J. Phys. A: Math. Gen. **22** (1989) 1975–1981. Printed in the UK

The probability distribution of the partition function of the random energy model

E Gardner† and B Derrida‡

† Department of Physics, University of Edinburgh, Edinburgh, EH9 3JZ, UK
‡ Service de Physique Théorique de Saclay§, F-91191 Gif-sur-Yvette Cedex, France

Received 13 January 1989

Abstract. We give the expression for both integer and non-integer moments of the partition function Z of the random energy model. In the thermodynamic limit, we find that the probability distribution $P(Z)$ can be decomposed into two parts. For $\log Z - \langle \log Z \rangle$ finite, the distribution is independent of N, the size of the system, whereas for $\log Z - \langle \log Z \rangle$ positive and of order N, the distribution is Gaussian. These two parts match in the region $1 \ll \log Z - \langle \log Z \rangle \ll N$ where the distribution is exponential.

1. Introduction

In spin-glass models, one is interested in the calculation of the average free energy, i.e. the average of the logarithm of the partition function Z. For most models, both this calculation and the calculation of the probability distribution of Z are extremely hard and cannot be done exactly. The random energy model (Derrida 1980, 1981, Derrida and Toulouse 1985) is simple enough to allow the calculation of the whole probability distribution $P(Z)$ of Z. This distribution (Derrida and Toulouse 1981) is of interest in particular in relation to the determination of the correct analytic continuation in the replica method (Parisi 1979, 1980, 1983, Mézard *et al* 1984a, b). The purpose of the present work is to give the expressions for both integer and non-integer moments of $P(Z)$ and to deduce from these moments the shape of $P(Z)$.

The random energy model is defined for a system of size N and consists of 2^N energy levels E_i distributed independently according to a probability distribution $\rho(E_i)$:

$$\rho(E_i) = \frac{1}{\sqrt{\pi N J^2}} \exp\left(-\frac{E_i^2}{NJ^2}\right). \tag{1}$$

The partition function Z is then given by

$$Z(\{E_i\}) = \sum_{i=1}^{2^N} \exp\left(-\frac{E_i}{T}\right). \tag{2}$$

In the present work we will derive the following result for the moments $\langle Z^\nu \rangle$ of the partition function

$$\langle Z^\nu \rangle = \int \ldots \int \prod_{i=1}^{2^N} \rho(E_i)\, \mathrm{d}E_i \qquad Z^\nu(\{E_i\}) \tag{3}$$

§ Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Énergie Atomique.

where $\langle \cdot \rangle$ denotes an average over disorder, and

$$\langle Z^{\nu}\rangle \sim 2^{N} \exp[N\nu^2 J^2/4T^2] \qquad \text{for} \begin{cases} T > T_c \text{ and } \nu > (T/T_c)^2 \\ T < T_c \text{ and } \nu > T/T_c \end{cases}$$

$$\langle Z^{\nu}\rangle \sim 2^{N\nu} \exp[N\nu J^2/4T^2] \qquad \text{for } T > T_c \text{ and } \nu < (T/T_c)^2 \tag{4b}$$

$$\langle Z^{\nu}\rangle \sim Z_0^{\nu} \frac{\Gamma(1-\nu T_c/T)}{\Gamma(1-\nu)} \qquad \text{for } T < T_c \text{ and } \nu < T/T_c \tag{4c}$$

where Z_0 is given by

$$Z_0 = \exp\left\{ N\frac{J}{T}\sqrt{\log 2} - \frac{T_c}{T}\left[\frac{1}{2}\log N - \log[-\Gamma(-T/T_c)] + \tfrac{1}{2}\log\left(\frac{\pi J^2}{T^2}\right)\right]\right\} \tag{5}$$

and where the critical temperature T_c is given by

$$T_c = J/2\sqrt{\log 2} \tag{6}$$

(see figure 1).

In formulae (4), the sign $\sim$ means the leading behaviour in the limit $N \to \infty$. In the limit $\nu \to 0$ and for $T < T_c$, formula (4*c*) gives the result for the average free energy of the random energy model (formula (31) of Derrida (1981)).

In § 2, we will derive the results given in formula (4). In § 3, we shall describe the shape of the probability distribution $P(Z)$ of the partition function.

