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Barrier heights in the Kauffman model

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Résumé. — Nous considérons deux versions du modèle de Kauffman (le modèle gelé et le modèle recuit) en présence de bruit. Quand on compare deux configurations soumises au même bruit thermique, on observe que, pour ces deux versions du modèle de Kauffman, au dessous d'une température critique T_c, le temps τ_2 qu'il faut pour que les deux configurations se rejoignent augmente exponentiellement avec la taille N du système. Cela définit une hauteur de barrière qui peut être calculée analytiquement pour le modèle recuit. Quand on compare plus que deux configurations, on observe que les temps τ_n pour qu'au moins deux configurations parmi n se rencontrent augmentent aussi exponentiellement. La pente de log τ_n en fonction de N semble être la même pour tout n et pour les deux modèles.

Abstract. — We consider two versions of the Kauffman model (the quenched and the annealed models) in presence of thermal noise. When we compare the time evolution of two configurations subjected to the same thermal noise, we find for both versions of the Kauffman model that below a critical temperature T_c, the time τ_2 for these two configurations to become identical increases exponentially with the system size N. This defines a barrier height which can be calculated analytically for the annealed model only. When we compare more than two configurations, we observe that the time τ_n it takes for at least two configurations among n to meet increases also exponentially. The slope of log τ_n *versus* the system size N seems to be the same for all n and for both models.

1. Introduction.

The dynamical properties of a large class of disordered systems at low temperature are dominated by the presence of many attractors or valleys [1-7]. The influence of each of these valleys on the global dynamics depends on the size of its basin of attraction [3-5], on the height of barriers [1, 2, 6] which separate it from the rest of phase space, on the dynamical properties of this particular valley. When each valley can be characterized by a simple order parameter [6] (like the magnetization for a ferromagnet) the valley structure can be understood by studying the time evolution of this order parameter. However for systems with a complex order parameter or for which no order parameter is known, one has to use other techniques [3-5] to describe the multivalley structure of the system.

A dynamical method has been proposed [7] to try to approach this problem.

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It consists in comparing the time evolution of two (or more than two) configurations subjected to the same thermal noise. For a large class of models [6-11] (spin models, automata) several phases were observed as a function of temperature.

A high temperature phase where the two configurations subjected to the same thermal noise meet very quickly. In that phase, the landscape in phase space consists of a single valley and no barrier prevents the two configurations from meeting.

A low temperature phase where the two configurations never meet (at least if the system size is large enough). Two reasons [12] have already been proposed to explain this behaviour at low temperature.

I) Either phase space consists of several valleys separated by high enough barriers and the two configurations do not meet because none of them is able to jump the barriers.

II) Or there is a tendency for the trajectories in phase space of two close trajectories to diverge (chaotic dynamics) and this prevents the two configurations from meeting.

Among the models studied up to now, several examples of I and II have been found : Ferromagnets belong to class I (the mechanism which keeps the two configurations different is that below T_c, one configuration is in the + phase and the other one in the − phase) and one can show [13, 9, 14] that the low temperature phase where two configurations subjected to the same noise do not meet coincides with the ferromagnetic phase. Automata [10, 15] and non symmetric spin glasses [9, 11, 12] belong to class II and the transition temperature below which the two configurations do not meet does not seem to be associated with the existence of any order parameter.

The main difference between cases I and II is that in case II, two configurations never meet even if their initial distance is very small [15] (there is only a non zero probability of meeting if the two initial conditions differ by a finite number of spins [16]) whereas in case I, they have a finite probability of meeting [6, 7] even if the initial conditions differ by a finite fraction of spins and this probability varies with the initial distance.

The low temperature phase where pairs of configurations never meet and the sharp phase transition to the high temperature phase exist only in infinite systems. In finite systems, two configurations always meet after a finite time. The difference between the high temperature and the low temperature phases is that in the low temperature phase, the time it takes for two configurations to meet increases exponentially with the system size. This exponential growth is reminiscent of what happens in systems such as ferromagnets [6] where barriers appear in the low temperature phase.

The goal of the present paper is to study the size dependence of the time it takes for two configurations to meet in the Kauffman model [17-19]. In section 2, we introduce the two (quenched and annealed) versions of the Kauffman model and we recall a few of their properties. In section 3, we describe the results of Monte Carlo simulations to measure the time τ_n when two configurations among n meet for the first time. We will see that in the low temperature phase these times τ_n increase exponentially with the number N of automata. This can be interpreted as the existence of barriers which prevent the two configurations of meeting. The rate of the exponential growth seems to be the same for all the times τ_n and for the two models (quenched and annealed). In section 4, we derive an analytic expression for the barrier height in the case $n = 2$ for the annealed model. This analytic result confirms the exponential growth observed in section 3.

2. The Kauffman model in presence of thermal noise.

In the present work we will consider two versions of the Kauffman model [17-19] with sequential dynamics and in presence of thermal noise [10, 15] : the quenched model and the annealed model.

Let us start by defining these two models (quenched and annealed). In both models, one considers a system of N Ising spins $S_i = \pm 1$ which evolve according to sequential dynamics. To each spin i, one assigns K input sites $j_1(i), \dots, j_K(i)$ chosen at random among the N sites (these K inputs are not necessarily different) and a Boolean function $f_i(= \pm 1)$ chosen at random among the 2^{2^K} possible functions of K variables.

