Directed Polymers in a Random Medium: $1/d$ Expansion.

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Abstract – The problem of direct polymers in a random medium has a mean-field theory similar to, but simpler in form than, spin glasses. We propose here a method which allows one to expand the free energy of directed polymers around the mean-field theory giving a $1/d$ expansion. Our first results concerning the ground-state energy, the free energy and the transition temperature are presented.

During the last ten years a lot of progress has been made in the mean-field theory of strongly disordered systems such as spin glasses [1]. It is now broadly accepted that the Parisi approach [2] with a broken replica symmetry is the best way of understanding spin glasses. Extending this approach to finite-dimensional systems presents many difficulties [3, 4] and at the moment the question of whether a spin glass phase with broken replica symmetry exists in finite dimension is still controversial [5, 6, 7]. One difficulty in trying to extend the Parisi solution to finite-dimensional systems is that the mean-field free energy is sufficiently complicated to make all expansions around it technically very difficult.

The problem of directed polymers in a random medium [8, 9] also has a mean-field theory [10] with broken replica symmetry. For that problem, however, the analytic expression of the free energy, both in the low- and the high-temperature phases, is extremely simple [10]. So one can try to expand around this mean-field solution and obtain a $1/d$ expansion.

Here, we describe a method which, we think, gives the $1d$ expansion for this problem and we present our first results obtained using this approach. Our method consists of building a family of tree models (n-tree approximations) which are better and better approximations to a finite-dimensional lattice (which is recovered in the limit $n \to \infty$).

Here firstly we define the model to be studied. Then we explain how the problem on a lattice in dimension d can be approximated by n tree models which can be solved exactly in the same way as in [10]. For d large, the calculation can be done for arbitrary n. We then obtain the $1/d$ expansion for the original lattice by taking the limit $n \to \infty$.

We consider here the model on a hypercubic lattice in dimension d. On each bond ij of the lattice, there is a random energy ε_{ij}. For convenience [11] (because this choice makes the

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calculation of several integrals which appear later possible) we choose the distribution $\rho(\varepsilon)$ to be

$$\rho(\varepsilon) = \exp[-\varepsilon]. \tag{1}$$

On this lattice, a directed walk of L steps which starts at a point $\boldsymbol{r} = (x_1, \ldots, x_d)$ is a walk which reaches any point $\boldsymbol{r}' = (x_1 + n_1, \ldots, x_d + n_d)$ with $n_1 + \ldots + n_d = L$. So, the walk is directed along the $(1, 1, \ldots, 1)$ direction [12]. The partition function $Z_L(\boldsymbol{r})$ of a walk of L steps starting at $\boldsymbol{r}$ is, by definition,

$$Z_L(\boldsymbol{r}) = \sum_w \exp[-E_w/T], \tag{2}$$

where in (2) the sum runs over the d^L directed walks w starting at $\boldsymbol{r}$ and the energy E_w is the sum of the energies ε_{ij} of the bonds ij visited by w.

One can write the following recursion for $Z_L(\boldsymbol{r})$:

$$Z_{L+1}(\boldsymbol{r}) = \sum_{i=1}^{d} a_i(\boldsymbol{r}) Z_L(\boldsymbol{r} + \boldsymbol{e}_i) \tag{3}$$

with

$$a_i(\boldsymbol{r}) = \exp(-\varepsilon_{\boldsymbol{r}, \boldsymbol{r} + \boldsymbol{e}_i}/T). \tag{4}$$

Since $Z_0(\boldsymbol{r}) = 1$, this recursion gives, in principle, all the $Z_L(\boldsymbol{r})$. One would like to compute $\langle \log Z_L(\boldsymbol{r}) \rangle / L$ for large L (where $\langle\ \rangle$ means the average over disorder).

The main difficulty in recursion (3) is that $Z_L(\boldsymbol{r} + \boldsymbol{e}_i)$ and $Z_L(\boldsymbol{r} + \boldsymbol{e}_j)$ both depend on $Z_{L-1}(\boldsymbol{r} + \boldsymbol{e}_i + \boldsymbol{e}_j)$ and thus are correlated.

Neglecting these correlations (*i.e.* considering the $Z_L(\boldsymbol{r} + \boldsymbol{e}_i)$ in (3) to be uncorrelated) is the same as replacing the lattice by a tree with d branches. The problem on a tree with an arbitrary number d of branches has been solved exactly [10] for any distribution $\rho(\varepsilon)$. The solution is the following: if one defines $G_1(\gamma)$ by

$$G_1(\gamma) = \gamma^{-1} \log\left(\sum_{i=1}^{d} \langle \exp[-\gamma\varepsilon_i] \rangle\right), \tag{5}$$

where ε_i is the energy of the i-th branch, the free energy $F_1(T)$ per unit length of the walk is given by

$$F_1(T) = \begin{cases} -G_1(1/T) & \text{if } T > 1/\gamma_{\min}, \\ -G_1(\gamma_{\min}) & \text{if } T < 1/\gamma_{\min}, \end{cases} \tag{6}$$

where $\gamma_{\min}$ is the value of γ which minimizes $G_1(\gamma)$ (because $\gamma G_1(\gamma)$ is convex, one can show that $G_1(\gamma)$ has a single minimum).