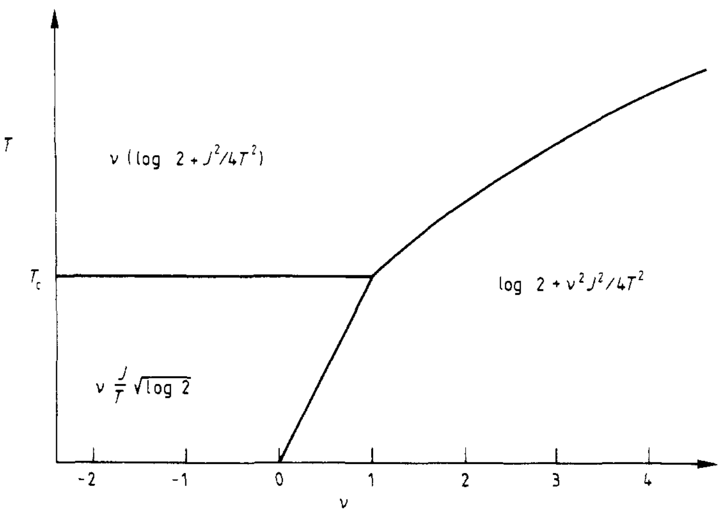

Figure 1. Phase diagram in the plane (ν, T) of the moments $\langle Z^{\nu}\rangle$ of the partition function of the random energy model.

2. The moments $\langle Z^{\nu}\rangle$

In order to calculate $\langle Z^{\nu}\rangle$, we will use the following integral representation

$$\langle Z^{\nu}\rangle = \frac{1}{\Gamma(n-\nu)} \int_0^{\infty} \mathrm{d}t\, t^{n-\nu-1}\left(-\frac{\partial}{\partial t}\right)^{n} \langle \exp(-tZ)\rangle \tag{7}$$

valid for any integer n larger than ν

$$n > \nu. \tag{8}$$

Formula (7) is true even without averaging. For $n = 0$, it is just the definition of the gamma function and for $n \geq 1$, it can easily be understood since $(-\partial/\partial t)^n \,\mathrm{e}^{-tZ} = (Z)^n \mathrm{e}^{-tZ}$. So the knowledge of all the moments $\langle Z^\nu \rangle$ reduces to the calculation of the following average $\langle \mathrm{e}^{-tZ} \rangle$ which is greatly simplified by the fact that the energies E_i are independent. If we define ϕ by

$$\exp(-\phi) = \langle \exp(-tZ) \rangle \tag{9}$$

then one has

$$\exp(-\phi/2^N) = f(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}y \exp[-(y^2 + t\,\mathrm{e}^{-\lambda y})] \tag{10}$$

where

$$\lambda = \sqrt{N}\, J/T. \tag{11}$$

The evaluation of $f(t)$ defined in (10) for different ranges of the value of $\log t$ has already been given by Derrida (1981).

For

$$-p\lambda^2/2 < \log t < -(p-1)\lambda^2/2 \tag{12}$$

one has

$$\begin{aligned} f(t) = 1 - t \exp\left(\frac{\lambda^2}{4}\right) + \frac{t^2}{2!} \exp(\lambda^2) + \ldots + (-1)^{p-1} \frac{t^{p-1}}{(p-1)!} \exp\left(\frac{(p-1)^2\lambda^2}{4}\right) \\ + \frac{1}{\sqrt{\pi}\,\lambda} \exp\left(\frac{-\log^2 t}{\lambda^2}\right)\left[\Gamma\left(\frac{2\log t}{\lambda^2}\right) - \frac{1}{\lambda^2}\Gamma''\left(\frac{2\log t}{\lambda^2}\right) + \mathrm{O}\left(\frac{1}{\lambda^4}\right)\right]. \end{aligned} \tag{13}$$

Thus to compute the moments $\langle Z^\nu \rangle$, we need to know for each ν and T, the range of values of $\log t$ which dominates the integral (7).

