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Fractional Moment Bounds and Disorder Relevance for Pinning Models

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Abstract: We study the critical point of directed pinning/wetting models with quenched disorder. The distribution $K(\cdot)$ of the location of the first contact of the (free) polymer with the defect line is assumed to be of the form $K(n) = n^{-\alpha-1}L(n)$, with $\alpha \geq 0$ and $L(\cdot)$ slowly varying. The model undergoes a (de)-localization phase transition: the free energy (per unit length) is zero in the delocalized phase and positive in the localized phase. For $\alpha < 1/2$ disorder is irrelevant: quenched and annealed critical points coincide for small disorder, as well as quenched and annealed critical exponents [3,28]. The same has been proven also for $\alpha = 1/2$, but under the assumption that $L(\cdot)$ diverges sufficiently fast at infinity, a hypothesis that is not satisfied in the $(1 + 1)$-dimensional wetting model considered in [12,17], where $L(\cdot)$ is asymptotically constant. Here we prove that, if $1/2 < \alpha < 1$ or $\alpha > 1$, then quenched and annealed critical points differ whenever disorder is present, and we give the scaling form of their difference for small disorder. In agreement with the so-called Harris criterion, disorder is therefore relevant in this case. In the marginal case $\alpha = 1/2$, under the assumption that $L(\cdot)$ vanishes sufficiently fast at infinity, we prove that the difference between quenched and annealed critical points, which is smaller than any power of the disorder strength, is positive: disorder is *marginally relevant*. Again, the case considered in [12,17] is out of our analysis and remains open.

The results are achieved by setting the parameters of the model so that the annealed system is localized, but close to criticality, and by first considering a quenched system of size that does not exceed the correlation length of the annealed model. In such a regime we can show that the expectation of the partition function raised to a suitably chosen power $\gamma \in (0, 1)$ is small. We then exploit such an information to prove that the expectation of the same fractional power of the partition function goes to zero with the size of the system, a fact that immediately entails that the quenched system is delocalized.

1. Introduction

Pinning/wetting models with quenched disorder describe the random interaction between a directed polymer and a one-dimensional *defect line*. In absence of interaction, a typical polymer configuration is given by $\{(n, S_n)\}_{n\geq 0}$, where $\{S_n\}_{n\geq 0}$ is a Markov Chain on some state space Σ (for instance, $\Sigma = \mathbb{Z}^d$ for $(1+d)$-dimensional directed polymers), and the initial condition S_0 is some fixed element of Σ which by convention we call 0. The defect line, on the other hand, is just $\{(n, 0)\}_{n\geq 0}$. The polymer-line interaction is introduced as follows: each time $S_n = 0$ (*i.e.*, the polymer touches the line at step n) the polymer gets an energy reward/penalty ϵ_n, which can be either positive or negative. In the situation we consider here, the ϵ_n's are independent and identically distributed (IID) random variables, with positive or negative mean h and variance $\beta^2 \geq 0$.

Up to now, we have made no assumption on the Markov Chain. The physically most interesting case is the one where the distribution $K(\cdot)$ of the first return time, call it τ_1, of S_n to 0 has a power-law tail: $K(n) := \mathbf{P}(\tau_1 = n) \approx n^{-\alpha-1}$, with $\alpha \geq 0$. This framework allows to cover various situations motivated by (bio)-physics: for instance, $(1+1)$-dimensional wetting models [12,17] ($\alpha = 1/2$; in this case $S_n \geq 0$, and the line represents an impenetrable wall), pinning of $(1+d)$-dimensional directed polymers on a columnar defect ($\alpha = 1/2$ if $d = 1$ and $\alpha = d/2-1$ if $d \geq 2$), and the Poland-Scheraga model of DNA denaturation (here, $\alpha \simeq 1.15$ [27]). This is a very active field of research, and not only from the point of view of mathematical physics, see . *e.g.* [11] and references therein. We refer to [20, Ch. 1] and references therein for further discussion.

The model undergoes a localization/delocalization phase transition: for any given value β of the disorder strength, if the average pinning intensity h exceeds some critical value $h_c(\beta)$ then the polymer typically stays tightly close to the defect line and the free energy is positive. On the contrary, for $h < h_c(\beta)$ the free energy vanishes and the polymer has only few contacts with the defect: entropic effects prevail. The annealed model, obtained by averaging the Boltzmann weight with respect to disorder, is exactly solvable, and near its critical point $h_c^{ann}(\beta)$ one finds that the annealed free energy vanishes like $(h - h_c^{ann}(\beta))^{\max(1,1/\alpha)}$ [16]. In particular, the annealed phase transition is first order for $\alpha > 1$ and second order for $\alpha < 1$, and it gets smoother and smoother as α approaches 0.

A very natural and intriguing question is whether and how randomness affects critical properties. The scenario suggested by the *Harris criterion* [26] is the following: disorder should be irrelevant for $\alpha < 1/2$, meaning that quenched critical point and critical exponents should coincide with the annealed ones if β is small enough, and relevant for $\alpha > 1/2$: they should differ for every $\beta > 0$. In the marginal case $\alpha = 1/2$, the Harris criterion gives no prediction and there is no general consensus on what to expect: renormalization-group considerations led Forgacs et al. [17] to predict that disorder is irrelevant (see also the recent [18]), while Derrida et al. [12] concluded for marginal relevance: quenched and annealed critical points should differ for every $\beta > 0$, even if the difference is zero at every perturbative order in β.

The mathematical understanding of these questions witnessed remarkable progress recently, and we summarize here the state of the art (prior to the present contribution).

(1) A lot is now known on the *irrelevant-disorder regime*. In particular, it was proven in [3] (see [28] for an alternative proof) that quenched and annealed critical points and critical exponents coincide for β small enough. Moreover, in [25] a small-disorder expansion of the free energy, worked out in [17], was rigorously justified.

(2) In the *strong-disorder regime*, for which the Harris criterion makes no prediction, a few results were obtained recently. In particular, in [29] it was proven that

for any given $\alpha > 0$ and, say, for Gaussian randomness, $h_c(\beta) \neq h_c^{ann}(\beta)$ for β large enough, and the asymptotic behavior of $h_c(\beta)$ for $\beta \to \infty$ was computed. These results were obtained through upper bounds on fractional moments of the partition function. Let us mention by the way that the fractional moment method allowed also to compute exactly [29] the quenched critical point of a *diluted wetting model* (a model with a built-in strong-disorder limit); the same result was obtained in [8] via a rigorous implementation of renormalization-group ideas. Fractional moment methods have proven to be useful also for other classes of disordered models [1,2,9,15].

(3) The *relevant-disorder regime* is only partly understood. In [24] it was proven that the free-energy critical exponent differs from the quenched one whenever $\beta > 0$ and $\alpha > 1/2$. However, the arguments in [24] do not imply the critical point shift. Nonetheless, the critical point shift issue has been recently solved for a *hierarchical version* of the model, introduced in [12]. The hierarchical model also depends on the parameter α, and in [21] it was shown that $h_c(\beta) - h_c^{ann}(\beta) \approx \beta^{2\alpha/(2\alpha-1)}$ for β small (upper and lower bounds of the same order are proven).

(4) In the *marginal case* $\alpha = 1/2$ it was proven in [3,28] that the difference $h_c(\beta) - h_c^{ann}(\beta)$ vanishes faster than any power of β, for $\beta \to 0$. Before discussing lower bounds on this difference, one has to be more precise on the tail behavior of $K(n)$, the probability that the first return to zero of the Markov Chain $\{S_n\}_n$ occurs at n: if $K(n) = n^{-(1+1/2)}L(n)$ with $L(\cdot)$ slowly varying (say, a logarithm raised to a positive or negative power), then the two critical points coincide for β small [3,28] if $L(\cdot)$ diverges sufficiently fast at infinity so that

$$\sum_{n=1}^{\infty} \frac{1}{nL(n)^2} < \infty. \tag{1.1}$$

The case of the $(1+1)$-dimensional wetting model [12] corresponds however to the case where $L(\cdot)$ behaves like a constant at infinity, and the result just mentioned does not apply.

The case $\alpha = 1/2$ is open also for the hierarchical model mentioned above.

In the present work we prove that if $\alpha \in (1/2, 1)$ or $\alpha > 1$ then quenched and annealed critical points differ for every $\beta > 0$, and $h_c(\beta) - h_c^{ann}(\beta) \approx \beta^{2\alpha/(2\alpha-1)}$ for $\beta \searrow 0$ (*cf.* Theorem 2.3 for a more precise statement). In the case $\alpha = 1/2$, while we do not prove that $h_c(\beta) \neq h_c^{ann}(\beta)$ in all cases in which condition (1.1) fails, we do prove such a result if the function $L(\cdot)$ vanishes sufficiently fast at infinity. Of course, $h_c(\beta) - h_c^{ann}(\beta)$ turns out to be exponentially small for $\beta \searrow 0$.

