A Letters Journal Exploring the Frontiers of Physics

April 2018

EPL, **122** (2018) 10001 www.epljournal.org
doi: 10.1209/0295-5075/122/10001

Focus Article

A new approach to computing the asymptotics of the position of Fisher-KPP fronts[(a)]

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received 19 February 2018; accepted in final form 7 May 2018
published online 21 May 2018

PACS 02.30.Jr – Partial differential equations
PACS 02.40.Xx – Singularity theory

Abstract – This paper presents a novel way of computing front positions in Fisher-KPP equations. Our method is based on an exact relation between the Laplace transform of the initial condition and some integral functional of the front position. Using singularity analysis, one can obtain the asymptotics of the front position up to the $\mathcal{O}(\log t/t)$ term. Our approach is robust and can be generalised to other front equations.

Introduction. – The goal of this letter is to present a novel way of computing the asymptotic position of a front propagating into an unstable phase. The typical equation we consider is the Fisher-KPP equation [1,2],

$$\partial_t h = \partial_x^2 h + h - h^2 \qquad \text{(Fisher-KPP)}, \tag{1}$$

but our method is general and can be adapted to a large class of other reaction-diffusion equations.

Equation (1) was introduced in 1937 independently by Fisher [1] and by Kolmogorov, Petrovski, Piscounov [2] in order to describe how a favourable mutation spreads in a population (there, $h(x,t)$ represents the fraction of the population with the mutation at position x and time t). This equation also appears in several other contexts [3], such as reaction-diffusion [4], growth [5], disordered systems [6], branching processes [7–10], high energy physics [11], etc. From the point of view of evolutionary biology, eq. (1) and its noisy version [4,12–18], are one of the most basic theoretical models to describe the evolution in a one-dimensional fitness landscape. In all these models, determining the speed of adaptation (and the effect of selection on genealogies [19,20]) is a central question [21–25].

It is remarkable that universal behaviours emerge both from equations like (1) or like its noisy version, in the sense that they do not depend on the precise form of the nonlinearities. In fact, some properties of the Fisher-KPP equation bear some similarities to properties seen in other models used to describe evolutionary biology [26,27]. It is therefore important to develop tools allowing to understand these universal behaviours.

An important feature of (1) is that the solution converges to a travelling wave: for an initial condition $h_0 \in [0,1]$ such that $h_0(x) \to 1$ as $x \to -\infty$ and $h_0(x) \to 0$ exponentially fast as $x \to \infty$, then

$$h(\mu_t + z, t) \to \omega_v(z), \qquad \frac{\mu_t}{t} \to v, \tag{2}$$

where μ_t is the position of the front (we will choose μ_t in such a way that $h(\mu_t, t) = \frac{1}{2}$ but other choices are possible), $v \geq 2$ is the asymptotic velocity and $\omega_v(z)$ is the travelling wave at velocity v, which is the unique solution to

$$\begin{aligned} &\omega_v'' + v\omega_v' + \omega_v - \omega_v^2 = 0, \\ &\omega_v(0) = \frac{1}{2}, \quad \omega_v(-\infty) = 1, \quad \omega_v(+\infty) = 0. \end{aligned} \tag{3}$$

The value of v depends only on the way the initial condition decays at infinity. If $h_0(x)$ decays as e^{-x} or faster,

[(a)] Contribution to the Focus Issue *Evolutionary Modeling and Experimental Evolution* edited by José Cuesta, Joachim Krug and Susanna Manrubia.