Then the configuration $\{S_i(t)\}$ of the system evolves in time according to the following rule (random sequentiel dynamics) : during each time interval Δt

$$\Delta t = 1/N \tag{1}$$

one chooses a spin i at random among the N spins and one updates this spin i as follows :

$$\begin{aligned} S_i(t+\Delta t) &= f_i(S_{j_1(i)}(t), \dots, S_{j_K(i)}(t)) \quad \text{with probability} \quad \frac{1}{2} + \frac{1}{2}\tanh\frac{1}{T} \\ &= -f_i(S_{j_1(i)}(t), \dots, S_{j_K(i)}(t)) \quad \text{with probability} \quad \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{T} \end{aligned} \tag{2}$$

where T is a parameter that we define as the temperature. (Other noisy dynamics for the Kauffman model have been considered in the literature which will not be discussed here [28]).

To implement these dynamics [27], one chooses at each time step $t \to t + \Delta t$, a random number $z(t)$ uniformly distributed between 0 and 1 and one obtains $S_i(t + \Delta t)$ by :

$$S_i(t+\Delta t) = \text{sign}\left[\frac{1}{2} + \frac{1}{2} f_i(S_{j_1(i)}(t), \dots, S_{j_K(i)}(t)) \tanh\frac{1}{T} - z(t)\right] \tag{3}$$

The only difference between the two models is that :

for the quenched model, the input sites $j_1(i) \dots j_K(i)$ and the Boolean function f_i of each site i are chosen once for all at time $t = 0$ and they do not change with time.

for the annealed model, the input sites and the Boolean functions are changed at each time step.

The annealed and the quenched version of the Kauffman model have already been considered in the case of parallel dynamics [20] (i.e. dynamics for which all the spins are updated at the same time) with deterministic rules ($T = 0$). It was shown [21-23] that some properties such as the time evolution of the distance between two initial configurations can be calculated exactly in the thermodynamic limit ($N \to \infty$) for the two models and that the expression of the distance is the same in both models.

The distance $D(t)$ between two configurations $\{S_i(t)\}$ and $\{S_i'(t)\}$ is defined as the fraction of spins which are different in the two configurations

$$D(t) = \frac{1}{2N} \sum_{i=1}^{N} |S_i(t) - S_i'(t)| \tag{4}$$

The reasons [21-23] which allowed one to calculate the distance exactly is that, in the limit $N \to \infty$, the tree of ancestors of each site i has no repetition with probability one. Therefore the different inputs of each site i are uncorrelated because they depend on branches which have no overlap. This reason remains true for sequential dynamics and in presence of thermal noise and thus one can extend the calculation of the distance to this case.

The distance between two configurations has often been considered in the case of deterministic dynamics (parallel dynamics at $T = 0$) [20-23] : The two configurations evolve according to the same rules and there is a unique definition of the distance.

For stochastic dynamics (sequential dynamics at non zero temperature) one has to decide how the stochastic forces which act on the two configurations are correlated and the value of the distance depends on this correlation [15]. As for other models [6-14], we choose here to use the same random force for the two configurations, i.e. to go from time t to time $t + \Delta t$, we update the same spin i and we use the same random number $z(t)$ for the two configurations :

$$\begin{aligned} S_i(t + \Delta t) &= \text{sign}\left[\frac{1}{2} + \frac{1}{2} f_i(S_{j_1(i)}(t), \ldots, S_{j_K(i)}(t)) \tanh \frac{1}{T} - z(t)\right] \\ S_i'(t + \Delta t) &= \text{sign}\left[\frac{1}{2} + \frac{1}{2} f_i(S'_{j_1(i)}(t), \ldots, S'_{j_K(i)}(t)) \tanh \frac{1}{T} - z(t)\right] \end{aligned} \tag{5}$$

Then one can show that the distance $D(t)$evolves according to the following equation

$$\frac{\mathrm{d}D(t)}{\mathrm{d}t} = \frac{1}{2}\left[1 - (1 - D(t))^K\right] \tanh \frac{1}{T} - D(t) \tag{6}$$

and $D(t)$ can be obtained from $D(0)$ by integrating this equation.

The origin of (6) is easy to understand. If the distance is $D(t)$ at time t, this means that the spin i which is updated at time t is identical in the two configurations with probability $1 - D(t)$ and is different with probability $D(t)$. This site has a probability $(1 - D(t))^K$ of having all its K inputs identical in the two configurations (case A) and $(1 - (1 - D(t))^K)$ of having at least one input different (case B). In case A, $S_i(t + \Delta t) = S_i'(t + \Delta t)$ with probability 1 and in case B, $S_i(t + \Delta t) = S_i'(t + \Delta t)$ with probability $\frac{1}{2} + \frac{1}{2}\left(1 - \tanh \frac{1}{T}\right)$ since with probability $\frac{1}{2}$ the function f_i takes the same value for the two different sets of inputs and with probability $\frac{1}{2}$ it does not and in this latter situation there is a probability $1 - \tanh \frac{1}{T}$ that $S_i(t + \Delta t) = S_i'(t + \Delta t)$ if $f_i(\{S_j\}) \neq f_i(\{S_j'\})$.

Therefore one can write

$$D(t + \Delta t) = D(t) + \frac{\varepsilon_1(t)}{N} - \frac{\varepsilon_2(t)}{N} \tag{7}$$

where

$$\begin{aligned} \varepsilon_1(t) &= 1 \text{ with probability } \frac{1}{2}\tanh\left(\frac{1}{T}\right)\left[1 - (1 - D(t))^K\right] \\ &= 0 \text{ with probability } 1 - \frac{1}{2}\tanh\left(\frac{1}{T}\right)\left[1 - (1 - D(t))^K\right] \end{aligned}$$

and

$$\begin{aligned} \varepsilon_2(t) &= 1 \text{ with probability } D(t) \\ &= 0 \text{ with probability } 1 - D(t) \end{aligned} \tag{8}$$

This gives, on average, (6) and since $D(t)$ is obtained by adding a large number of $\varepsilon_1(t)$ and $\varepsilon_2(t)$, $D(t)$ is equal to its average with probability 1. From (6), one can expect to observe two phases.

