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Laboratoire Kastler Brossel, CNRS, Ecole normale supérieure & Collège de France

The resonant Fermi gas

Lecture 1 — **Zero-range limit, three-body problem and dynamical symmetry**

Lecture 2 — **Many-body physics: methods and basic properties**

Lecture 3 — **2-body and 3-body contacts**

Contents

Lecture 1

Zero-range limit, three-body problem and dynamical symmetry

Refs:

- with Y. Castin: Lect. Notes Phys. **836**, 127 (2012); PRL **97**, 150401 (2006); PRA **74**, 053604 (2006)
- S. Tan, arXiv:cond-mat/0412764
- V. Efimov, Sov. J. Nucl. Phys. **12**, 589 (1971); Nucl. Phys. **A210**, 157 (1973).
- Y. Castin, Comptes Rendus Physique **5**, 407 (2004)

Chapter 1

Short-range resonant interactions and the universal zero-range limit

Part 1 : Simplified atomic physics

Fermionic atoms in 2 internal states : $\uparrow$, $\downarrow$
(hyperfine)

$N = N_\uparrow + N_\downarrow$. $N_\uparrow$ and $N_\downarrow$ are conserved

$\Longrightarrow$ Two species $\begin{cases} N_\uparrow & \text{red fermions} \\ N_\downarrow & \text{blue fermions} \end{cases}$

red and blue are distinguishable .

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e.g. $N_\uparrow = N_\downarrow = 2$:

$$\Psi(\underbrace{\vec{r}_1, \vec{r}_2}_{\text{antisym.}}, \underbrace{\vec{r}_3, \vec{r}_4}_{\text{antisym.}}) = -\Psi(\vec{r}_2, \vec{r}_1, \vec{r}_3, \vec{r}_4) = -\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_4, \vec{r}_3)$$

Interaction: $\underbrace{V_{\uparrow\downarrow}(r)}_{\boxed{V(r)}}$, $\underbrace{V_{\uparrow\uparrow}(r), \quad V_{\downarrow\downarrow}(r)}_{\text{negligible effect}}$

$\boxed{\begin{array}{l}\text{Trap:} \\ U(\vec{r})\end{array}}$

$$\boxed{-\frac{\hbar^2}{2m}\sum_{i=1}^{N} \Delta_{\vec{r}_i}\Psi + \sum_{i=1}^{N} U(\vec{r}_i)\Psi + \sum_{\substack{i \leq N_\uparrow \\ j > N_\uparrow}} V(r_{ij})\Psi = E\Psi}$$

$r_{ij} := \|\vec{r}_j - \vec{r}_i\|$

Part 2: 2-body problem

3D

2 fermions in *different internal states*, $\uparrow$ and $\downarrow$

[$\Leftrightarrow$ 2 *distinguishable* particles]

$$\psi(\vec{r_1}, \vec{r_2})$$

no symmetry constraint
when exchanging $\vec{r_1}$ and $\vec{r_2}$

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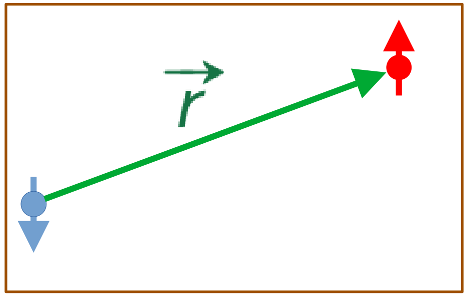

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \quad , \quad r = \| \vec{r} \|$$

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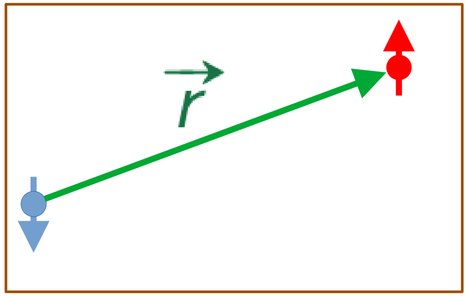

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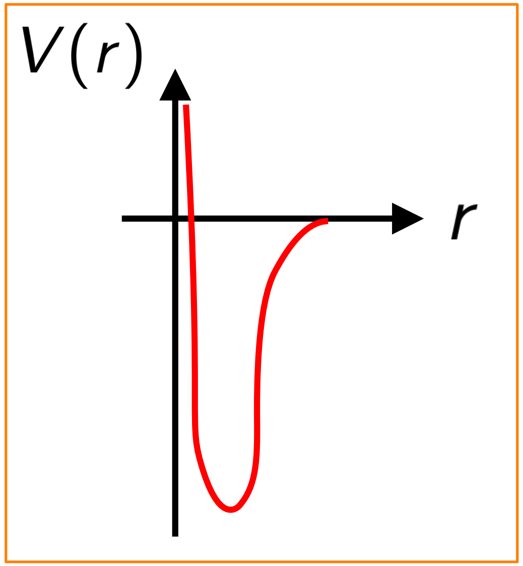
interaction potential $V(r)$
in free space



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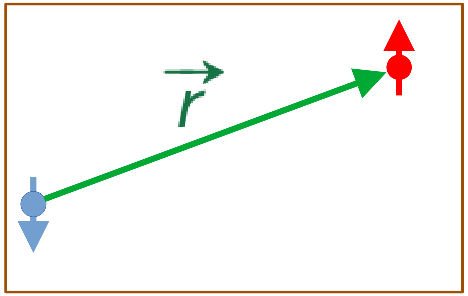

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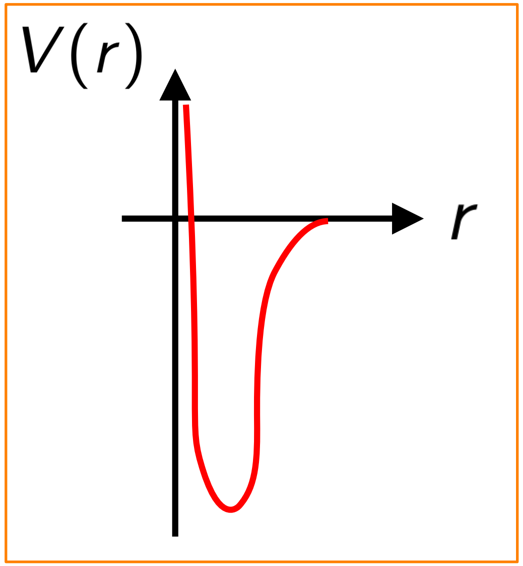

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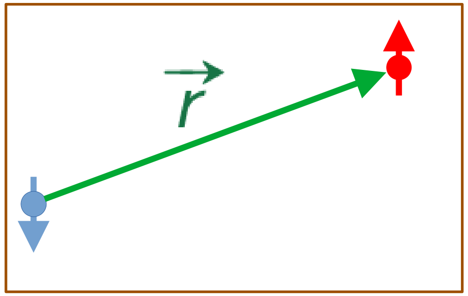

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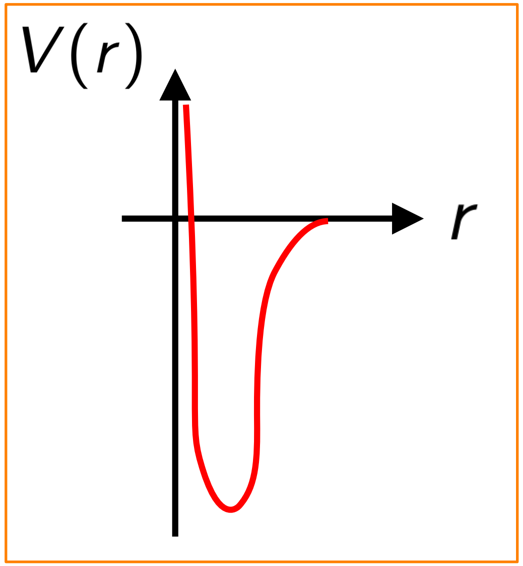

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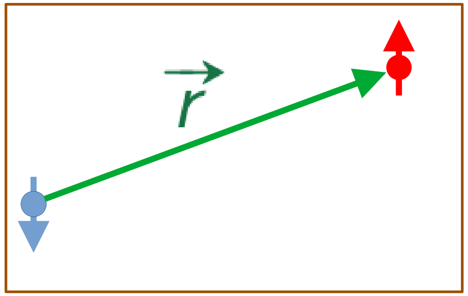

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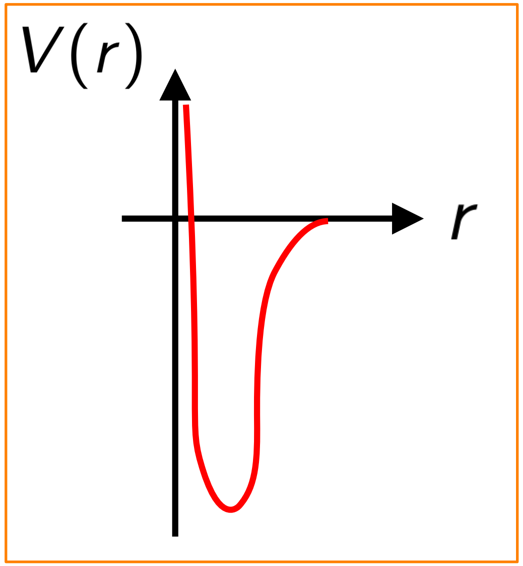

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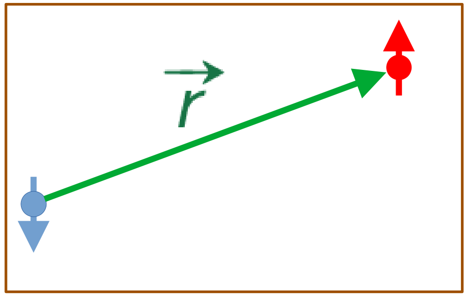

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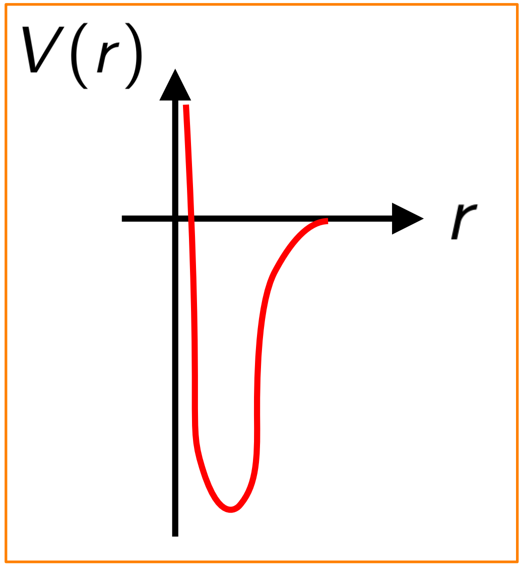

$$-\frac{\hbar^2}{2m}\underbrace{\left(\Delta_{\vec{r_1}} + \Delta_{\vec{r_2}}\right)}_{2\,\Delta_{\vec{r}} \,+\, \frac{1}{2}\Delta_{\vec{c}}}\psi + V(r)\,\psi = E\,\psi$$

$$\vec{c} = \frac{\vec{r_1}+\vec{r_2}}{2}$$

$$\psi(\vec{r_1}\,,\vec{r_2}) = \psi(\vec{r})\;e^{i\,\vec{K}\cdot\vec{c}}$$

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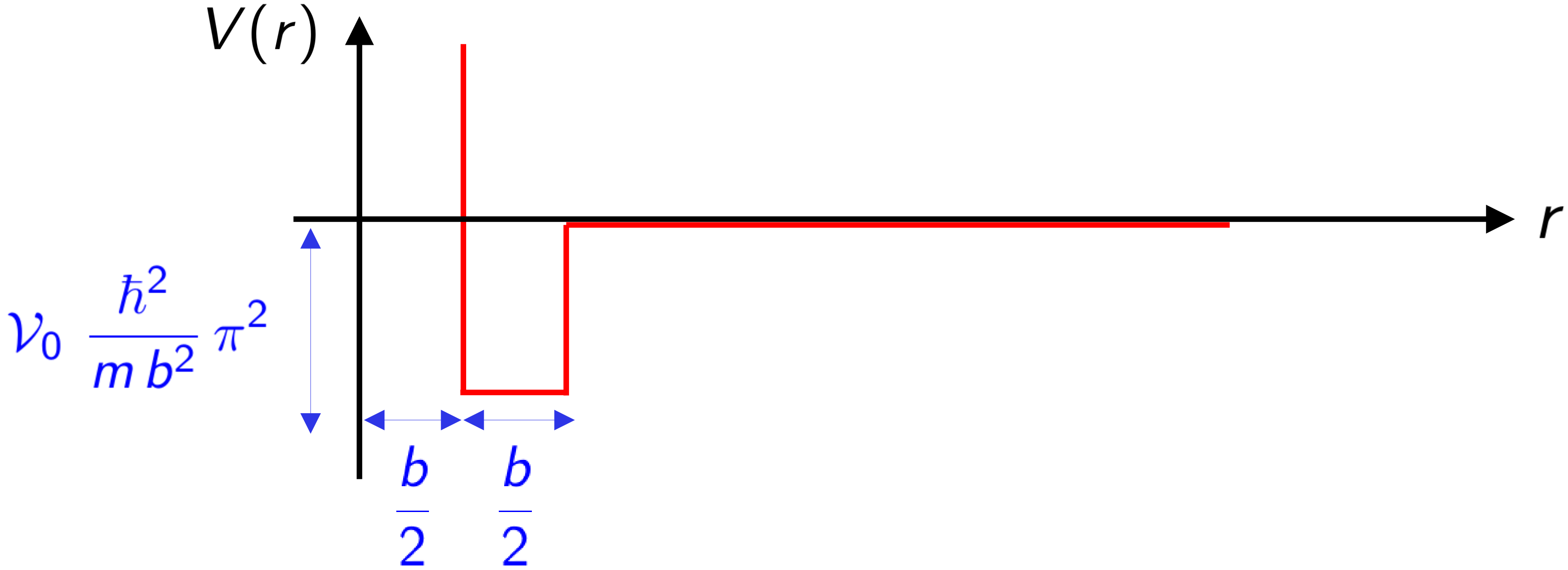

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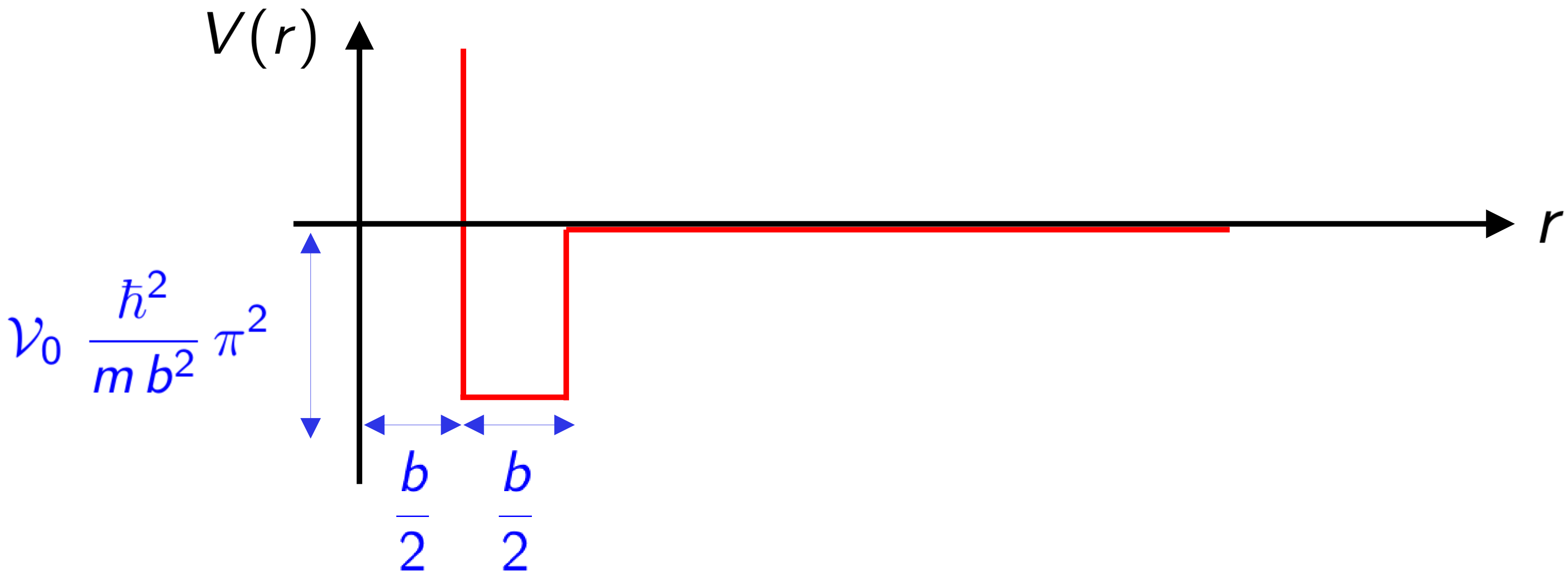

Dimers (= bound states): $E < 0$ $\psi(\vec{r})$ normalizable

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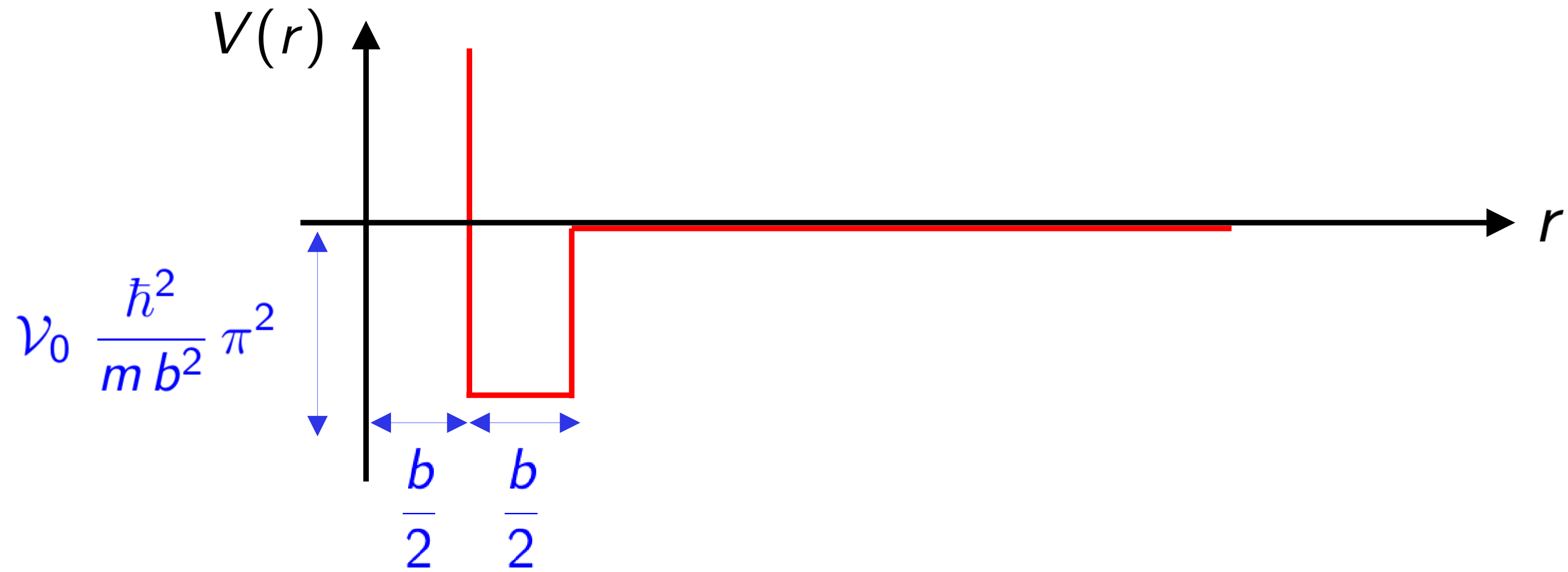

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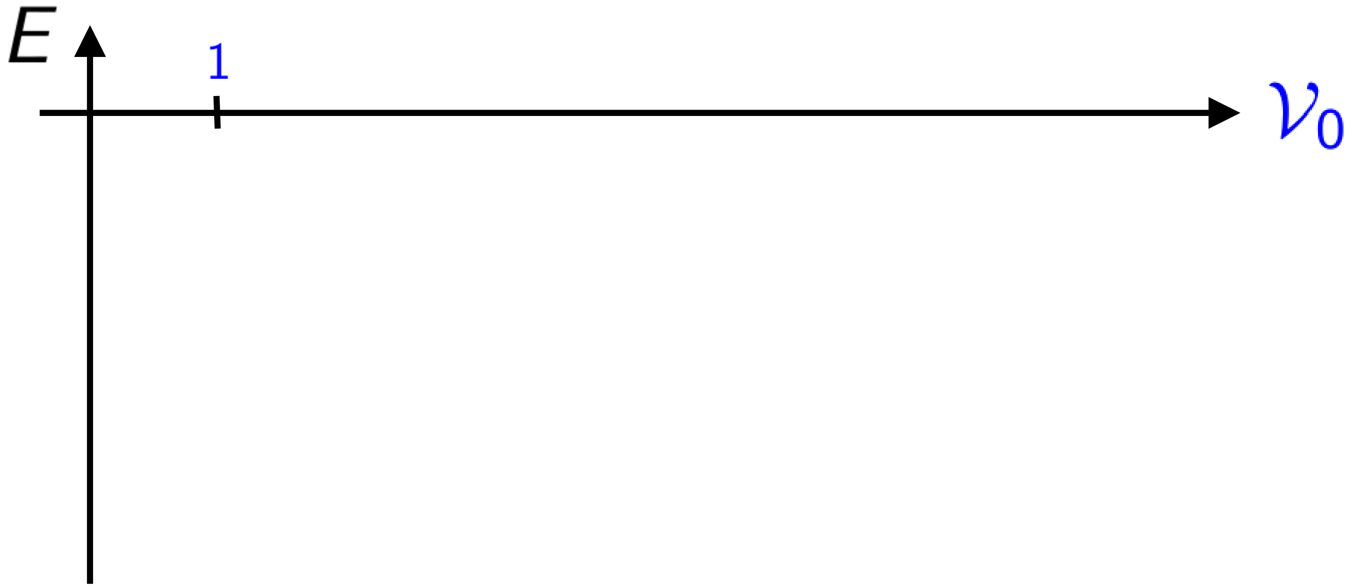

