the random variable noise_t represents the thermal noise and the map (2) allows paths from any configuration to any other configuration in phase space. The definition of valleys is much more difficult with stochastic dynamics. For finite systems, the valley structure depends on the timescale: in the limit $t \to \infty$, the system is able to explore the whole phase space and therefore one observes a single valley. On the contrary, for large systems and at finite t, one expects rather well defined valleys corresponding to the possible phases of the system. So one expects the number and the size of the valleys to depend on the timescale.

In the present work, we will consider a very simplified model of stochastic dynamics: the annealed random map model which is defined by the following rules.

(1) Phase space consists of M points.

(2) At each time step t, the map $\mathscr{C}_t \to G(\mathscr{C}_t, \text{noise}_t)$ is a random map of this set of M points into itself ($G(\mathscr{C}, \text{noise}_t)$ and $G(\mathscr{C}', \text{noise}_t)$ are uncorelated if $\mathscr{C} \neq \mathscr{C}'$).

(3) The maps at time t and t' are uncorrelated.

The valley structure of the quenched (deterministic) version of this model for which the map $\mathscr{C}_t \to \mathscr{C}_{t+1}$ is random but remains fixed at all times has been studied recently (Derrida and Flyvbjerg 1987a).

Since the notion of valleys depends on time for stochastic dynamics, we will use the following definition of valleys: we submit two different initial configurations $\mathscr{C}_0$ and $\mathscr{C}'_0$ to the same noise, noise$_t$, and we say that they belong to the same valley at time t if $\mathscr{C}_t = \mathscr{C}'_t$. (Of course if two configurations meet at some time t, they remain identical at any later time.) This definition was already used numerically in several problems (Derrida and Weisbuch 1987) and gave well defined dynamical transitions.

For the annealed random map model, to submit two configurations to the same noise, noise$_t$, means simply that the same map (2) is used for the time evolution of $\mathscr{C}_t$ and $\mathscr{C}'_t$. The goal of the present work is to calculate the statistical properties of the sizes of these valleys analytically.

Let us start with the simple case of two randomly chosen configurations $\mathscr{C}_0$ and $\mathscr{C}'_0$. If one defines A_t the probability that $\mathscr{C}_t = \mathscr{C}'_t$ and B_t the probability that $\mathscr{C}_t \neq \mathscr{C}'_t$, one has

$$\begin{aligned} A_{t+1} &= A_t + \frac{1}{M} B_t \\ B_{t+1} &= \left(1 - \frac{1}{M}\right) B_t \end{aligned} \tag{3}$$

with the initial condition $A_{-1} = 0$ and $B_{-1} = 1$. This gives

$$1 - A_t = B_t = \left(1 - \frac{1}{M}\right)^{t+1}. \tag{4}$$

For three random initial configurations, one can define A_t the probability that they are identical at time t, B_t the probability that two of them are identical but differ from the third one and C_t the probability that they are all different. One then gets

$$\begin{aligned} A_{t+1} &= A_t + \frac{1}{M} B_t + \frac{1}{M^2} C_t \\ B_{t+1} &= \left(1 - \frac{1}{M}\right) B_t + \frac{3}{M}\left(1 - \frac{1}{M}\right) C_t \\ C_{t+1} &= \left(1 - \frac{1}{M}\right)\left(1 - \frac{2}{M}\right) C_t \end{aligned} \tag{5}$$

with the initial condition $A_{-1} = B_{-1} = 0$ and $C_{-1} = 1$. The solution is then

$$\begin{aligned} A_t &= 1 - \frac{3}{2}\left(1 - \frac{1}{M}\right)^{t+1} + \frac{1}{2}\left[\left(1 - \frac{1}{M}\right)\left(1 - \frac{2}{M}\right)\right]^{t+1} \\ B_t &= \frac{3}{2}\left(1 - \frac{1}{M}\right)^{t+1}\left[1 - \left(1 - \frac{2}{M}\right)^{t+1}\right] \\ C_t &= \left[\left(1 - \frac{1}{M}\right)\left(1 - \frac{2}{M}\right)\right]^{t+1}. \end{aligned} \tag{6}$$

One can of course generalise equations (3)-(5) to the case of n random initial conditions. If one defines $P_i^{(n)}(t)$ the probability that, at time t, the n configurations have been grouped into i clusters, i.e. that only i configurations are still different, the time evolution of $P_i^{(n)}(t)$ is given by

$$P_i^{(n)}(t+1)=\sum_{j\geq i}\frac{M!}{(M-i)!}\frac{1}{M^j}\rho_{ij}P_j^{(n)}(t) \tag{7}$$

where the matrix ρ_{ij} is given by

$$\begin{aligned}&\rho_{j,j}=\rho_{1,j}=1 && \forall j\geq 1\\ &\rho_{i,j+1}=\rho_{i-1,j}+i\rho_{i,j} && \forall j;\ \forall i\geq 2.\end{aligned} \tag{8}$$