A high temperature phase $T > T_c$ where the only fixed point of (6) is $D = 0$ and where $D(t) \to 0$ as $t \to \infty$. In that phase the effect of noise is strong enough to make the two configurations meet.

A low temperature phase $T < T_c$ where the fixed point $D = 0$ is unstable and where

$D(t) \to D^*$ the attractive fixed point of (6). The transition temperature is given by the condition

$$\frac{\mathrm{d}D(t)}{\mathrm{d}t}\bigg|_{D(t)=0} = \frac{K}{2}\tanh\frac{1}{T_\mathrm{c}} - 1 = 0 \tag{9}$$

i.e. for $K > 2$

$$T_\mathrm{c} = \frac{2}{\log\left[\dfrac{K+2}{K-2}\right]} \tag{10}$$

whereas for $K \leq 2$, there is no transition.

Equation (6) and its consequences (phase transition, a high temperature phase where the distance vanishes and a low temperature phase where it does not) are valid for the two models (quenched and annealed) in the thermodynamic limit ($N \to \infty$) only.

For finite systems, the transition temperature is no longer a sharp transition and two configurations subjected to the same thermal noise always become identical after a finite time. The main difference between the high temperature and the low temperature phase is the size dependence of the time it takes for two configurations to meet. We will see in the next section that for $T < T_\mathrm{c}$, this time increases exponentially with the system size whereas above T_c, it increases logarithmically.

3. The time it takes for two configurations to meet.

The dynamical phase transition at a temperature T_c given by (10) exists only in the limit of an infinite system for both the annealed and the quenched models. For a finite system, the reason (independence of the inputs of almost all sites) which led to (6) is no longer valid (After a time $t \sim \log N/\log K$, the number of ancestors in the tree of ancestors of each site i is $K^t \sim N$ and there must be repetitions in this tree). Thus one does not expect the time evolution of the distance to be given by (6) for a finite system. In fact, for a finite system, the distance always vanishes in the long time limit if the temperature is non zero. This is because for the finite system, the only steady state is a steady state with two identical configurations.

It is interesting to study how the time τ_2 it takes for two configurations to meet depends on the system size N. The time τ_2 depends on the network (choice of the inputs $j_r(i)$ and of the functions f_i), on the random history (the choice of the spin i to update and of the random number $z(t)$ used in Eq. (5)). Therefore τ_2 is a random variable and it varies with the sample (for each new sample, we choose a new network, new random initial conditions and a new history).

In figure 1 we show $\langle \log \tau_2 \rangle$, the logarithm of τ_2 averaged over 5000 different samples for the Kauffman model in its low temperature phase. The calculations were done for $K = 4$ and $\tanh\dfrac{1}{T} = 0.7$ whereas the transition temperature T_c would correspond to $\tanh\dfrac{1}{T_\mathrm{c}} = 0.5$. The plot for the quenched model is shown in figure 1a and for the annealed model in figure 1b.

We see that for both models, if one excludes the results for the smallest sizes, $\langle \log \tau_2 \rangle$ increases linearly with N. The results are similar for the two models. The slopes have close values. One cannot however simulate big enough systems to be sure that these slopes are equal for both models.

So as N increases, it takes longer and longer for the two configurations to meet. One can say that there is a barrier which has a height increasing linearly with the system size N and which prevents the two configurations to meet.

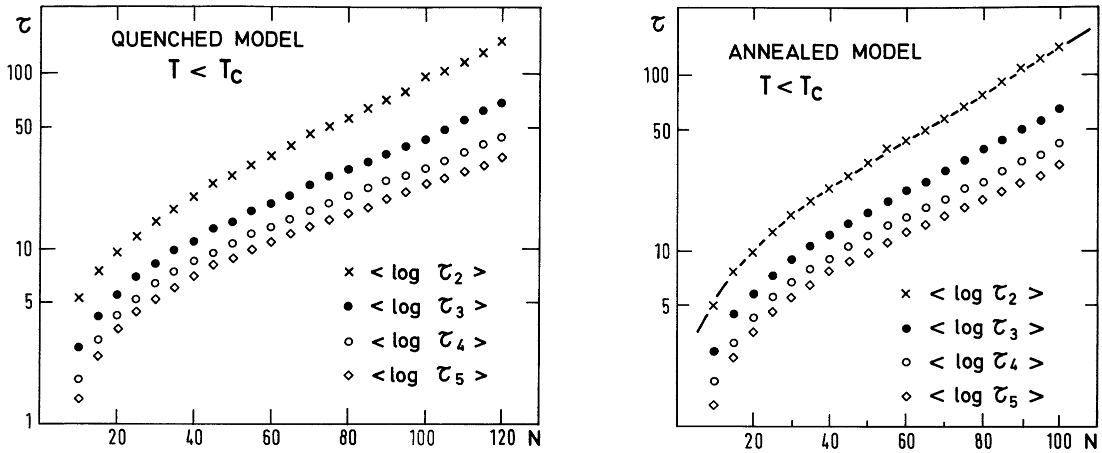

Fig. 1. — (a) Quenched model in the low temperature phase : $K = 4$ and $\tanh \frac{1}{T} = 0.7$. Survival time $\langle \log \tau_n \rangle$ for $n = 2, 3, 4$ and 5 configurations as a function of the number N of automata. The times τ_n increase exponentially with N. The slope measures the height of the repulsion barrier. The average is done over 5000 samples. (b) Annealed model : the same as figure 1a. The curve is the analytic expression obtained in section 4 of $\langle \log \tau_2 \rangle$.

