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Exact results for the one dimensional asymmetric exclusion model

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The asymmetric exclusion model describes a system of particles hopping in a preferred direction with hard core repulsion. These particles can be thought of as charged particles in a field, as steps of an interface, as cars in a queue. Several exact results concerning the steady state of this system have been obtained recently. The solution consists of representing the weights of the configurations in the steady state as products of non-commuting matrices.

1. Introduction

Systems of particles with hard core repulsion and stochastic dynamics have been studied for a long time both in the mathematical [1–6] and the physical literature [7–19]. They are among the simplest examples of systems out of equilibrium. The goal of this paper is to review the content of a recent work [20] describing a matrix approach which allows one to solve exactly the one dimensional asymmetric exclusion model with open boundaries.

The system we consider is a one dimensional chain of N sites and each site is either occupied by one particle or empty. A configuration $\{\tau_1, \tau_2, \ldots, \tau_N\}$ of the system is characterised by N binary variables $\tau_i = 1$ or 0 ($\tau_i = 1$ if site i is occupied by a particle and $\tau_i = 0$ if it is empty). During an infinitesimal interval of time dt, each particle in the system has a probability dt of jumping to its neighbouring site on the right provided that this site is empty (otherwise the particle does not move). Moreover, if site $i = 1$ is empty, there is a probability α dt that a new particle is introduced into the system (at site $i = 1$) and if site $i = N$ is occupied, there is a probability β dt that the particle on site $i = N$ will leave the system. Given these rules, it is easy to write equations for the time evolution of any correlation function: for example, one has for $2 \leq i \leq N-1$

$$\frac{\mathrm{d}\langle \tau_i \rangle}{\mathrm{d}t} = \langle \tau_{i-1}(1-\tau_i)\rangle - \langle \tau_i(1-\tau_{i+1})\rangle \tag{1}$$

or for $2 \leqslant i \leqslant N-2$

$$\frac{\mathrm{d}\langle \tau_i \tau_{i+1} \rangle}{\mathrm{d}t} = \langle \tau_{i-1}(1-\tau_i)\tau_{i+1} \rangle - \langle \tau_i \tau_{i+1}(1-\tau_{i+2}) \rangle , \tag{2}$$

whereas at the boundaries one has

$$\frac{\mathrm{d}\langle \tau_1 \rangle}{\mathrm{d}t} = \alpha \langle (1-\tau_1) \rangle - \langle \tau_1 (1-\tau_2) \rangle , \tag{3}$$

$$\frac{\mathrm{d}\langle \tau_N \rangle}{\mathrm{d}t} = \langle \tau_{N-1}(1-\tau_N) \rangle - \beta \langle \tau_N \rangle , \tag{4}$$

where $\langle \ \rangle$ denotes an average on the history.

Once these relations are written, one can in principle calculate the time evolution of any quantity of interest. The problem however is that the computation of the one point functions $\langle \tau_i \rangle$ requires the knowledge of the two point functions $\langle \tau_i \tau_j \rangle$, which themselves require the knowledge of higher correlation functions. This is a situation quite common in statistical mechanics where one can write relationships between different correlation functions but there is an infinite hierarchy of equations which makes the problem hard to solve.

Even the properties of the steady state are difficult to obtain, as can be seen from (1)–(4), where the steady state condition amounts to replacing the left hand side of these equations by zero.

Our approach to describe the steady state of this system is inspired by a technique [21,22] which has previously been applied to the problems of directed lattice animals [23] and quantum antiferromagnetic spin chains [24,25]. For the asymmetric exclusion process the present approach simplifies the derivation of known results [16–19] and facilitates their generalisation.

The idea is to write the weights $f_N(\tau_1, \ldots, \tau_N)$ of the configurations in the steady state as

$$f_N(\tau_1, \ldots, \tau_N) = \langle W | \prod_{i=1}^{N} [\tau_i D + (1-\tau_i) E] | V \rangle , \tag{5}$$

where D, E are matrices, $\langle W|$, $|V\rangle$ are vectors (we use the standard bra–ket notation of quantum mechanics) and τ_i are the occupation variables. In other words in the product we use matrix D whenever $\tau_i = 1$ and E whenever $\tau_i = 0$. In general, since the matrices D and E do not commute, the weights $f_N(\tau_1, \ldots, \tau_N)$ are complicated functions of the configuration $\{\tau_1, \ldots, \tau_N\}$. As the weights $f_N(\tau_1, \ldots, \tau_N)$ are usually not normalised, the probability $p_N(\tau_1, \ldots, \tau_N)$ of a configuration $\{\tau_1, \ldots, \tau_N\}$ in the steady state is given by

$$p_N(\tau_1, \ldots, \tau_N) = f_N(\tau_1, \ldots, \tau_N) \Big(\sum_{\tau_1=1,0} \cdots \sum_{\tau_N=1,0} f_N(\tau_1, \ldots, \tau_N) \Big)^{-1} . \tag{6}$$