We wish to emphasize that, although the Harris criterion is expected to be applicable to a large variety of disordered models, rigorous results are very rare: let us mention however [10,14].

Starting from the next section, we will forget the full Markov structure of the polymer, and retain only the fact that the set of points of contact with the defect line, $\tau := \{n \geq 0 : S_n = 0\}$, is a renewal process under the law $\mathbf{P}$ of the Markov Chain.

2. Model and Main Results

Let $\tau := \{\tau_0, \tau_1, \ldots\}$ be a renewal sequence started from $\tau_0 = 0$ and with inter-arrival law $K(\cdot)$, *i.e.*, $\{\tau_i - \tau_{i-1}\}_{i \in \mathbb{N} := \{1,2,\ldots\}}$ are IID integer-valued random variables with law

$\mathbf{P}(\tau_1 = n) = K(n)$ for every $n \in \mathbb{N}$. We assume that $\sum_{n\in\mathbb{N}} K(n) = 1$ (the renewal is recurrent) and that there exists $\alpha > 0$ such that

$$K(n) = \frac{L(n)}{n^{1+\alpha}} \tag{2.1}$$

with $L(\cdot)$ a function that varies slowly at infinity, *i.e.*, $L : (0,\infty) \to (0,\infty)$ is measurable and such that $L(rx)/L(x) \to 1$ when $x \to \infty$, for every $r > 0$. We refer to [6] for an extended treatment of slowly varying functions, recalling just that examples of $L(x)$ include $(\log(1+x))^b$, any $b \in \mathbb{R}$, and any (positive, measurable) function admitting a positive limit at infinity (in this case we say that $L(\cdot)$ is *trivial*). Dwelling a bit more on nomenclature, $x \mapsto x^\rho L(x)$ is a *regularly varying function of exponent* ρ, so $K(\cdot)$ is just the restriction to the natural numbers of a regularly varying function of exponent $-(1+\alpha)$.

We let $\beta \geq 0$, $h \in \mathbb{R}$ and $\omega := \{\omega_n\}_{n\geq 1}$ be a sequence of IID centered random variables with unit variance and finite exponential moments. The law of ω is denoted by $\mathbb{P}$ and the corresponding expectation by $\mathbb{E}$.

For $a, b \in \{0, 1, \dots\}$ with $a \leq b$ we let $Z_{a,b,\omega}$ be the partition function for the system on the interval $\{a, a+1, \dots, b\}$, with zero boundary conditions at both endpoints:

$$Z_{a,b,\omega} = \mathbf{E}\left(e^{\sum_{n=a+1}^{b}(\beta\omega_n + h)\mathbf{1}_{\{n\in\tau\}}} \mathbf{1}_{\{b\in\tau\}} \middle| a \in \tau \right), \tag{2.2}$$

where $\mathbf{E}$ denotes expectation with respect to the law $\mathbf{P}$ of the renewal. One may rewrite $Z_{a,b,\omega}$ more explicitly as

$$Z_{a,b,\omega} = \sum_{\ell=1}^{b-a} \sum_{i_0=a<i_1<\cdots<i_\ell=b} \prod_{j=1}^{\ell} K(i_j - i_{j-1}) e^{h\ell + \beta\sum_{j=1}^{\ell}\omega_{i_j}}, \tag{2.3}$$

with the convention that $Z_{a,a,\omega} = 1$. Notice that, when writing $n \in \tau$, we are interpreting τ as a subset of $\mathbb{N} \cup \{0\}$ rather than as a sequence of random variables. We will write for simplicity $Z_{N,\omega}$ for $Z_{0,N,\omega}$ (and in that case the conditioning on $0 \in \tau$ in (2.2) is superfluous since $\tau_0 = 0$). In absence of disorder ($\beta = 0$), it is convenient to use the notation

$$Z_N(h) := \mathbf{E}\left(e^{h\sum_{n=1}^{N}\mathbf{1}_{\{n\in\tau\}}} \mathbf{1}_{\{N\in\tau\}} \right) = \mathbf{E}\left(e^{h|\tau\cap\{1,\dots,N\}|} \mathbf{1}_{\{N\in\tau\}} \right), \tag{2.4}$$

for the partition function.

We mention that the recurrence assumption $\sum_{n\in\mathbb{N}} K(n) = 1$ entails no loss of generality, since one can always reduce to this situation via a redefinition of h (*cf.* [20, Ch. 1]).

As usual the *quenched free energy* is defined as

$$\mathrm{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega}. \tag{2.5}$$

It is well known (*cf.* for instance [20, Ch. 4]) that the limit (2.5) exists $\mathbb{P}(\mathrm{d}\omega)$-almost surely and in $\mathbb{L}^1(\mathbb{P})$, and that it is almost-surely independent of ω. Another well-established fact is that $\mathrm{F}(\beta, h) \geq 0$, which immediately follows from $Z_{N,\omega} \geq K(N)\exp(\beta\omega_N + h)$. This allows to define, for a given $\beta \geq 0$, the critical point $h_c(\beta)$ as

$$h_c(\beta) := \sup\{h \in \mathbb{R} : \mathrm{F}(\beta, h) = 0\}. \tag{2.6}$$

It is well known that $h > h_c(\beta)$ corresponds to the *localized phase* where typically τ occupies a non-zero fraction of $\{1, \ldots, N\}$ while, for $h < h_c(\beta)$, $\tau \cap \{1, \ldots, N\}$ contains with large probability at most $O(\log N)$ points [23]. We refer to [20, Chs. 7 and 8] for further literature and discussion on this point.

In analogy with the quenched free energy, the *annealed free energy* is defined by

$$\textsc{f}^{ann}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{N,\omega} = \textsc{f}(0, h + \log \mathrm{M}(\beta)), \tag{2.7}$$

with

$$\mathrm{M}(\beta) := \mathbb{E}(e^{\beta\omega_1}). \tag{2.8}$$

We see therefore that the annealed free energy is just the free energy of the pure model ($\beta = 0$) with a different value of h. The pure model is exactly solvable [16], and we collect here a few facts we will need in the course of the paper.

Theorem 2.1 [20, Th. 2.1]. *For the pure model $h_c(0) = 0$. Moreover, there exists a slowly varying function $\widehat{L}(\cdot)$ such that for $h > 0$ one has*

$$\textsc{f}(0, h) = h^{1/\min(1,\alpha)} \widehat{L}(1/h). \tag{2.9}$$

In particular,

(1) *if $\mathbf{E}(\tau_1) = \sum_{n\in\mathbb{N}} n\, K(n) < \infty$ (for instance, if $\alpha > 1$) then $\widehat{L}(1/h) \overset{h\searrow 0}{\sim} 1/\mathbf{E}(\tau_1)$;*
(2) *if $\alpha \in (0, 1)$, then $\widehat{L}(1/h) = C_\alpha h^{-1/\alpha} R_\alpha(h)$, where C_α is an explicit constant and $R_\alpha(\cdot)$ is the function, unique up to asymptotic equivalence, that satisfies $R_\alpha(b^\alpha L(1/b)) \overset{b\searrow 0}{\sim} b$.*

As a consequence of Theorem 2.1 and (2.7), the annealed critical point is simply given by

$$h_c^{ann}(\beta) := \sup\{h : \textsc{f}^{ann}(\beta, h) = 0\} = -\log \mathrm{M}(\beta). \tag{2.10}$$

Via Jensen's inequality one has immediately that $\textsc{f}(\beta, h) \leq \textsc{f}^{ann}(\beta, h)$ and as a consequence $h_c(\beta) \geq h_c^{ann}(\beta)$, and the point of the present paper is to understand when this last inequality is strict. In this respect, let us recall that the following is known: if $\alpha \in (0, 1/2)$, then $h_c(\beta) = h_c^{ann}(\beta)$ for β small enough [3,28]. Also for $\alpha = 1/2$ it has been shown that $h_c(\beta) = h_c^{ann}(\beta)$ if $L(\cdot)$ diverges sufficiently fast (see below). Moreover, assuming that $\mathbb{P}(\omega_1 > t) > 0$ for every $t > 0$, one has that for every $\alpha > 0$ and $L(\cdot)$ there exists $\beta_0 < \infty$ such that $h_c(\beta) \neq h_c^{ann}(\beta)$ for $\beta > \beta_0$ [29]: quenched and annealed critical points differ for *strong disorder*. The strategy we develop here addresses the complementary situations: $\alpha > 1/2$ and *small disorder* (and also the case $\alpha = 1/2$ as we shall see below).