This gives in particular

$$\rho_{j-1,j}=j(j-1)/2 \tag{9}$$

$$\rho_{j-2,j}=j(j-1)(j-2)(3j-5)/24. \tag{10}$$

The recursion (8) can be understood by saying that for $j+1$ configurations to regroup themselves into i configurations, either the first j of them go into $i-1$ configurations and the $(j+1)$th must go elsewhere or the first j go into i configurations and the $(j+1)$th has i possible choices. Of course the initial condition becomes, in the case of n configurations,

$$P_i^{(n)}(-1)=\delta_{i,n}. \tag{11}$$

(Choosing n configurations at random at $t=0$ is the same as choosing n different configurations at time $t=-1$.) It is clear from (4) and (6) that, for M large, the natural timescale is M. If one defines

$$\tau=t/M \qquad Q_i^{(n)}(\tau)=P_i^{(n)}(t) \tag{12}$$

the time evolution of the $Q_i^{(n)}$ is governed for M large by

$$\frac{\mathrm{d}Q_i^{(n)}}{\mathrm{d}\tau}=-\tfrac{1}{2}i(i-1)Q_i^{(n)}+\tfrac{1}{2}i(i+1)Q_{i+1}^{(n)} \tag{13}$$

with the initial condition

$$Q_i^{(n)}(0)=\delta_{i,n}. \tag{14}$$

Equation (13) can be obtained easily by expanding (7) up to power M^{-1} for large M. It expresses the fact that, for large M, the dominant effect is that pairs of configurations meet at a given time. The events for which three or more configurations meet at a given time would give higher-order contributions in the M^{-1} expansion.

Equation (13) can be solved by recursion starting with $Q_n^{(n)}$:

$$Q_n^{(n)}(\tau)=\exp[-\tfrac{1}{2}n(n-1)\tau] \tag{15}$$

and one gets

$$Q_i^{(n)}(\tau)=\frac{n!(n-1)!}{i!(i-1)!}\sum_{p=i}^{n}(-1)^{p+i}(2p-1)\frac{(p+i-2)!}{(p-i)!}\frac{\exp[-\frac{1}{2}p(p-1)\tau]}{(n+p-1)!(n-p)!}. \tag{16}$$

This expression can easily be derived from (13) by using the Laplace transform with respect to τ.

Let us define $Z_i(\tau)$ by

$$Z_i(\tau)=\lim_{n\to\infty} Q_i^{(n)}(\tau). \tag{17}$$

(Here the limit $n\to\infty$ has to be taken after $M\to\infty$ so that $n\ll M^{1/2}$.) $Z_i(\tau)$ is the probability that there are exactly i valleys in the system since it is the probability of finding exactly i different configurations starting with an arbitrarily large number n of different initial configurations. From (16), it follows that

$$Z_i(\tau)=\frac{1}{i!(i-1)!}\sum_{p=i}^{\infty}(-1)^{p+i}(2p-1)\frac{(p+i-2)!}{(p-i)!}\exp[-\tfrac{1}{2}p(p-1)\tau]. \tag{18}$$

The series (18) are obviously convergent for any time $\tau>0$. Physically, we know also that all the $Z_i(\tau)$ vanish as $\tau\to 0$. It is, however, much less obvious by looking at (18) that all the $Z_i(\tau)$ given by (18) vanish as $\tau\to 0$. This can, however, be shown using the fact that $Z_1(\tau)$ can be rewritten as

$$Z_1(\tau)=\prod_{p=1}^{\infty}(1-e^{-p\tau})^3. \tag{19}$$

Expression (19) can be obtained from (18) using Jacobi's theta functions (Bateman 1953). By definition of the function θ_1:

$$\theta_1(v, e^{-\tau/2})=2\,e^{-\tau/8}\sum_{n=0}^{\infty}(-1)^n\exp[-\tfrac{1}{2}n(n+1)\tau]\sin[(2n+1)\pi v] \tag{20}$$

we see that

$$Z_1(\tau)=\lim_{v\to 0}\frac{\theta_1(v, e^{-\tau/2})}{2\pi\,e^{-\tau/8}v}. \tag{21}$$

The zeros of θ_1 are all known (Bateman 1953) and the function θ_1 can be written as

$$\theta_1(v, e^{-\tau/2})=2\,e^{-\tau/8}\sin(\pi v)\prod_{n=1}^{\infty}(1-e^{-n\tau})[1-2\,e^{-n\tau}\cos(\pi v)+e^{-2n\tau}]. \tag{22}$$

Clearly (21) and (22) give expression (19).