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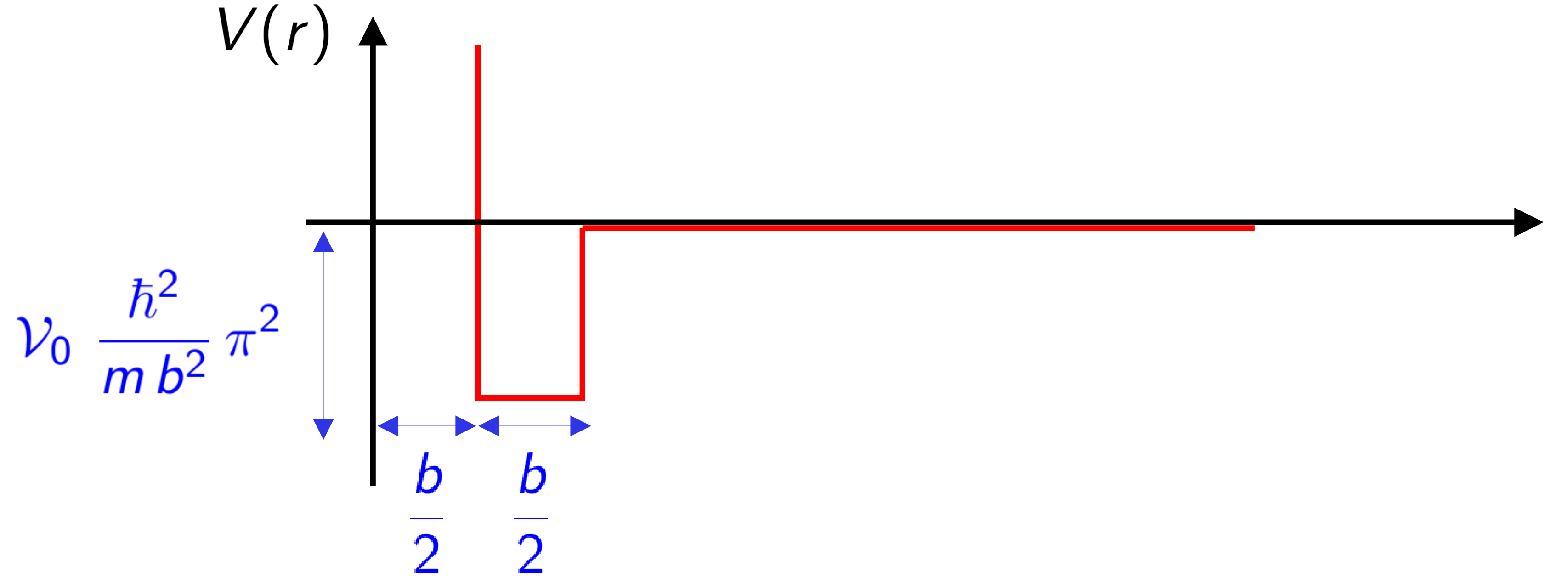

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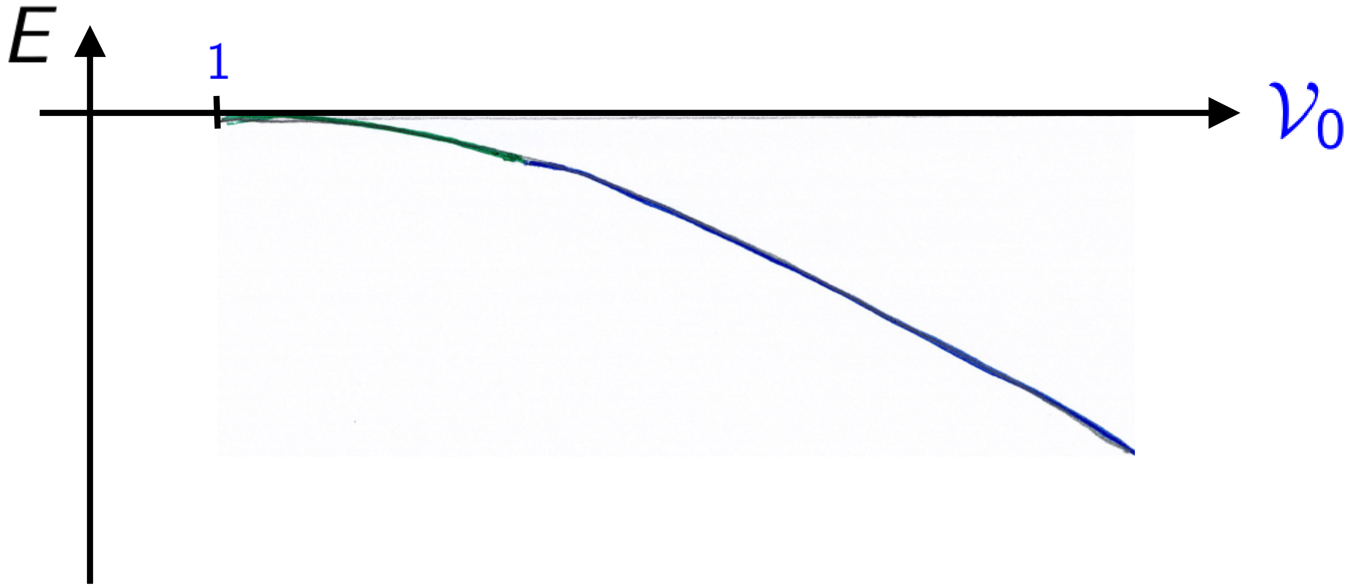

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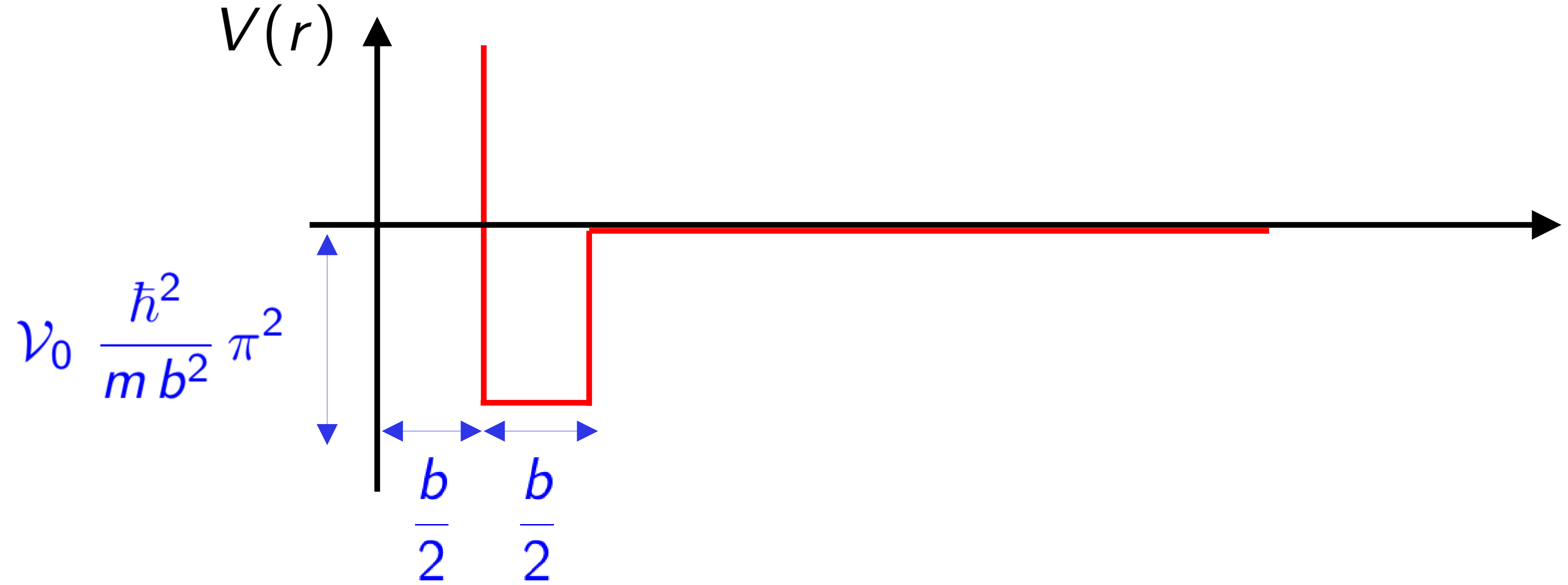

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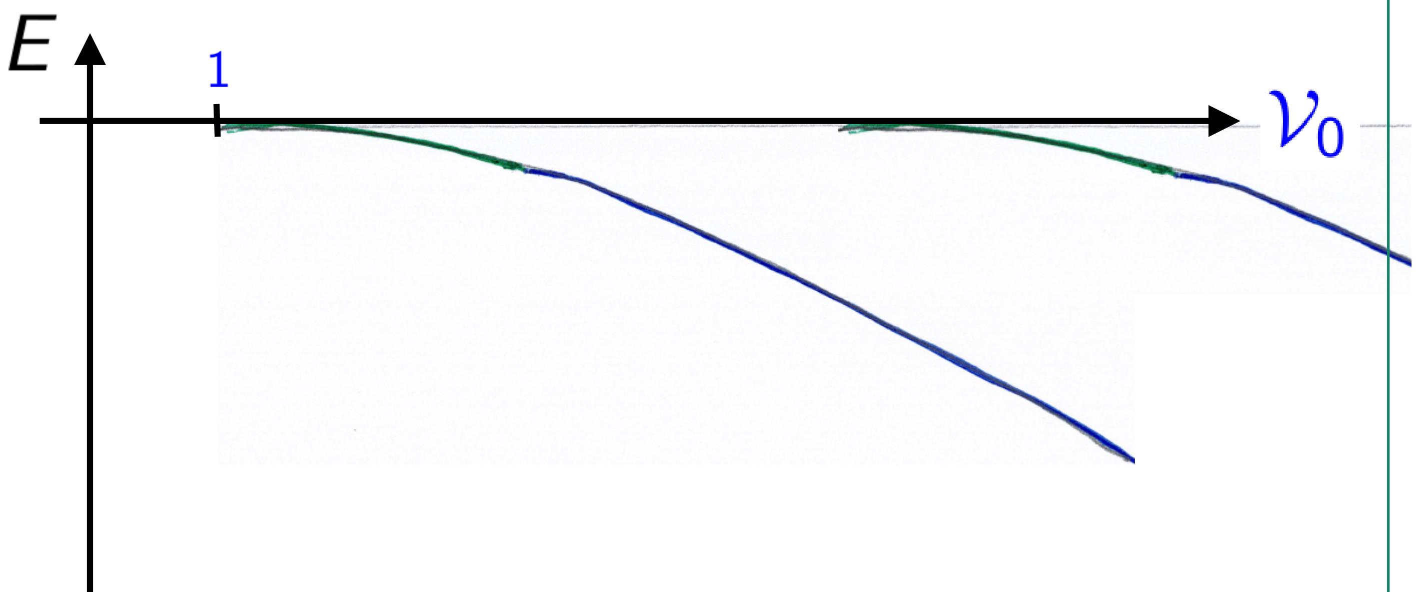

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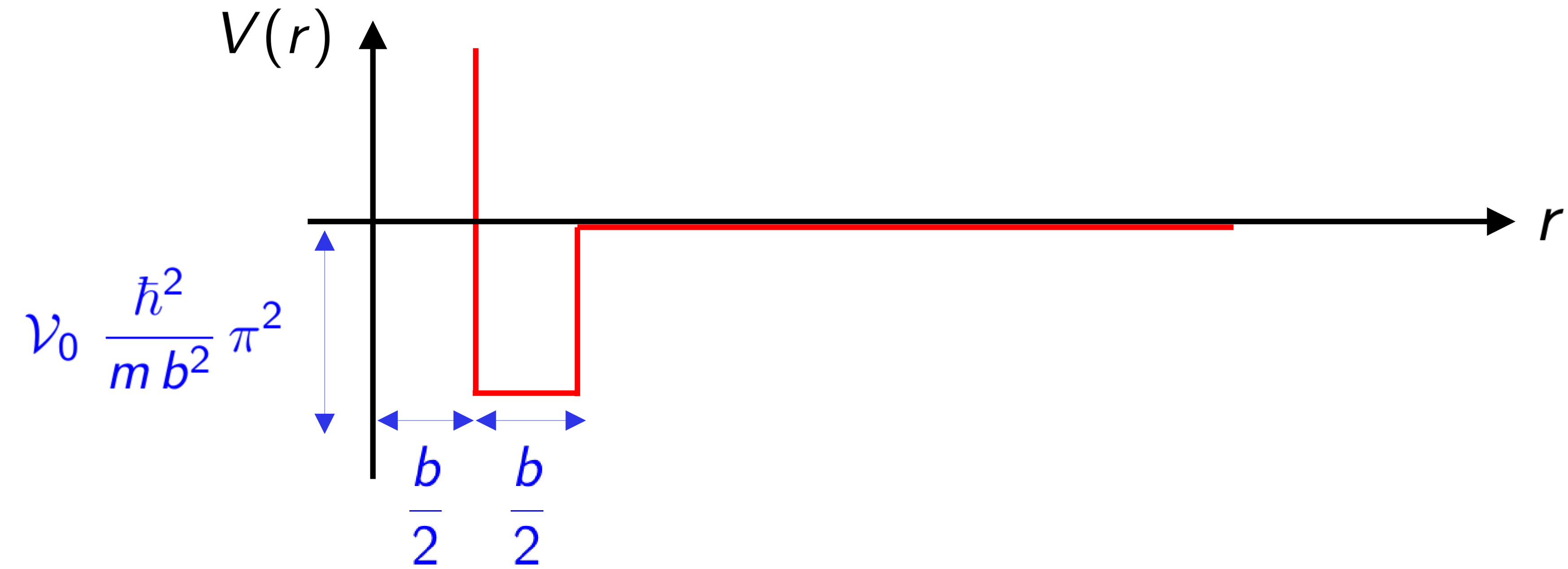

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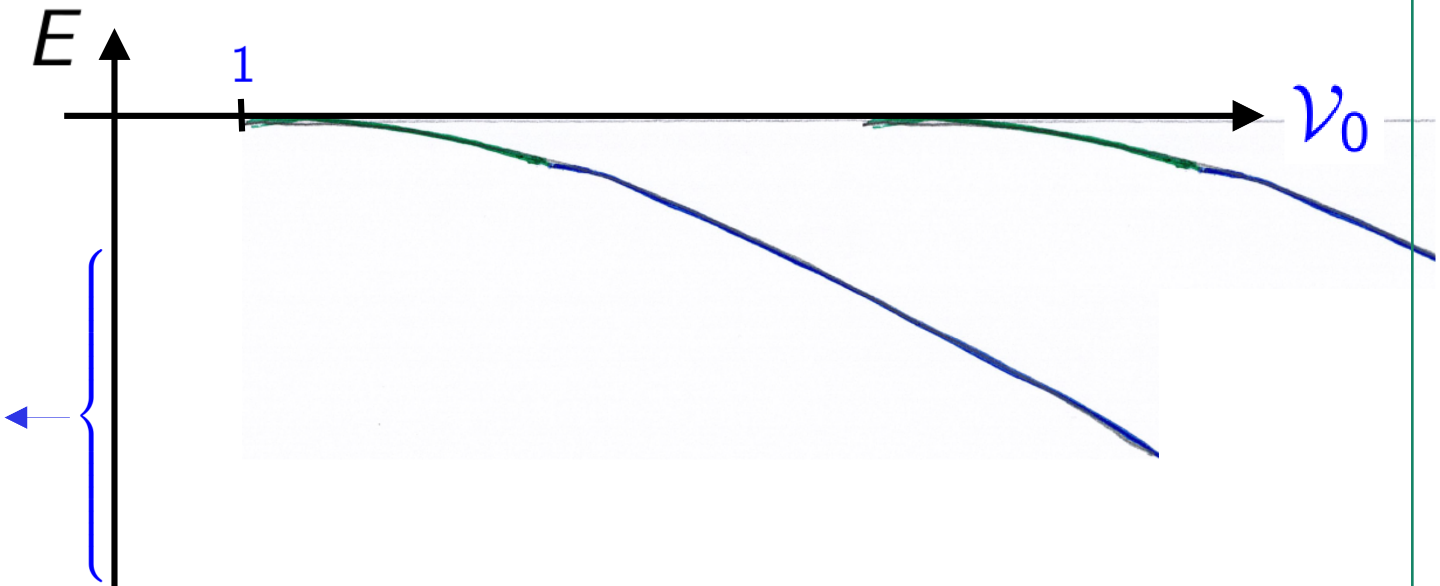

deeply bound: $|E| \sim \frac{\hbar^2}{m\,b^2}$

size $\sim b$

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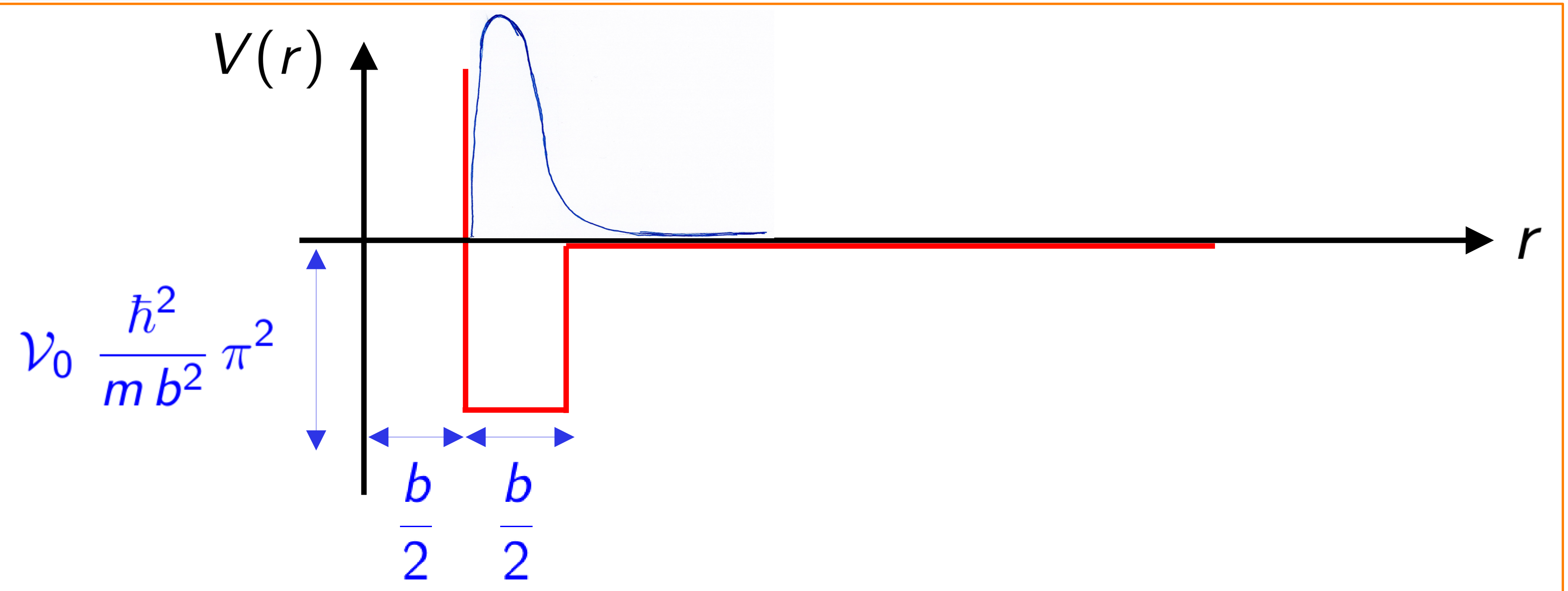

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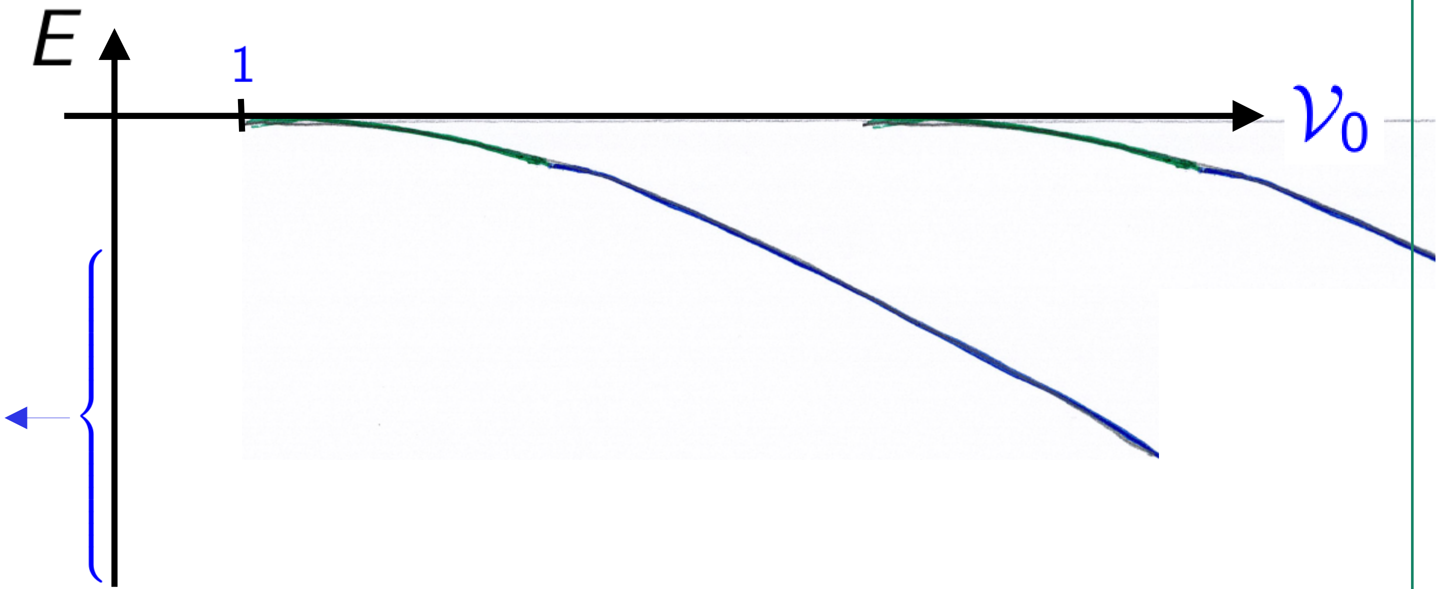