Our first result concerns the case $\alpha > 1$:

Theorem 2.2. *Let $\alpha > 1$. There exists $a > 0$ such that for every $\beta \leq 1$,*

$$h_c(\beta) - h_c^{ann}(\beta) \geq a\beta^2. \tag{2.11}$$

Moreover, $h_c(\beta) > h_c^{ann}(\beta)$ for every $\beta > 0$.

Since $h_c(\beta) \le h_c(0) = 0$ and $h_c^{ann}(\beta) \stackrel{\beta \searrow 0}{\sim} -\beta^2/2$, we conclude that the inequality (2.11) is, in a sense, of the optimal order in β. Note that $h_c(\beta) \le h_c(0)$ is just a consequence of Jensen's inequality:

$$
\begin{aligned}
Z_{N,\omega} &= Z_N(h) \frac{\mathbf{E}\left(e^{\sum_{n=1}^N (\beta\omega_n + h)\mathbf{1}_{\{n\in\tau\}}} \mathbf{1}_{\{N\in\tau\}} \right)}{\mathbf{E}\left(e^{h\sum_{n=1}^N \mathbf{1}_{\{n\in\tau\}}} \mathbf{1}_{\{N\in\tau\}} \right)} \\
&\ge Z_N(h) \exp\left[\beta \sum_{n=1}^N \omega_n \frac{\mathbf{E}\left(\mathbf{1}_{\{n\in\tau\}} e^{h|\tau\cap\{1,\dots,N\}|} \mathbf{1}_{\{N\in\tau\}} \right)}{\mathbf{E}\left(e^{h|\tau\cap\{1,\dots,N\}|} \mathbf{1}_{\{N\in\tau\}} \right)} \right], \qquad (2.12)
\end{aligned}
$$

from which $\mathrm{F}(\beta, h) \ge \mathrm{F}(0, h)$ and therefore $h_c(\beta) \le h_c(0)$ immediately follows from $\mathbb{E}(\omega_n) = 0$. This can be made sharper in the sense that from the explicit bound in [20, Th. 5.2(1)] one directly extracts also that $h_c(\beta) \le -b\beta^2$ for a suitable $b \in (0, 1/2)$ and every $\beta \le 1$, so that $-h_c(\beta)/\beta^2 \in (b, 1/2 - a)$. We recall also that the (strict) inequality $h_c(\beta) < h_c(0)$ has been established in great generality in [4].

In the case $\alpha \in (1/2, 1)$ we have the following:

Theorem 2.3. *Let $\alpha \in (1/2, 1)$. For every $\varepsilon > 0$ there exists $a(\varepsilon) > 0$ such that*

$$h_c(\beta) - h_c^{ann}(\beta) \ge a(\varepsilon)\, \beta^{(2\alpha/(2\alpha-1))+\varepsilon}, \qquad (2.13)$$

for $\beta \le 1$. Moreover, $h_c(\beta) > h_c^{ann}(\beta)$ for every $\beta > 0$.

To appreciate this result, recall that in [3,28] it was proven that

$$h_c(\beta) - h_c^{ann}(\beta) \le \widetilde{L}(1/\beta)\beta^{2\alpha/(2\alpha-1)}, \qquad (2.14)$$

for some (rather explicit, *cf.* in particular [3]) slowly varying function $\widetilde{L}(\cdot)$. Notably, $\widetilde{L}(\cdot)$ is trivial if $L(\cdot)$ is. The conclusion of Theorem 2.3 can actually be strengthened and we are able to replace the right-hand side of (2.13) with $\bar{L}(1/\beta)\beta^{2\alpha/(2\alpha-1)}$ with $\bar{L}(\cdot)$ another slowly varying function, but on one hand $\bar{L}(\cdot)$ does not match the bound in (2.14) and on the other hand it is rather clear that it reflects more a limit of our technique than the actual behavior of the model; therefore, we decided to present the simpler argument leading to the slightly weaker result (2.13).

The case $\alpha = 1/2$ is the most delicate, and whether quenched and annealed critical points coincide or not crucially depends on the slowly varying function $L(\cdot)$. In [3,28] it was proven that, whenever

$$\sum_{n\ge 1} \frac{1}{n\, L(n)^2} < \infty, \qquad (2.15)$$

there exists $\beta_0 > 0$ such that $h_c(\beta) = h_c^{ann}(\beta)$ for $\beta \le \beta_0$, and that when the same sum diverges then $h_c(\beta) - h_c^{ann}(\beta)$ is bounded *above* by some function of β which vanishes faster than any power for $\beta \searrow 0$. For instance, if $L(\cdot)$ is asymptotically constant then

$$h_c(\beta) - h_c^{ann}(\beta) \le c_1\, e^{-c_2/\beta^2}, \qquad (2.16)$$

for $\beta \le 1$. While we are not able to prove that quenched and annealed critical points differ as soon as condition (2.15) fails (in particular not when $L(\cdot)$ is asymptotically constant), our method can be pushed further to prove this if $L(\cdot)$ vanishes sufficiently fast at infinity:

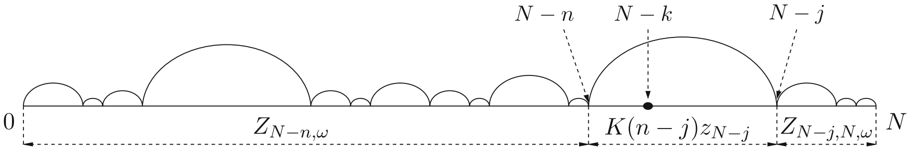

Fig. 1. The decomposition of the partition function is simply obtained by fixing a value of k and summing over the values of the last contact (or renewal epoch) before $N-k$ and the first after $N-k$. In the drawing the two contacts are respectively $N-n$ and $N-j$ and arcs of course identify steps between successive contacts

Theorem 2.4. *Assume that for every* $n \in \mathbb{N}$,

$$K(n) \le c \frac{n^{-3/2}}{(\log n)^{\eta}}, \tag{2.17}$$

for some $c > 0$ *and* $\eta > 1/2$. *Then for every* $0 < \varepsilon < \eta - 1/2$ *there exists* $a(\varepsilon) > 0$ *such that*

$$h_c(\beta) - h_c^{ann}(\beta) \ge a(\varepsilon) \exp\left(- \frac{1}{\beta^{\frac{1}{\eta-1/2-\varepsilon}}} \right). \tag{2.18}$$

Moreover, $h_c(\beta) > h_c^{ann}(\beta)$ *for every* $\beta > 0$.

2.1. Fractional moment method. In order to introduce our basic idea and, effectively, start the proof, we need some additional notation. We fix some $k \in \mathbb{N}$ and we set for $n \in \mathbb{N}$

$$z_n := e^{h+\beta\omega_n}. \tag{2.19}$$

Then, the following identity holds for $N \ge k$:

$$Z_{N,\omega} = \sum_{n=k}^{N} Z_{N-n,\omega} \sum_{j=0}^{k-1} K(n-j)\, z_{N-j} Z_{N-j,N,\omega}. \tag{2.20}$$

This is simply obtained by decomposing the partition function (2.2) according to the value $N-n$ of the last point of τ which does not exceed $N-k$ (whence the condition $0 \le N-n \le N-k$ in the sum), and to the value $N-j$ of the first point of τ to the right of $N-k$ (so that $N-k < N-j \le N$). It is important to notice that $Z_{N-j,N,\omega}$ has the same law as $Z_{j,\omega}$ and that the three random variables $Z_{N-n,\omega}$, z_{N-j} and $Z_{N-j,N,\omega}$ are independent, provided that $n \ge k$ and $j < k$.

Let $0 < \gamma < 1$ and $A_N := \mathbb{E}[(Z_{N,\omega})^{\gamma}]$, with $A_0 := 1$. Then, from (2.20) and using the elementary inequality

$$(a_1 + \cdots + a_n)^{\gamma} \le a_1^{\gamma} + \cdots + a_n^{\gamma}, \tag{2.21}$$

which holds for $a_i \ge 0$, one deduces

$$A_N \le \mathbb{E}\left[z_1^{\gamma}\right] \sum_{n=k}^{N} A_{N-n} \sum_{j=0}^{k-1} K(n-j)^{\gamma} A_j. \tag{2.22}$$

The basic principle is the following:

Proposition 2.5. *Fix β and h. If there exists $k \in \mathbb{N}$ and $\gamma < 1$ such that*

$$\rho := \mathbb{E}\left[z_1^\gamma\right] \sum_{n=k}^{\infty} \sum_{j=0}^{k-1} K(n-j)^\gamma A_j \le 1, \tag{2.23}$$

then $\mathrm{F}(\beta, h) = 0$. Moreover if $\rho < 1$ there exists $C = C(\rho, \gamma, k, K(\cdot)) > 0$ such that

$$A_N \le C\,(K(N))^\gamma\,, \tag{2.24}$$

for every N.