The derivation of (5) and (6) can be obtained by several techniques (a replica method [13], the GREM [13], the analogy with travelling waves [10]). It would be too long to repeat these derivations here. Let us just mention that (5) and (6) express the fact that above $T_c = 1/\gamma_{\min}$ the quenched and annealed free energies are equal, at $T_c = 1/\gamma_{\min}$ the entropy vanishes and below T_c the free energy is equal to the ground-state energy. For $\rho(\varepsilon)$ given by (1), one has

$$G_1(\gamma) = \gamma^{-1}[\log d - \log(\gamma + 1)]. \tag{7}$$

Using (6) and (7) one can expand $F_1(T)$ in powers of $1/d$. For example one gets that for the tree problem

$$T_c = (ed)^{-1} + 2(ed)^{-2} + 9(ed)^{-3}/2 + 32(ed)^{-4}/3 + 625(ed)^{-5}/24 + O(d^{-6}). \tag{8}$$

To extend these tree results to a finite-dimensional lattice, one has to include che correlations between the $Z_L(r + e_i)$ in (3). In order to do this, we make a sequence of approximations (the n-tree approximation), which are built by first iterating the recursion (3) exactly n times on the d-dimensional lattice and by then neglecting the correlations between the Z_L. So for $n = 2$, one gets

$$Z_{L+2}(r) = \sum_{i=1}^{d} a_i(r) a_i(r + e_i) Z_L(r + 2e_i) + \sum_{i=1}^{d}\sum_{j>i}^{d} [a_i(r) a_j(r + e_i) + a_j(r) a_i(r + e_j)] Z_L(r + e_i + e_j) \tag{9}$$

and one neglects the correlations between the Z_L on the r.h.s. of (9). So the original lattice is replaced by a 2 tree.

This new tree problem can be solved as the previous problem. The only difference is that instead of d branches, we now have $d + d(d - 1)/2$ branches. On d branches (1st term in (9)) there is an energy which is the sum of two independent random energies ε, whereas on each of the remaining $d(d - 1)/2$ branches (2nd term in (9)), there is an effective energy which represents the contribution of the two paths forming this branch. As for the 1 tree, the solution is obtained by defining a function $G_2(\gamma)$

$$G_2(\gamma) = (2\gamma)^{-1} \log \left[\sum_{i=1}^{d} \left\langle \exp\left[- \gamma(\varepsilon_i + \varepsilon_i') \right] \right\rangle + \sum_{i<j} \left\langle (\exp\left[- (\varepsilon_i + \varepsilon_j')/T \right] + \exp\left[- (\varepsilon_j + \varepsilon_i')/T \right])^{rT} \right\rangle \right], \tag{10}$$

where all the energies which appear in (10) are distributed according to $\rho(\varepsilon)$. The factor $1/2$ in (10) comes from the fact that if one iterates (9) L times, the length of the polymer is $2L$. From (10), one can calculate γ_{min} and then obtain the free energy $F_2(T)$ as in (6). Now γ_{min} is a function of T because T appears explicitly in (10) (but not in (5)).

As n increases, the formulae which generalize (5) and (10) to give $G_n(\gamma)$ become more and more complicated. For example $G_3(\gamma)$ is given by

$$G_3(\gamma) = (3\gamma)^{-1} \log [d\langle \exp[- \gamma\varepsilon]\rangle^3 + d(d - 1) \langle (f_3(T))^{rT} \rangle + d(d - 1)(d - 2)/6\langle (f_4(T))^{rT} \rangle], \tag{11}$$

where f_3 and f_4 are the partition functions associated with diagrams drawn in fig. 1 (e.g. f_3 is the sum over the 3 walks of 3 steps which connect the 2 marked points of the 3rd diagram). The calculation of $\langle (f_3(T))^{rT} \rangle$ requires averaging over 7 bonds which is already a very hard problem. So although increasing n should give successively better approximations to the original lattice, the $G_n(\gamma)$ quickly become so complicated that there is no hope of doing any calculation for large n.
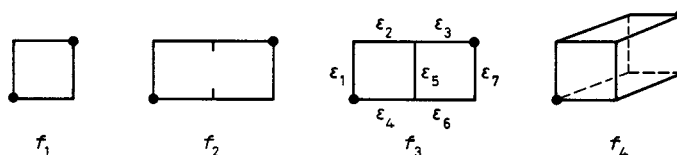

Fig. 1. – f_i is the sum of the weights of all walks connecting the two marked points on the diagram i. For example $f_3 = \exp[- (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/T] + \exp[- (\varepsilon_4 + \varepsilon_5 + \varepsilon_3)/T] + \exp[- (\varepsilon_4 + \varepsilon_6 + \varepsilon_7)/T]$.