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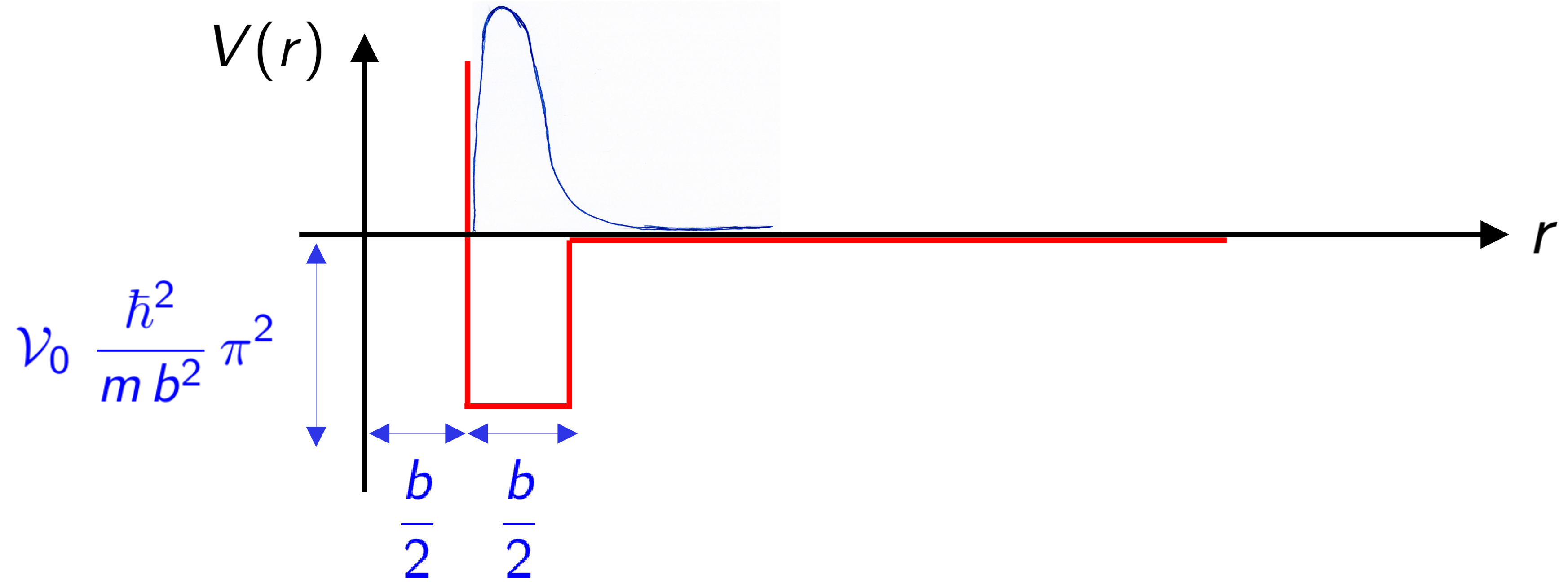

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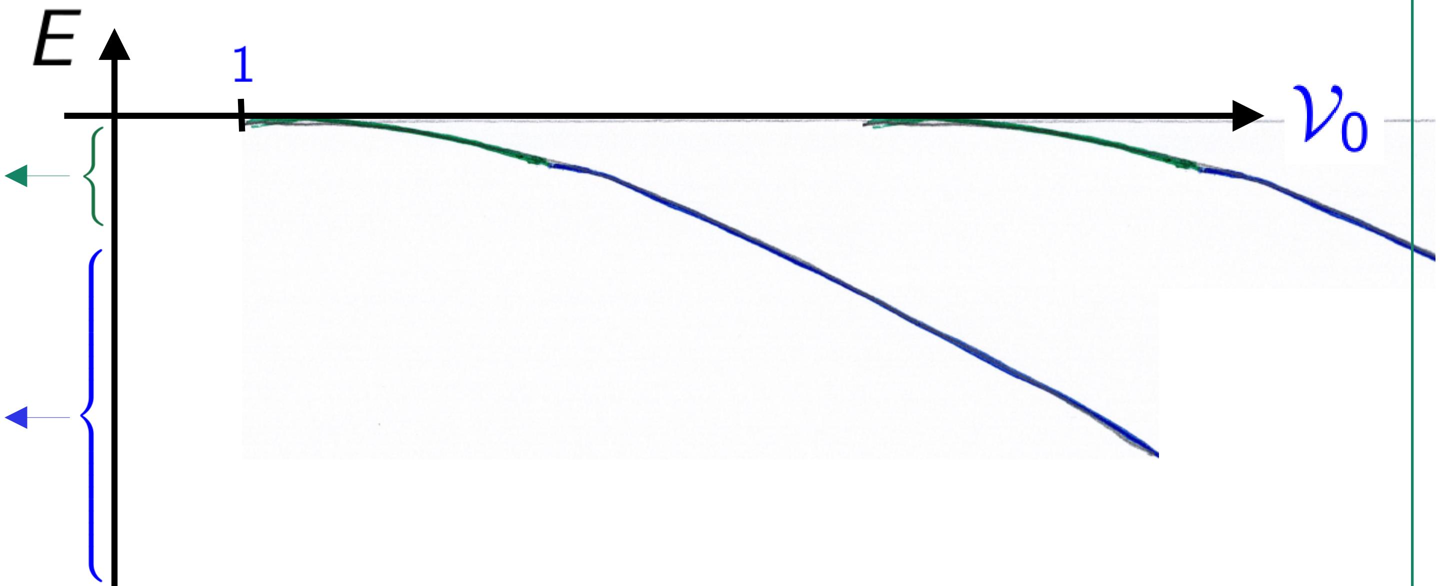

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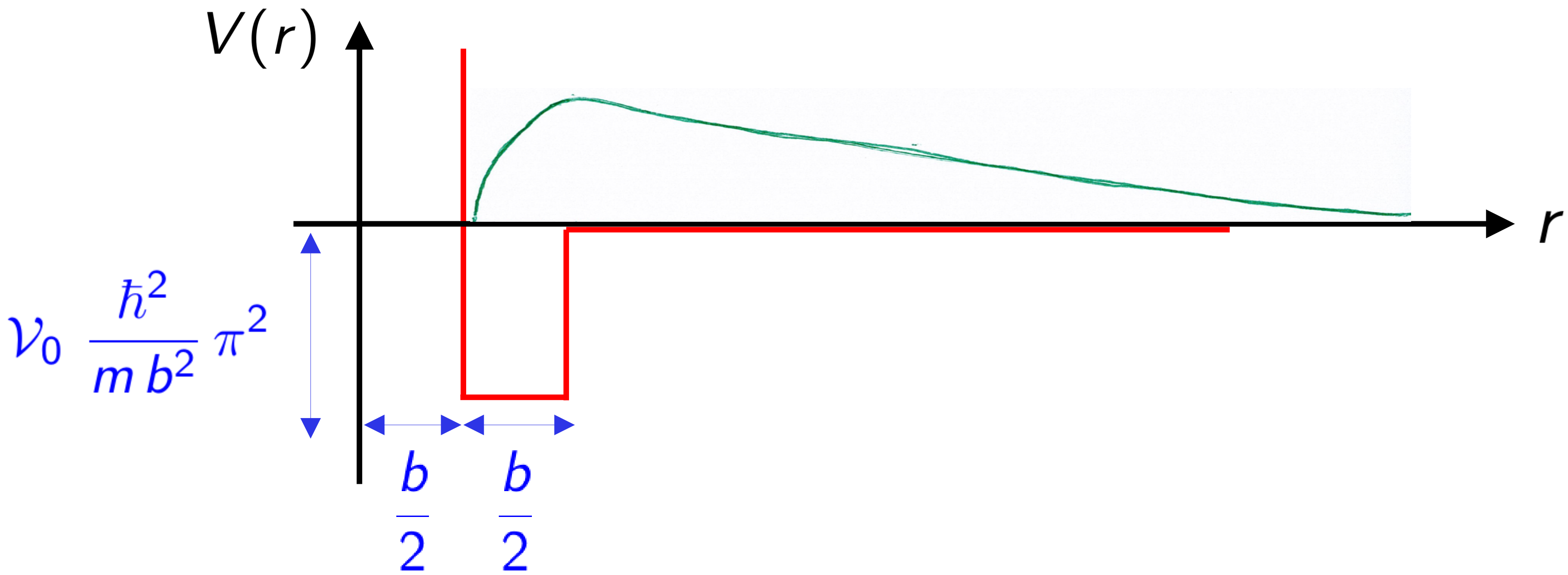

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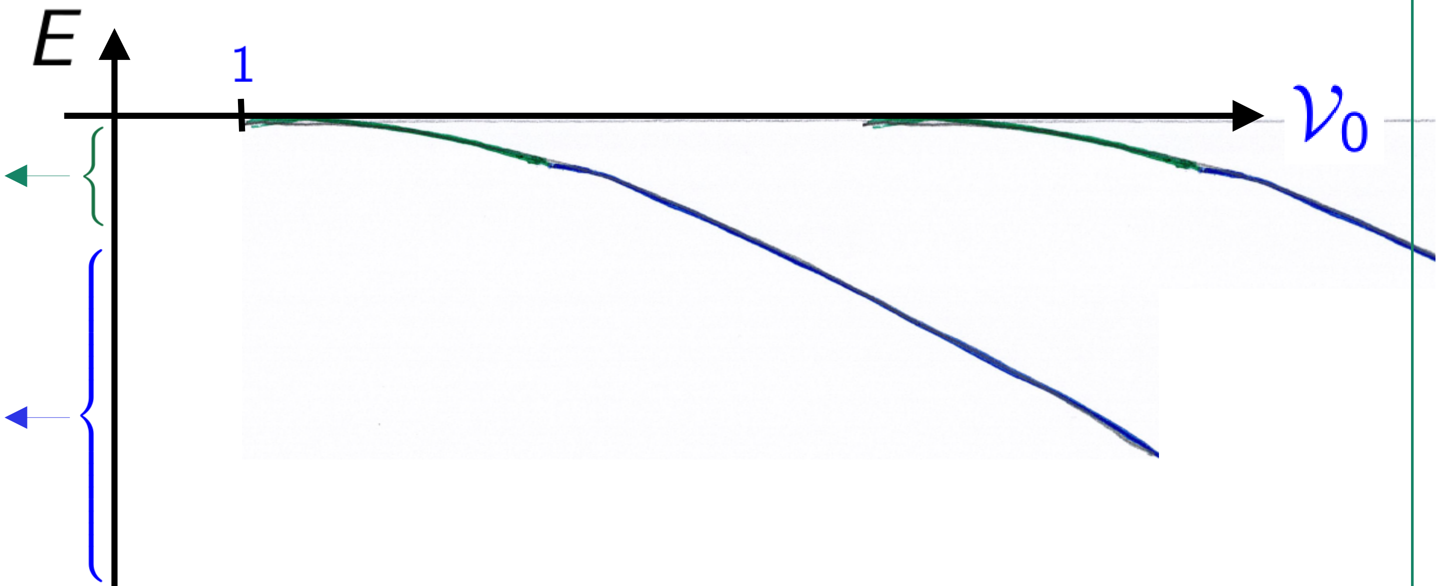

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Scattering states: $E > 0$. $\qquad E = \frac{\hbar^2 k^2}{m}$

$$\psi(\vec{r}) \underset{r\to\infty}{\simeq} e^{i\,\vec{k}\cdot\vec{r}} + f_k(\hat{r})\,\frac{e^{ikr}}{r} \qquad \left[\,\hat{r} := \frac{\vec{r}}{r}\,\right]$$

$$a := -\lim_{k\to 0} f_k(\hat{r})$$ scattering length

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$$\boxed{E \sim -\frac{\hbar^2}{ma^2}} \qquad \boxed{\psi(r) \longrightarrow \mathcal{N}\,\frac{e^{-r/a}}{r}}$$

for any shape of $V(r)$

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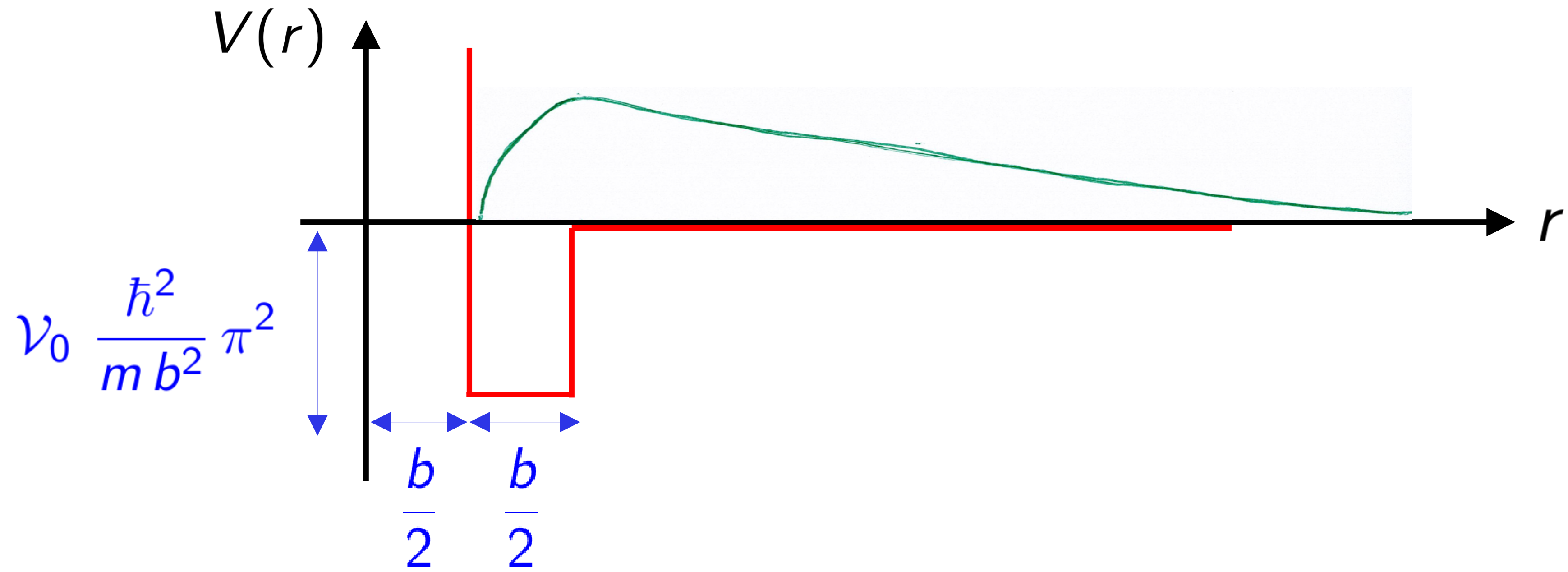

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Zero-range limit: $b \to 0$, a fixed.

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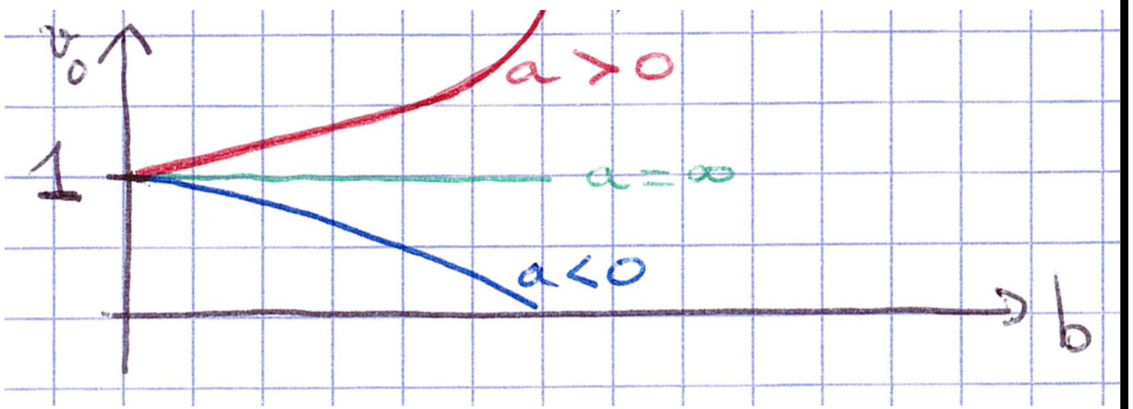

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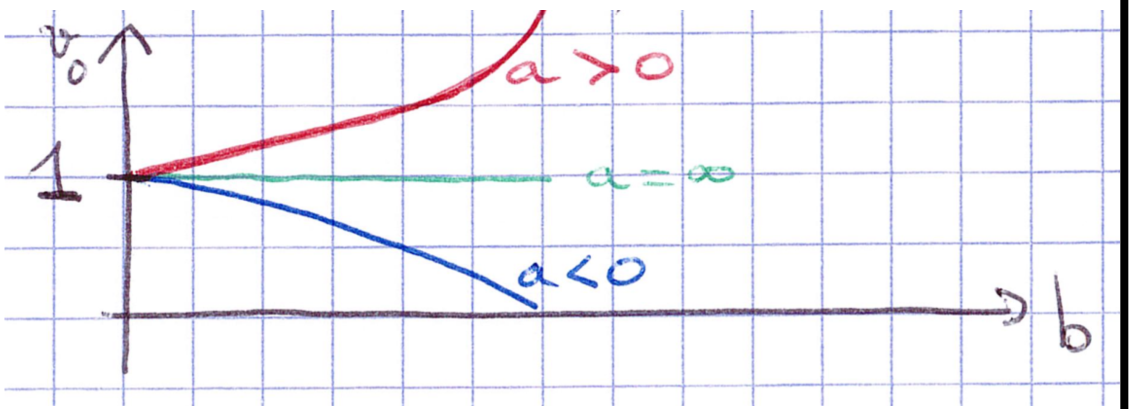

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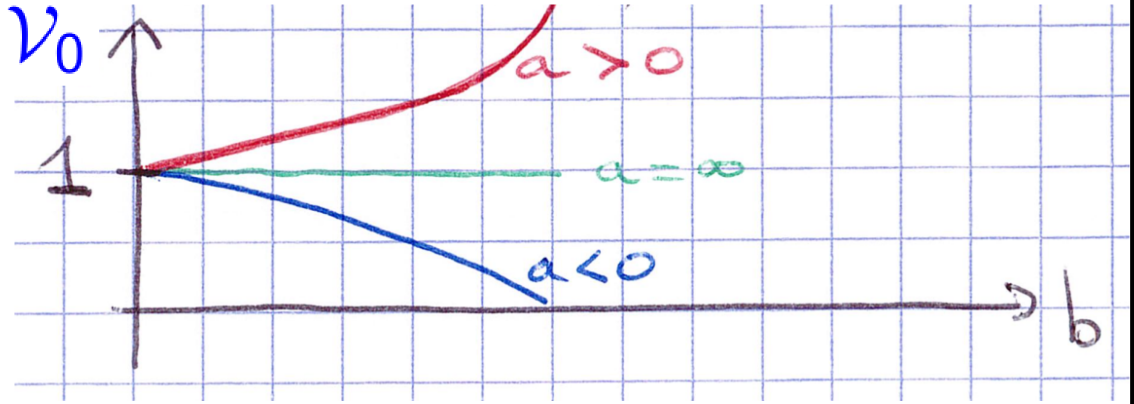

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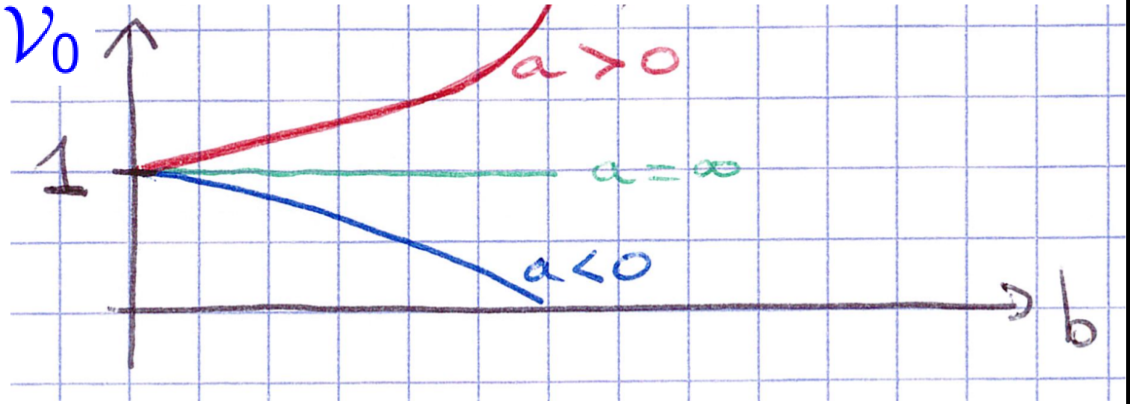

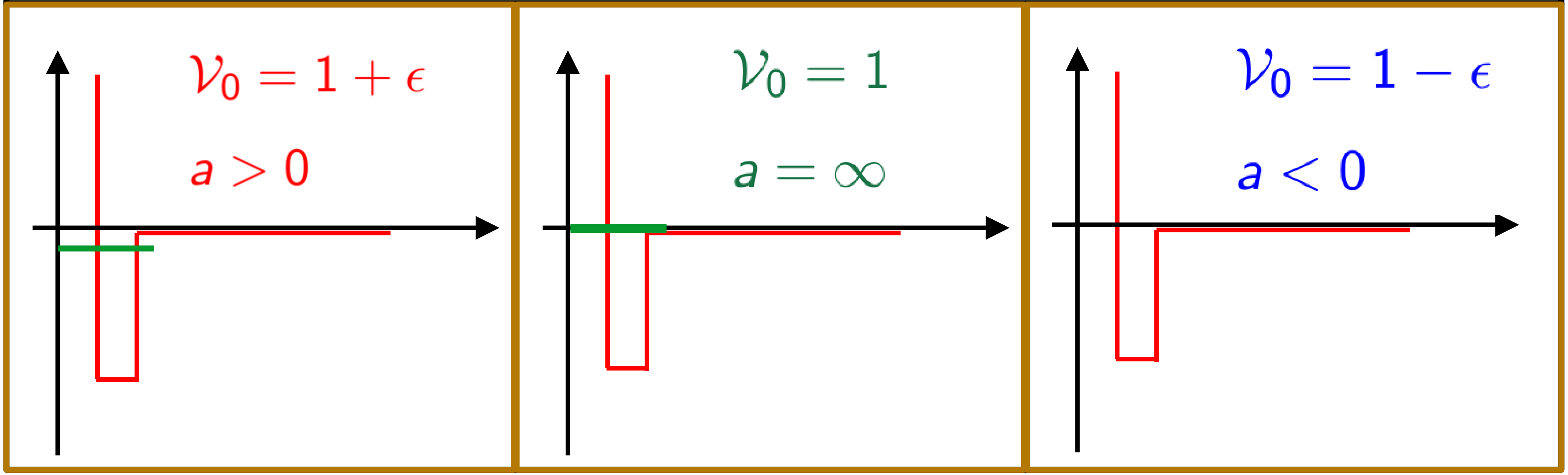

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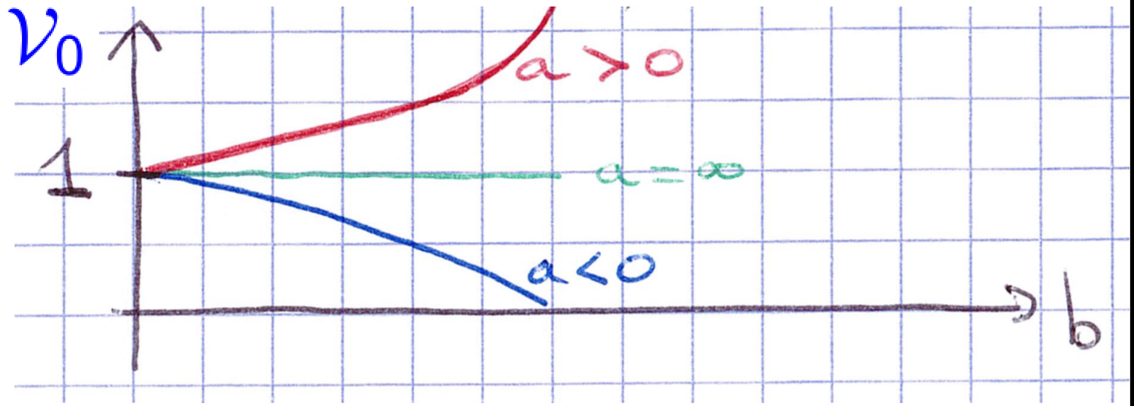