Of course, in view of the results we want to prove, the main result of Proposition 2.5 is the first one. The second one, namely (2.24), is however of independent interest and may be used to obtain path estimates on the process (using for example the techniques in [23] and [20, Ch. 8]).

Proof of Proposition 2.5. Let $\bar{A} := \max\{A_0, A_1, \ldots, A_{k-1}\}$. From (2.22) it follows that for every $N \ge k$

$$A_N \le \rho \max\{A_0, \ldots, A_{N-k}\}, \tag{2.25}$$

from which one sees by induction that, since $\rho \le 1$, for every n one has $A_n \le \bar{A}$. The statement $\mathrm{F}(\beta, h) = 0$ follows then from Jensen's inequality:

$$\mathrm{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N\gamma} \mathbb{E} \log (Z_{N,\omega})^\gamma \le \lim_{N\to\infty} \frac{1}{N\gamma} \log A_N = 0. \tag{2.26}$$

In order to prove (2.24) we introduce

$$Q_k(n) := \begin{cases} \mathbb{E}[z_1^\gamma] \sum_{j=0}^{k-1} K(n-j)^\gamma A_j, & \text{if } n \ge k, \\ 0 & \text{if } n = 1, \ldots k-1. \end{cases} \tag{2.27}$$

Since $\rho = \sum_n Q_k(n)$, the assumption $\rho < 1$ tells us that $Q_k(\cdot)$ is a sub-probability distribution and it becomes a probability distribution if we set, as we do, $Q_k(\infty) := 1 - \rho$. Therefore the renewal process $\widetilde{\tau}$ with inter-arrival law $Q_k(\cdot)$ is *terminating*, that is $\widetilde{\tau}$ contains, almost surely, only a finite number of points. A particularity of terminating renewals with regularly varying inter-arrival distribution is the asymptotic equivalence, up to a multiplicative factor, of inter-arrival distribution and mass renewal function ([20, Th. A.4]), namely

$$u_N \overset{N\to\infty}{\sim} \frac{1}{(1-\rho)^2} Q_k(N), \tag{2.28}$$

where $u_N := \mathbf{P}(N \in \widetilde{\tau})$ and it satisfies the renewal equation $u_N = \sum_{n=1}^{N} u_{N-n} Q_k(n)$ for $N \ge 1$ (and $u_0 = 1$). Since $Q_k(n) = 0$ for $n = 1, \ldots, k-1$, for the same values of n we have $u_n = 0$ too. Therefore the renewal equation may be rewritten, for $N \ge k$, as

$$u_N = \sum_{n=1}^{N-k} u_{N-n} Q_k(n) + Q_k(N). \tag{2.29}$$

Let us observe now that if we set $\widetilde{A}_N := A_N \mathbf{1}_{N \ge k}$ then (2.22) implies that for $N \ge k$,

$$\widetilde{A}_N \le \sum_{n=1}^{N-k} \widetilde{A}_{N-n} Q_k(n) + P_k(N), \quad \text{with} \quad P_k(N) := \sum_{n=0}^{k-1} A_n Q_k(N-n), \tag{2.30}$$

and observe that $P_k(N) \le c\, Q_k(N)$, with c that depends on ρ, γ, k and $K(\cdot)$ (and on h and β, but these variables are kept fixed). Therefore

$$\frac{\widetilde{A}_N}{c} \le \sum_{n=1}^{N-k} \frac{\widetilde{A}_{N-n}}{c} Q_k(n) + Q_k(N), \tag{2.31}$$

for $N \ge k$. By comparing (2.29) and (2.31), and by using (2.28) and $Q_k(N) \overset{N\to\infty}{\sim} K(N)^\gamma \mathbb{E}[z_1^\gamma] \sum_{j=0}^{k-1} A_j$, one directly obtains (2.24). □

2.2. Disorder relevance: sketch of the proof. Let us consider for instance the case $\alpha > 1$, which is technically less involved than the others, but still fully representative of our strategy. Take (β, h) such that β is small and $h = h_c^{ann}(\beta) + \Delta$, with $\Delta = a\beta^2$. We are therefore considering the system inside the annealed localized phase, but close to the annealed critical point (at a distance Δ from it), and we want to show that $\textsc{f}(\beta, h) = 0$. In view of Proposition 2.5, it is sufficient to show that ρ in (2.23) is sufficiently small, and we have the freedom to choose a suitable k. Specifically, we choose k to be of the order of the correlation length of the annealed system: $k = 1/\textsc{f}^{ann}(\beta, h) = 1/\textsc{f}(0, \Delta) \approx \text{const.}/(a\beta^2)$, where the last estimate holds since the phase transition of the annealed system is first order for $\alpha > 1$. Note that k diverges for β small.

For the purpose of this informal discussion, assume that $K(n) = c\, n^{-(1+\alpha)}$, *i.e.*, the slowly varying function $L(\cdot)$ is constant. The sum over n in the right-hand side of (2.23) is then immediately performed and (up to a multiplicative constant) one is left with estimating

$$\sum_{j=0}^{k-1} \frac{A_j}{(k-j)^{(1+\alpha)\gamma - 1}}. \tag{2.32}$$

One can choose $\gamma < 1$ such that $(1+\alpha)\gamma - 1 > 1$ and it is actually not difficult to show that $\sup_{j<k} A_j$ is bounded by a constant uniformly in k. On one hand in fact $A_j \le [\mathbb{E} Z_{j,\omega}]^\gamma = [Z_j(\Delta)]^\gamma$, where the first step follows from Jensen's inequality and the second one from the definition of the model (recall (2.4)). On the other hand for $j < k$, *i.e.*, for j smaller than the correlation length of the annealed model, one has that the annealed partition function $Z_j(\Delta)$ is bounded above by a constant, *independently of how small* Δ *is*, *i.e.*, of how large the correlation length is. This just establishes that the quantity in (2.32) is bounded, so we need to go beyond and show that A_j is small: this of course is not true unless j is large, but if we restrict the sum in (2.32) to $j \ll k$ what we obtain is small, since the denominator is approximately $k^{(1+\alpha)\gamma - 1}$, that is k to a power larger than 1.

In order to control the terms for which $k - j$ is of order 1 a new ingredient is clearly needed, and we really have to estimate the fractional moment of the partition function without resorting to Jensen's inequality. To this purpose, we apply an idea which was introduced in [21]. Specifically, we change the law $\mathbb{P}$ of the disorder in such a way that under the new law, $\widetilde{\mathbb{P}}$, the system is delocalized and $\widetilde{\mathbb{E}}(Z_{j,\omega})^{\gamma}$ is small. The change of measure corresponds to tilting negatively the law of $\omega_i, i \le j$, *cf.* (A.1), so that the system is more delocalized than under $\mathbb{P}$. The non-trivial fact is that with our choice $\Delta = a\beta^2$ and $j \le 1/\mathrm{F}(0, \Delta)$, one can guarantee on one hand that $Z_{j,\omega}$ is typically small under $\widetilde{\mathbb{P}}$, and on the other that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are close (their mutual density is bounded, in a suitable sense), so that the same statement about $Z_{j,\omega}$ holds also under the original measure $\mathbb{P}$. At this point, we have that all terms in (2.32) are small: actually, as we will see, the whole sum is as small as we wish if we choose a small. The fact that $\mathrm{F}(\beta, h) = 0$ then follows from Proposition 2.5.

As we have mentioned above, the case $\alpha \in [1/2, 1)$ is not much harder, at least on a conceptual level, but this time it is not sufficient to establish bounds on A_j that do not depend on j: the exponent in the denominator of the summand in (2.32) is in any case smaller than 1 and one has to exploit the decay in j of A_j: with respect to the $\alpha > 1$ case, here one can exploit the decay of $\mathbf{P}(j \in \tau)$ as j grows, while such a quantity converges to a positive constant if $\alpha > 1$. Once again the case of $j \ll k$ can be dealt with by direct annealed estimates, while when one gets close to k a finer argument, direct generalization of the one used for the $\alpha > 1$ case, is needed.

3. The Case $\alpha > 1$

In order to avoid repetitions let us establish that, in this and the next sections, $R_i, i = 1, 2, \ldots$ denote (large) constants, $L_i(\cdot)$ are slowly varying functions and C_i positive constants (not necessarily large).