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the asymptotic velocity is $v = 2$ [2,8,28]. For an initial condition that (roughly) decays as $e^{-\gamma x}$ with $\gamma \in (0,1)$, the velocity is $v = \gamma + \gamma^{-1}$ [9]. Since the work of Bramson [8,9], the leading sublinear terms for the position are known: for example if the initial condition satisfies

$$h_0(x) \sim Ax^\nu e^{-\gamma x},$$

then

$$\begin{cases} \mu_t = (\gamma+\gamma^{-1})t + \nu \log t + a + o(1), & \gamma \in (0,1), \\ \mu_t = 2t + \dfrac{\nu-1}{2}\log t + a + o(1), & \gamma = 1, \nu > -2, \\ \mu_t = 2t - \dfrac{3}{2}\log t + \log\log t + a + o(1), & \gamma = 1, \nu = -2, \\ \mu_t = 2t - \dfrac{3}{2}\log t + a + o(1), & \gamma = 1, \nu < -2, \\ & \text{or } \gamma > 1, \end{cases} \tag{4}$$

where a is some constant. The last line also holds for an initial condition decaying faster than any exponential. Twenty years later, Ebert and van Saarloos [29] computed the next term for μ_t in the case where the initial condition decays fast enough:

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + o(t^{-\frac{1}{2}}). \tag{5}$$

It is remarkable that although the constant a is unknown and depends on the precise shape of $h_0(x)$, the next-order term can be determined and is universal. This corrective term was rigorously established recently [30], but only for initial conditions such that $h_0(x) = 0$ if x is large enough.

So far, results (4) and (5) were obtained either by probabilistic methods, exploiting the link between (1) and branching processes [7–9,31] or by computing precisely how the shape $h(\mu_t + x, t)$ of the centred front converges to the travelling wave [2,12,28–30,32–35].

In a recent work [36,37], we have presented a new approach, different from previous ones, to compute the position of the front in an equation looking like (1), in a particular case where the nonlinear term is replaced by a free boundary condition: the problem was to find h and ν_t such that

$$\begin{cases} \partial_t h = \partial_x^2 h + h, & \text{for } x > \nu_t, \\ h(\nu_t, t) = 1 \quad \text{and} \quad \partial_x h(\nu_t, t) = 0. \end{cases} \tag{6}$$

This approach was based on an exact relation between an integral involving ν_t and the Laplace transform of the initial condition $h_0(x)$. It allowed us to show in [36,37] that the results (4) and (5), known for the FKPP equation (1), remain valid for the free boundary problem (6). We further established the necessary conditions on the initial condition under which the Ebert van Saarloos term holds and obtained next-order terms.

In the present paper, we show that our method is much more general and that it works even in the presence of nonlinear terms in the equation. In particular, it can be applied to (1). Introduce

$$\varphi(r,t) := \int_{\mathbb{R}} \mathrm{d}z\, h(\mu_t + z, t)^2 e^{rz}, \tag{7}$$

and define

$$\Psi(r) := \int_{\mathbb{R}} \mathrm{d}x\, h_0(x) e^{rx}. \tag{8}$$

Then, all our results will be obtained from the following equality (derived in the next section): for any $r < 1$ small enough so that (8) converges,

$$\Psi(r) = \int_0^\infty \mathrm{d}t\, \varphi(r,t) e^{r\mu_t - (r^2+1)t}. \tag{9}$$

Notice from (7) that $\varphi(r,t)e^{r\mu_t}$ is independent of μ_t. Therefore, (9) holds in fact for an arbitrary choice of μ_t and, by itself, it is not sufficient to determine the position of the front. However, when μ_t is the position of the front (defined as above by $h(\mu_t, t) = \frac{1}{2}$), we then have

$$\varphi(r,t) \to \hat{\varphi}(r) \quad \text{with } \hat{\varphi}(r) := \int \mathrm{d}z\, \omega_v(z)^2 e^{rz}, \tag{10}$$

for r small enough, and we can evaluate the speed of that convergence. Then, with (9) and (10), we will determine the first terms of the large t asymptotics of μ_t.