We have now to express ϕ as a function of t or t as a function of ϕ using (10) and (13) in order to obtain $\langle Z^\nu \rangle$ through the integral (7). Since the expression of $f(t)$ changes with the range of values of $\log t$, one has to distinguish several cases. It turns out that depending on the temperature T and on ν, the part of the integral (7) which dominates is either $\phi \sim 1$ or ϕ exponentially small in N: For $T < T_\mathrm{c}$, $\nu < T/T_\mathrm{c}$ and for $T > T_\mathrm{c}$, $\nu < (T/T_\mathrm{c})^2$ the part which dominates is $\phi \sim 1$, whereas for $T < T_\mathrm{c}$, $\nu > T/T_\mathrm{c}$ and for $T > T_\mathrm{c}$, $\nu > (T/T_\mathrm{c})^2$ it is ϕ exponentially small with N.

For $T > T_\mathrm{c}$ and $\phi \sim 1$, formula (13) should be used in the range

$$\log t < -\lambda^2/2 \tag{14}$$

and this gives

$$t = \phi \exp(-\lambda^2/4)/2^N. \tag{15}$$

Clearly since $\phi \sim 1$, (15) is consistent with (14) as long as $T > T_\mathrm{c}$. Then using (15) in (7) one gets from the contribution of the range $\phi \sim 1$

$$\langle Z^\nu \rangle \sim [2^N \exp(\lambda^2/4)]^\nu. \tag{16}$$

For $T < T_\mathrm{c}$ and $\phi \sim 1$, one has to use (13) in the case

$$-\lambda^2/2 < \log t < 0 \tag{17}$$

and this gives

$$\frac{\phi}{2^N} = -\frac{1}{\sqrt{\pi}\,\lambda} \exp\left(-\frac{\log^2 t}{\lambda^2}\right)\Gamma\left(\frac{2\log t}{\lambda^2}\right) \tag{18}$$

which, since $\phi \sim 1$, becomes

$$\log t = -\log Z_0 + (T_c/T)\log\phi \tag{19}$$

where Z_0 is given (5).

Then the contribution to $\langle Z^\nu\rangle$ from the range $\phi \sim 1$ is

$$\begin{aligned}\langle Z^\nu\rangle &\sim \frac{1}{\Gamma(n-\nu)}\int_0^\infty t^{n-\nu-1}\left(-\frac{\partial}{\partial t}\right)^n \langle e^{-tZ}\rangle\, dt \\ &= \frac{1}{\Gamma(n-\nu)} Z_0^\nu \frac{T_c}{T}\int_0^\infty d\phi^{(T_c/T)(n-\nu)-1}\left(-\frac{T}{T_c}\phi^{1-T_c/T}\frac{\partial}{\partial\phi}\right)^n \exp(-\phi)\end{aligned} \tag{20}$$

and after some integrations by parts one obtains

$$\langle Z^\nu\rangle = Z_0^\nu \frac{T_c}{T}\frac{\Gamma(-\nu T_c/T)}{\Gamma(-\nu)} = Z_0^\nu \frac{\Gamma(1-\nu T_c/T)}{\Gamma(1-\nu)}. \tag{21}$$

Expressions (16) and (21) come only from the contribution of the range $\phi \sim 1$. Let us now discuss the range ϕ exponentially small in N. If ϕ is exponentially small with N, one can replace $\langle \exp - tZ\rangle$ in (7) by

$$\langle \exp(-tZ)\rangle = \exp(-\phi) = 1-\phi = 1-2^N \log f(t). \tag{22}$$

So $f(t)$ is close to 1 and for the range

$$-p\lambda^2/2 < \log t < -(p-1)\lambda^2/2 \tag{23a}$$

one can write from (13)

$$\log f(t) = \sum_{q=1}^{p-1} t^{q-1}\alpha_q + \frac{1}{\sqrt{\pi}\,\lambda}\exp\left(-\frac{\log^2 t}{\lambda^2}\right)\Gamma\left(\frac{2\log t}{\lambda^2}\right) + \ldots \tag{23b}$$

where the coefficients α_q come from the polynomial term in (13).