We see from (19) that $Z_1(\tau)$ and all its derivatives vanish at $\tau=0$. Therefore since from (13) one has

$$Z_{i+1}(\tau)=\frac{1}{i(i+1)}\left(2\frac{dZ_i}{d\tau}+i(i-1)Z_i\right) \tag{23}$$

one can check that all the $Z_i(\tau)$ vanish at $\tau=0$.

The expressions (18), (19) and (23) of the $Z_i(\tau)$ give the probability of finding exactly i valleys at time $M\tau$. One can then try to determine the statistical properties of the weights of these valleys. By definition, the weight W_S of valley S is the probability that a randomly chosen configuration belongs to the valley S. To study the statistical properties of these weights, it is convenient to introduce the probabilities $X_{n_1,n_2,\ldots,n_k}(\tau)$ that, starting at $t=0$ with $n_1+n_2+\ldots+n_k$ randomly chosen configurations, the configurations 1 to n_1 have become identical at time $M\tau$, the configurations $n_1+1,\ldots,n_1+n_2$ have become identical at time $M\tau$ but still differ from the first n_1 configurations, etc, and the configurations $(n_1+\ldots+n_{K-1}+1)\ldots(n_1+\ldots+n_K)$ are identical at time $M\tau$ but still differ from the first $n_1+\ldots+n_{K-1}$ configurations. The time evolution of the $X_{n_1,n_2,\ldots,n_K}(\tau)$ is given by

$$X_{n_1,\ldots,n_K(\tau)} = \int_0^\tau \mathrm{d}\tau' \exp[-\tfrac{1}{2}N(N-1)(\tau-\tau')] \times\left(\frac{n_1(n_1-1)}{2} X_{n_1-1,n_2,\ldots,n_K}(\tau')+\ldots+\frac{n_K(n_K-1)}{2} X_{n_1,n_2,\ldots,n_K-1}(\tau')\right) \tag{24}$$

where

$$N = n_1+n_2+\ldots+n_K. \tag{25}$$

The initial condition is

$$X_{n_1,n_2,\ldots,n_K}(0) = \delta_{n_1,1}\delta_{n_2,1}\ldots\delta_{n_K,1}\delta_{K,N}. \tag{26}$$

Formula (24) expresses the fact that the N configurations remain different from time 0 to time $M(\tau-\tau')$ when a pair of configurations becomes identical. Differentiating with respect to τ, one gets

$$\mathrm{d}X_{n_1,\ldots,n_K}/\mathrm{d}\tau = \tfrac{1}{2}n_1(n_1-1)X_{n_1-1,n_2,\ldots,n_K}+\ldots +\tfrac{1}{2}n_K(n_K-1)X_{n_1,n_2,\ldots,n_K-1}-\tfrac{1}{2}N(N-1)X_{n_1,\ldots,n_K}. \tag{27}$$

From the solution of (24), we can deduce the statistical properties of the weights of the valleys. If one defines $f(W_1,\ldots,W_K)$ by

$$X_{n_1,n_2,\ldots,n_K} = \int_0^1\ldots\int_0^1 \mathrm{d}W_1\ldots\mathrm{d}W_K\, W_1^{n_1}\ldots W_K^{n_K} f(W_1,\ldots,W_K). \tag{28}$$

$f(W_1,\ldots,W_K)$ is the probability that, if one chooses K valleys among all the possible valleys, the first one has weight W_1, the second one has weight W_2 and the Kth one has weight W_K. Then (28) expresses the fact that the first n_1 configurations fall into the first valley, etc, and the last n_K configuration fall into the Kth valley.

One can show that the $f(W_1,\ldots,W_K)$ are given by

$$f(W_1,\ldots,W_K) = \sum_{i\geq K}\frac{(i-1)!\,i!}{(i-K)!}Z_i(\tau)\int_0^1\ldots\int_0^1 \mathrm{d}Y_1\ldots\mathrm{d}Y_i\,\delta(1-Y_1-\ldots-Y_i) \times[\delta(W_1-Y_1)\ldots\delta(W_K-Y_K)] \tag{29}$$

where the $Z_i(\tau)$ are given by (19) and (23) or (18). In order to prove (29) from (27), one can calculate the $X_{n_1,\ldots,n_K}$ from (28) and (29)

$$X_{n_1,\ldots,n_K}(\tau) = n_1!\ldots n_K!\sum_{i\geq K}^{\infty}\frac{i!(i-1)!}{(N+i-1)!(i-K)!}Z_i(\tau) \tag{30}$$

and show that (30) satisfies (27) when the $Z_i(\tau)$ are given by (19) and (23).