Proof of Theorem 2.2. Fix $\beta_0 > 0$ and let $\beta \le \beta_0$, $h = h_c^{ann}(\beta) + a\beta^2$ and $\gamma < 1$ sufficiently close to 1 so that

$$(1 + \alpha)\gamma > 2. \tag{3.1}$$

It is sufficient to show that the sum in (2.23) can be made arbitrarily small (for some suitable choice of k) by choosing a small, since $\mathbb{E}[z_1^{\gamma}]$ can be bounded above by a constant independent of a (for a small).

We choose $k = k(\beta) = 1/(a\beta^2)$, so that $\beta = 1/\sqrt{ak(\beta)}$. In order to avoid a plethora of $\lfloor\cdot\rfloor$, we will assume that $k(\beta)$ is integer. Note that $k(\beta)$ is large if β or a are small.

First of all note that, thanks to Eqs. (A.21) and (A.24), the sum in the r.h.s. of (2.23) is bounded above by

$$\sum_{j=0}^{k(\beta)-1} \frac{L_1(k(\beta) - j)\, A_j}{(k(\beta) - j)^{(1+\alpha)\gamma - 1}}. \tag{3.2}$$

We split this sum as

$$S_1 + S_2 := \sum_{j=0}^{k(\beta)-1-R_1} \frac{L_1(k(\beta) - j)\, A_j}{(k(\beta) - j)^{(1+\alpha)\gamma - 1}} + \sum_{j=k(\beta)-R_1}^{k(\beta)-1} \frac{L_1(k(\beta) - j)\, A_j}{(k(\beta) - j)^{(1+\alpha)\gamma - 1}}. \tag{3.3}$$

To estimate S_1, note that by Jensen's inequality $A_j \le (\mathbb{E}Z_{j,\omega})^{\gamma} \le C_1$ with C_1 a constant independent of j as long as $j < k(\beta)$. Indeed, from (2.2) and the definition of the annealed critical point one sees that (recall (2.4))

$$\mathbb{E}Z_{j,\omega} = Z_j(a\beta^2) = \mathbf{E}\left(e^{a\beta^2|\tau\cap\{1,\ldots,j\}|}\mathbf{1}_{\{j\in\tau\}}\right), \tag{3.4}$$

and the last term is clearly smaller than e. Therefore, using again (A.21)

$$S_1 \le \frac{L_2(R_1)}{R_1^{(1+\alpha)\gamma-2}}, \tag{3.5}$$

which can be made small with R_1 large in view of the choice (3.1). As for S_2, one has

$$S_2 \le C_2 \max_{k(\beta)-R_1\le j<k(\beta)} A_j. \tag{3.6}$$

We apply now Lemma A.1 (note also the definition in (A.1)) with $N = j$ and $\lambda = 1/\sqrt{j}$ so that we have

$$A_j \le \left[\mathbb{E}_{j,1/\sqrt{j}}\left(Z_{j,\omega}\right)\right]^{\gamma} \exp\left(c\gamma/(1-\gamma)\right), \tag{3.7}$$

for $1/\sqrt{j} \le \min(1, (1-\gamma)/\gamma)$, that is for a sufficiently small, since we are in any case assuming $j \ge k(\beta) - R_1$.

We are therefore left with showing that $\mathbb{E}_{j,1/\sqrt{j}}\left[Z_{j,\omega}\right]$ is small for the range of j's we are considering. For such an estimate it is convenient to recall (2.10) and to observe that for any given values of β, h and λ and for any j,

$$\mathbb{E}_{j,\lambda}[Z_{j,\omega}] = \mathbf{E}\left[\left(\exp\left(h - h_c^{ann}(\beta)\right)\frac{\mathrm{M}(\beta-\lambda)}{\mathrm{M}(\beta)\mathrm{M}(-\lambda)}\right)^{|\tau\cap\{1,\dots,j\}|}\mathbf{1}_{\{j\in\tau\}}\right]. \tag{3.8}$$

In order to exploit such a formula let us observe that

$$\frac{\mathrm{M}(\beta-\lambda)}{\mathrm{M}(\beta)\mathrm{M}(-\lambda)} = \exp\left[-\int_0^\beta \mathrm{d}x \int_{-\lambda}^0 \mathrm{d}y\, \frac{\mathrm{d}^2}{\mathrm{d}t^2}\log \mathrm{M}(t)\Big|_{t=x+y}\right] \le e^{-C_3\beta\lambda}, \tag{3.9}$$

which holds for $0 < \lambda \le \beta \le \beta_0$ and $C_3 := \min_{t\in[-\beta_0,\beta_0]} \mathrm{d}^2(\log \mathrm{M}(t))/\,\mathrm{d}t^2 > 0$. If a is sufficiently small, for $j \le k(\beta) = 1/(a\beta^2)$ we have

$$a\beta^2 - \frac{C_3\beta}{\sqrt{j}} \le \frac{1}{k(\beta)}\left[1 - \frac{C_3}{\sqrt{a}}\right] \le -\frac{C_3}{2k(\beta)\sqrt{a}}. \tag{3.10}$$

As a consequence,

$$\begin{aligned}\max_{k(\beta)-R_1\le j<k(\beta)} &\mathbb{E}_{j,1/\sqrt{j}}(Z_{j,\omega}) \\ &\le e^{C_3\sqrt{a}\beta^2R_1/2}\mathbf{E}\left[\exp\left(-\frac{C_3}{2\sqrt{a}k(\beta)}\left|\tau\cap\{1,\dots,k(\beta)\}\right|\right)\right].\end{aligned} \tag{3.11}$$

The right-hand side in (3.11) can be made small by choosing a small (and this is uniform on $\beta \le \beta_0$) because of

$$\lim_{c\to+\infty}\limsup_{N\to\infty}\mathbf{E}\left(e^{-(c/N)|\tau\cap\{1,\dots,N\}|}\right) = 0, \tag{3.12}$$

that we are going to prove just below. Putting everything together, we have shown that both S_1 and S_2 can be made small via a suitable choice of R_1 and a, and the theorem is proven.

To prove (3.12), since the function under expectation is bounded by 1 it is sufficient to observe that

$$\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{\{n\in\tau\}} \stackrel{N\to\infty}{\longrightarrow} \frac{1}{\sum_{n\in\mathbb{N}} nK(n)} = \frac{1}{\mathbf{E}(\tau_1)} > 0, \tag{3.13}$$

almost surely (with respect to $\mathbf{P}$) by the classical Renewal Theorem (or by the strong law of large numbers).

The claim $h_c(\beta) > h_c^{ann}(\beta)$ for every β follows from the arbitrariness of β_0. □

4. The Case $1/2 < \alpha < 1$

Proof of Theorem 2.3. To make things clear, we fix now $\varepsilon > 0$ small and $0 < \gamma < 1$ such that

$$\gamma\left\{(1+\alpha)+(1-\varepsilon^2)\left[1-\alpha+(\varepsilon/2)(\alpha-1/2)\right]\right\} > 2, \tag{4.1}$$

and

$$\gamma\left[(1+\alpha)+(1-\varepsilon^2)(1-\alpha)\right] > 2-\varepsilon^2. \tag{4.2}$$

Moreover we take $\beta \le \beta_0$ and

$$h = h_c^{ann}(\beta)+\Delta := h_c^{ann}(\beta)+a\beta^{\frac{2\alpha}{2\alpha-1}(1+\varepsilon)}. \tag{4.3}$$

We notice that it is crucial that $(\alpha - 1/2) > 0$ for (4.1) to be satisfied. We will take ε sufficiently small (so that (4.1) and (4.2) can occur) *and then, once ε and γ are fixed, a* also small. We set moreover

$$k(\beta) := \frac{1}{\textsc{f}(0,\Delta)} \tag{4.4}$$

and we notice that $k(\beta)$ can be made large by choosing a small, uniformly for $\beta \le \beta_0$. As in the previous section, we assume for ease of notation that $k(\beta) \in \mathbb{N}$ (and we write just k for $k(\beta)$).

Our aim is to show that $\textsc{f}(\beta, h) = 0$ if a is chosen sufficiently small in (4.3). We recall that, thanks to Proposition 2.5, the result is proven if we show that (3.2) is $o(1)$ for k large. In order to estimate this sum, we need a couple of technical estimates which are proven at the end of this section (Lemma 4.2) and in Appendix 5 (Lemma 4.1).

Lemma 4.1. *Let $\alpha \in (0, 1)$. There exists a constant C_4 such that for every $0 < h < 1$ and every $j \le 1/\textsc{f}(0, h)$,*

$$Z_j(h) \le \frac{C_4}{j^{1-\alpha}L(j)}. \tag{4.5}$$

In view of $Z_j(h_c(0)) = Z_j(0) = \mathbf{P}(j \in \tau)$ and (A.8), this means that as long as $j \le 1/\textsc{f}(0, h)$ the partition function of the homogeneous model behaves essentially like in the (homogeneous) critical case.