However, for large d and for $\rho(\varepsilon)$ given by (1), one can expand the expressions for the G_n in powers of $1/d$ for arbitrary n. G_n becomes simpler because one expects $\gamma_{\min} = 1/T_c = O(d)$ (see (8)) and therefore the averages of the functions f_3 and f_4 can be simplified. Let us take as an example the case of $\langle (f_3(T))^{\gamma T} \rangle$. f_3 consists of the sum of 3 terms. When one averages over the 7 energies, the leading contribution $O(\gamma^{-3})$ to f_3 (when γ is large and $T = O(\gamma^{-1})$) comes from the situations where one of these 3 terms is much larger than the other 2. Then at order γ^{-5}, one finds the contribution of pairs of walks which differ by only 2 bonds. Then pairs of walks which differ by 3 bonds contribute at order γ^{-6}, and so on.

The general rule is that the contribution to G_n coming from all the paths which pass through K bonds is γ^{-K}. So the first contribution to G_n comes from the d^n single walks of n steps, then from pairs of walks which differ by only 2 bonds, etc. So

$$G_n(\gamma) = \frac{1}{n\gamma} \log \left\{ d^n \langle \exp[-\gamma\varepsilon] \rangle^n + (n-1)\frac{d^{n-1}(d-1)}{2} \cdot \right.$$
$$\left. \cdot \langle \exp[-\gamma\varepsilon] \rangle^{n-2} [\langle (f_1(T))^{\gamma T} \rangle - 2\langle \exp[-\gamma\varepsilon] \rangle^2] + O(\gamma^{-3}) \right\}. \qquad (12)$$

The function f_1 is the sum over the 2 walks of diagram 1 of fig. 1. (The next order would involve f_2.) In the sum which appears on the r.h.s. of (12), the leading term is of order 1 (because $\gamma \sim d$), the 2nd term is of order γ^{-2}. Notice that $2\langle \exp[-\gamma\varepsilon] \rangle^2$ has to be subtracted from $\langle (f_1(T))^{\gamma T} \rangle$ because of situations where the weight of one of the two paths dominates the other. So, to compute all the G_n to a given order in γ^{-1}, a finite number of integrals ($\langle \exp[-\gamma\varepsilon] \rangle$, $\langle (f_1(T))^{\gamma T} \rangle$, $\langle (f_2(T))^{\gamma T} \rangle$ ...) are needed.

Following the same rules which lead from (5) to (6), one gets from (12) the free energy $F_n(T)$ in the low-temperature phase:

$$F_n(T) = (ed)^{-1} + (ed)^{-2} + 3(ed)^{-3}/2 - (ed)^{-3}(g(T\,ed) - 3)(n-1)/n + O(d^{-4}), \qquad (13)$$

where

$$g(t) = t \int\limits_0^\infty \mathrm{d}u\,(2 + tu)\,((1 + \exp[-u])^t - 1) \qquad (14)$$

and $e = 2.71828...$ In the high-temperature phase, the free energy is the annealed free energy $F_n(T) = T \log[(T+1)/dT]$ for all n. Equation (14) implies in particular that the specific heat is linear at low temperature $C_n(T) \sim T/d$. So the low-temperature phase is no longer fully frozen.

From (12), the transition temperature can be obtained (by writing $\gamma_{\min}(T_c) = 1/T_c$).

$$T_c = (ed)^{-1} + 2(ed)^{-2} + (ed)^{-3}[9/2 + (n-1)g'(1)/n] + O(d^{-4}), \qquad (15)$$

where $g'(1) = 7.527827...$ see (14). These expansions can be pushed further. The diagrams which appear in this expansion have previously been used for directed percolation[12]. When doing these $1/d$ expansions, we observed that each term has a regular n-dependence and has a limit when $n \to \infty$. For example, the ground-state energy obtained by this method is in the limit $n \to \infty$

$$E_{\mathrm{GS}} = (ed)^{-1} + (ed)^{-2} + 9(ed)^{-3}/2 + (110/3 - 3e)(ed)^{-4} + (11297/24 - 74e)(ed)^{-5} + O(d^{-6}). \qquad (16)$$

In this paper we have obtained $1/d$ expansions based on n-tree approximations. The main assumption for the method to work is that the limits $n \to \infty$ and $d \to \infty$ commute. At the

moment we have no proof of this. However, in all the calculations we have done so far, each coefficient of the $1/d$ expansion has a limit when $n \to \infty$. Also for some cases (which we will describe elsewhere) one can show that these two limits commute.

The approach used here for the free energy can be extended to obtain various quantities: overlaps, geometrical properties. It can also be used for other distributions $\rho(\varepsilon)$ (leading to noninteger powers of $1/d$). These results will be presented in a future work.

The method presented here allows one to obtain the properties of the low-temperature phase by expanding around the mean-field solution. It would be interesting to see whether one can estimate the upper critical dimension from these expansions. Also it would be nice to extend the present approach to other disordered systems such as spin glasses for which, up to now, $1/d$ expansions can only be done in the high-temperature phase [14, 15].

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