This results is very reminiscent of the case of the (mean field) ferromagnet for which $\langle \log \tau_2 \rangle$ increases linearly with the system size[6] in the low temperature phase. In the case of the mean field ferromagnet, the exponential increase of τ_2 was due to the free energy barrier the system has to overcome in order to go from the + phase to the − phase.

For the Kauffman model, one cannot define a free energy (there is no hamiltonian and no Boltzmann weight to describe the equilibrium). Nevertheless, the results concerning τ_2 of figure 1 show clearly the existence of a barrier between the two configurations.

One can extend the above calculation to the comparison of more than 2 configurations. If one considers n configurations subjected to the same noise, one can define [6] a time τ_n as the first time when two among these n configurations meet. In the study of mean field ferromagnets [6], the size dependence of τ_3 was very different from τ_2. Since for a mean field ferromagnet, there are only two phases, the time τ_3 is always short because two configurations at least fall into the same valley and thus meet very quickly.

The values of $\langle \log \tau_3 \rangle$, $\langle \log \tau_4 \rangle$ and $\langle \log \tau_5 \rangle$ are shown in figure 1a (quenched model) and figure 1b (annealed model) for $K = 4$ and $\tanh \frac{1}{T} = 0.7$. We see that the behaviour of τ_3, τ_4 and τ_5 is very similar to the behaviour of τ_2. They all increase with N and the results of figure 1 seem to indicate that for large N all the $\langle \log \tau_n \rangle$ increase with N with similar slopes. Here again, N is not large enough to decide wether these slopes are equal or slightly different.

These results indicate that the structure of the attractors in the Kauffman model is very different from what it is in the mean field ferromagnet [6]. If one starts with 2, 3, ..., n (any finite number n) configurations, there are always barriers between all the possible pairs which prevent the configurations of meeting.

On can notice that the numerical results of figure 1 indicate that the averages $\langle \log \tau_n \rangle$ have very close values in the quenched and the annealed model with the tendency for the times to be slightly longer in the annealed than in the quenched model. We could not find any argument which would show that these times should be identical or different. Only at

$T = 0$ and for finite N, $\langle \log \tau_n \rangle$ should be infinite for the quenched model because one can build samples with more than n distinct attractors, whereas $\langle \log \tau_n \rangle$ should be finite for the annealed model. Also the slopes (for N large) seem to be the same for all n and for both models.

We will see in the next section that one can get an analytic expression of $\langle \log \tau_2 \rangle$ for the annealed model and that this expression (curve in Fig. 1b) agrees with the results of the Monte Carlo simulations. We were not able to find an analytic expression for the other times τ_n in the annealed model $n \geq 3$ and in the quenched model $n \geq 2$. So the possibility that the slopes of all the $\langle \log \tau_n \rangle$ are the same is purely based on the numerical results shown in figures 1.

It is interesting to notice that a model (the annealed random map model) for which all the times τ_n can be calculated was studied recently [24] and that it was found in that case that all the slopes are identical with even a further relation

$$\tau_n = \frac{2\,\tau_2}{n(n-1)} \tag{11}$$

On the basis of the numerical results presented in figure 1, we see that the Kauffman model (in both its annealed and quenched versions) seems to behave in a very similar way. Even the result (obtained for the random map [24]) $\langle \log \tau_n \rangle = \langle \log \tau_2 \rangle - \log \left(\frac{n(n-1)}{2} \right)$ seems to be not a too bad approximation for the Kauffman model.

It takes very long for different configurations subjected to the same noise to meet only in the low temperature phase. For $T \geq T_c$, one does not expect an exponential growth of the times τ_n.

For the quenched model, we repeated our Monte Carlo calculations for $K = 4$ for two temperatures $\tanh \frac{1}{T_c} = 0.5$ and $\tanh \frac{1}{T} = 0.3$ i.e. $T > T_c$.

At T_c, figure 2 shows $\langle \log \tau_n \rangle$ *versus* $\log N$. For large N, we see that the $\langle \log \tau_n \rangle$ seem to increase linearly with $\log N$. So the τ_n are power laws (which could be expected since $T = T_c$ is a critical point)

$$\tau_n \sim N^z$$

and the exponent $z \simeq 0.50$ seems to be the same for all n for the quenched model. (When we tried to plot $\langle \log \tau_n \rangle$ *versus* N or $\exp(\langle \log \tau_n \rangle)$ *versus* $\log N$, our results were much more curved for large N and this indicates that the power law $\tau_n \sim N^z$ is the best fit of our numerical data).

For $T > T_c$, $\exp(\langle \log \tau_n \rangle)$ seems to increase like $\log N$ for large N (Fig. 3). This can be understood easily : the distance (6) between 2 configurations decreases exponentially with time. The time it takes for two configurations to meet is of the order of the time such that the two configurations differ by a finite number of spins (0, 1, 2 spins)i.e. the distance is of order $1/N$. Hence $\exp - \alpha \tau_2 \sim 1/N$ leads to $\tau_2 \sim \log N$.

4. Analytic expression of the barrier height for the annealed model.

In this section we will show that one can obtain an analytic expression of the slope $\langle \log \tau_2 \rangle$ *versus* the number of automata N for the annealed model in the low temperature phase. The calculation can be reduced to the problem of a random walk in a one dimensional landscape in presence of a trap.