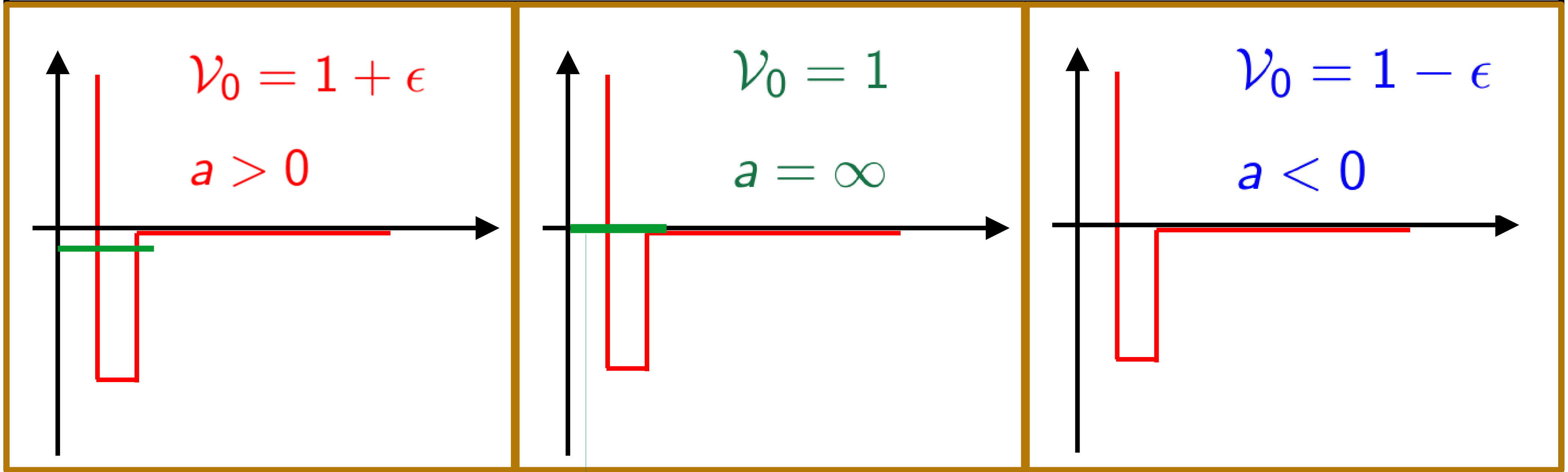

Zero-energy state $\phi_0(r)$

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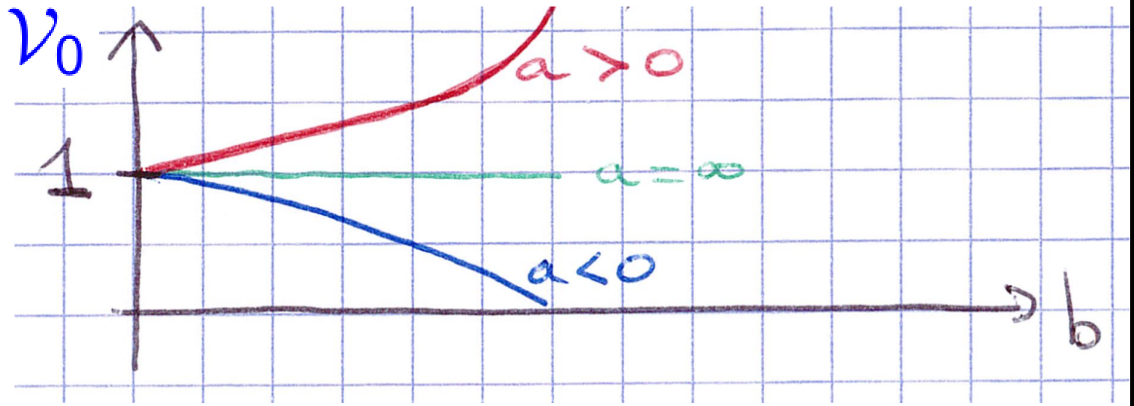

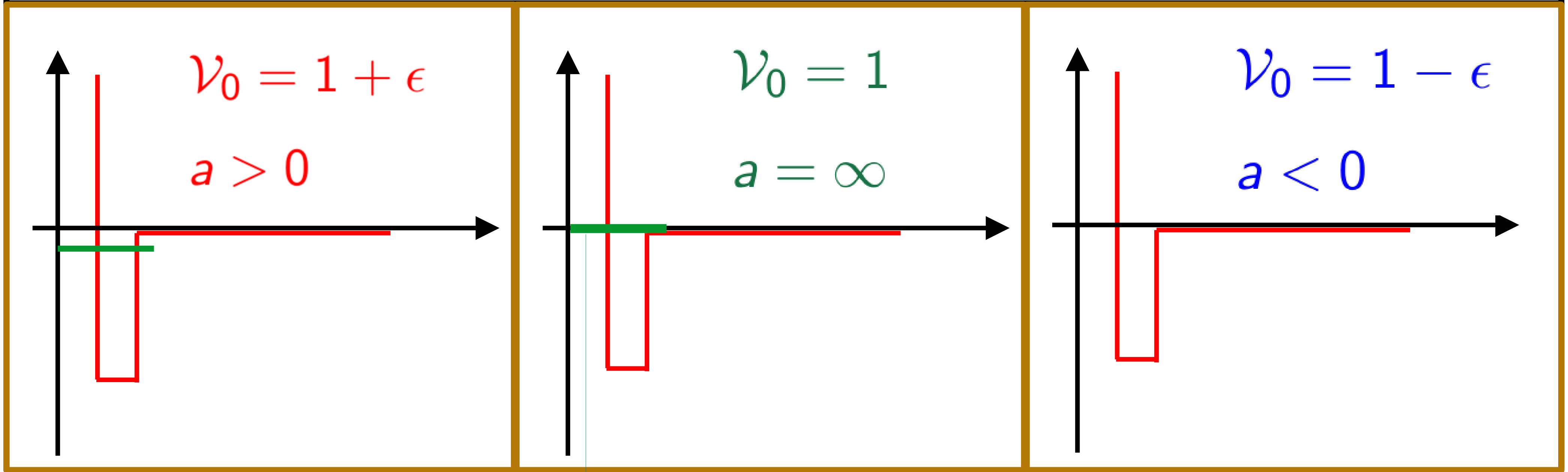

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$$-\frac{\hbar^2}{m}\Delta\phi_0 + V(r)\,\phi_0 = 0$$

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- $r \neq 0$: $\quad -\frac{\hbar^2}{m}\Delta\psi(\vec{r}) = E\,\psi(\vec{r})$
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"Contact Condition" (CC)

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$a = \infty$: scattering cross-section $\boxed{\sigma_k = \frac{4\pi}{k^2}}$ "unitary limit" $\left(\begin{array}{c}\text{max value allowed}\\ \text{by optical theorem}\end{array}\right)$

Part 3 : N-body problem

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Zero-range limit & ZRM

Take $N = 4$, $N_\uparrow = N_\downarrow = 2$

(just to alleviate notations)

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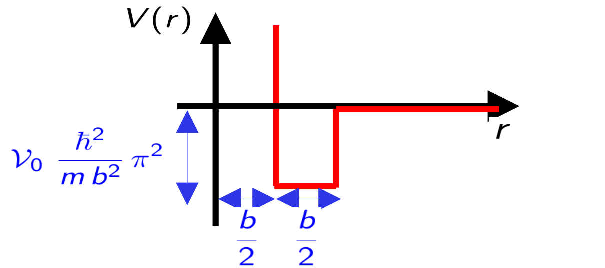

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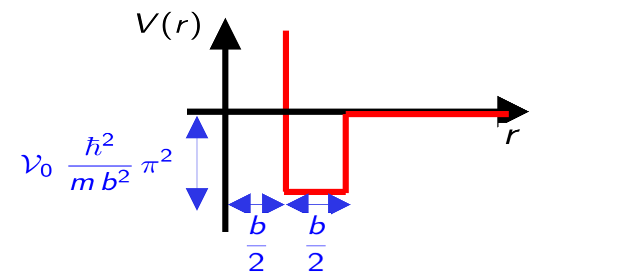

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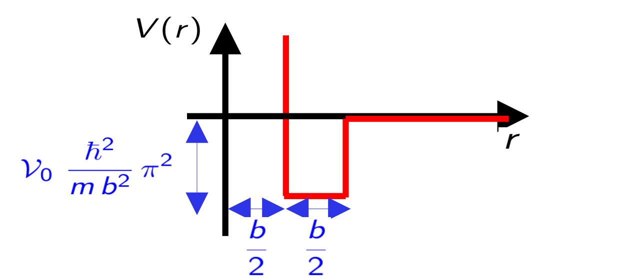

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Universality: This holds for any shape of $V(r)$
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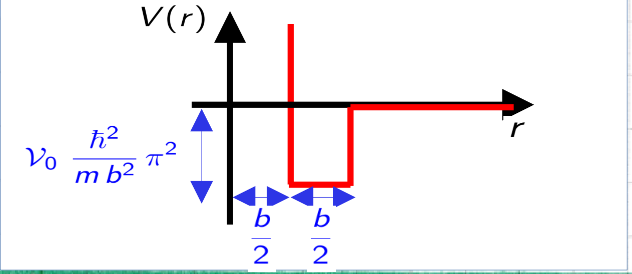

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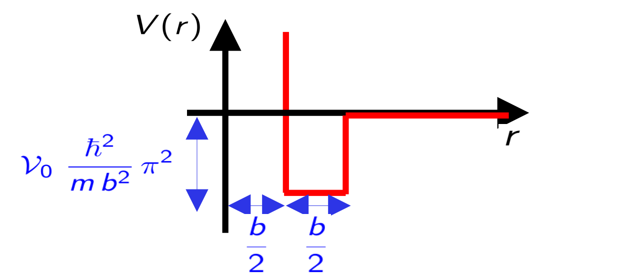

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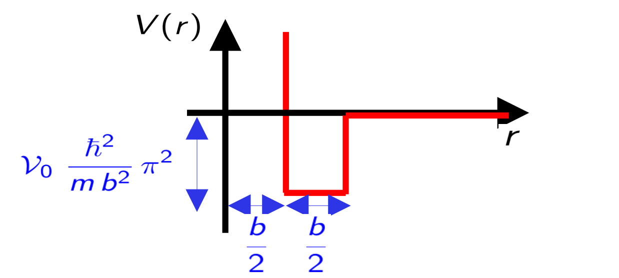

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$\Longrightarrow \quad -\frac{\hbar^2}{m}\Delta_{\vec{r}}\Psi + V(r)\Psi \simeq 0 \quad \Longrightarrow \quad \Psi(\vec{r}_1, \dots, \vec{r}_4) \underset{r \lesssim b}{\simeq} \underbrace{\phi_0(r)}_{\underset{r > b}{\simeq} \frac{1}{r} - \frac{1}{a}} \times A$

$\Longrightarrow$ ZRM

Homogeneous gas

$N_\uparrow = N_\downarrow$ ("unpolarized" / "balanced")

Volume $\mathcal{V}$. Density $n = \dfrac{N}{\mathcal{V}}$.

$\mathcal{V} \to \infty\,, \; N \to \infty$

n fixed

interparticle distance $\sim d = n^{-1/3}$

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$b \ll d$

$b \ll \lambda_T \sim \dfrac{\hbar}{\sqrt{m\,k_B\,T}}$

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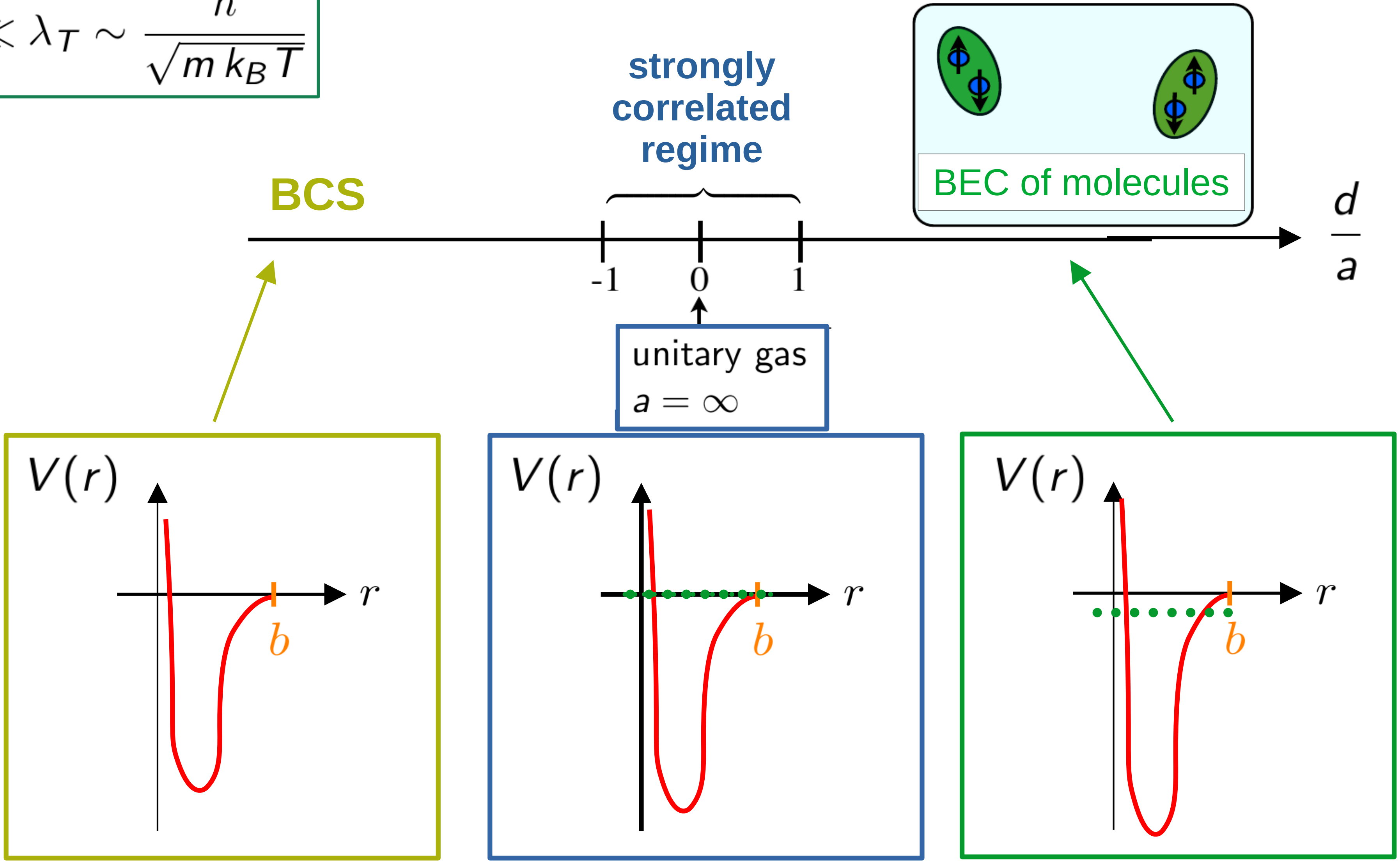

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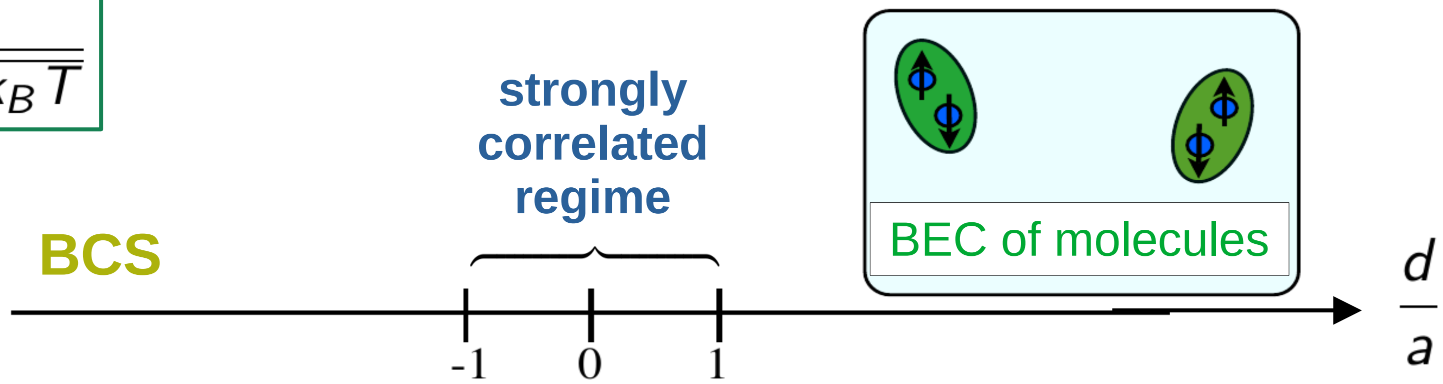

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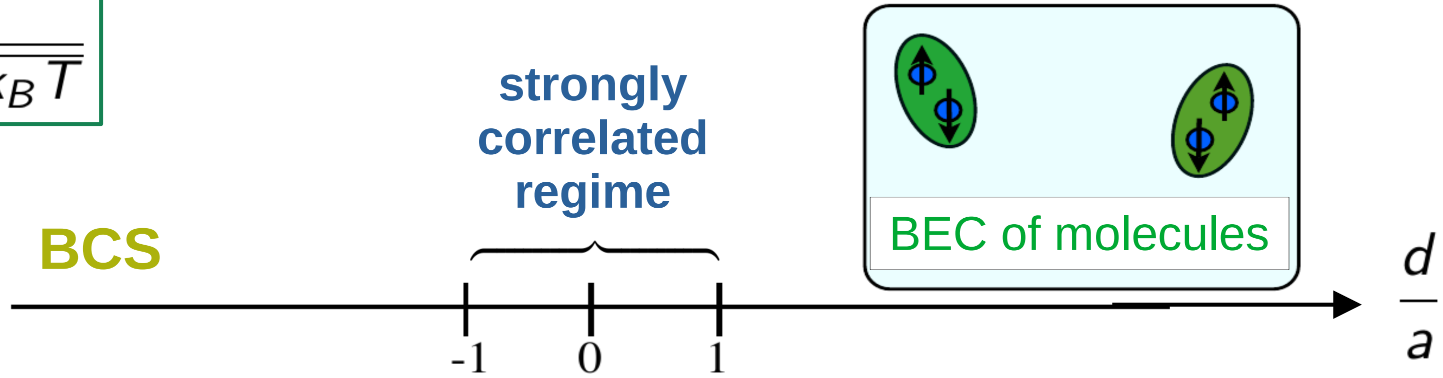


$\boxed{T = 0}$

Ideal Fermi gas

$$n_{\vec{k},\uparrow} = n_{\vec{k},\downarrow} = \begin{cases} 1, & k < k_F \\ 0, & k > k_F \end{cases}$$

Homogeneous gas

$N_\uparrow = N_\downarrow$ ("unpolarized" / "balanced")

Volume $\mathcal{V}$. Density $n = \dfrac{N}{\mathcal{V}}$.