Derivation of (9). – From its definition (7), it is obvious that $\varphi(r,t)e^{r\mu_t}$ is independent of the choice of μ_t. Thus, it is sufficient to establish (9) for $\mu_t = 0$. Define, for r small enough,

$$g(r,t) = \int_{\mathbb{R}} \mathrm{d}x\, h(x,t) e^{rx}. \tag{11}$$

(Of course $\Psi(r) = g(r,0)$ from (8).) Then, from (1) and (7) with $\mu_t = 0$ one has

$$\partial_t g(r,t) = (1+r^2) g(r,t) - \varphi(r,t), \tag{12}$$

where we integrated by parts $\int \mathrm{d}x\, \partial_x^2 h\, e^{rx}$. One can solve (12) to get

$$g(r,t) = e^{(1+r^2)t}\left[\Psi(r) - \int_0^t \mathrm{d}s\, \varphi(r,s) e^{-(1+r^2)s}\right].$$

There only remains to show that

$$g(r,t)e^{-(1+r^2)t} \to 0 \quad \text{as } t \to \infty \tag{13}$$

to obtain (9). The solution $h(x,t)$ to (1) is smaller than $L(x,t)$, the solution to the linearised equation $\partial_t L(x,t) = \partial_x^2 L(x,t) + L(x,t)$ with $L(x,0) = h_0(x)$. For any β

$$\begin{aligned} L(x,t) &= \int_{\mathbb{R}} \mathrm{d}y\, h_0(y) e^t \, \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}, \\ &= \int_{\mathbb{R}} \mathrm{d}y\, h_0(y) e^{(1+\beta^2)t - \beta(x-y)} \, \frac{e^{-\frac{(x-y-2\beta t)^2}{4t}}}{\sqrt{4\pi t}}. \end{aligned}$$

Using the definition (8), this gives

$$L(x,t) \le \frac{e^{(1+\beta^2)t}}{\sqrt{4\pi t}} e^{-\beta x} \Psi(\beta). \tag{14}$$

We choose β such that $\Psi(\beta) < \infty$. Then, we write that $h(x,t) \le \min[1, L(x,t)]$. By using the bound (14), one has

$$h(x,t) \le \begin{cases} 1, & \text{if } x < d_{\beta,t}, \\ \dfrac{e^{(1+\beta^2)t}}{\sqrt{4\pi t}} e^{-\beta x}\Psi(\beta), & \text{if } x > d_{\beta,t}, \end{cases} \tag{15}$$

where $d_{\beta,t}$ is the position where the second bound is also equal to 1. Then, for $r < \beta$, one gets from (11)

$$g(r,t) \le \Big(\frac{1}{r} + \frac{1}{\beta - r}\Big) e^{r d_{\beta,t}}.$$

Using $e^{r d_{\beta,t}} = \Big(\frac{e^{(1+\beta^2)t}}{\sqrt{4\pi t}}\Psi(\beta)\Big)^{r/\beta}$, this leads for $t > 1$ to

$$g(r,t) \le C e^{r(\beta+\beta^{-1})t} \tag{16}$$

for some constant C. Choose furthermore $\beta \le 1$. With $r < \beta$, one checks that $r(\beta + \beta^{-1}) < 1 + r^2$, and one concludes that (13) and (9) hold for all $r < 1$ such that $r < \sup\big[\beta; \Psi(\beta) < \infty\big]$.

Velocity selection. – Let us first see how (9) and (10) allow to recover the asymptotic velocity $v = \lim_{t\to\infty} \mu_t/t$ of the front, and how this velocity depends on the initial condition.

First assume that $\Psi(r)$ in (8) is singular as $r \nearrow \gamma \le 1$, meaning (roughly speaking) that $h_0(x)$ decays as $e^{-\gamma x}$. Then, obviously, the right-hand-side of (9) must also be singular as $r \nearrow \gamma$. This singularity can only come from the large t part of the integral, where $\varphi(r,t)$ is nearly equal to $\hat\varphi(r)$ according to (10). The only mechanism for (9) to become singular at $r = \gamma$ is that $r\mu_t - (r^2+1)t \to -\infty$ for $r = \gamma - \epsilon$ and $r\mu_t - (r^2+1)t \to +\infty$ for $r = \gamma + \epsilon$ (with $\epsilon > 0$ small). This means that $\mu_t \sim vt$ with v such that $\gamma v - (\gamma^2 + 1) = 0$, which is the expected relation between the decay rate γ and the velocity v when $\gamma < 1$.