Lemma 4.2. *There exists* $\varepsilon_0 > 0$ *such that, if* $\varepsilon \leq \varepsilon_0$ *(*ε *being the same one which appears in* (4.3)*),*

$$\mathbb{E}_{j,1/\sqrt{j}}[Z_{j,\omega}] \leq \frac{C_5}{j^{1-\alpha+(\varepsilon/2)(\alpha-1/2)}} \tag{4.6}$$

for some constant C_5 *(depending on* ε *but not on* β *or* a*), uniformly in* $0 \leq \beta \leq \beta_0$ *and in* $k^{(1-\varepsilon^2)} \leq j < k$.

In order to bound above (3.2), we split it as

$$S_3 + S_4 := \sum_{j=0}^{\lfloor k^{(1-\varepsilon^2)} \rfloor} \frac{L_1(k-j)\, A_j}{(k-j)^{(1+\alpha)\gamma-1}} + \sum_{j=\lfloor k^{(1-\varepsilon^2)} \rfloor+1}^{k-1} \frac{L_1(k-j)\, A_j}{(k-j)^{(1+\alpha)\gamma-1}}. \tag{4.7}$$

For S_3 we use simply $A_j \leq (\mathbb{E}Z_{j,\omega})^\gamma = [Z_j(\Delta)]^\gamma$ and Lemma 4.1, together with (A.21) and (A.24):

$$S_3 \leq \frac{L_3(k)}{k^{[(1+\alpha)\gamma-1]}} \frac{1}{k^{(1-\varepsilon^2)((1-\alpha)\gamma-1)}}, \tag{4.8}$$

where $L_3(\cdot)$ can depend on ε but not on a. The second condition (4.2) imposed on γ guarantees that S_3 is arbitrarily small for k large, *i.e.*, for a small.

As for S_4, we use Lemma A.1 with $N = j$ and $\lambda = 1/\sqrt{j}$ to estimate A_j (recall the definition in (A.1)). We get

$$A_j \leq \left[\mathbb{E}_{j,1/\sqrt{j}}(Z_{j,\omega})\right]^\gamma \exp(c\gamma/(1-\gamma)), \tag{4.9}$$

provided that $1/\sqrt{j} \leq \min(1, (1-\gamma)/\gamma)$, which is true for all $j \geq k^{1-\varepsilon^2}$ if a is small. Then, provided we have chosen $\varepsilon \leq \varepsilon_0$, Lemma 4.2 gives for every $k^{(1-\varepsilon^2)} < j < k$,

$$A_j \leq \frac{C_6}{j^{[1-\alpha+(\varepsilon/2)(\alpha-1/2)]\gamma}}. \tag{4.10}$$

Note that C_6 is large for ε small (since from (4.1)–(4.2) it is clear that γ must be close to 1 for ε small) but it is independent of a. As a consequence, using (A.22),

$$\begin{aligned} S_4 &\leq \max_{k^{(1-\varepsilon^2)} \leq j < k} A_j \times \sum_{r=1}^{k} \frac{L_1(r)}{r^{(1+\alpha)\gamma-1}} \leq \max_{k^{(1-\varepsilon^2)} \leq j < k} A_j \times \frac{L_4(k)}{k^{(1+\alpha)\gamma-2}} \\ &\leq C_6\, L_4(k)\, k^{2-(1+\alpha)\gamma-(1-\varepsilon^2)[1-\alpha+(\varepsilon/2)(\alpha-1/2)]\gamma}. \end{aligned} \tag{4.11}$$

Then, the first condition (4.1) imposed on γ guarantees that S_4 tends to zero when k tends to infinity. □

Proof of Lemma 4.2. Using (3.8) together with the observation (3.9), the definition of Δ and of $k = k(\beta)$ in terms of $\mathrm{F}(0, \Delta)$ (plus the behavior of $\mathrm{F}(0, \Delta)$ for Δ small described in Theorem 2.1 (2)) one sees that for $j \leq k(\beta)$,

$$\mathbb{E}_{j,1/\sqrt{j}}[Z_{j,\omega}] \leq \mathbf{E}\left(e^{-C_7 \frac{\beta}{\sqrt{j}} |\tau \cap \{1,\dots,j\}|} \mathbf{1}_{\{j \in \tau\}}\right), \tag{4.12}$$

uniformly for $0 \le \beta \le \beta_0$. If moreover $j \ge k^{(1-\varepsilon^2)}$ one has

$$\frac{\beta}{\sqrt{j}} \ge \frac{C_8}{j^{1/2+(\alpha-1/2)(1+2\varepsilon^2)/(1+\varepsilon)}} \ge \frac{C_8}{j^{\alpha-(\varepsilon/2)(\alpha-1/2)}}, \tag{4.13}$$

with C_8 independent of a for a small. The condition that ε is small has been used, say, to neglect ε^2 with respect to ε. Going back to (4.12) and using Proposition A.2 one has then

$$\mathbb{E}_{j,1/\sqrt{j}}[Z_{j,\omega}] \le \frac{C_9}{j^{1-\alpha+(\varepsilon/2)(\alpha-1/2)}} \tag{4.14}$$

with C_9 depending on ε but not on a. □

5. The Case $\alpha = 1/2$

Proof of Theorem 2.4. The proof is not conceptually different from that of Theorem 2.3, but here we have to carefully keep track of the slowly varying functions, and we have to choose $\gamma(< 1)$ as a function of k. Under our assumption (2.17) on $L(\cdot)$, it is easy to deduce from Theorem 2.1 (2) that (say, for $0 < \Delta < 1$)

$$\textsc{f}(0, \Delta) = \Delta^2 \widehat{L}(1/\Delta) \ge C(c, \eta)\Delta^2 \,|\log \Delta|^{2\eta}. \tag{5.1}$$

We take $\beta \le \beta_0$ and

$$h = h_c^{ann}(\beta) + \Delta := h_c^{ann}(\beta) + a \exp\left(-\beta^{-1/(\eta-1/2-\varepsilon)}\right), \tag{5.2}$$

and, as in the last section, $k = 1/\textsc{f}(0, \Delta) = \Delta^{-2}/\widehat{L}(1/\Delta)$. We note also that (for $a < 1$)

$$\beta \ge |\log \Delta|^{-\eta+1/2+\varepsilon}. \tag{5.3}$$

We set $\gamma = \gamma(k) = 1 - 1/(\log k)$. As γ is k–dependent one cannot use (A.21) and (A.24) without care to pass from (2.23) to (3.2), since one could in principle have γ-dependent (and therefore k-dependent) constants in front. Therefore, our first aim will be to (partly) get rid of γ in (2.23). We notice that for any $j \le k - 1$, for k such that $\gamma(k) \ge 5/6$,

$$\sum_{n=k}^{\infty} K(n-j)^{\gamma} \le \sum_{n=k-j}^{k^6} K(n) \exp\left[(3/2 \log n - \log L(n))/\log k\right] + \sum_{n=k^6+1}^{\infty} [K(n)]^{5/6}. \tag{5.4}$$

Now, properties of slowly varying functions guarantee that the quantity in the exponential in the first sum is bounded (uniformly in j and k). As for the second sum, (A.21) guarantees it is smaller than $k^{-6/5}$ for k large. Since by Lemma 4.1 the A_j are bounded by a constant in the regime we are considering, when we reinsert this term in (2.23) and we sum over $j < k$ we obtain a contribution which vanishes at least like $k^{-1/5}$ for $k \to \infty$. We will therefore forget from now on the second sum in (5.4).

Therefore one has

$$\rho \leq C_{10} \sum_{n=k}^{\infty} \sum_{j=0}^{k-1} K(n-j) A_j \leq C_{11} \sum_{j=0}^{k-1} \frac{L(k-j) A_j}{(k-j)^{1/2}}, \tag{5.5}$$

where we have safely used (A.21) to get the second expression and now γ appears only (implicitly) in the fractional moment A_j but not in the constants C_i.