At time t, one can characterize the two configurations by the number n of automata which differ between the two configurations (i.e. the distance $D(t) = n/N$). With sequential

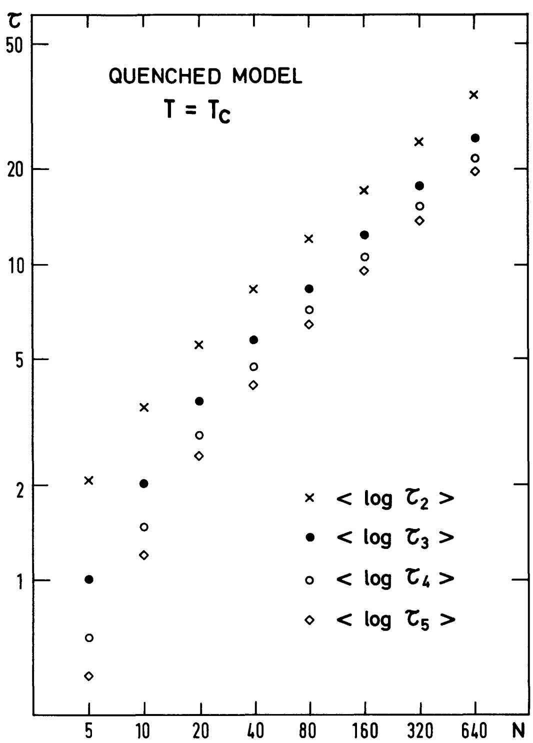

Fig. 2.

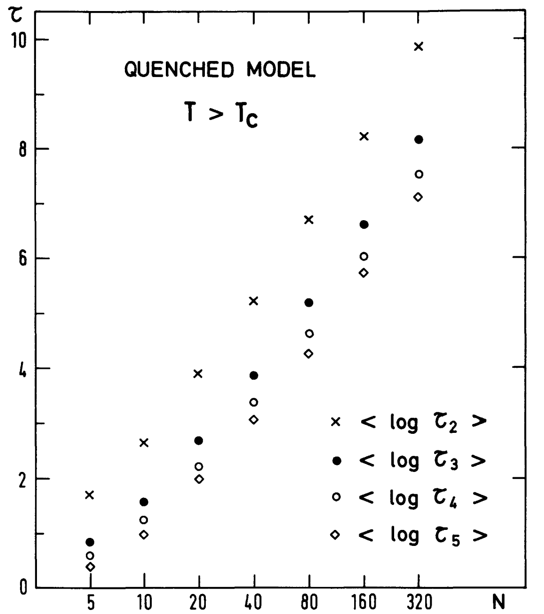

Fig. 3.

Fig. 2. — Quenched model at T_c : $K = 4$ and $\tanh \frac{1}{T} = 0.5$. Survival time $\langle \log \tau_n \rangle$ for $n = 2, 3, 4$ and 5 configurations as a function of the number N of automata. The time τ_n increase like $N^{0.5}$. The average is over 1000 samples.

Fig. 3. — Quenched model in the high temperature phase : $K = 4$ and $\tanh \frac{1}{T} = 0.3$. The same as figure 2. The survival time increases logarithmically with the number N automata. The average is over 5000 samples.

dynamics, only one automaton can change during a time step $\Delta t = 1/N$: the variable n performs a one dimensional random walk.

Let us call $B\left(\frac{n}{N}\right)$ the probability that a site, which is updated at time $t + \frac{1}{N}$ is different in the two configurations, when the distance at time t is $\frac{n}{N}$. We have seen in (8) that

$$B(D) = \frac{1}{2} \tanh\left(\frac{1}{T}\right) [1 - (1 - D)^K] \tag{12}$$

With $D(t) = \frac{n}{N}$, $D\left(t + \frac{1}{N}\right)$ can be equal to $\frac{n+1}{N}$, $\frac{n-1}{N}$ or $\frac{n}{N}$ with the following probabilities

$$\begin{aligned} a(n) &= \frac{N-n}{N} B\left(\frac{n}{B}\right) \\ b(n) &= \frac{n}{N}\left[1 - B\left(\frac{n}{N}\right)\right] \\ c(n) &= 1 - a(n) - b(n) \end{aligned} \tag{13}$$

The probability $P_t(n)$ that two configurations differ by n spins at time t evolves according to the Master equation :

$$P_{t+1/N}(n) = a(n-1)\,P_t(n-1) + b(n+1)\,P_t(n+1) + c(n)\,P_t(n) \tag{14}$$

One can check that $n = 0$ is a trap : this means that two identical configurations remain identical for ever.

Because the variable n is bounded ($0 \leq n \leq N$), one can consider that equation (14) is valid for $0 \leq n \leq N$ with the following boundary conditions $P_t(-1) = P_t(N+1) = 0$.

Equation (14) is valid for both the quenched and annealed models when $t = 0$. For $t > 0$, the inputs of each site i become correlated in the quenched model and the Master equation becomes wrong. On the contrary, for the annealed model, since at each time step all the functions and inputs are changed, the Master equation (14) remains valid at any time $t > 0$. The calculations which follow are based on (14) and therefore are valid for the annealed model only.

Using (13), the Master equation (14) can be written as

$$P_{t+\frac{1}{N}}(n) - P_t(n) = R(n+1) - R(n) \tag{15}$$

where

$$R(n) = b(n)\,P_t(n) - a(n-1)\,P_t(n-1) \tag{16}$$

(At the boundaries, $R(0) = R(N+1) = 0$).

The Master equation -(14) or (15)- describes also the evolution of a population of particles in a landscape. Particle positions are $n = 0, 1, \ldots, N$ and $P_t(n)$ is the density at time t and position n. Each particle can jump from n to $n+1$ or $n-1$ with probability $a(n)$ and $b(n)$. Within this picture, $R(n)$ defined in (16) is the flux of particles between n and $n-1$.