When $\Psi(r)$ in (8) has no singularity up to $r = 1$ (meaning that the initial condition decays "fast"), the velocity of the front cannot be larger than 2 (otherwise, there would a singularity at some $\gamma < 1$ solution to $\gamma v = \gamma^2 + 1$) so it must be equal to 2 as there are no positive travelling waves of speed less than 2; this is also a well-known fact of the Fisher-KPP equation.

Higher-order corrections. – We have just seen that the position of the singularity determines the velocity: $\mu_t \approx vt$; we are now going to see that the nature of the singularity gives the next-order terms in μ_t. Let us illustrate this method by focusing on the Ebert-van Saarloos term (5). All the other asymptotics given in (4), including the Bramson logarithmic term can be obtained in a similar way.

Assume, for simplicity, that the initial condition decays fast enough for $\Psi(r)$ as given by (8) to be analytic at $r = 1$. Then, from (4),

$$\mu_t = 2t - \frac{3}{2}\log t + a + o(1), \tag{17}$$

where a is some unknown constant that depends on the nonlinearity in the equation and on the initial condition. We now apply our method to evaluate the $o(1)$.

As a first attempt, let us look at what happens as $r \nearrow 1$ in (9) when $\varphi(r,t)$ is replaced by its limit $\hat\varphi(r)$ and μ_t is given by $2t - \frac{3}{2}\log t + a$ for $t > t_0$, without any further corrective terms. Then, with these substitutions, $\Psi(1-\epsilon)$ would be equal to

$$f(\epsilon) + \hat\varphi(1-\epsilon) e^{(1-\epsilon)a} \int_{t_0}^{\infty} \mathrm{d}t\, \frac{e^{-\epsilon^2 t}}{t^{\frac{3}{2}}} e^{\epsilon \frac{3}{2}\log t}, \tag{18}$$

where $f(\epsilon)$, which corresponds to the integral from 0 to t_0, is obviously analytic. On the other hand, the integral above is an incomplete Gamma function, which one can expand in powers of ϵ to obtain

$$A + B\epsilon + 6\sqrt{\pi}\epsilon^2 \log\epsilon + C\epsilon^2 + \mathcal{O}(\epsilon^3),$$

where A, B and C depend on t_0, but where the singular term in $\epsilon^2 \log\epsilon$ does not. (See also the last section before the conclusion.)

Such a singular term cannot be actually present in the expansion of $\Psi(1-\epsilon)$, because we know (from our choice of initial condition) that Ψ is analytic at $r = 1$. As in the linear case [37], the only possibility for the $\epsilon^2\log\epsilon$ term to disappear, is that it is cancelled by another $\epsilon^2 \log\epsilon$ term coming from the $o(1)$ in (17). One finds that this $o(1)$ term must be given, to leading order, by the Ebert and van Saarloos term:

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + \cdots. \tag{19}$$

Repeating the same procedure, one can notice that inserting $\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}}$ into (9) leads to a $\epsilon^3 \log\epsilon$ singular term in the expansion. By a careful small ϵ expansion, one finds as that this term is cancelled by choosing

$$\mu_t = 2t - \frac{3}{2}\log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + \cdots, \tag{20}$$

and so on: each new term in the large t expansion of μ_t allows to remove a singularity in the small ϵ expansion of Ψ, but introduces a new, weaker, singularity.

Remark that we started this analysis by requiring that $\Psi(r)$ is analytic at $r = 1$. In fact, this hypothesis is not needed: to obtain (19), the only requirement is that there is no $\epsilon^2\log\epsilon$ term in the expansion of $\Psi(1-\epsilon)$:

$$\Psi(1-\epsilon) = A + B\epsilon + o(\epsilon^2 \log\epsilon)$$

for some constants A and B. (From (8), this condition is satisfied if the initial condition decays a bit faster than $x^{-3}e^{-x}$.) Similarly, the $(\log t)/t$ term of (20) requires that there is no $\epsilon^3 \log \epsilon$ term in $\Psi(r)$, that is that the initial condition decays a bit faster than $x^{-4}e^{-x}$.