Of course, from looking at (5) it is not obvious that such matrices D, E and vectors $\langle W|$, $|V\rangle$ exist. We shall see, however, that it is possible to choose these matrices and vectors so that $f_N(\tau_1, \ldots, \tau_N)$ given by (5) are indeed the actual weights in the steady state.

Before presenting explicit forms for the matrices and vectors involved in (5) let us discuss the advantages of this approach. If one defines the matrix C,

$$C = D + E , \tag{7}$$

it is clear that $\langle \tau_i \rangle_N$ defined by

$$\langle \tau_i \rangle_N = \sum_{\tau_1=1,0} \cdots \sum_{\tau_N=1,0} \tau_i f_N(\tau_1, \ldots, \tau_N) \Big(\sum_{\tau_1=1,0} \cdots \sum_{\tau_N=1,0} f_N(\tau_1, \ldots, \tau_N) \Big)^{-1} \tag{8}$$

can be calculated through the following formula:

$$\langle \tau_i \rangle_N = \frac{\langle W | C^{i-1} D C^{N-i} | V \rangle}{\langle W | C^N | V \rangle} . \tag{9}$$

Similarly, higher correlation functions will have simple expressions in terms of these matrices. For example, when $i < j$, $\langle \tau_i \tau_j \rangle_N$ is equal to

$$\langle \tau_i \tau_j \rangle_N = \frac{\langle W | C^{i-1} D C^{j-i-1} D C^{N-j} | V \rangle}{\langle W | C^N | V \rangle} . \tag{10}$$

Therefore, if we have convenient forms for the matrices D, E and the vectors $\langle W|$, $|V\rangle$ so that matrix elements of any power of $C = D + E$ have simple expressions, then formulae for the density profile $\langle \tau_i \rangle_N$ and for higher correlations $\langle \tau_i \cdots \tau_j \rangle_N$ will follow easily.

One can show [20] that if the matrices D, E and the vectors $\langle W|$, $|V\rangle$ satisfy (11)–(13),

$$D|V\rangle = (1/\beta)|V\rangle , \tag{11}$$

$$DE = D + E , \tag{12}$$

$$\langle W|E = (1/\alpha)\langle W| \,, \tag{13}$$

then (5) does give the steady state.

We shall not repeat here the proof that (11)–(13) are sufficient conditions to give the weights in the steady state. It is however easy to check that the relations (1)–(4) will be satisfied in the steady state provided that the corresponding identities hold,

$$DE(D + E) = (D + E)DE \,, \tag{14}$$

$$DED(D + E) = (D + E)DDE \,, \tag{15}$$

$$\alpha\langle W|E(D + E) = \langle W|DE \,, \tag{16}$$

$$DE|V\rangle = \beta(D + E)D|V\rangle \,, \tag{17}$$

and that these relations are immediate consequences of the algebraic rules (11)–(13). Another easy check that (11)–(13) do give the right steady state is to look at some special configurations. If one takes the case of a configuration where the first p sites are empty and the last $N - p$ are occupied, it is easy to show that in the steady state one must have

$$\langle W|E^{p-1}DED^{N-p-1}|V\rangle = \alpha\langle W|E^pD^{N-p}|V\rangle + \beta\langle W|E^pD^{N-p}|V\rangle \tag{18}$$

since this expresses that during a time interval dt the probability of entering the leaving the configuration are the same. Here again, this equality appears as a very simple consequence of the algebraic rule (11)–(13).

There is one line ($\alpha + \beta = 1$) where one can choose commuting matrices D and E which solve (11)–(13). This can be seen by writing

$$\begin{aligned}(1/\alpha + 1/\beta)\langle W|V\rangle &= \langle W|D + E|V\rangle = \langle W|DE|V\rangle = \langle W|ED|V\rangle \\ &= (1/\alpha\beta)\langle W|V\rangle \,. \end{aligned} \tag{19}$$

As $\langle W|V\rangle \neq 0$, this clearly implies that $\alpha + \beta = 1$. This is a well known special case where the steady state is factorised ($f_N(\tau_1, \ldots, \tau_N)$ depends only on $\Sigma_i \tau_i$ and all connected correlations vanish). Under this condition ($\alpha + \beta = 1$), one can choose the matrices D and E to be unidimensional, with $D = \beta^{-1}$ and $E = \alpha^{-1}$.