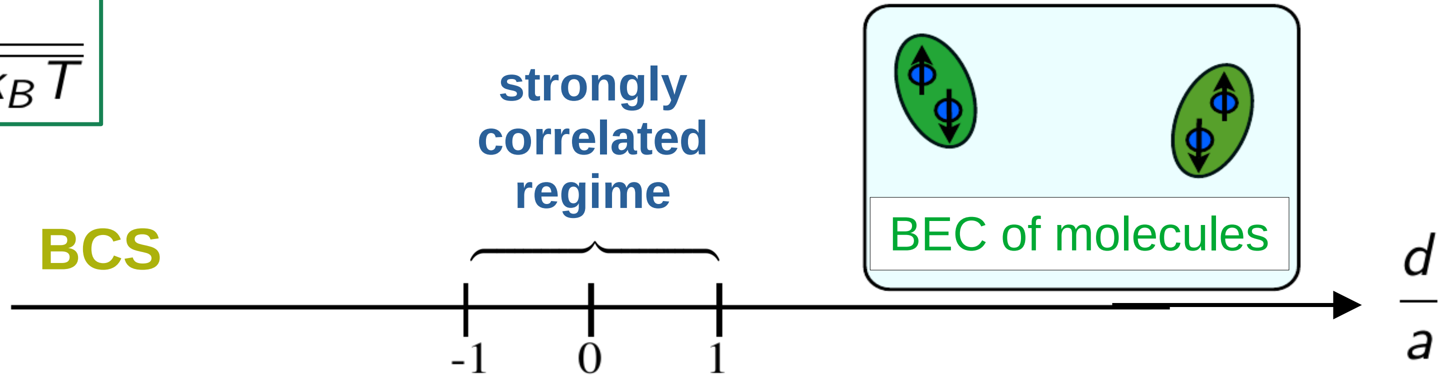
$\mathcal{V} \to \infty$, $N \to \infty$
n fixed

interparticle distance $\sim d = n^{-1/3}$

Zero-range limit:

$b \ll d$

$b \ll \lambda_T \sim \dfrac{\hbar}{\sqrt{m\, k_B T}}$



$$k_F = (3\pi^2 n)^{1/3}$$

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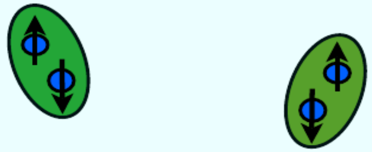
strongly correlated regime

BCS



BEC of molecules

-1 0 1 $\quad \dfrac{d}{a}$

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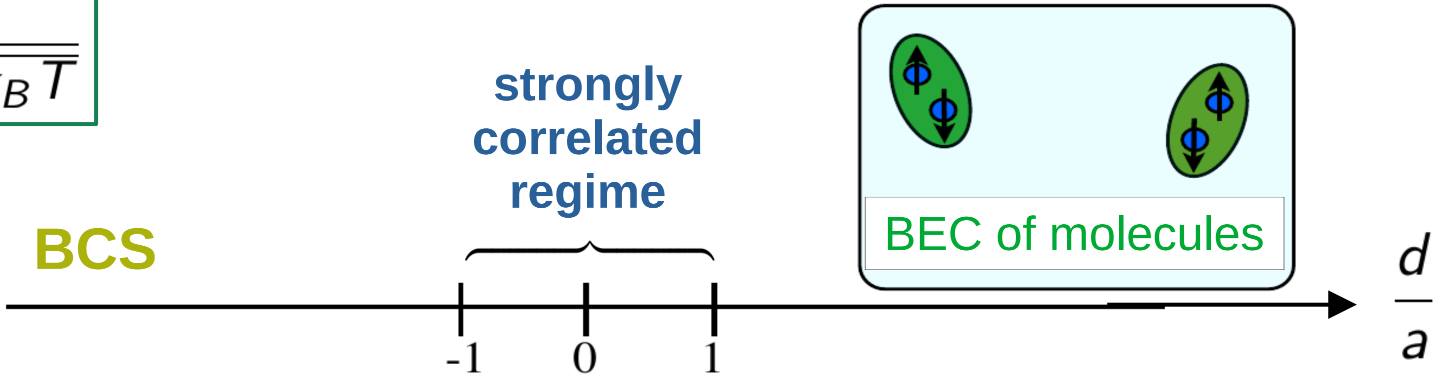

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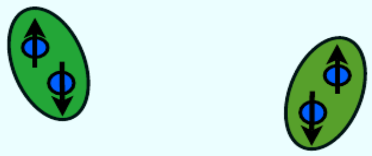

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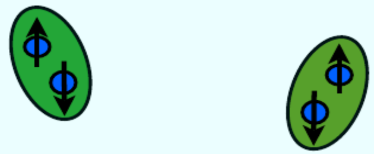

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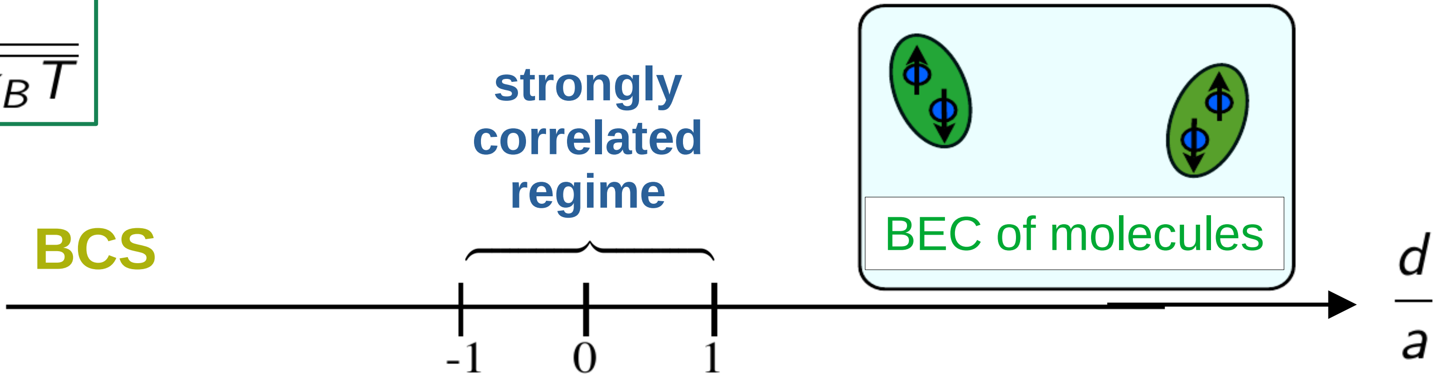

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$\xi \simeq 0.37$ [QMC; Experiments]

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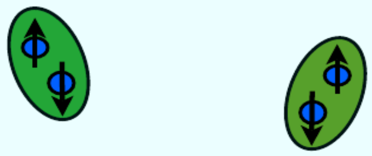

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For typical collisions in the gas: $k \approx k_F \Rightarrow \sigma_k \approx \dfrac{4\pi}{k_F^2} \approx 1.3\, d^2$

$\boxed{\sigma_k \sim d^2}$ like in a liquid

Chapter 2

Solution of the unitary 3-body problem

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- V. Efimov, Yad. Fiz. **12**, 1080 (1970) [Sov. J. Nucl. Phys. **12**, 589 (1971)]; Nucl. Phys. **A210**, 157 (1973)
- S. Tan, arXiv:cond-mat/0412764
- FW & Y. Castin, PRL **97**, 150401 (2006)

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$N_\uparrow = 2\ ,\ N_\downarrow = 1$

$$\boxed{\Psi(\vec{r_1}, \vec{r_2}, \vec{r_3}) \;=\; -\Psi(\vec{r_2}, \vec{r_1}, \vec{r_3})}$$

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ZRM

$\boxed{r := r_{13} \to 0}$

fixed $\vec{c} := \dfrac{\vec{r_1} + \vec{r_3}}{2}$

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$$\Psi(\vec{r_1}, \vec{r_2}, \vec{r_3}) \underset{r\to 0}{=} \left(\frac{1}{r} - \underbrace{\frac{1}{a}}_{=0} \right) \mathcal{A}(\vec{c}\,; \vec{r_2}) + O(r)$$

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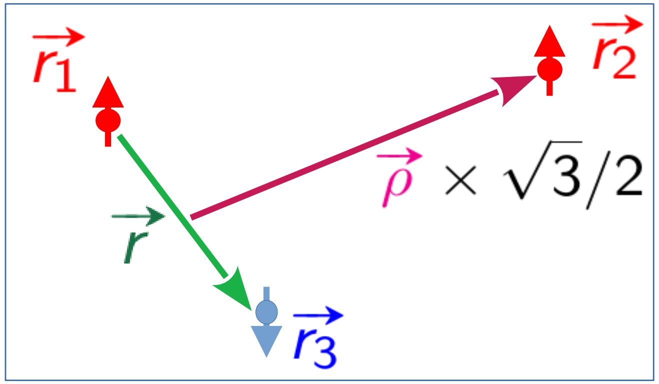

Chapter 2

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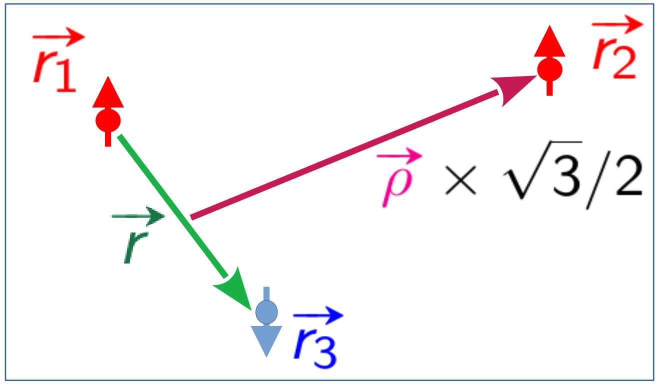

$$(\hat{P}_{12}\, \psi)(\vec{r}, \vec{\rho}) = -\psi(\vec{r}, \vec{\rho})$$

where $\hat{P}_{12}$ exchanges particles 1 and 2

Chapter 2 — Solution of the unitary 3-body problem

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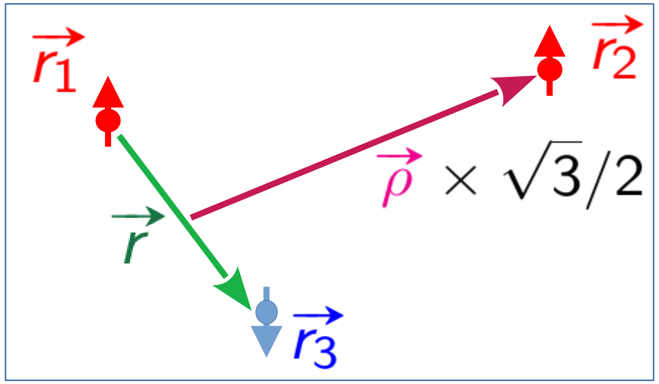

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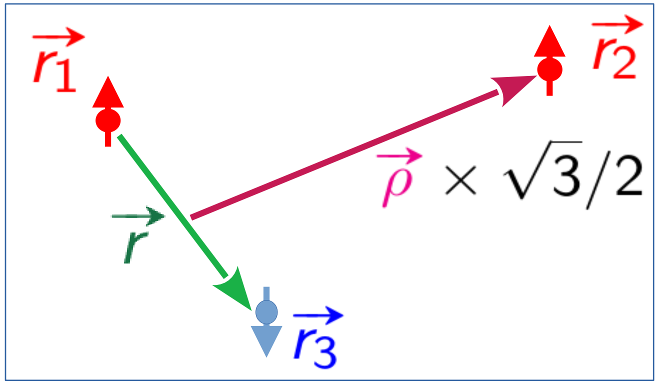

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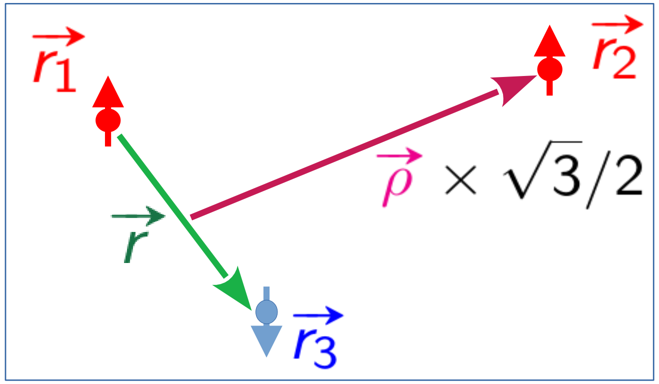

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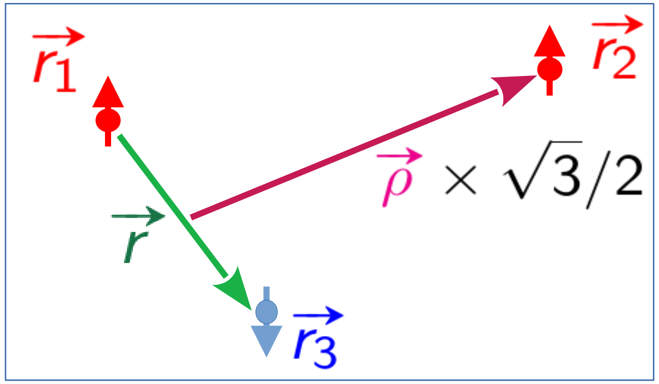

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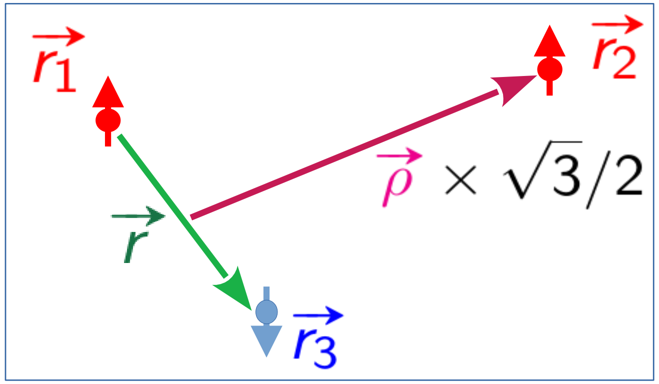

$$(\hat{P}_{12}\,\psi)(\vec{r},\vec{\rho}) = -\psi(\vec{r},\vec{\rho})$$

where $\hat{P}_{12}$ exchanges particles 1 and 2

$$\vec{C} = \frac{\vec{r_1} + \vec{r_2} + \vec{r_3}}{3} \qquad \Psi(\vec{r_1},\vec{r_2},\vec{r_3}) = \psi(\vec{r},\vec{\rho})\ \psi_{\mathrm{CM}}(\vec{C}\,)$$

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Jacobi coordinates

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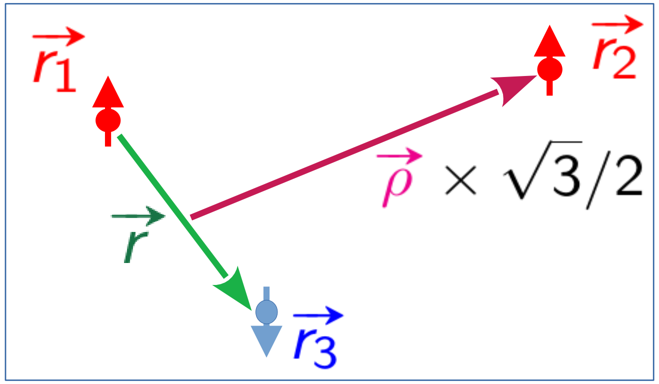

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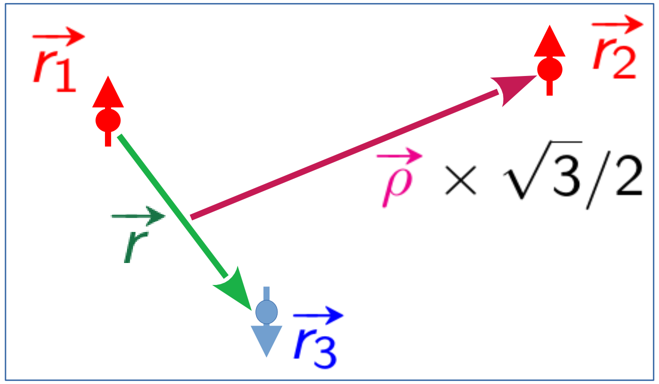

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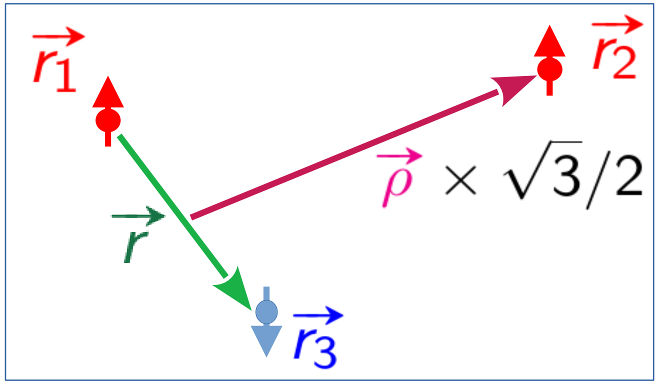

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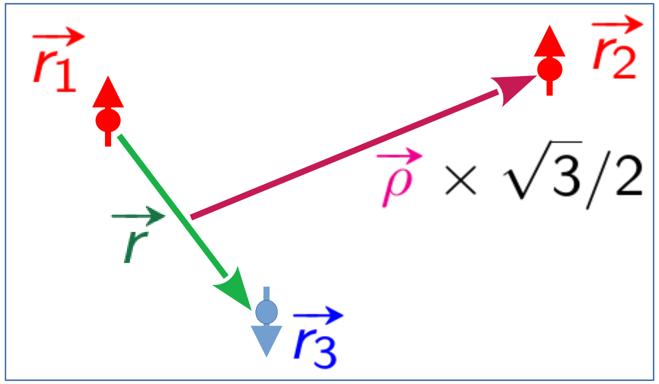

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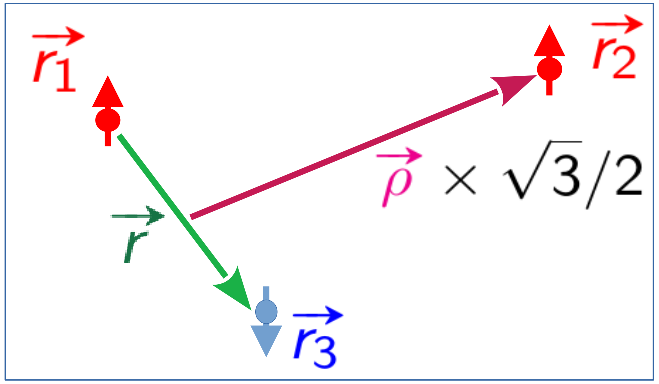

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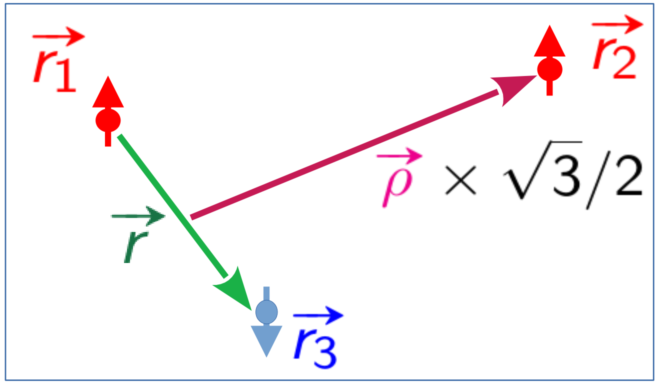

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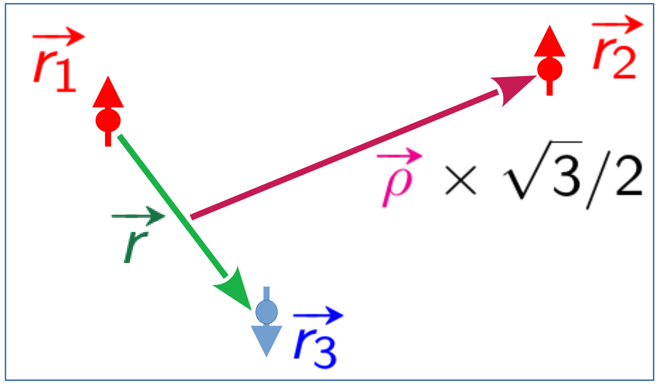

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$$\varphi\left(\frac{\pi}{2}\right) = 0 \;\Leftarrow\; \left(\begin{array}{c}\psi \text{ finite} \\ \text{for } \rho\to 0\end{array}\right)$$

$$\psi(\vec{r},\vec{\rho}) \underset{r\to 0}{=} \frac{1}{r}\, A(\vec{\rho}) \;+\; O(r) \qquad \text{(CC)}$$

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$$\varphi'(0) - \frac{4}{\sqrt{3}}\,(-1)^\ell\,\varphi\left(\frac{\pi}{3}\right) = 0 \qquad \varphi\left(\frac{\pi}{2}\right) = 0$$

$$\text{(Schrö)}\begin{cases} -\varphi''(\alpha) + \dfrac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha) \\ -\dfrac{\hbar^2}{m}\left[F''(R) + \dfrac{1}{R}\,F'(R)\right] + \left(\dfrac{\hbar^2\,s^2}{m\,R^2} + \dfrac{m\,\omega^2}{4}R^2\right) F(R) = E\,F(R) \end{cases}$$

$$\varphi'(0) - \frac{4}{\sqrt{3}} (-1)^{\ell} \varphi\left(\frac{\pi}{3}\right) = 0$$

$$\varphi\left(\frac{\pi}{2}\right) = 0$$

$$-\varphi''(\alpha) + \frac{\ell(\ell+1)}{\cos^2 \alpha} \varphi(\alpha) = s^2 \varphi(\alpha)$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R} F'(R)\right] + \left(\frac{\hbar^2 s^2}{m R^2} + \frac{m \omega^2}{4} R^2\right) F(R) = E F(R)$$

$$\varphi'(0) - \frac{4}{\sqrt{3}}(-1)^{\ell}\,\varphi\left(\frac{\pi}{3}\right) = 0$$

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$$\boxed{s} \longleftarrow \left\{ \begin{array}{l} \boxed{\varphi'(0) - \frac{4}{\sqrt{3}} (-1)^{\ell} \varphi\left(\frac{\pi}{3}\right) = 0 \qquad \varphi\left(\frac{\pi}{2}\right) = 0 \\ -\varphi''(\alpha) + \frac{\ell(\ell+1)}{\cos^2 \alpha} \varphi(\alpha) = s^2 \varphi(\alpha)} \end{array} \right.$$

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$$\boxed{\ell = 0} \qquad \boxed{\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]}$$

$$\Rightarrow \quad \boxed{s\,\cos\left(\frac{s\,\pi}{2}\right) + \frac{4}{\sqrt{3}}\,\sin\left(\frac{s\,\pi}{6}\right) = 0}$$

smallest solution: $\boxed{s_{\ell=0} = 2.166221977\ldots}$

$$\boxed{s} \longleftarrow \left\{ \begin{array}{l} \varphi'(0) - \dfrac{4}{\sqrt{3}}\,(-1)^{\ell}\,\varphi\left(\dfrac{\pi}{3}\right) = 0 \qquad \varphi\left(\dfrac{\pi}{2}\right) = 0 \\[2ex] -\varphi''(\alpha) + \dfrac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha) \end{array} \right.$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}\,F'(R)\right] + \left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)F(R) = E\,F(R)$$

$\boxed{\ell = 0}$ $\boxed{\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]}$

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$\boxed{\ell = 1}$ $\boxed{s_{\ell=1} = 1.772724267\ldots}$

$$\boxed{s} \longleftarrow \left\{ \begin{array}{ll} \varphi'(0) - \frac{4}{\sqrt{3}}\,(-1)^{\ell}\,\varphi\left(\frac{\pi}{3}\right) = 0 & \varphi\left(\frac{\pi}{2}\right) = 0 \\ -\varphi''(\alpha) + \frac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha) & \end{array} \right.$$

$$-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R}\,F'(R)\right] + \left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right)F(R) = E\,F(R)$$