Once again, it is convenient to split this sum into

$$S_5 + S_6 := \sum_{j=0}^{k/R_2} \frac{A_j\, L(k-j)}{(k-j)^{1/2}} + \sum_{j=(k/R_2)+1}^{k-1} \frac{A_j\, L(k-j)}{(k-j)^{1/2}}, \tag{5.6}$$

with R_2 a large constant. To bound S_5 we simply use Jensen inequality to estimate A_j. Lemma 4.1 gives that for all $j \leq k$,

$$A_j \leq \frac{C_{12}}{j^{\gamma/2} L(j)^{\gamma}} \leq \frac{C_{13}}{\sqrt{j} L(j)}, \tag{5.7}$$

where the second inequality comes from our choice $\gamma = 1 - 1/(\log k)$. Knowing this, we can use (A.21) to compute S_5 and get

$$S_5 \leq \frac{C_{14}}{\sqrt{R_2}} \frac{L(k(1-1/R_2))}{L(k/R_2)}. \tag{5.8}$$

We see that S_5 can be made small choosing R_2 large. It is important for the following to note that it is sufficient to choose R_2 large but independent of k; in particular, for k large at R_2 fixed the last factor in (5.8) approaches 1 by the property of slow variation of $L(\cdot)$. As for S_6,

$$S_6 \leq C_{15} \max_{k/R_2 < j < k} A_j \times \sqrt{k}\, L(k). \tag{5.9}$$

In order to estimate this maximum, we need to refine Lemma 4.2:

Lemma 5.1. *There exists a constant $C_{16} := C_{16}(R_2)$ such that for $\gamma = 1 - 1/(\log k)$ and $k/R_2 < j < k$,*

$$A_j \leq C_{16} \left(L(j) \sqrt{j}\, (\log j)^{2\varepsilon} \right)^{-1}. \tag{5.10}$$

Given this, we obtain immediately

$$S_6 \;\leq\; C_{17}(R_2) \left[\log \left(\frac{k}{R_2} \right) \right]^{-2\varepsilon}. \tag{5.11}$$

It is then clear that S_6 can be made arbitrarily small with k large, *i.e.*, with a small. □

Proof of Lemma 5.1. Once again, we use Lemma A.1 with $N = j$ but this time $\lambda = (j \log j)^{-1/2}$. Recalling that $\gamma = 1 - 1/(\log k)$ we obtain

$$A_j \le \left[\mathbb{E}_{j,(j \log j)^{-1/2}}(Z_{j,\omega})\right]^{\gamma} \exp\left(c \frac{\log k}{\log j}\right), \tag{5.12}$$

for all j such that $(j \log j)^{1/2} \ge \log k$. The latter condition is satisfied for all $k/R_2 < j < k$ if k is large enough. Note that, since $j > k/R_2$, the exponential factor in (5.12) is bounded by a constant $C_{18} := C_{18}(R_2)$.

Furthermore, for $j \le k$, Eqs. (3.8), (3.9) combined give

$$\mathbb{E}_{j,(j \log j)^{-1/2}}[Z_{j,\omega}] \le Z_j \left(-C_{19}\beta (j \log j)^{-1/2}\right), \tag{5.13}$$

for some positive constant C_{19}, provided a is small (here we have used (5.1) and the definition $k = 1/\textsc{f}(0, \Delta)$).

In view of $j \ge k/R_2$, the definition of k in terms of β and assumption (2.17), we see that

$$\beta \ge C_{20} (\log j)^{(-\eta+1/2+\varepsilon)} \ge \frac{C_{21}}{c} L(j) (\log j)^{1/2+\varepsilon}, \tag{5.14}$$

so that the r.h.s. of (5.13) is bounded above by

$$Z_j \left(-C_{21} \frac{L(j)}{c\sqrt{j}} (\log j)^{\varepsilon}\right) \le C_{22} \frac{(\log j)^{-2\varepsilon}}{L(j)\sqrt{j}}, \tag{5.15}$$

where in the last inequality we used Lemma A.2. The result is obtained by re-injecting this in (5.12), and using the value of $\gamma(k)$. □

Appendix A. Frequently Used Bounds

A.1. Bounding the partition function via tilting. For $\lambda \in \mathbb{R}$ and $N \in \mathbb{N}$ consider the probability measure $\mathbb{P}_{N,\lambda}$ defined by

$$\frac{\mathrm{d}\mathbb{P}_{N,\lambda}}{\mathrm{d}\mathbb{P}}(\omega) = \frac{1}{\mathrm{M}(-\lambda)^N} \exp\left(-\lambda \sum_{i=1}^{N} \omega_i\right), \tag{A.1}$$

where $\mathrm{M}(\cdot)$ was defined in (2.8). Note that under $\mathbb{P}_{N,\lambda}$ the random variables ω_i are still independent but no longer identically distributed: the law of ω_i, $i \le N$ is tilted while ω_i, $i > N$ are distributed exactly like under $\mathbb{P}$.

Lemma A.1. *There exists $c > 0$ such that, for every $N \in \mathbb{N}$ and $\gamma \in (0, 1)$,*

$$\mathbb{E}\left[(Z_{N,\omega})^{\gamma}\right] \le \left[\mathbb{E}_{N,\lambda}\left(Z_{N,\omega}\right)\right]^{\gamma} \exp\left(c\left(\frac{\gamma}{1-\gamma}\right)\lambda^2 N\right), \tag{A.2}$$

for $|\lambda| \le \min(1, (1-\gamma)/\gamma)$.

Proof. We have

$$\begin{aligned}\mathbb{E}\left[(Z_{N,\omega})^{\gamma}\right] &= \mathbb{E}_{N,\lambda}\left[(Z_{N,\omega})^{\gamma}\,\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_{N,\lambda}}(\omega)\right]\\ &\le \left[\mathbb{E}_{N,\lambda}\left(Z_{N,\omega}\right)\right]^{\gamma}\left(\mathbb{E}_{N,\lambda}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_{N,\lambda}}(\omega)\right)^{1/(1-\gamma)}\right]\right)^{1-\gamma}\\ &= \left[\mathbb{E}_{N,\lambda}\left(Z_{N,\omega}\right)\right]^{\gamma}\left(\mathrm{M}(-\lambda)^{\gamma}\mathrm{M}\left(\lambda\gamma/(1-\gamma)\right)^{1-\gamma}\right)^{N}, \qquad \text{(A.3)}\end{aligned}$$

where in the second step we have used Hölder inequality and the last step is a direct computation. The proof is complete once we observe that $0 \le \log \mathrm{M}(x) \le cx^2$ for $|x| \le 1$ if c is the maximum of the second derivative of $(1/2)\log \mathrm{M}(\cdot)$ over $[-1, 1]$. □

A.2. Estimates on the renewal process. With the notation (2.4) one has

Proposition A.2. *Let $\alpha \in (0,1)$ and $r(\cdot)$ be a function diverging at infinity and such that*

$$\lim_{N\to\infty}\frac{r(N)L(N)}{N^{\alpha}} = 0. \qquad \text{(A.4)}$$

For the homogeneous pinning model,

$$Z_N(-N^{-\alpha}L(N)r(N)) \overset{N\to\infty}{\sim} \frac{N^{\alpha-1}}{L(N)\,r(N)^2}. \qquad \text{(A.5)}$$

To prove this result we use:

Proposition A.3 ([13, Theorems A & B]). *Let $\alpha \in (0,1)$. There exists a function $\sigma(\cdot)$ satisfying*

$$\lim_{x\to+\infty}\sigma(x) = 0, \qquad \text{(A.6)}$$

and such that for all $n, N \in \mathbb{N}$,

$$\left|\frac{\mathbf{P}(\tau_n = N)}{nK(N)} - 1\right| \le \sigma\left(\frac{N}{a(n)}\right), \qquad \text{(A.7)}$$

where $a(\cdot)$ is an asymptotic inverse of $x \mapsto x^{\alpha}/L(x)$.

Moreover,

$$\mathbf{P}(N\in\tau) \overset{N\to\infty}{\sim} \left(\frac{\alpha\sin(\pi\alpha)}{\pi}\right)\frac{N^{\alpha-1}}{L(N)}. \qquad \text{(A.8)}$$

We observe that by [6, Th. 1.5.12] we have that $a(\cdot)$ is regularly varying of exponent $1/\alpha$, in particular $\lim_{n\to\infty} a(n)/n^b = 0$ if $b > 1/\alpha$. We point out also that (A.8) has been first established for $\alpha \in (1/2, 1)$ in [19].