The steady state $P_{\text{eq}}(n)$ satisfies $P_{t+\frac{1}{N}}(n) = P_t(n)$. Using (15),

$$R(n) = R(0) = 0 \quad \text{for all } n. \tag{17}$$

Then (16) gives

$$P_{\text{eq}}(n) = \frac{a(n-1)}{b(n)}\,P_{\text{eq}}(n-1) \tag{18}$$

Because $a(0) = 0$, the steady state solution is

$$P_{\text{eq}}(n) = \delta_{n,0} \tag{19}$$

This means that the steady state corresponds to two configurations identical with probability one.

At initial time $t_0 = 0$ the two initial configurations $\{S_i(0)\}$ and $\{S_i'(0)\}$ are chosen at random, and the initial probabilities are

$$P_0(n) = \frac{1}{2^N}\,\frac{N!}{n!\,(N-n)!} \tag{20}$$

Then, at time t, $P_t(n)$ can be obtained from (20) by iterating (14). The probability that the two configurations meet at time t (in other words, the probability that $\tau_2 = t$) is given by :

$$P_t(0) - P_{t-\frac{1}{N}}(0)$$

Therefore, the average of log τ_2 is given by :

$$\langle \log \tau_2 \rangle = \sum_{m=1}^{\infty} [P_{\frac{m}{N}}(0) - P_{\frac{m-1}{N}}(0)] \log \left(\frac{m}{N} \right) \tag{21}$$

We could not find a direct method of calculating $\langle \log \tau_2 \rangle$, but we can calculate all the moments $\langle \tau_2^k \rangle$, with $k \geq 1$. Then $\langle \log \tau_2 \rangle$ can be obtained from these moments in the limit $k \to 0$.

The initial distribution, $P_0(n)$ is normalized and from (14), one can easily check that it remains normalized at $t > 0$. Therefore

$$P_t(0) = 1 - \sum_{n=1}^{N} P_t(n) \tag{22}$$

By definition, the moment $\langle \tau_2^k \rangle$ is given by

$$\langle \tau_2^k \rangle = \sum_{m=1}^{\infty} \left(\frac{m}{N} \right)^k [P_{\frac{m}{N}}(0) - P_{\frac{m-1}{N}}(0)] \tag{23}$$

Then using (22) :

$$\langle \tau_2^k \rangle = \frac{1}{N^k} \sum_{n=1}^{N} \sum_{m=0}^{\infty} P_{\frac{m}{N}}(n)[(m+1)^k - m^k] \tag{24}$$

If we define $S_k(n)$ by

$$S_k(n) = \sum_{m=0}^{\infty} P_{\frac{m}{N}}(n)[(m+1)^k - m^k] \tag{25}$$

(24) can be written as :

$$\langle \tau_2^k \rangle = \frac{1}{N^k} \sum_{n=1}^{N} S_k(n) \tag{26}$$

The sum in (25) is finite for $n \geq 1$. In fact, the Master equation has only one eigenvalue equal to 1, with eigenvector P_{eq} (19), and the modulus of the other eigenvalues are less than 1. Then, for large time, $P_t(n)$ decreases exponentially ($n \geq 1$) and this implies that the series (25) converges for $n \geq 1$. Let us notice that $S_k(0)$ is infinite.

The main advantage of (26) is that S_k is the solution of a linear recursion. Using the Master equation (14) and the definition (25) of S_k, we obtain :

$$\begin{aligned} S_k(n) - [a(n-1)\, S_k(n-1) + b(n+1)\, S_k(n+1) + c(n)\, S_k(n)] \\ = \sum_{m=0}^{\infty} [(m+1)^k - m^k][P_{\frac{m}{N}}(n) - P_{\frac{m+1}{N}}(n)] \\ = \sum_{m=1}^{\infty} P_{\frac{m}{N}}(n)[(m+1)^k - 2\,m^k + (m-1)^k] + P_0(n) \end{aligned} \tag{27}$$

Using the binomial formula, one can see that

$$(m+1)^k - 2\,m^k + (m-1)^k = \sum_{i=1}^{k-1} (-1)^{k-1-i} \binom{k}{i} [(m+1)^i - m^i] \tag{28}$$

From (27) and (28), we see that S_k is the solution of :

$$S_k(n) - [a(n-1)\,S_k(n-1) + b(n+1)\,S_k(n+1) + c(n)\,S_k(n)] = F_k(n) \tag{29}$$

where

$$F_k(n) = \sum_{i=1}^{k-1} \left[(-1)^{k-1-i}\binom{k}{i}\,S_i(n)\right] + (-1)^{k-1}\,P_0(n) \tag{30}$$

$S_k(n)$ is finite only for $n \geq 1$ and S_k is the steady state of the Master equation (29), with a right hand side : $F_k(n)$. In F_k, we find only $P_0(n)$ and $S_i(n)$ for $i < k$. So we can try to solve this hierarchy of equations by starting with $k = 1$, $k = 2$, etc. For example,

$$F_1(n) = P_0(n),$$
$$F_2(n) = 2\,S_1(n) - P_0(n), \text{ etc.}$$

If we interpret $S_k(n)$ as a stationnary density of particles, $F_k(n)$ corresponds to an external flux of particles in n, and compensates the loss of particles which get trapped at $n = 0$.

The total population $\sum_{n=1}^{N} S_k(n)$ is $N^k\langle\tau_2^k\rangle$. For $k = 1$, we could have anticipated this result as $F_1(n) = P_0(n)$: each time that one particle is added to the system, one particle gets trapped in $n = 0$. The averàge survival time (divided by N) is equal to the number of particles in the system.