For $p=1$, the condition that ϕ is exponentially small with N is equivalent to $2^N \exp - (\log^2 t)/\lambda^2 \ll 1$, i.e.

$$\log t < -\lambda\sqrt{N\log 2} = -N/(2TT_c) \tag{24}$$

whereas for $p>1$, this condition becomes $2^N\alpha_1 t \ll 1$, i.e.

$$\log t < -\frac{N}{4}\left(\frac{1}{T_c^2}+\frac{1}{T^2}\right). \tag{25}$$

Assume that one wants to calculate the moment $\langle Z^\nu\rangle$ for $p-1<\nu<p$. One can choose $n=p$ in (7), and one sees that when replacing $\langle\exp(-tZ)\rangle$ by (22) and (23) in (7), the polynomial part in t disappears and the dominant contribution comes from the last term in (23). The integral (7) for ϕ exponentially small becomes

$$\int_{-\infty}^{-NJ^2/2TT_c} d\log t\,\exp[(n-\nu)\log t]\left(-\frac{\partial}{\partial t}\right)^n \frac{2^N}{\sqrt{\pi}\,\lambda}\exp\left(-\frac{\log^2 t}{\lambda^2}\right)\Gamma\left(\frac{2\log t}{\lambda^2}\right). \tag{26}$$

This integral is dominated by its saddle point

$$\log t = -\nu N J^2/2T^2 = -\nu\lambda^2/2 \tag{27}$$

and leads to the expression

$$\langle Z^\nu \rangle = 2^N \exp\left(\frac{N\nu^2 J^2}{4T^2}\right). \tag{28}$$

The condition for (28) to be valid is that the saddle point (27) satisfies conditions (24) and (25), i.e. that

$$\nu > T/T_c \qquad \text{for } \nu < 1 \tag{29}$$

and

$$\nu > \tfrac{1}{2}(T^2/T_c^2 + 1) \qquad \text{for } \nu > 1. \tag{30}$$

So the range ϕ exponentially small with N leads to (28) provided that (29) and (30) are satisfied. Combining (16), (21) and (28) with conditions (29) and (30), one gets the result (4). (One should notice that for $T > T_c$, even if the saddle point (27) exists if (30) is satisfied, it is only for $\nu > T^2/T_c^2$ that (28) dominates (16)).

One can recover the average free energy (Derrida 1981) (4*c*) and (5) for $T < T_c$ by taking the limit $\nu \to 0$:

$$\langle \log Z \rangle = \lim_{\nu \to 0} \frac{\langle Z^\nu \rangle - 1}{\nu} = \log(Z_0) + \Gamma'(1)(1 - T_c/T). \tag{31}$$

The term $-\frac{1}{2}\log N$ coming from $\log Z_0$ agrees with the prediction of Galves *et al* (1989).

3. The shape of $P(Z)$

From expressions (4*a*) and (4*c*), one can show that the probability distribution $P(Z)$ of Z has the form

$$P(Z)\mathrm{d}Z = g\left(\frac{Z}{Z_0}\right) \mathrm{d}\left(\frac{Z}{Z_0}\right) \qquad \text{for } \frac{Z}{Z_0} \text{ finite} \tag{32a}$$

and

$$P(Z)\mathrm{d}Z = 2^N \sqrt{\frac{T^2}{NJ^2\pi}} \exp\left(-\frac{T^2}{NJ^2}\log^2 Z\right) \mathrm{d}\log Z$$

$$\text{for } \frac{Z}{Z_0} \text{ exponentially large in } N \tag{32b}$$

where the function $g(x)$ satisfies

$$\int x^\nu g(x)\,\mathrm{d}x = \begin{cases} \dfrac{\Gamma(1-\nu T_c/T)}{\Gamma(1-\nu)} & \text{if } \nu < T/T_c \quad (33a) \\ \infty & \text{if } \nu > T/T_c. \quad (33b) \end{cases}$$

Depending whether $\nu < T/T_c$ or $\nu > T/T_c$, it is (32*a*) or (32*b*) which dominates the integral $\int Z^\nu P(Z)\,\mathrm{d}Z$ and one recovers (4*c*) or (4*a*), respectively.

Except for $T = T_c/2$, where $g(x)$ is given by

$$g(x) = \frac{1}{2\sqrt{\pi}} x^{-3/2} \exp\left(-\frac{1}{4x}\right) \tag{34}$$

we did not find an explicit expression of $g(x)$. However, it is possible to obtain the tail ($x \to \infty$) of $g(x)$ since it is related to the divergence of $\langle Z^\nu \rangle$ as $\nu \to T/T_c$:

$$g(x) \sim x^{-T/T_c-1}\left(-\frac{1}{\Gamma(-T/T_c)}\right) \qquad \text{for } x \to \infty. \tag{35}$$

This tail for $g(x)$ implies (from (32a)) that

$$\langle Z^\nu \rangle = \int Z^\nu g\left(\frac{Z}{Z_0}\right) \mathrm{d}\left(\frac{Z}{Z_0}\right) \sim \frac{Z_0^{T/T_c}}{T/T_c - \nu}\left(-\frac{1}{\Gamma(-T/T_c)}\right) \qquad \text{as } \nu \to (T/T_c)^- \tag{36}$$

in agreement with (4c).