The expressions (28) and (29) of the $X_{n_1,\ldots,n_K}(\tau)$ and of the $f(W_1,\ldots,W_K)$ can be simplified in many ways. Let us just give here two alternative expressions that we found particularly simple. Expressions (30) can be replaced by

$$X_{n_1,n_2,\ldots n_K}(\tau) = n_1!\,n_2!\ldots n_K!(-1)^{K-1} \times\sum_{n=K-1}^{n=N-1}(-1)^n\frac{(2n+1)(N-K)!(n+K-1)!}{(N+n)!(N-1-n)!(n-K+1)!}\exp[-\tfrac{1}{2}n(n+1)\tau]. \tag{31}$$

This result could have been obtained in a straightforward way. If one introduces the Laplace transform in τ of $X_{n_1,n_2,\ldots,n_K}(\tau)$:

$$\hat{X}_{n_1,n_2,\ldots,n_K}(\beta) = \int_0^{+\infty} e^{-\beta\tau} X_{n_1,n_2,\ldots,n_K}(\tau)\, d\tau. \tag{32}$$

Inserting it into equation (27), one finds (by induction)

$$\hat{X}_{n_1,n_2,\ldots,n_K}(\beta) = n_1!\,n_2!\ldots n_K!\,\frac{(N-K)!}{2^{N-K}} \prod_{n=K-1}^{n=N-1} \frac{1}{\beta+\frac{1}{2}n(n+1)}. \tag{33}$$

Taking the inverse transform, one gets (31).

Expression (29) can also be rewritten as

$$f(W_1,\ldots,W_K) = Z_K(\tau)K!(K-1)!\delta(1-W_1-\ldots-W_K) + \sum_{i\geqslant K+1}^{\infty} Z_i(\tau)\frac{i!(i-1)!}{(i-K)!(i-K-1)!}(1-W_1-\ldots-W_K)^{i-K-1}. \tag{34}$$

This shows that $f(W_1,\ldots,W_K)$ depends only on the sum $W_1+\ldots+W_K$.

In the present work we have obtained the analytic expressions (19) and (23) of the probability $Z_i(\tau)$ of finding i valleys at time τ and the probability distribution $f(W_1, W_2,\ldots,W_K)$ of the weights of the valleys. From the knowledge of f, one can deduce other properties like the probability distribution $\pi(Y)$ of Y defined by

$$Y = \sum_S W_S^2 \tag{35}$$

where the sum runs over all the valleys. One gets

$$\pi(Y) = Z_1(\tau)\delta(Y-1) + \sum_{i=2}^{\infty}(i-1)!Z_i(\tau) \times \int_0^1 \ldots \int_0^1 dW_1\ldots dW_i\,\delta(1-W_1-\ldots-W_i)\delta(Y-W_1^2-\ldots-W_i^2). \tag{36}$$

From this expression, it is clear that for this model too $\pi(Y)$ is singular at $Y = 1, \frac{1}{2},\ldots,1/n$ (Derrida and Flyvbjerg 1987a, b, Gutfreund *et al* 1988) and that the valley structure of the model studied here has sample-to-sample fluctuations which are qualitatively similar to those of the mean-field spin glasses (Mezard *et al* 1984a, b), and of the quenched random map model (Derrida and Flyvberg 1987a).

The annealed random map studied here can also be viewed as a very simplified mean-field model of aggregation (Ernst 1986, Herrmann 1986 and references therein). The M possible initial conditions are the M particles. Each time that two particles or clusters meet, they stick and stay together for ever. The number of clusters decreases with time and their sizes increase. The results of the present work deal with the latest stage of this aggregation process, i.e. when macroscopic clusters (of size of order M) appear. Our calculation gives that there is a finite probability of finding several macroscopic clusters (probability $Z_i(\tau)$ of finding i clusters at time $M\tau$) and their weights W_S fluctuate from sample to sample with a probability distribution given by (29) and (34). It would of course be interesting to know what these results become for more complicated aggregation models and then to relate the properties of the macroscopic clusters to what is already known on the aggregation in finite systems (Lushnikov 1978, Van Dongen and Ernst 1987a, b).

We would like to thank M E Ernst and J Vannimenus for useful discussions on the analogy with aggregation problems.

References

Bateman H 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill) pp 355-7
Derrida B and Flyvbjerg H 1987a *J. Physique* **48** 971
—— 1987b *J. Phys. A: Math. Gen.* **20** 5273
Derrida B and Weisbuch G 1987 *Europhys. Lett.* **4** 657
Ernst M H 1986 *Fractals in Physics* ed L Pietronero and E Tosatti (Amsterdam: Elsevier) p 289
Gutfreund H, Reger J and Young P 1988 *Preprint*
Herrmann H J 1986 *Phys. Rep.* **136** 153
Lushnikov A A 1978 *J. Colloid Interface Sci.* **65** 276
Mezard M, Parisi G, Sourlas N, Toulouse G and Virasoro M 1984a *Phys. Rev. Lett.* **52** 1146
—— 1984b *J. Physique* **45** 843
Van Dongen P G J and Ernst M H 1987a *J. Stat. Phys.* **49** 879
—— 1987b *J. Stat. Phys.* **49** 927