As (16), we define :

$$R_k(n) = b(n)\,S_k(n) - a(n-1)\,S_k(n-1) \tag{31}$$

($R_k(n)$ is the flux between n and $n-1$). Now, (29) is equivalent to

$$R_k(n) - R_k(n+1) = F_k(n) \tag{32}$$

At the boundary : $R_k(N) = F_k(N)$. Then

$$R_k(n) = \sum_{i=n}^{N} F_k(i) \tag{33}$$

(31) gives :

$$\begin{aligned} S_k(n) &= \frac{R_k(n)}{b(n)} + \frac{a(n-1)}{b(n)}\,S_k(n-1) \\ &= \frac{R_k(n)}{b(n)} + \frac{a(n-1)}{b(n)}\frac{R_k(n-1)}{b(n-1)} + \cdots + \frac{a(n-1)\ldots a(1)}{b(n)\ldots b(2)}\frac{R_k(1)}{b(1)} \end{aligned} \tag{34}$$

Defining

$$E(n) = \frac{a(n-1)\ldots a(1)}{b(n)\ldots b(2)\,b(1)} \tag{35}$$

(34) can be written as

$$\begin{aligned} S_k(n) &= E(n) \sum_{i=1}^{n} \frac{R_k(i)}{b(i)\,E(i)} \\ &= E(n) \sum_{i=1}^{n} \frac{1}{b(i)\,E(i)} \sum_{j=i}^{N} F_k(j) \end{aligned} \tag{36}$$

This solution is not particular to our specific problem. It would be valid for any 1-D random walk, with a trap. (The trap appears in (36) because $b(1) \neq 0$ and $a(0) = 0$). We see that $\log E(n)$ plays the role of a potential. In fact, if we were studying a system with a hamiltonian : $H(n)$ the probabilities of jumping would be :

$$\begin{aligned} a(n) &= \mathrm{e}^{-\frac{1}{2T}[H(n+1)-H(n)]} \\ b(n) &= \mathrm{e}^{-\frac{1}{2T}[H(n-1)-H(n)]} \end{aligned} \tag{37}$$

These expressions satisfy the detailed balance and, in this case

$$E(n) = \text{cte } \mathrm{e}^{-H(n)/T}$$

Now, we can look for the specific form of $a(n)$, $b(n)$ and $E(n)$.
With the definitions (35) and (13)

$$\begin{aligned} \log E(n) &= \sum_{i=i}^{n-1} \log \frac{a(i)}{b(i)} - \log b(n) \\ &= \sum_{i=1}^{n-1} \log \left(\frac{1-i/N}{i/N} \frac{B(i/N)}{1-B(i/N)} \right) - \log b(n) \end{aligned} \tag{38}$$

For large N, $\frac{n}{N}$ becomes a continuous variable. $E(n)$ has the scaling shape :

$$\frac{1}{N} \log E(n) = f(n/N) + O\left(\frac{1}{N}\right)$$

where

$$f(D) = \int_0^D \log \left(\frac{1-x}{x} \frac{B(x)}{1-B(x)} \right) \mathrm{d}x \tag{39}$$

This integral can be calculated. We will not give its analytic expression because it would be too complicated. We are just interested here by the extrema of $f(D)$. The shape of $f(D)$ depends on the function $B(x)$.

$B(D) = \frac{1}{2} \tanh \left(\frac{1}{T}\right) [1-(1-D)^K]$ has an attractive point : $D^* = B(D^*)$. In the chaotic phase $(T < T_c)$, $D^* > 0$; $f(D)$ increases for $D < D^*$ and decreases for $D > D^*$. Then $f(D^*)$ is the maximum of $f(D)$. Now, we can calculate, in the chaotic phase

$$S_k(n) = E(n) \sum_{i=1}^{n} \frac{1}{b(i) E(i)} \sum_{j=i}^{N} F_k(j) \tag{40}$$

$E(n) = \exp\left(N f\left(\frac{n}{N}\right)\right)$ is very peaked around $n \sim ND^*$ for large N and $\frac{1}{b(i) E(i)}$ is very peaked around $i \simeq 1$. Then $S_k(n)$ is very large for $n \sim ND^*$. For these n, in (40), the sum over i can be restricted to i close to 1. In the definition (30), $F_k(j)$ is a sum of $P_0(j)$ and $S_i(j)$ with $i < k$.

$P_0(j) = \frac{1}{2^N} \binom{N}{j}$ is very peaked around $j \simeq \frac{N}{2}$ and $S_i(j)$ around $j \simeq ND^*$: Thus we can replace $\sum_{j=i}^{N} F_k(j)$ by $\sum_{j=1}^{N} F_k(j)$. Then, for n close to ND^*,

$$S_k(n) = E(n)\left(\sum_{j=1}^{N} F_k(j)\right)\left(\sum_{i \text{ around } 1} \frac{1}{b(i)\,E(i)}\right) \tag{41}$$

With $B(i/N) \simeq B'(0)\dfrac{i}{N}$ for i close to 1, (13) and (35) gives

$$E(i)\,b(i) = B'(0)^{i-1}$$

Then the solution $S_k(n)$ is

$$S_k(n) = \left[\sum_{j=1}^{N} F_k(j)\right] E(n)\frac{B'(0)}{B'(0)-1} \tag{42}$$

The moment $\langle \tau_2^k \rangle$ are given by :

$$\langle \tau_2^k \rangle = \frac{1}{N^k}\sum_{n=1}^{N} S_k(n) = \frac{1}{N^k}\frac{B'(0)}{B'(0)-1}\left[\sum_{j=1}^{N} F_k(j)\right]\left[\sum_{n=1}^{N} E(n)\right] \tag{43}$$