It is also interesting to notice that in the region $1 \ll \log Z - \log Z_0 \ll N$, the two expressions (32a) and (32b) agree and give

$$P(Z) \sim \frac{1}{Z_0}\left(\frac{Z}{Z_0}\right)^{-1-T_c/T}\left(-\frac{1}{\Gamma(-T/T_c)}\right) \tag{37}$$

(equation (37) can either be obtained from (32a) and (35) by choosing $\log Z - \log Z_0 \gg 1$ or from (32b) and (5) by choosing $\log Z - \log Z_0 \ll N$).

So the distribution $P(Z)$ has the exponential shape (37) in the intermediate regime $1 \ll \log Z - \log Z_0 \ll N$ between the range $\log Z - \log Z_0 = O(1)$ given by (32a) and the range $\log Z - \log Z_0 = O(N)$ given by (32b).

4. Conclusion

In the present work we have obtained the expressions (4) of the moments $\langle Z^\nu \rangle$ for the random energy model. The expression of $\langle Z^\nu \rangle$, for large N depends both on T and ν and one obtains a phase diagram (figure 1) in the plane (ν, T) of the same nature as that found (Kondor 1983) for the Sherrington-Kirkpatrick model using the replica method.

For the random energy model the replica method (Gross and Mézard 1984) can also be applied and the results (4) can be recovered: the replica symmetric solution gives (4a) and (4b). Expression (4c) is obtained from the solution with one replica breaking although the Parisi ansatz does not define a way of obtaining the finite-size effect. One should notice that the replica symmetric solution would give (4c) even for $\nu < T/T_c$ and this would imply that $P(Z)$ is normalised to 2^N. In order to obtain the correct normalisation 1 of $P(Z)$, the part of $P(Z)$ determined from the solution with one replica breaking (32a) must be included.

It would be interesting to extend the calculation of the moments $\langle Z^\nu \rangle$ and of the shape of $P(Z)$ to the generalised random energy model (Derrida and Gardner 1986) and to the p spin models (Gardner 1985) in order to see the effect of correlations of the energies.

5. Epilogue (BD, December 1988)

Elizabeth wrote this paper in 1985 in the form which is presented here. I have just added a few sentences and changed a few words to complete it. The reason why we did not publish it earlier was that I had not fully understood the derivation that Elizabeth gave of equation (28). I tried several times to obtain an alternative argument without success. It was only while trying once more to complete this work in December 1988 that I fully understood Elizabeth's idea. Although I postponed the publication of this work year after year and that Elizabeth was convinced that her proof was sufficient, she kindly accepted to wait until I could agree with what she had written.

References

Derrida B 1980 *Phys. Rev. Lett.* **45** 79

—— 1981 *Phys. Rev.* B **24** 2613

Derrida B and Gardner E 1986 *J. Phys. C: Solid State Phys.* **19** 2253, 5783

Derrida B and Toulouse G 1981 *IV Brazilian Symp. on Theoretical Physics* ed E Ferreira and B Koller p 143

—— 1985 *J. Physique Lett.* **46** 223

Galves A, Martinez S and Picco P 1989 *J. Stat. Phys.* **54** 515

Gardner E 1985 *Nucl. Phys.* B **257** [FS14] 747

Gross D J and Mézard M 1984 *Nucl. Phys.* B **240** [FS12] 431

Kondor I 1983 *J. Phys. A: Math. Gen.* **16** L127

Mézard M, Parisi G, Sourlas N, Toulouse G and Virasoro M 1984a *Phys. Rev. Lett.* **523** 1246

—— 1984b *J. Physique* **45** 843

Parisi G 1979 *Phys. Rev. Lett.* **43** 1754

—— 1980 *J. Phys. A: Math. Gen.* **13** L115, 1101, 1887

—— 1983 *Phys. Rev. Lett.* **50** 1946