Proof of Proposition A.2. We put for simplicity of notation $v(N) := N^\alpha/L(N)$. Decomposing Z_N with respect to the cardinality of $\tau \cap \{1, \ldots, N\}$,

$$\begin{aligned} Z_N(-r(N)/v(N)) &= \sum_{n=1}^{N} \mathbf{P}\left(|\tau \cap \{1, \ldots, N\}| = n, N \in \tau\right) e^{-n\, r(N)/v(N)} \\ &= \sum_{n=1}^{N} \mathbf{P}(\tau_n = N) e^{-n\, r(N)/v(N)} \\ &= \sum_{n=1}^{\frac{v(N)}{\sqrt{r(N)}}} \mathbf{P}(\tau_n = N) e^{-n\, \frac{r(N)}{v(N)}} + \sum_{n=\frac{v(N)}{\sqrt{r(N)}}+1}^{N} \mathbf{P}(\tau_n = N) e^{-n\, \frac{r(N)}{v(N)}}. \quad \text{(A.9)} \end{aligned}$$

Observe now that one can rewrite the first term in the last line of (A.9) as

$$(1 + o(1))K(N) \sum_{n=1}^{v(N)/\sqrt{r(N)}} n\, e^{-n\, r(N)/v(N)}, \tag{A.10}$$

and $o(1)$ is a quantity which vanishes for $N \to \infty$ (this follows from Proposition A.3, which applies uniformly over all terms of the sum in view of $\lim_N r(N) = \infty$). Thanks to condition (A.4), one can estimate this sum by an integral:

$$\sum_{n=1}^{v(N)/\sqrt{r(N)}} n\, e^{-n\, r(N)/v(N)} = \frac{v(N)^2}{r(N)^2}(1 + o(1)) \int_0^\infty \mathrm{d}x\, x\, e^{-x} = \frac{v(N)^2}{r(N)^2}(1 + o(1)).$$

As for the second sum in (A.9), observing that $\sum_{n\in\mathbb{N}} \mathbf{P}(\tau_n = N) = \mathbf{P}(N \in \tau)$, we can bound it above by

$$\mathbf{P}(N \in \tau)e^{-\sqrt{r(N)}}. \tag{A.11}$$

In view of (A.8), the last term is negligible with respect to $N^{\alpha-1}/(L(N)\, r(N)^2)$ and our result is proved. □

Proof of Lemma 4.1. Recalling the notation (2.4), point (2) of Theorem 2.1 (see in particular the definition of $\widehat{L}(\cdot)$) and (A.8), we see that the result we are looking for follows if we can show that for every $c > 0$ there exists $C_{23} = C_{23}(c) > 0$ such that

$$\mathbf{E}\left[e^{c|\tau\cap\{1,\ldots,N\}|L(N)/N^\alpha} \,\middle|\, N \in \tau\right] \le C_{23}, \tag{A.12}$$

uniformly in N. Let us assume that $N/4 \in \mathbb{N}$; by Cauchy-Schwarz inequality the result follows if we can show that

$$\mathbf{E}\left[e^{2c|\tau\cap\{1,\ldots,N/2\}|L(N)/N^\alpha} \,\middle|\, N \in \tau\right] \le C_{24}. \tag{A.13}$$

Let us define $X_N := \max\{n = 0, 1, \ldots, N/2 : n \in \tau\}$ (last renewal epoch up to $N/2$). By the renewal property we have

$$\begin{aligned} &\mathbf{E}\left[e^{2c|\tau\cap\{1,\ldots,N/2\}|L(N)/N^\alpha} \,\middle|\, N \in \tau\right] \\ &= \sum_{n=0}^{N/2} \mathbf{E}\left[e^{2c|\tau\cap\{1,\ldots,N/2\}|L(N)/N^\alpha} \,\middle|\, X_N = n\right] \mathbf{P}\left(X_N = n \,|\, N \in \tau\right). \quad \text{(A.14)} \end{aligned}$$

If we can show that for every $n = 0, 1, \dots, N/2$,

$$\mathbf{P}\left(X_N = n \,|\, N \in \tau\right) \le C_{25}\mathbf{P}\left(X_N = n\right), \tag{A.15}$$

then we are reduced to proving (A.13) with $\mathbf{E}[\cdot|N \in \tau]$ replaced by $\mathbf{E}[\cdot]$.

Let us then observe that

$$\begin{aligned} \mathbf{P}\left(X_N = n,\ N \in \tau\right) &= \mathbf{P}(n \in \tau)\mathbf{P}\left(\tau_1 > (N/2) - n,\ N - n \in \tau\right) \\ &= \mathbf{P}(n \in \tau) \sum_{j=(N/2)-n+1}^{N-n} \mathbf{P}(\tau_1 = j)\mathbf{P}\left(N - n - j \in \tau\right). \end{aligned} \tag{A.16}$$

We are done if we can show that

$$\sum_{j=(N/2)-n+1}^{N-n} \mathbf{P}(\tau_1 = j)\mathbf{P}\left(N - n - j \in \tau\right) \le C_{26}\mathbf{P}\left(N \in \tau\right) \sum_{j=(N/2)-n+1}^{\infty} \mathbf{P}(\tau_1 = j), \tag{A.17}$$

because the mass renewal function $\mathbf{P}(N \in \tau)$ cancels when we consider the conditioned probability and, recovering $\mathbf{P}(n \in \tau)$ from (A.16) we rebuild $\mathbf{P}(X_N = n)$. We split the sum in the left-hand side of (A.17) in two terms. By using (A.8) (but just as upper bound) and the fact that the inter-arrival distribution is regularly varying we obtain

$$\begin{aligned} &\sum_{j=(3N/4)-n+1}^{N-n} \mathbf{P}(\tau_1 = j)\mathbf{P}\left(N - n - j \in \tau\right) \\ &\le C_{27}\frac{L(N)}{N^{1+\alpha}} \sum_{j=(3N/4)-n+1}^{N-n} \frac{1}{(N-n-j+1)^{1-\alpha}L(N-n-j+1)} \\ &= C_{27}\frac{L(N)}{N^{1+\alpha}} \sum_{j=1}^{N/4} \frac{1}{j^{1-\alpha}L(j)} \le \frac{C_{28}}{N}. \end{aligned} \tag{A.18}$$

Since the right-hand side of (A.17) is bounded below by $1/N$ times a suitable constant (of course if n is close to $N/2$ this quantity is sensibly larger) this first term of the splitting is under control. Now the other term: since the renewal function is regularly varying

$$\sum_{j=(N/2)-n+1}^{(3N/4)-n} \mathbf{P}(\tau_1 = j)\mathbf{P}\left(N - n - j \in \tau\right) \le C_{29}\mathbf{P}\left(N \in \tau\right) \sum_{j=(N/2)-n+1}^{(3N/4)-n} \mathbf{P}(\tau_1 = j), \tag{A.19}$$

that gives what we wanted.

It remains to show that (A.13) holds without conditioning. For this we use the asymptotic estimate $-\log \mathbf{E}[\exp(-\lambda\tau_1)] \overset{\lambda\searrow 0}{\sim} c_\alpha \lambda^\alpha L(1/\lambda)$, with $c_\alpha = \int_0^\infty r^{-1-\alpha} (1-\exp(-r))\,\mathrm{d}r = \Gamma(1-\alpha)/\alpha$, and the Markov inequality to get that if $x > 0$,

$$\mathbf{P}\left(|\tau \cap \{1, \dots, N\}|L(N)/N^\alpha > x\right) = \mathbf{P}\left(\tau_n < N\right) \le \exp\left(-\frac{1}{2}c_\alpha\lambda^\alpha L(1/\lambda)n + \lambda N\right), \tag{A.20}$$

with n the integer part of $xN^\alpha/L(N)$ and $\lambda \in (0, \lambda_0)$ for some $\lambda_0 > 0$. If one chooses $\lambda = y/N$, y a positive number, then for $x \geq 1$ and N sufficiently large (depending on λ_0 and y) we have that the quantity at the exponent in the right-most term in (A.20) is bounded above by $-(c_\alpha/3)y^\alpha x + y$. The proof is then complete if we select y such that $(c_\alpha/3)y^\alpha > 2c$ (c appears in (A.13)) since if X is a non-negative random variable and q is a real number $\mathbf{E}[\exp(qX)] = 1 + q\int_0^\infty e^{qx}\mathbf{P}(X > x)\,\mathrm{d}x$.

A.3. Some basic facts about slowly varying functions. We recall here some of the elementary properties of slowly varying functions which we repeatedly use, and we refer to [6] for a complete treatment of slow variation.

The first two well-known facts are that, if $U(\cdot)$ is slowly varying at infinity,

$$\sum_{n\geq N} \frac{U(n)}{n^m} \overset{N\to\infty}{\sim} U(N)\frac{N^{1-m}}{m-1}, \tag{A.21}$$

if $m > 1$ and

$$\sum_{n=1}^{N} \frac{U(n)}{n^m} \overset{N\to\infty}{\sim} U(N)\frac{N^{1-m}}{1-m}, \tag{A.22}$$

if $m < 1$ (*cf.* for instance [20, Sect. A.4]). The second two facts are that (*cf.* [6, Th. 1.5.3])

$$\inf_{n\geq N} U(n)n^m \overset{N\to\infty}{\sim} U(N)\,N^m, \tag{A.23}$$

if $m > 0$, and

$$\sup_{n\geq N} U(n)n^m \overset{N\to\infty}{\sim} U(N)\,N^m, \tag{A.24}$$

if $m < 0$. □

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