$\boxed{\ell = 0}$ $\boxed{\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]}$

$$\Rightarrow \boxed{s \cos\left(\frac{s\,\pi}{2}\right) + \frac{4}{\sqrt{3}} \sin\left(\frac{s\,\pi}{6}\right) = 0}$$

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$$\boxed{s} \longleftarrow \begin{cases} \boxed{\varphi'(0) - \frac{4}{\sqrt{3}} (-1)^{\ell} \varphi\left(\frac{\pi}{3}\right) = 0} \quad \boxed{\varphi\left(\frac{\pi}{2}\right) = 0} \\ \boxed{-\varphi''(\alpha) + \frac{\ell(\ell+1)}{\cos^2\alpha} \varphi(\alpha) = s^2 \varphi(\alpha)} \end{cases}$$

1-body Schrö. eq.
fictitious particle
in 2D

$$\boxed{-\frac{\hbar^2}{m}\left[F''(R) + \frac{1}{R} F'(R)\right] + \left(\frac{\hbar^2 s^2}{m R^2} + \frac{m\,\omega^2}{4} R^2\right) F(R) = E\,F(R)}$$

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$\boxed{\ell = 1}$ $\boxed{s_{\ell=1} = 1.772724267\ldots}$

$$\Rightarrow \boxed{s\,\cos\left(\frac{s\,\pi}{2}\right) + \frac{4}{\sqrt{3}}\,\sin\left(\frac{s\,\pi}{6}\right) = 0}$$

smallest solution: $\boxed{s_{\ell=0} = 2.166221977\ldots}$

$$\boxed{s} \longleftarrow \left\{ \begin{array}{l} \varphi'(0) - \dfrac{4}{\sqrt{3}}\,(-1)^{\ell}\,\varphi\left(\dfrac{\pi}{3}\right) = 0 \qquad \varphi\left(\dfrac{\pi}{2}\right) = 0 \\ -\varphi''(\alpha) + \dfrac{\ell(\ell+1)}{\cos^2\alpha}\,\varphi(\alpha) = s^2\,\varphi(\alpha) \end{array} \right.$$

1-body Schrö. eq.
fictitious particle
in 2D

$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}\,F'(R)\right]}_{\text{2D Laplacian}} + \left(\frac{\hbar^2\,s^2}{m\,R^2} + \frac{m\,\omega^2}{4}\,R^2\right) F(R) = E\,F(R)$$

$\boxed{\ell = 0}$ $\boxed{\varphi(\alpha) = \sin\left[s\left(\frac{\pi}{2} - \alpha\right)\right]}$

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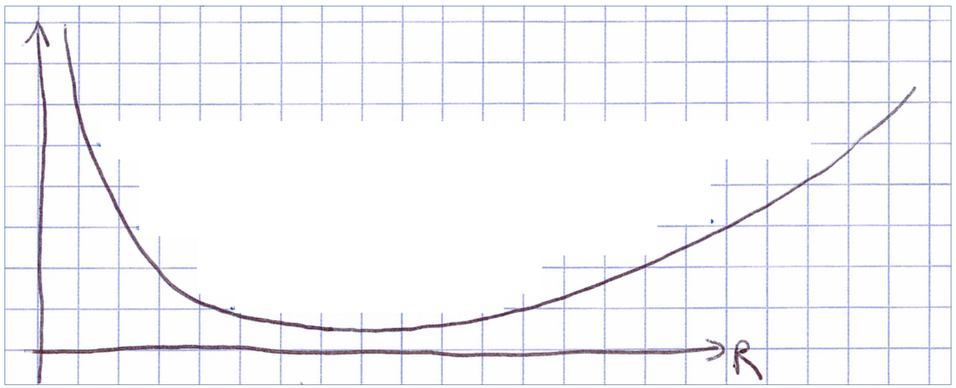

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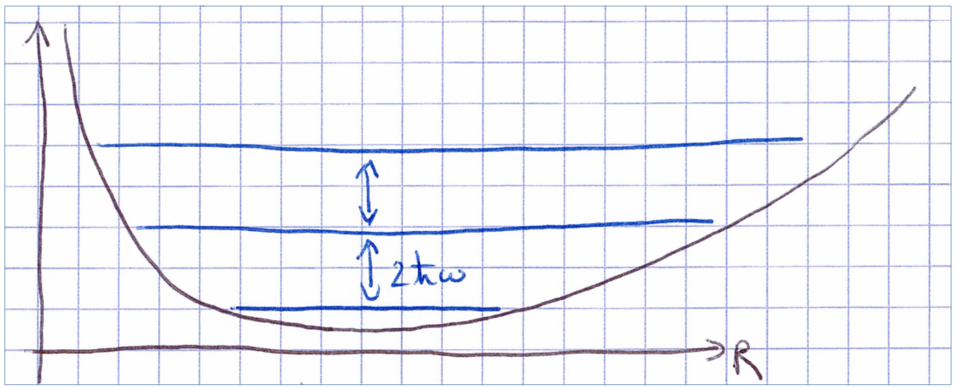
$$\boxed{E = (s + 1 + 2q)\,\hbar\omega} \qquad q = 0, 1, 2\ldots$$



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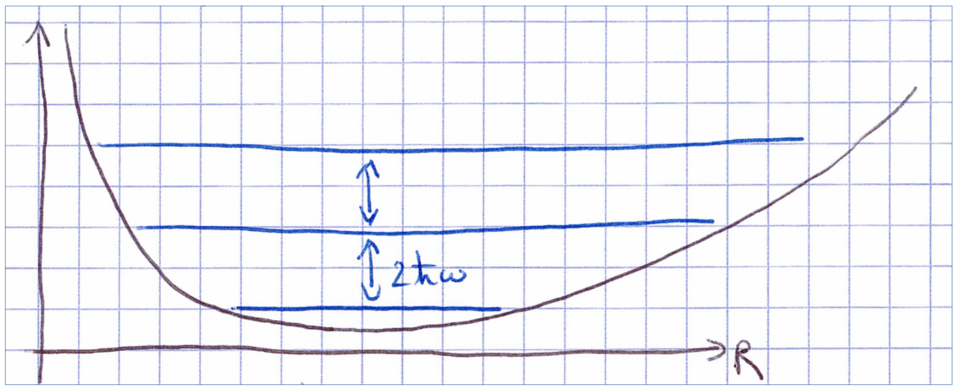
$\boxed{E = (\color{magenta}{s} + 1 + 2q)\,\hbar\omega}$ $\quad q = 0, 1, 2\ldots$

ground state: $\ell = 1, \quad E = 2.7727\ \hbar\omega$



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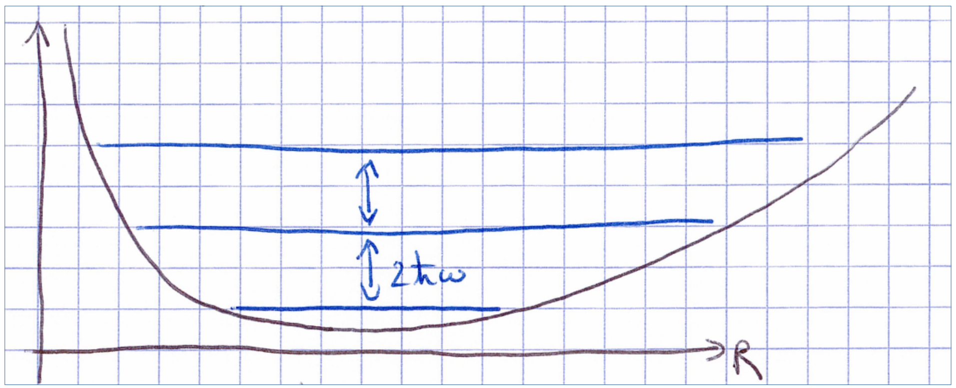

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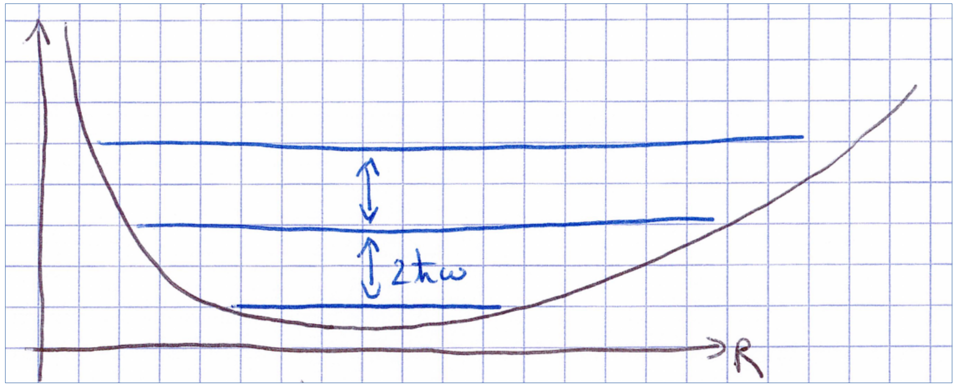

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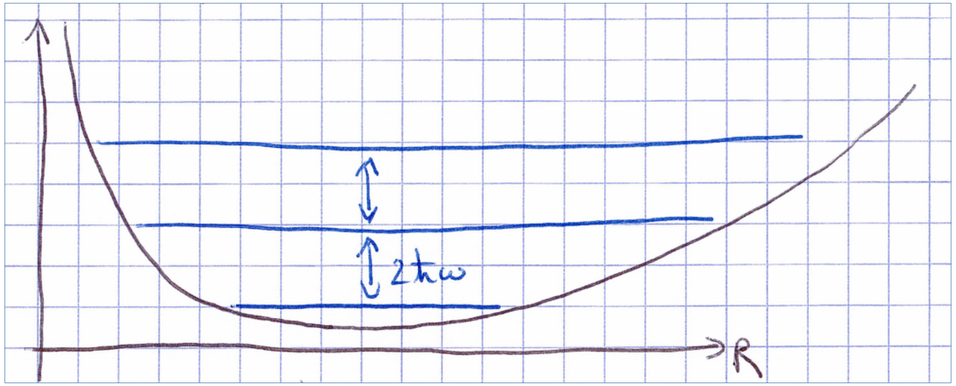
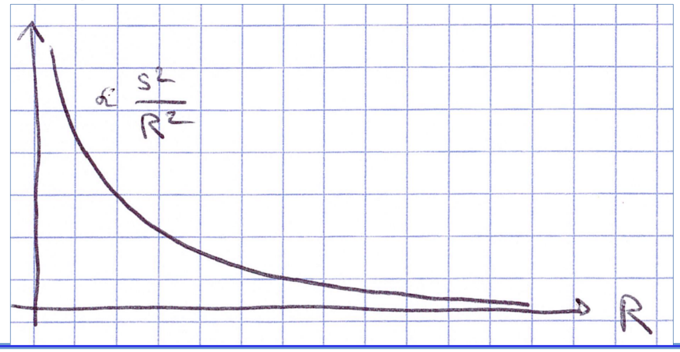


Free space: $\omega = 0$

$U_{\text{eff}}(R)$



$E = 0$:

$F(R) = R^s$

<u>1-body Schrö. eq.</u>
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$$-\frac{\hbar^2}{m}\underbrace{\left[F''(R) + \frac{1}{R}F'(R)\right]}_{\text{2D Laplacian}} + \underbrace{\left(\frac{\hbar^2 s^2}{m R^2} + \frac{m\omega^2}{4}R^2\right)}_{U_{\text{eff}}(R)} F(R) = E\,F(R)$$

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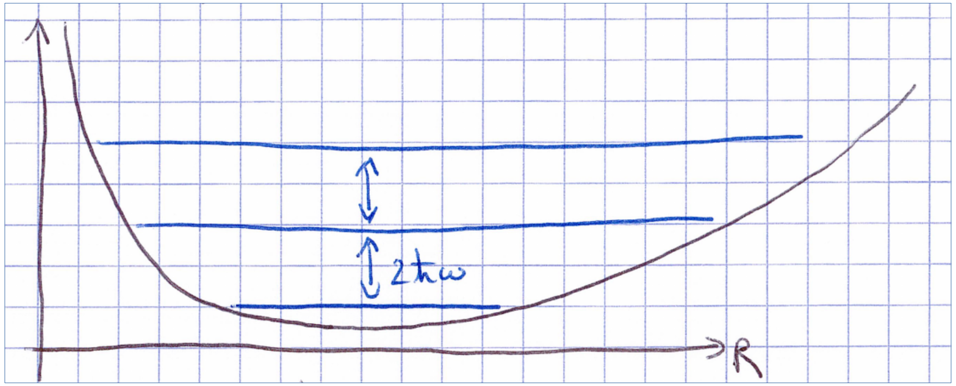
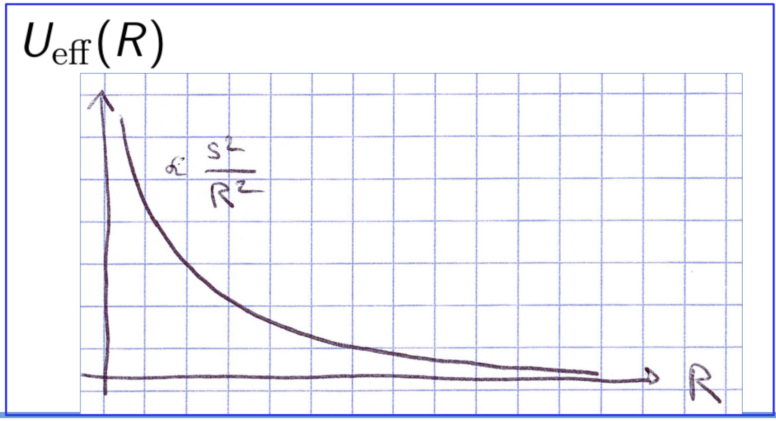


<u>Free space:</u> $\omega = 0$



$$\boxed{\begin{aligned} &E = 0: \\ &F(R) = R^s \end{aligned}}$$

All this remains true for arbitrary N
Only values of s are different

1-body Schrö. eq.
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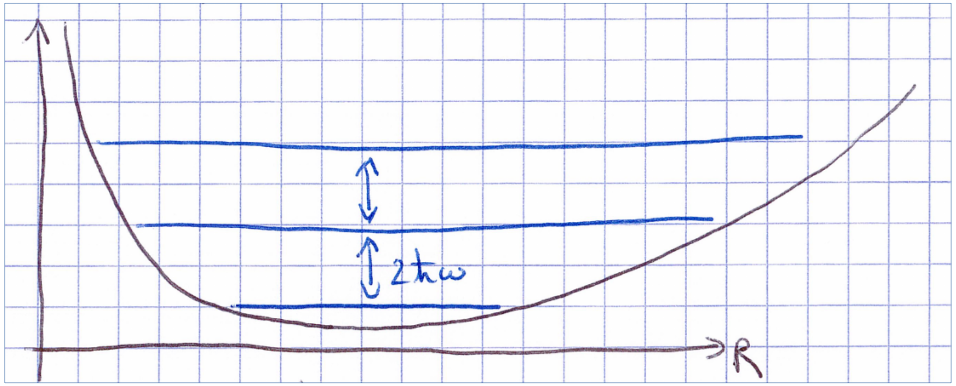
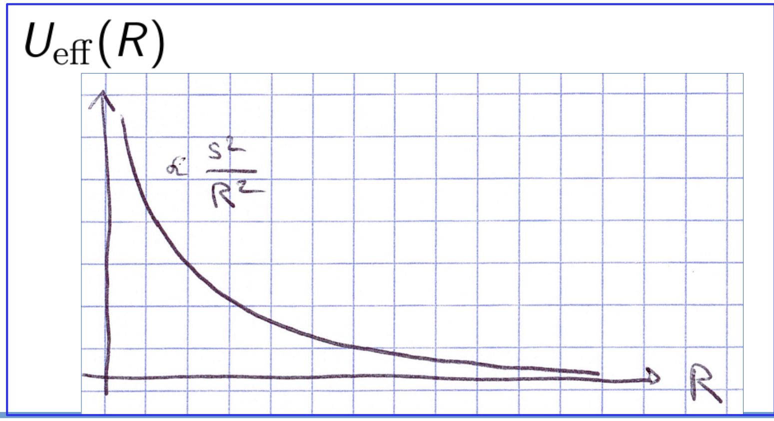


Free space: $\omega = 0$



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Chapter 3

Symmetry properties of the unitary N-body problem

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- Y. Castin, Comptes Rendus Physique **5**, 407 (2004)
- S. Tan, arXiv:cond-mat/0412764
- FW & Y. Castin, PRA **74**, 053604 (2006)

Chapter 3

Symmetry properties of the unitary N-body problem

ZRM , $a = \infty$

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$\Psi(\vec{r_1}, \ldots, \vec{r_N})$

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$\boxed{\mathrm{ZRM} \quad , \quad a = \infty}$ $\qquad \Psi(\vec{r_1}, \ldots, \vec{r_N})$

$$\begin{cases} -\dfrac{\hbar^2}{2m} \displaystyle\sum_{i=1}^{N} \Delta_{\vec{r_i}} \Psi + \sum_{i=1}^{N} U(r_i)\, \Psi = \mathcal{E}\, \Psi \\ \\ \end{cases}$$

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Isotropic harmonic trap:

$$U(r) = \frac{1}{2}\, m\, \omega^2\, r^2$$

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$\vec{\Omega}$: $3N - 4$ angles

parameterizing $\left(\dfrac{\vec{r_i} - \vec{r_1}}{R}\right)_{2 \le i \le N}$

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$\left[\; N=3:\ \vec{\Omega}=(\alpha,\hat{r},\hat{\rho})\;\right]$

$$\Psi(\vec{r_1},\dots,\vec{r_N}) = \psi_{\rm CM}(\vec{C})\,\frac{F(R)}{R^{\frac{3N-5}{2}}}\,\phi(\vec{\Omega})$$

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(CC) for $\phi(\vec{\Omega})$

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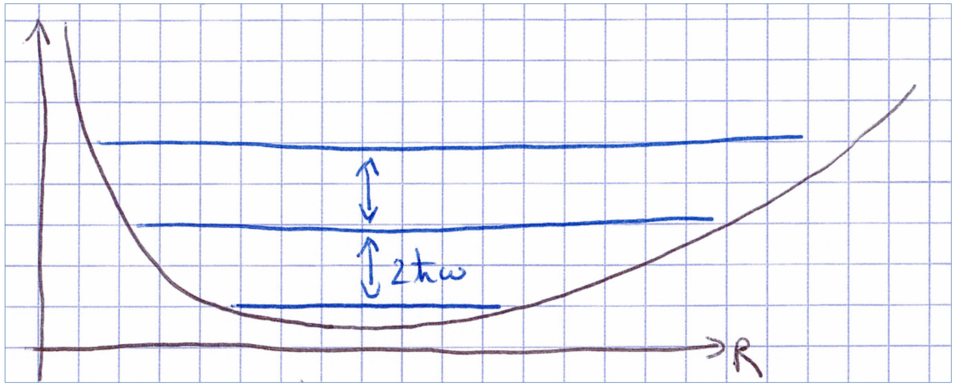

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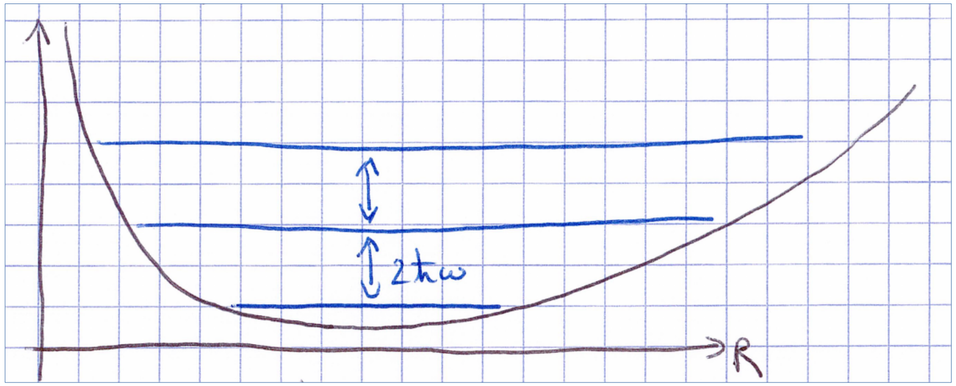

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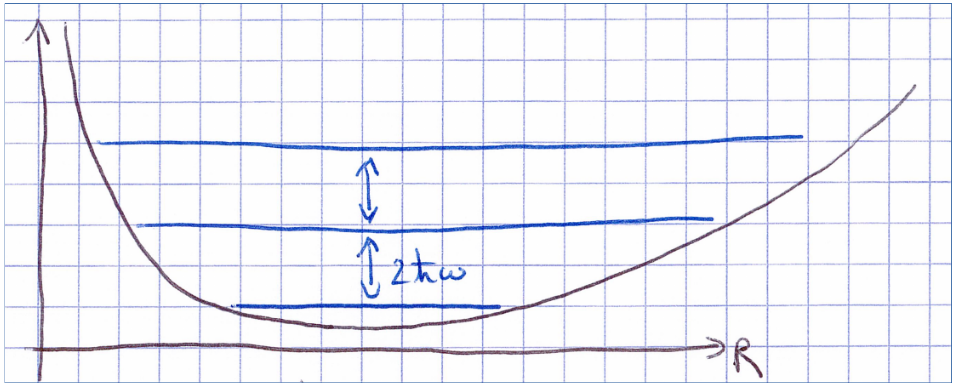

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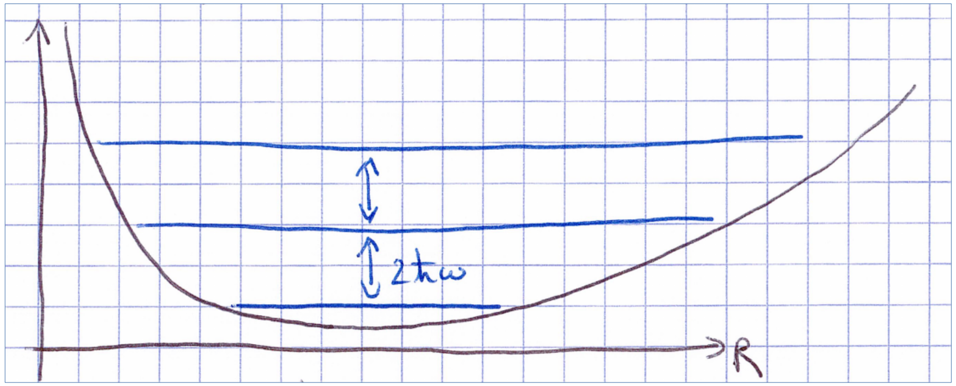

$$\boxed{s(N_\uparrow, N_\downarrow) := \text{lowest } s} \qquad s(2,1) = 1.7727$$