The previous remark also shows that for $\alpha + \beta \neq 1$, the size of the matrices D, E must be greater than one. The next question is whether one can find finite dimensional matrices that will satisfy (11)–(13). It turns out that one can prove that this is impossible (if D and E were finite dimensional matrices, the relation $DE = D + E$ would imply that $D = E(1-E)^{-1}$ which itself would imply that the matrices D and E commute). So the only possibility left is to use infinite dimensional matrices. There are several possible choices for the matrices D, E and vectors $\langle W|$, $|V\rangle$ that satisfy (11)–(13). One particular simple choice, which has proved useful in extensions of this approach to other cases, is

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \tag{20}$$

$$\langle W| = \left(1 \quad \left(\frac{1-\alpha}{\alpha}\right) \quad \left(\frac{1-\alpha}{\alpha}\right)^2 \quad \cdots\right), \qquad |V\rangle = \begin{pmatrix} 1 \\ \left(\frac{1-\beta}{\beta}\right) \\ \left(\frac{1-\beta}{\beta}\right)^2 \\ \vdots \end{pmatrix}. \tag{21}$$

This choice makes the particle–hole symmetry of the problem apparent since the matrices D and E have very similar forms and the boundary conditions α and β only appear in the vectors $\langle W|$ and $|V\rangle$. For this choice (20) of D, E the elements of C^N (where $C = D + E$ and N denotes the Nth power of matrix C) are given by

$$(C^N)_{nm} = \binom{2N}{N+n-m} - \binom{2N}{N+n+m}. \tag{22}$$

Expression (22) can be obtained by noting that $(C^N)_{nm}$ is proportional to the probability that a random walker who starts at site $2m$ of a semi-infinite chain with absorbing boundary at the origin, is at site $2n$ after $2N$ steps of a random walk. This probability may be calculated by the method of images.

An apparent disadvantage of this choice (20), (21) is that, due to the form of $\langle W|$ and $|V\rangle$, one has to sum geometric series to obtain the correlation functions and these series diverge in some range of α, β (in fact $\alpha + \beta \leqslant 1$). However, at least for finite N, all expressions are rational functions of α, β so that in principle one can obtain results for $\alpha + \beta \leqslant 1$ by analytic continuation from those for $\alpha + \beta > 1$.

In many cases calculations may be done directly from (11)–(13) without recourse to particular forms of the matrices. For example, one can calculate $\langle W|C^n|V\rangle$ (where $C = D + E$) for all values of α and β,

$$\frac{\langle W|C^N|V\rangle}{\langle W|\,|V\rangle} = \sum_{p=0}^{n} \frac{p(2n-1-p)!}{n!(n-p)!} \, \frac{\beta^{-p-1} - \alpha^{-p-1}}{\beta^{-1} - \alpha^{-1}} \,. \tag{23}$$

The derivation of (23) is given in [20]. The first few cases can easily be checked directly: $\langle W|C^N|V\rangle = 1$ for $N = 0$; $\alpha^{-1} + \beta^{-1}$ for $N = 1$; $\alpha^{-2} + \alpha^{-1}\beta^{-1} + \beta^{-2} + \alpha^{-1} + \beta^{-1}$ for $N = 2$; $\alpha^{-3} + \alpha^{-2}\beta^{-1} + \alpha^{-1}\beta^{-2} + \beta^{-3} + 2(\alpha^{-2} + \alpha^{-1}\beta^{-1} + \beta^{-2} + \alpha^{-1} + \beta^{-1})$ for $N = 3$, against direct calculation from (11)–(13).

2. Some results

Once the matrix elements of C are known, expressions for several quantities can be derived. For example, in the steady state, the current through the bond i is simply $J = \langle \tau_i(1 - \tau_{i+1})\rangle$, because during a time dt, the probability that a particle jumps from i to $i + 1$ is $\tau_i(1 - \tau_{i+1})\,\mathrm{d}t$. Therefore, J is given by

$$J = \frac{\langle W|C^{i-1}DEC^{N-i-1}|V\rangle}{\langle W|C^N|V\rangle} = \frac{\langle W|C^{N-1}|V\rangle}{\langle W|C^N|V\rangle} \,, \tag{24}$$

where we have used the fact (12) that $DE = C$. This expression is independent of i, as expected in the steady state. From the large N behaviour of the matrix elements $\langle W|C^N|V\rangle$ given by (23) one can show [20] that there are three different phases where the current J is given by