At the beginning of the current section, we have replaced $\varphi(r,t)$ in (9) by its limit $\hat{\varphi}(r)$ to obtain (18). It is now time to justify this simplification. The term we neglected until now is

$$\Delta(r) = \int_0^\infty \mathrm{d}t \left[\varphi(r,t) - \hat{\varphi}(r)\right] e^{r\mu_t - (r^2+1)t}. \tag{21}$$

We claim that

$$\Delta(1-\epsilon) = \tilde{A} + \tilde{B}\epsilon + \tilde{C}\epsilon^2 + \mathcal{O}(\epsilon^3), \tag{22}$$

which means that the first singularity in the small $\epsilon > 0$ expansion of $\Delta(1-\epsilon)$ is smaller than ϵ^3. Then, the result (20) still holds as it was obtained by suppressing a singularity $\epsilon^3 \log \epsilon$, bigger than ϵ^3.

To justify (22), we argue in the next section that, when μ_t is defined as the position where the front is 1/2, one has

$$\varphi(r,t) = \hat{\varphi}(r) + \mathcal{O}\Big(\frac{1}{t}\Big). \tag{23}$$

Then, inserting (23) and Bramson's estimate (17) for the position μ_t of the front into (21), one obtains

$$\Delta(1-\epsilon) = \int_1^\infty \mathrm{d}t \, \frac{e^{-\epsilon^2 t + \frac{3}{2}\epsilon \log t}}{t^{3/2}} \times \mathcal{O}\Big(\frac{1}{t}\Big).$$

One checks directly that the integral on the right-hand side satisfies (22).

Justification of (23). – With μ_t being the position where the front is 1/2, define

$$\delta(x,t) = h(\mu_t + x, t) - \omega_2(x)$$

(recall that ω_2 is the travelling wave at velocity 2). One obtains from (1) and (3) that

$$\begin{aligned}\partial_t \delta &= \partial_x^2 \delta + 2\partial_x \delta + (1 - 2\omega_2)\delta - (2 - \dot{\mu}_t)(\partial_x \delta + \omega_2') - \delta^2 \\ &\approx \partial_x^2 \delta + 2\partial_x \delta + (1 - 2\omega_2)\delta - (2 - \dot{\mu}_t)\omega_2',\end{aligned}$$

where one has neglected two second-order terms (recall that $\delta \to 0$ and $2 - \dot{\mu}_t \to 0$). With $\mu_t \approx 2t - \frac{3}{2}\log t$, one expects $(2 - \dot{\mu}_t) \sim 3/(2t)$ for large times. This means that

$$\delta(x,t) \sim \frac{3}{2t}\eta(x), \qquad \text{as } t \to \infty,$$

with $\eta(x)$ the unique solution to

$$\eta'' + 2\eta' + (1 - 2\omega_2)\eta = \omega_2', \qquad \eta(0) = 0, \qquad \eta(\pm\infty) = 0.$$

(The $\partial_t \delta = \mathcal{O}(t^{-2})$ term is also negligible compared to δ, so that δ satisfies a inhomogeneous second-order linear equation. We eliminate other solutions by using $\delta(0,t) = 0$, and $\delta(\pm\infty, t) = 0$.)

One checks that $\eta(x) \sim -Ax^3 e^{-x}$ for large x. We now compute the difference

$$\begin{aligned}\varphi(r,t) - \hat{\varphi}(r) &= \int \mathrm{d}x \, e^{rx} \Big[h(\mu_t + x, t)^2 - \omega_2(x)^2\Big] \\ &= \int \mathrm{d}x \, e^{rx} \delta(x,t) \Big[h(\mu_t + x, t) + \omega_2(x)\Big] \\ &\sim \frac{3}{2t} \int \mathrm{d}x \, e^{rx} \eta(x) \Big[h(\mu_t + x, t) + \omega_2(x)\Big].\end{aligned}$$

In the integral, $\omega_2(x)$ and $h(\mu_t + x, t)$ both decay roughly like e^{-x} for large x (more precisely, a standard result for the Fisher-KPP equation is that $\omega_2(x) \sim Bxe^{-x}$ for some constant B). With $\eta(x) \sim -Ax^3 e^{-x}$, we see that the integral converges if $0 < r < 2$. In particular, it converges for r around 1 and so we obtain (23).