In the case, $k = 1$,

$$F_k(n) = P_0(n) = \frac{1}{2^N}\binom{N}{n} \tag{44}$$

then the average time is :

$$\langle \tau_2 \rangle = \frac{1}{N}\left(\sum_{n=1}^{N} E(n)\right)\frac{B'(0)}{B'(0)-1} \tag{45}$$

$$\text{and} \quad \langle \tau_2^k \rangle = \frac{1}{N^{k-1}}\left(\sum_{j=1}^{N} F_k(j)\right)\langle \tau_2 \rangle \tag{46}$$

Using, the definition (30) of $F_k(j)$

$$\langle \tau_2^k \rangle = \langle \tau_2 \rangle \left(\sum_{i=0}^{k-1}\left(\frac{1}{N}\right)^{k-1-i}\binom{k}{i}\langle \tau_2^i \rangle\right) \tag{47}$$

For large N, (47) gives

$$\begin{aligned} \langle \tau_2^k \rangle &= k\,\langle \tau_2 \rangle \langle \tau_2^{k-1} \rangle \\ \text{then} \quad \langle \tau_2^k \rangle &= k!\,\langle \tau_2 \rangle^k \end{aligned} \tag{48}$$

This relation between the moment $\langle \tau_2^k \rangle$ means that the probability law of τ_2, the time it takes for two configurations to meet, is a exponential law. This distribution of time is the same as the one found recently on supercritical contact process [26]. Then $\langle \log \tau_2 \rangle$ is given by :

$$\langle \log \tau_2 \rangle = \log \langle \tau_2 \rangle - \gamma \tag{49}$$

where γ is the Euler's constant : 0.577. Using the expression (39), we find :

$$\langle \tau_2 \rangle \sim \frac{C}{\sqrt{N}}\,\mathrm{e}^{Nf(D^*)} \tag{50}$$

for large N.

Then

$$\frac{1}{N}\langle \log \tau_2 \rangle = f(D^*) \tag{51}$$

where $f(D^*)$ is the maximum of :

$$f(D) = \int_0^D \mathrm{d}x \log\left(\frac{1-x}{x} \tanh\left(\frac{1}{T}\right) \frac{1-(1-x)^K}{2-\tanh\left(\frac{1}{T}\right)[1-(1-x)^K]} \right) \tag{52}$$

In the chaotic phase τ_2 increases exponentially with N. The exponential rate $f(D^*)$ is similar to a free energy barrier height.

With the parameters chosen of the figure 1 ($K = 4$ and $\tanh \frac{1}{T} = 0.7$), this barrier height is : $f(D^*) = 0.0408$.

If in figure 1b we measure the slopes of the $\langle \log \tau_n \rangle$ obtained by Monte Carlo simulations for the annealed model, we find 0.031 for τ_2 ; 0.028 for τ_3 ; 0.025 for τ_4 and 0.023 for τ_5. For the quenched model (Fig. 1a), we find 0.026 for τ_2 ; 0.023 for τ_3 ; 0.021 for τ_4 and 0.019 for τ_5. The difference between the predicted value 0.0408 and the measure 0.031 for τ_2 is due to finite size effects. From equation (50), we expect :

$$\frac{\mathrm{d}}{\mathrm{d}N}\langle \log \tau_2 \rangle = f(D^*) - \frac{1}{2N} \tag{53}$$

The curve on the figure 1b is $\langle \log \tau_2 \rangle$ computed with the Master equation (14) and (21) for finite N. For $N \leq 100$, it agrees with the simulations. For larger N, the slope of this curve increases to the predicted value $f(D^*)$. This means that at $N \leq 100$, τ_2 is not yet in its asymptotic regime. Since the measured value of the slope 0.031 for τ_2 has not yet reached its asymptotic value 0.0408, we believe that the slopes of all the other τ_n for the annealed and quenched models are also far from convergence. Therefore the measured values 0.031, 0.028, 0.025, 0.022 of the slopes of the $\langle \log \tau_n \rangle$ for the annealed model and 0.026, 0.023, 0.021, 0.019 for the quenched model do not allow to conclude that these slopes would remain different in the large N limit.

Up to equation (36), the results are valid at all temperatures. From (36), it would be possible to show that in the high temperature phase $\langle \tau_2 \rangle$ increases like $\log N$ whereas at T_c, $\langle \tau_2 \rangle$ is a power law in N.

5. Conclusion.

In the present work, we have seen that the times τ_n for at least two configurations among n to meet increase exponentially with the number N of automata for the (two versions of the) Kauffman model in the low temperature phase. The exponential rate of increase of these times τ_n seems to depend neither on the model (annealed or quenched) nor on n. In the case $n = 2$ for the annealed model, we have obtained an analytic expression of the barrier height.

When we compare these results with those of similar calculations done for the mean field ferromagnet [6], we see that the main difference is that for the Kauffman model, all the times τ_n ($n \geq 2$) seem to increase exponentially with N whereas for the ferromagnet only τ_2 increased exponentially with N (because in the mean field ferromagnet there exist only two valleys).

The results of the present work are much closer to what was found for the annealed random maps model [20] for which the ratios τ_2/τ_n have a finite limit $n(n-1)/2$ as $N \to \infty$. In the annealed random map model [24] it was possible to obtain a much more detailed information on the weights of the attractors since one could calculate the full probability distribution of these weights. In the Kauffman model, it seems to be much more difficult to do the same because we have not been able to find a method which allows one to calculate the times τ_n for $n \geq 3$ even in the annealed case.

We see that the method which consists in comparing several configurations subjected to the same noise can be used to define and measure barrier heights in a large class of systems without any need of a free energy. One can imagine other definitions [25] of barrier heights which would not rely on the trick of using the same noise for two or more configurations. It would be, of course, very interesting, to compare all these possible definitions of barrier heights in order to know under what conditions the idea of using the same noise for several configurations gives a good measure of barriers.

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