$\boxed{\textbf{Excursion: } s \text{ for } N \to \infty}$ $\quad s\left(\frac{N}{2}, \frac{N}{2}\right) \underset{N\to\infty}{\sim} \quad ?$

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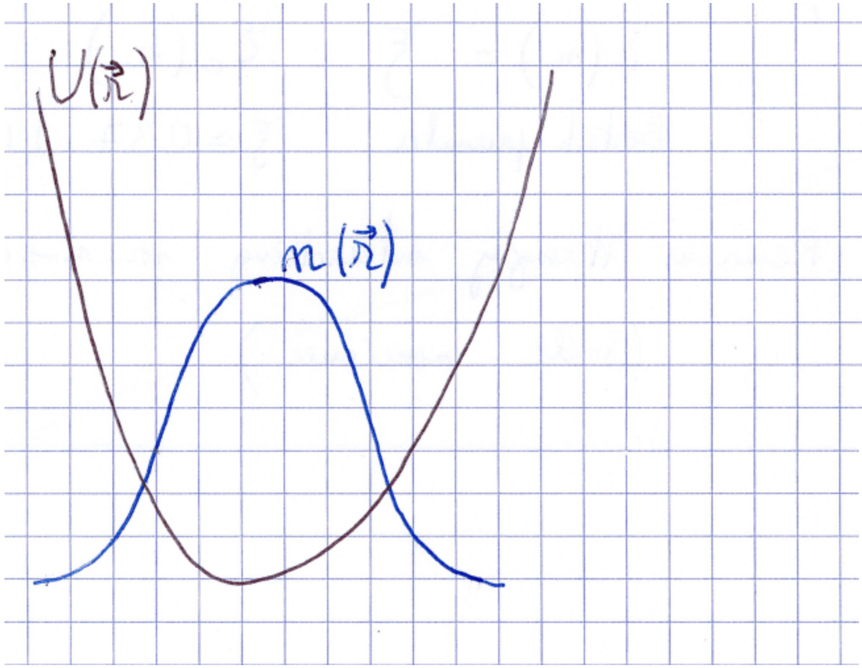

<u>Local Density Approximation</u> (= hydrostatics)

Locally : $\simeq$ homogeneous gas,

$$\mu_{local}(\vec{r}) = \mu_0 - U(\vec{r})$$

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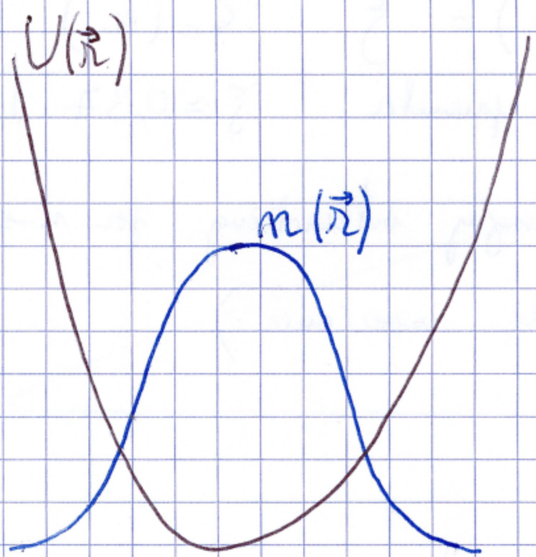

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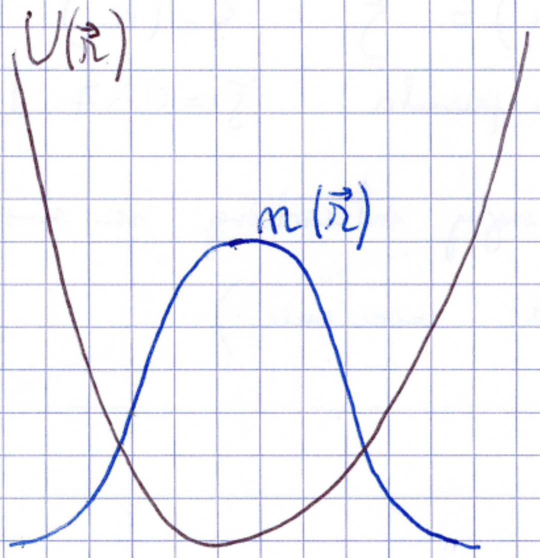

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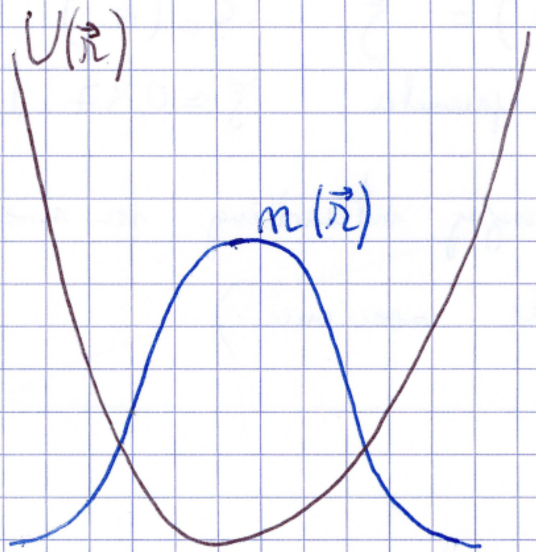

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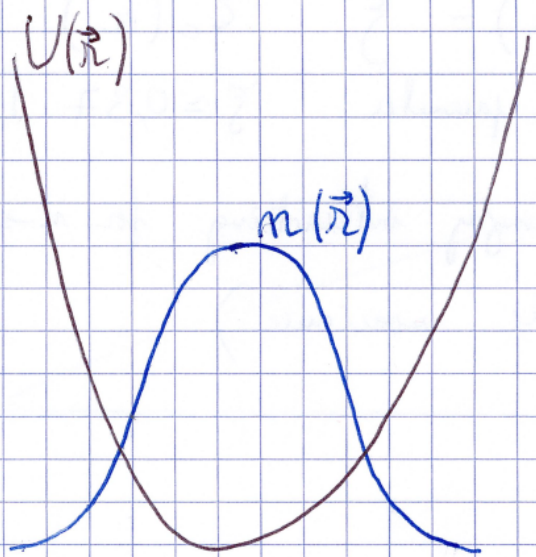

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NB: Free space, $E = 0$: $\psi(\vec{r_1}, \ldots, \vec{r_N}) = R^{\,s - \frac{3N-5}{2}}\ \phi(\vec{\Omega})$

Part 2: Short-distance scaling law

Arbitrary a and $U(\vec{r})$.

$\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ eigenstate of ZRM.

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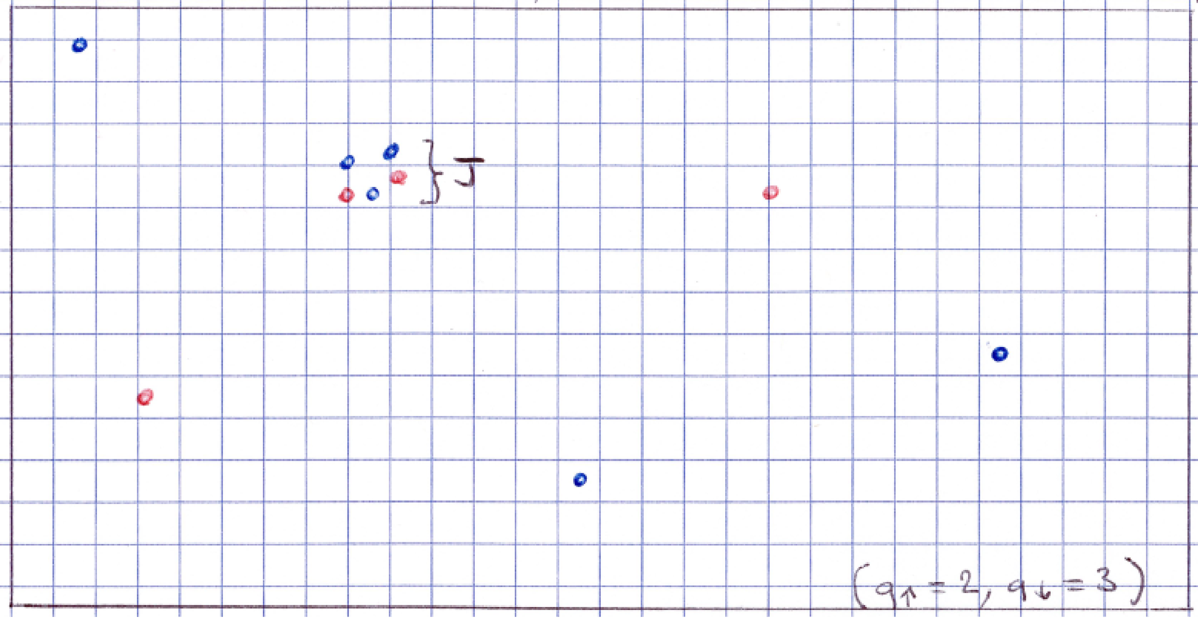

$J \subset \{1, 2, \dots, N\}$

with $\begin{cases} q_\uparrow & \text{spin-}\uparrow \\ q_\downarrow & \text{spin-}\downarrow \end{cases}$

$q = q_\uparrow + q_\downarrow \geq 3$

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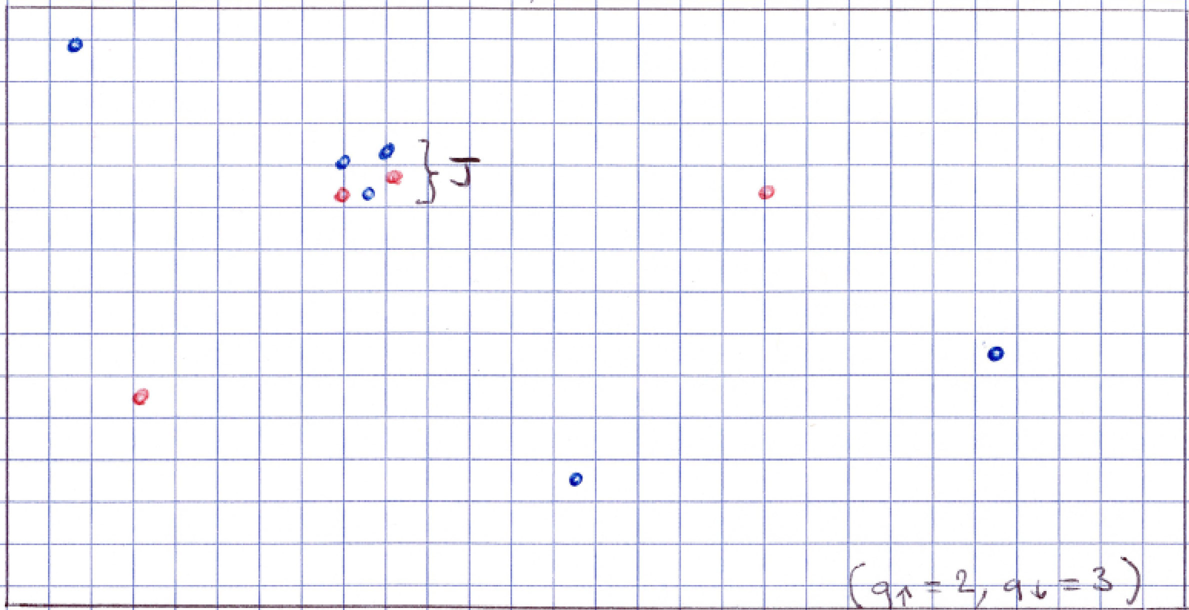

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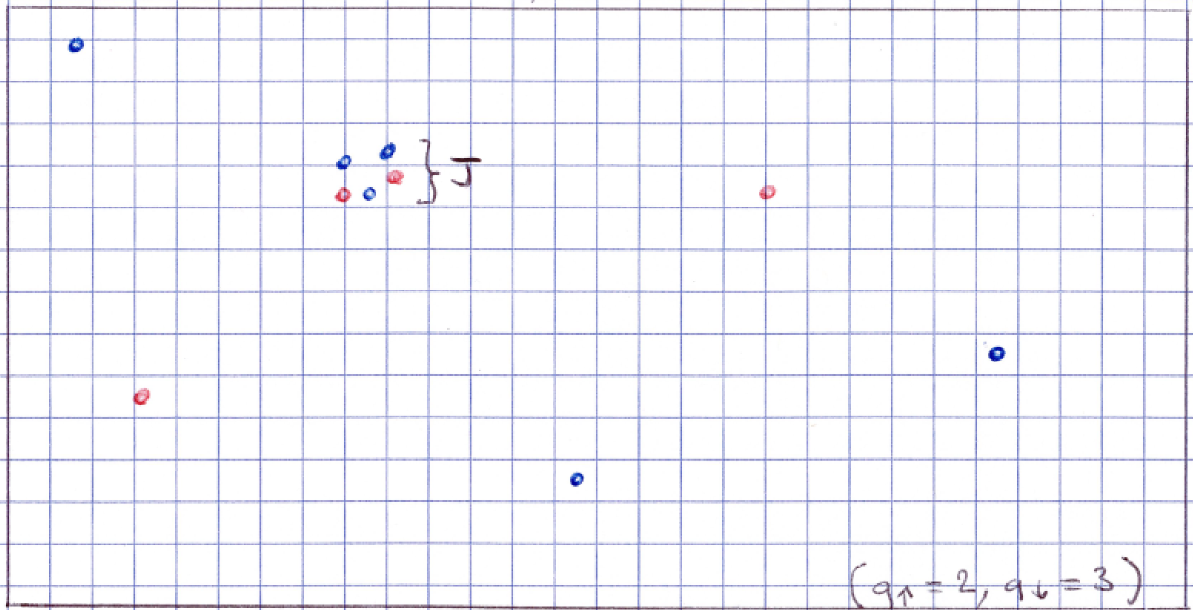

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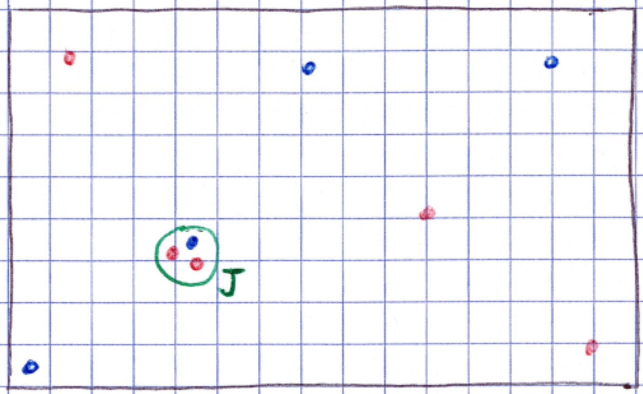

$$\Psi(\vec{r}_1, \ldots, \vec{r}_N) \underset{R_J \to 0}{\propto} R_J^{\,s(2,1)-2} = R_J^{\,-0.23}$$

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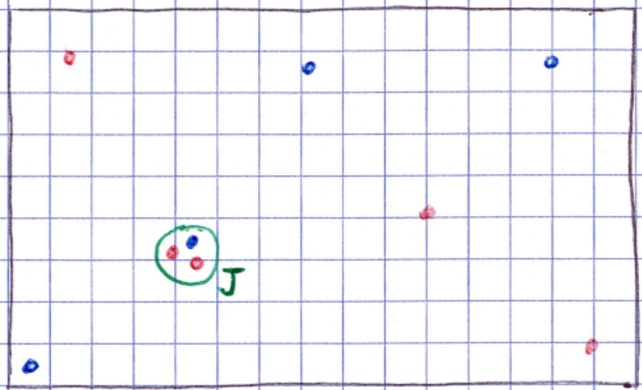
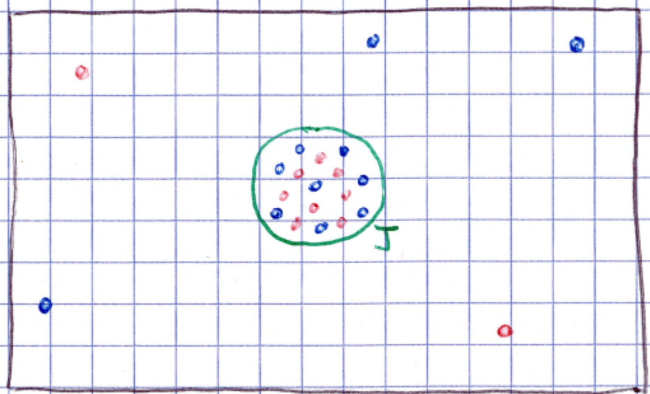
$(q_\uparrow, q_\downarrow) = (2, 1)$:



$$\Psi(\vec{r}_1, \dots, \vec{r}_N) \underset{R_J \to 0}{\propto} R_J^{\,s(2,1)-2} = R_J^{-0.23}$$

$(q_\uparrow, q_\downarrow) = \left(\frac{q}{2}, \frac{q}{2}\right)$

$q \gg 1$



$$\Psi(\vec{r}_1 \dots \vec{r}_N) \underset{R_J \to 0}{\propto} R_J^{\,\underbrace{s\left(\frac{q}{2}, \frac{q}{2}\right)}_{\simeq \sqrt{\xi}\left(\frac{3q}{4}\right)^{\frac{4}{3}}} - \frac{3q-5}{2}}$$

Part 3 : The Castin mode

Back to $\boxed{a = \infty}$. Isotropic harmonic trap, $\omega(t)$.

$\vec{X} := (\vec{r}_1, \dots, \vec{r}_N)$.

ZRM, t-dependent : $\Psi(\vec{X}, t)$

$$\begin{cases} \bullet \; i\hbar \frac{\partial}{\partial t} \Psi(\vec{X}, t) = -\frac{\hbar^2}{2m} \Delta_{\vec{X}} \Psi(\vec{X}, t) + \frac{1}{2} m \, \omega(t)^2 X^2 \times \Psi(\vec{X}, t) \\ \bullet \; \boxed{\Psi(\vec{X}, t) \text{ satisfies } (CC) \text{ for } a = \infty} \end{cases}$$

$$\hookrightarrow \Psi(\vec{X}, t) \underset{r \to 0}{=} \frac{1}{r} A(\vec{c}; \vec{r}_2, \vec{r}_4 \dots \vec{r}_N; t) + O(r)$$

Consider : $\omega(t) = \begin{cases} \omega_0, & t < 0 \\ \text{arbitrary}, & t > 0. \end{cases}$

Scaling solution [Y. Castin, 2004]

if $\Psi(\vec{X}, t = 0)$ stationary state of ZRM (ω_0)

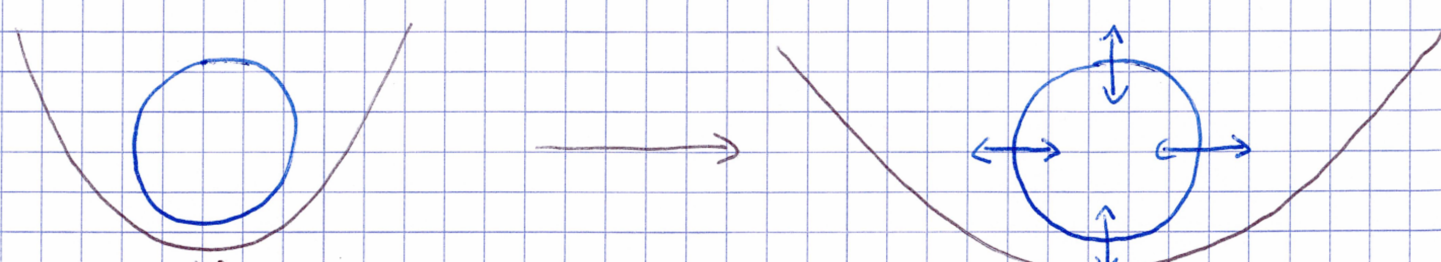
then : $$\boxed{\Psi(\vec{X}, t) = \Psi\left(\frac{\vec{X}}{\lambda(t)}, 0\right) e^{i \frac{\dot{\lambda}(t)}{\lambda(t)} X^2 \frac{m}{2\hbar}} \frac{e^{i\theta(t)}}{\lambda(t)^{3N/2}}}$$

with $\lambda(t)$ s.t. $$\ddot{\lambda} = -\frac{d}{d\lambda}\left[\frac{\omega_0^2}{2\lambda^2} + \frac{\omega(t)^2}{2}\lambda^2\right] \quad (*)$$

$$[\lambda(0) = 1, \quad \dot{\lambda}(0) = 0]$$

Proof : $\begin{cases} \bullet \; (CC) : \text{ satisfied by } \Psi(\vec{X}, 0) \Rightarrow \text{also by } \Psi(\frac{\vec{X}}{\lambda}, 0) \\ \bullet \; \text{Schrödinger eq : explicit computation (same than non-interacting case)} \end{cases}$

Breathing mode : $\omega(t) = \omega_1, \quad t > 0$.