(i) for $\alpha \geqslant \frac{1}{2}$ and $\beta \geqslant \frac{1}{2}$

$$J = \tfrac{1}{4} \,; \tag{25}$$

(ii) for $\alpha < \frac{1}{2}$ and $\beta > \alpha$

$$J = \alpha(1 - \alpha); \tag{26}$$

(iii) for $\beta < \frac{1}{2}$ and $\alpha > \beta$

$$J = \beta(1 - \beta) \,. \tag{27}$$

Thus, the phase diagram consists of three phases: $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$; $\alpha < \frac{1}{2}$, $\beta > \alpha$;

$\beta < \frac{1}{2}$, $\alpha > \beta$. This is exactly the phase diagram predicted by the mean field theory [11,16,19].

From the knowledge of the matrix elements $\langle W|C^N|V\rangle$, one can obtain [20] exact expressions for all equal time correlation functions. For example the profile $\langle \tau_i \rangle_N$ is given by

$$\langle \tau_i \rangle_N = \sum_{p=0}^{n-1} \frac{2p!}{p!(p+1)!} \frac{\langle W|C^{N-1-p}|V\rangle}{\langle W|C^N|V\rangle} + \frac{\langle W|C^{i-1}|V\rangle}{\langle W|C^N|V\rangle} \sum_{p=2}^{n+1} \frac{(p-1)(2n-p)!}{n!(n+1-p)!} \beta^{-p} , \tag{28}$$

where $n = N - i$. Several limiting behaviours (N large, i large) are discussed in [20]. In the case $\alpha = \beta = 1$, one can even perform the sum in (28) to obtain [16]

$$\langle \tau_i \rangle_N = \frac{1}{2} + \frac{N-2i+1}{4} \frac{(2i)!}{(i!)^2} \frac{(N!)^2}{(2N+1)!} \frac{(2N-2i+2)!}{[(N-i+1)!]^2} . \tag{29}$$

3. Conclusion and extensions

The matrix method of solving the asymmetric exclusion process is much simpler than previous approaches [16–19]. The problem is reduced to finding the elements of products of matrices that satisfy the algebra (11)–(13). This gives a fast way of obtaining analytic expressions of the profiles $\langle \tau_i \rangle_N$ and correlations $\langle \tau_{i} \cdots \tau_{i_n} \rangle_N$.

This approach can be extended to several situations:

– The partially asymmetric exclusion problem where particles can jump either to the right with probability p or to the left with probability $q = 1 - p$. In that case one can show [20] that replacing (12) by

$$pDE - qED = D + E , \tag{30}$$

still gives the steady state.

– The case of second class particles and of shocks: Using the matrix approach, one can solve exactly a two species problem [26] (with K_1 first class particles and K_2 second class particles on a ring of N sites). During each time interval dt, each first class particle has a probability dt of jumping to its right provided that its nearest neighbour on the right is empty or occupied by a second class particle. In the latter case, the second class particle jumps simultaneously to the left. Also, during the time dt each second class particle has a probability dt of jumping to the right if its right neighbour is empty. One can show that the weights in the steady state can be written as

$$\mathrm{tr}(X_1 X_2 \cdots X_N)\,, \tag{31}$$

where $X_i = D$ if site i is occupied by a first class particle, $X_i = A$ if it is occupied by a second class particle and $X_i = E$ if it is empty. The algebra satisfied by the matrices D, A and E is then

$$DE = D + E\,, \qquad DA = A\,, \qquad AE = A\,. \tag{32}$$

Using this description, one can obtain exact expressions of shock profiles [4,5,26,27].

– One can also extend the approach to calculate exactly more general steady state properties than equal time correlation functions. The first result of this kind [28] is the exact expression of the diffusion constant Δ for the system of K particles on a ring of N sites in the fully asymmetric case (each particle jumps to its right neighbour with probability dt when the right neighbour is empty). If Y_t denotes the distance travelled by a particle after time t, one can show that in the long time limit

$$\langle Y_t \rangle \simeq vt\,, \qquad \langle Y_t^2 \rangle - \langle Y_t \rangle^2 \simeq \Delta t\,, \tag{33}$$

where the exact expressions of v and Δ are

$$v = \frac{N-K}{N-1}\,, \qquad \Delta = \frac{(2N-3)!}{(2K-1)!(2N-2K-1)!} \left(\frac{(K-1)!(N-K)!}{(N-1)!} \right)^2 . \tag{34}$$

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