A small ϵ expansion. – To illustrate the methods used in the present paper to obtain the asymptotic expansion of μ_t, we give here (without going into the details of the computation) the small ϵ expansion of

$$I = \int_0^\infty \mathrm{d}t \, e^{-\epsilon^2 t + (1-\epsilon)(\mu_t - 2t)},$$

where μ_t is an arbitrary function such that, as $t \to \infty$,

$$\mu_t = 2t - \frac{3}{2}\log t + a + \frac{b}{\sqrt{t}} + \frac{c \log t + d}{t} + o(t^{-1}),$$

for arbitrary constants a, b, c, d. One finds

$$\begin{aligned} I = {} & A_0 + A_1\epsilon + 2e^a(b + 3\sqrt{\pi})\epsilon^2 \log \epsilon + A_2\epsilon^2 \\ & - 3e^a(b + 3\sqrt{\pi})\epsilon^3 \log^2 \epsilon \\ & + e^a \Big[\Big(15 - \frac{8}{3}c - 18\log 2\Big)\sqrt{\pi} \\ & - (3\gamma_E + 2a - 1)(b + 3\sqrt{\pi})\Big]\epsilon^3 \log \epsilon \\ & + A_3\epsilon^3 + o(\epsilon^3),\end{aligned}$$

with γ_E the Euler constant. Notice that the singular terms only depend on the asymptotic behaviour of μ_t, while the regular terms A_0, A_1, ... depend on the whole function μ_t. For instance, $A_0 = \int_0^\infty \mathrm{d}t \, e^{\mu_t - 2t}$ and $A_1 = -e^a 2\sqrt{\pi} + \int_0^\infty \mathrm{d}t \, e^{\mu_t - 2t}(2t - \mu_t)$. The value of A_0 is obvious, the value of A_1 is maybe less obvious, and A_2 and A_3 have complicated expressions.

To remove the singularities in the expansion of I, the only possible choice is $b = -3\sqrt{\pi}$ and $c = \frac{9}{8}(5 - 6\log 2)$.

Conclusion. – In this letter, we have presented a new method to study the Fisher-KPP equation. It relies on a single relation (9) between the initial condition h_0 (through Ψ) and the position μ_t of the front. A careful analysis of the singularities in (9) leads to the large time asymptotics of the position of the front.

In [36,37], we already used a similar method to study, respectively, a linear front equation with a free boundary or on the lattice. The $(\log t)/t$ term was first identified, for the lattice case in [38]. The main progress of the present work is to show that this method is not limited to linear fronts, but works also in the nonlinear case. Our main relation in [37] was simpler than (9) because the term $\varphi(r,t)$ was absent. However, we argue that $\varphi(r,t)$ converges fast enough as $t \to \infty$ for the large time analysis in [37] to apply equally in the present setting, for the Fisher-KPP equation.

The method presented here is robust, and can be adapted to a wide variety of front equations. For instance, one could apply it to

$$\partial_t h = \partial_x^2 h + h - F(h),$$

with the h^2 term replaced by an arbitrary nonlinearity $F(h)$ (satisfying some conditions such as $F(h) > 0$, $F(0) = F(1) = 0$). In fact, $F(h)$ could even be a functional of h rather than a function, for instance for the nonlocal Fisher-KPP [39]

$$\partial_t h = \partial_x^2 h + h - h\rho * h,$$

where $\rho > 0$ is some well-behaved kernel with $\int \rho = 1$. One could also work with equations discrete in space and/or time [12,36].

For the noisy Fisher-KPP equation, the leading corrections to the front velocity are also known to be universal in a weak noise expansion [12–14,16]. It would be interesting to know whether a generalization of the method presented here could allow to recover these universal velocity corrections, and to predict new ones.

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JB was partially supported by ANR grants ANR-14-CE25-0014 (ANR GRAAL) and ANR-14-CE25-0013 (ANR NONLOCAL). ÉB and BD were partially supported by ANR grant ANR-16-CE93-0003 (ANR MALIN)

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