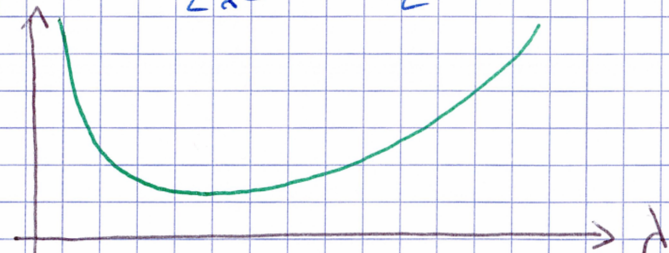


ω_0 $\qquad$ ω_1

$\boxed{\lambda(t) \text{ periodic, frequency } 2\omega_1}$

indeed : $(*) \Leftrightarrow$ a classical particle

in $V_{eff}(\lambda) = \frac{\omega_0^2}{2\lambda^2} + \frac{\omega_1^2}{2}\lambda^2$



$\frac{\omega_0^2}{\lambda^2} \Leftrightarrow$ centrifugal potential

$\leadsto$ just harmonic oscillator (ω_1)

$$\boxed{n(\vec{r}, t) = n\left(\frac{\vec{r}}{\lambda(t)}, 0\right) \times \frac{1}{\lambda(t)^3}}$$

$\Rightarrow$ Breathing motion, frequency $= \boxed{2\omega_1}$, $\boxed{\text{undamped !}}$

Experimental observation : Wuhan, Kaijun Jiang et al.
PRL 132, 243403 (2024)
PRA 111, 053317 (2025)

frequency: $\omega_B \simeq 2.010 \times \omega_1$

damping : $\frac{\Gamma_B}{\omega_B} = 2 \times 10^{-3}$,

mainly due to residual anisotropy

+ technical noise, also damping CM mode
[Min + Peng, PRA 2026]

Consequence : in hydrodynamics,

2 viscosities : $\begin{cases} \bullet \text{ shear} & \text{(no effect here)} \\ \bullet \text{ bulk } \zeta & (T > T_c) \end{cases}$

entropy : $\dot{S} = \int d^3r \, \frac{\zeta}{T} \|\vec{\nabla} \cdot \vec{v}\|^2 \geq 0$

Undamped breathing : $S(t)$ periodic

$\Rightarrow S(t) =$ constant $\Rightarrow \boxed{\zeta = 0}$

[D.T. Son,
PRL **98**, 020604 (2007)]

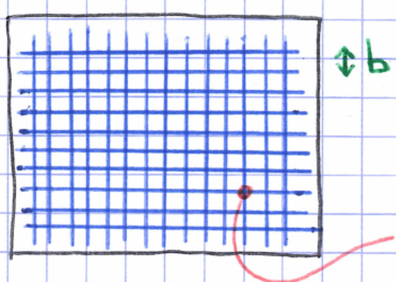
Lecture 2 — Many-body physics: methods and basic properties

Cubic box, volume $\mathcal{V} = L^3$, periodic boundary conditions.

$$\langle \vec{r} | \vec{k} \rangle = \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{\mathcal{V}}} .$$

Chapter 0 — Lattice Model

discretize space :



$\updownarrow b$ = lattice spacing

$\vec{r} \in (b\mathbb{Z})^3$

$$\vec{k} \in \left(\frac{2\pi}{L}\mathbb{Z}\right)^3 \cap \mathcal{D} =: \mathcal{D}_L \qquad \Big| \qquad \mathcal{D} := \left[-\frac{\pi}{b}, \frac{\pi}{b}\right[^3$$

$$-\frac{\hbar^2}{2m}\sum_{i=1}^{N} \Delta_{\vec{r}_i} \Psi + \sum_{i \leq N_\uparrow < j} V(\vec{r}_i - \vec{r}_j)\Psi = E\Psi$$

V :

$$\boxed{V(\vec{r}_2 - \vec{r}_1) = \frac{g_0}{b^3}\,\delta_{\vec{r}_1, \vec{r}_2}}$$

(on-site : range $\sim b$)

$\Delta_{\vec{r}}$:

$$-\frac{\hbar^2}{2m}\Delta_{\vec{r}}\, e^{i\vec{k}\cdot\vec{r}} := \varepsilon_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}$$

- Option 1 : $$\Delta_{\vec{r}} f(\vec{r}) = \sum_{\alpha=1}^{3} \frac{f(\vec{r} + b\vec{u}_\alpha) + f(\vec{r} - b\vec{u}_\alpha) - 2f(\vec{r})}{b^2}$$

$$\Rightarrow \quad \varepsilon_{\vec{k}} = -\frac{\hbar^2}{mb^2}\sum_{\alpha=1}^{3}\left[\cos(k_\alpha b) - 1\right]$$

- Option 2 : $$\varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m} .$$

Second quantization :

$\hat{\psi}_\sigma(\vec{r})$ annihilates a particle of spin $\sigma \in \{\uparrow, \downarrow\}$ in state $|\vec{r}\rangle$

$\hat{c}_{\vec{k},\sigma}$ ——————— $|\vec{k}\rangle$.

$\hat{n}_\sigma(\vec{r}) = \hat{\psi}^+_\sigma(\vec{r})\, \hat{\psi}_\sigma(\vec{r})$. $\qquad \hat{n}_{\vec{k},\sigma} = \hat{c}^+_{\vec{k},\sigma}\, \hat{c}_{\vec{k},\sigma}$

$$\hat{H} = \sum_{\sigma \in \{\uparrow,\downarrow\}} \sum_{\vec{k} \in \mathcal{D}_L} \varepsilon_{\vec{k}}\, \hat{n}_{\vec{k},\sigma} + g_0 \sum_{\vec{r}} b^3\, \hat{n}_\uparrow(\vec{r})\, \hat{n}_\downarrow(\vec{r})$$

For option 1 : $\hat{H} = -t \sum_{\substack{\vec{r},\vec{r}' \\ \|\vec{r}-\vec{r}'\| = b}} \hat{\psi}^+_\sigma(\vec{r})\, \hat{\psi}_\sigma(\vec{r}') + U \sum_{\vec{r}} \hat{n}_\uparrow(\vec{r})\hat{n}_\downarrow(\vec{r}) + 6t\hat{N}$

$\boxed{t = \frac{\hbar^2}{2mb^2}}$ $\qquad \boxed{U = g_0 b^3}$ $\qquad$ (Hubbard model)

2-body problem . $\psi(\vec{r})$. $\quad -\frac{\hbar^2}{m}\Delta_{\vec{r}}\psi + V(\vec{r})\psi = E\cdot\psi$

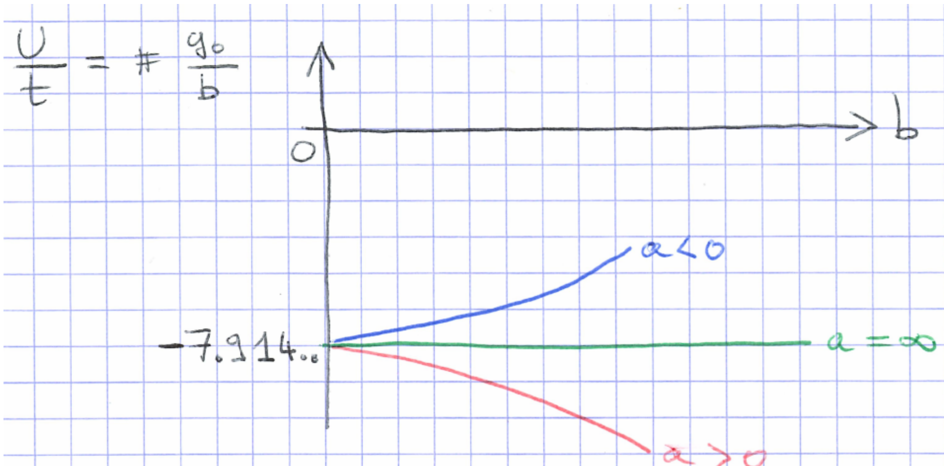
$(L \to \infty)$ $\qquad$ Scattering states $\rightsquigarrow$ a :

$$\frac{1}{g_0} = \frac{m}{4\pi\hbar^2 a} - I \quad , \quad I = \int_{\mathcal{D}} \frac{d^3k}{(2\pi)^3} \frac{1}{2\varepsilon_{\vec{k}}} .$$

Continuum limit $\qquad$ [$\Leftrightarrow$ zero-range lim]

$b \to 0$, fixed a .

$\boxed{I = \# \frac{m}{\hbar^2} \cdot \frac{1}{b}}$ $\quad \Rightarrow \quad g_0 \underset{b\to 0}{\sim} -\frac{1}{I} = -\# \frac{\hbar^2}{m} b$



filling factor: $n b^3 \propto b^3 \xrightarrow[b\to 0]{} 0$.

Chapter 1 — Many-body methods

Part 1: Virial expansion

Unpolarized gas. (T, n, a)

Grand-canonical: $n \to \mu$.

$$\boxed{\hat{H}\,|\nu\rangle = E_\nu\,|\nu\rangle} \qquad \boxed{\hat{N}\,|\nu\rangle = N_\nu\,|\nu\rangle}.$$

Observable $\hat{Q}$. $\quad \langle \nu|\hat{Q}|\nu\rangle =: Q_\nu$.

$$Q = \langle \hat{Q} \rangle = \frac{\sum_\nu Q_\nu\, e^{-\beta(E_\nu - \mu N_\nu)}}{\sum_\nu e^{-\beta(E_\nu - \mu N_\nu)}} =: \frac{\tilde{Q}}{Z}.$$

$$\boxed{\tilde{Q} = \sum_{N=0}^{\infty} \underbrace{\left(\sum_{\nu / N_\nu = N} Q_\nu\, e^{-\beta E_\nu} \right)}_{\tilde{Q}_N} e^{\beta\mu\cdot N}} \qquad Z = \sum_N Z_N\, e^{\beta\mu\cdot N}$$

$$Q = \sum_{N=0}^{\infty} Q_N\, e^{\beta\mu\cdot N}$$

$Q_N \longrightarrow$ simple combination of $\begin{cases} (\tilde{Q}_0, \dots, \tilde{Q}_N) \\ (Z_0, \dots, Z_N) \end{cases}$

$\Longrightarrow \ \leq N$-body problem.

$$Q(\mu, T, a) \underset{\mu \to -\infty}{=} \sum_{N=0}^{\infty} Q_N(T, a)\, e^{\beta\mu N}$$

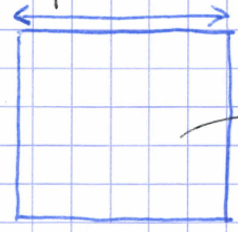
Can take $V \to \infty$.

Physical justification: $\quad n\lambda_T^3 \xrightarrow[\beta\mu \to -\infty]{} 0$.

$$e^{\beta\mu} \ll 1 \implies \lambda_T \ll d.$$

interacting gas $\simeq$ classical ideal gas, except at distances $\lesssim \lambda_T$.

$\lambda_T \ll l \ll d$



Probability (N particles) $\sim \big(\underbrace{n\,l^3}_{\ll 1}\big)^N$

$\boxed{\text{Part 2: Simple variational wavefunctions}}$

$$\min_{\Psi \in \text{Ansatz}} \langle \Psi | \hat{H} | \Psi \rangle \geq E_{exact} \qquad (T=0)$$

(↳ lattice model)

$\boxed{\text{BCS}}$ $\quad N_\uparrow = N_\downarrow = M$. $\qquad (N = 2M)$

$$\Psi(\underbrace{\vec r_1, \dots, \vec r_M}_{\uparrow}, \underbrace{\vec r_{M+1}, \dots, \vec r_N}_{\downarrow})$$

Antisymmetrizing: $$(\hat A \Psi)(\vec r_1, \dots, \vec r_N) := \sum_{P_\uparrow, P_\downarrow} (-1)^{\varepsilon(P_\uparrow)+\varepsilon(P_\downarrow)} \psi(\vec r_{P_\uparrow(1)}, \dots, \vec r_{P_\uparrow(M)}, \vec r_{P_\downarrow(1)}, \dots, \vec r_{P_\downarrow(M)})$$

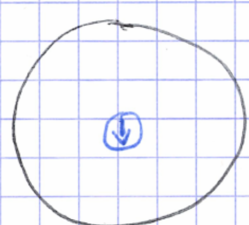
permutations of $\{1, \dots, M\}$

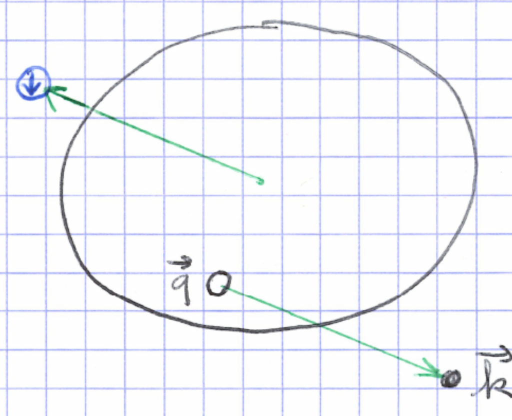
$$\Psi_{BCS} \overset{(\#)}{=} \hat A \tilde\Psi, \qquad \tilde\Psi(\vec r_1, \dots, \vec r_N) = \phi(\vec r_1 - \vec r_{M+1})\,\phi(\vec r_2 - \vec r_{M+2}) \cdots \phi(\vec r_M - \vec r_N)$$

$\boxed{\text{Chevy}}$ $\quad$ Polaron : $N_\downarrow = 1$, $\quad N_\uparrow \to \infty$ $\left(\text{fixed } \frac{N_\uparrow}{V}\right)$

$$|FS_\uparrow\rangle = \prod_{\substack{\vec k \\ k<k_F}} \hat c^\dagger_{\vec k,\uparrow} |0\rangle .$$

$$|\Psi_{Chevy}\rangle = \alpha_0\, \hat c^\dagger_{\vec 0,\downarrow} |FS_\uparrow\rangle + \sum_{\vec k,\vec q} \alpha_{\vec k,\vec q}\, \hat c^\dagger_{\vec q-\vec k,\downarrow}\, \hat c^\dagger_{\vec k,\uparrow}\, \hat c_{\vec q,\downarrow} |FS_\uparrow\rangle$$



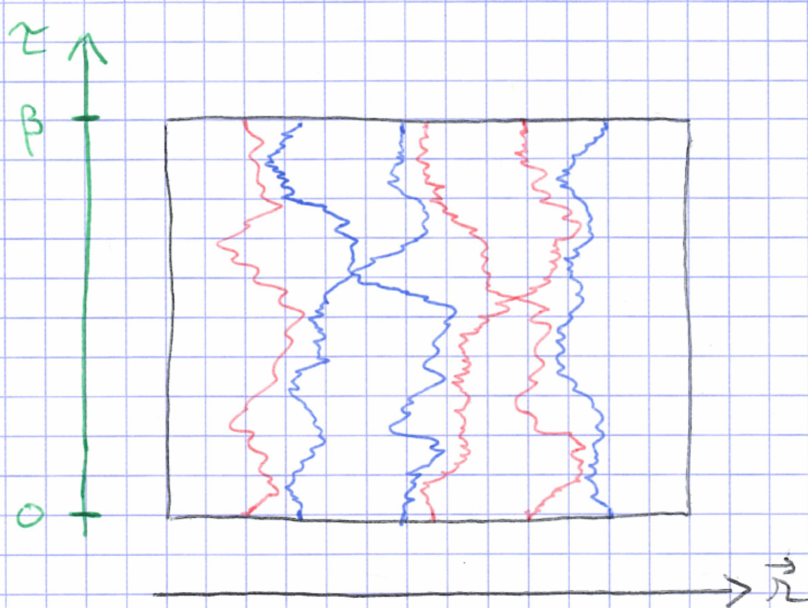


Part 3: Quantum Monte Carlo (conventional)

$$\boxed{\begin{array}{c}\text{Quantum problem} \\ \vec{r} \in [0,L]^3\end{array}} \xrightarrow[(Z)]{\text{Mapping}} \text{"Classical" problem } (\vec{r}, \tau) \in [0,L]^3 \times [0,\beta]$$

τ ↑ imaginary time

• Mapping 1: Path integrals. $\vec{r}_1(\tau), \dots, \vec{r}_N(\tau)$



PIQMC

• Mapping 2: Auxiliary field $\phi(\vec{r}, \tau)$

AFQMC / Determinant QMC

$$Q = \langle \hat{Q} \rangle = \frac{\sum_{\mathcal{C}} Q(\mathcal{C})\, w(\mathcal{C})}{\sum_{\mathcal{C}} w(\mathcal{C})}$$

⟶ generically: $w(\mathcal{C})$ can be < 0.

$$\Big| \sum_{\mathcal{C}} w(\mathcal{C}) \Big| \ll \sum_{\mathcal{C}} |w(\mathcal{C})|$$

$$t_{CPU} \sim \exp(\# \beta \mathcal{V}).$$

$\boxed{\text{Sign problem}}$ ☹

⟶ special cases: $w(\mathcal{C}) \geq 0, \ \forall \mathcal{C}$

↳ unpolarized gas + lattice model.

[sign problem = physical ?..]

Approximations to remove sign problem : Fixed-node

$$\mathcal{N}[\Psi] := \left\{ (\vec{r}_1, \dots, \vec{r}_N) \ / \ \Psi(\vec{r}_1, \dots, \vec{r}_N) = 0 \right\}.$$

$$\min_{\{\Psi / \mathcal{N}[\Psi] = \mathcal{N}[\Psi_{\text{Ansatz}}]\}} \langle \Psi | \hat{H} | \Psi \rangle$$

$\Psi_{\text{Ansatz}} \to \begin{cases} \Psi_{BCS} & \text{(unpolarized gas in continuous space)} \\ \Psi_{\text{ideal}} & \text{(polarized gas)} \end{cases}$

Part 4: Diagrammatic approaches

$$\hat{H}' := \hat{H} - \mu_\uparrow \hat{N}_\uparrow - \mu_\downarrow \hat{N}_\downarrow .$$

Lattice model: $$\hat{H}' = \underbrace{\sum_{\vec{k},\sigma} (\varepsilon_{\vec{k}} - \mu_\sigma)\, \hat{n}_{\vec{k},\sigma}}_{\hat{H}'_0} + g_0 \sum_{\vec{r}} b^3\, \hat{n}_\uparrow(\vec{r})\, \hat{n}_\downarrow(\vec{r})$$

$$Q(g_0; T, \mu_\uparrow, \mu_\downarrow) = \langle \hat{Q} \rangle_{H'} \equiv \frac{\mathrm{Tr}\left[\hat{Q}\, e^{-\beta \hat{H}'}\right]}{\mathrm{Tr}\left[e^{-\beta \hat{H}'}\right]} .$$

E.g.: $\hat{Q} = \hat{n}_\uparrow(\vec{0})$. $Q = n_\uparrow$

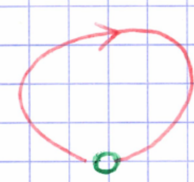
[$n_\sigma(T, \mu_\uparrow, \mu_\downarrow)$ is Eq. of state]

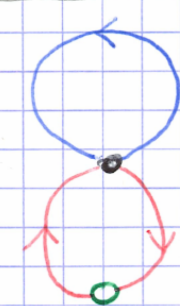
Diagrams for the lattice model

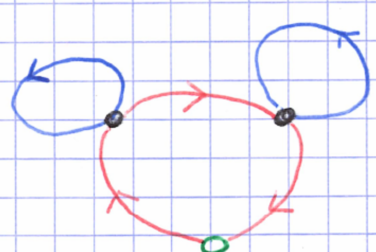
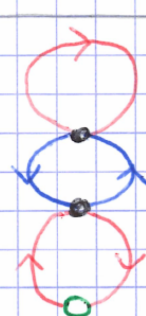
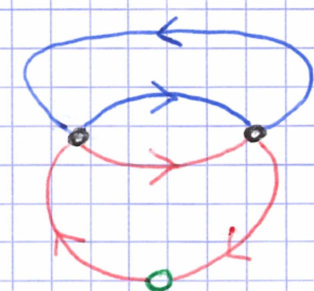
Bare expansion in powers of g_0

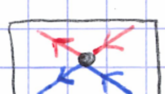
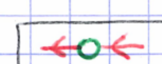
$$Q(g_0) = \sum_{N=0}^{\infty} \underbrace{Q_N\, g_0^N}_{a_N} .$$

For $Q = n_\uparrow$:

$a_0 =$ 

$a_1 =$ 

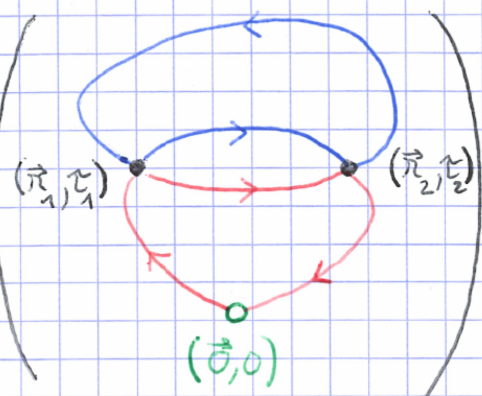
$a_2 =$  +  + 

Rules: • all connected diagrams, with N  and 1 

- Contribution :

Define: $\begin{cases} \hat{\psi}_\sigma(\vec{r},\tau) = e^{\tau H_0'}\,\hat{\psi}_\sigma(\vec{r})\,e^{-\tau H_0'} \\ \hat{\psi}^{+}_\sigma(\text{---}) = \text{---}\;\hat{\psi}^{+}_\sigma(\vec{r})\;\text{---} \end{cases}$ $\Bigg|$ $G_{0,\sigma}(\vec{r},\tau) = \begin{cases} -\langle \hat{\psi}_\sigma(\vec{r},\tau)\hat{\psi}^{+}_\sigma(\vec{0},0)\rangle_{H_0'}\,, & \tau>0 \\ +\langle \hat{\psi}^{+}_\sigma(\vec{0},0)\hat{\psi}_\sigma(\vec{r},\tau)\rangle_{H_0'}\,, & \tau<0 \end{cases}$

Contribution



$$= (-1)^{N_\ell}(-g_0)^2 \sum_{\vec{r}_1} b^3 \int_0^\beta d\tau_1 \sum_{\vec{r}_2} \int_0^\beta d\tau_2$$

$$G_{0\uparrow}(\vec{r}_1,\tau_1)\,G_{0\uparrow}(\vec{r}_2-\vec{r}_1,\tau_2-\tau_1)\,G_{0\uparrow}(-\vec{r}_2,-\tau_2)$$

$$G_{0\downarrow}(\vec{r}_2-\vec{r}_1,\tau_2-\tau_1)\,G_{0\downarrow}(\vec{r}_1-\vec{r}_2,\tau_1-\tau_2)$$

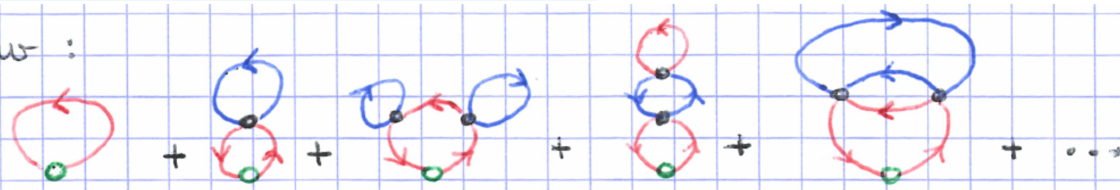
$\boxed{N_\ell = \#\ \text{internal closed loops} = 1}$

Can take $v \longrightarrow \infty$ (at fixed μ_σ)

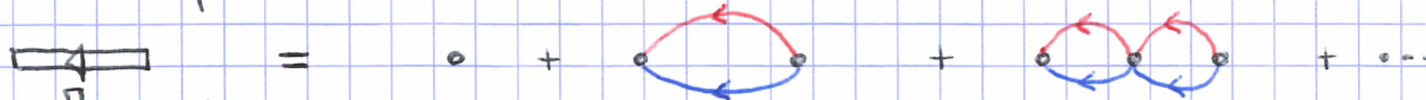
$\displaystyle\sum_{\vec{r}_i \in \text{infinite lattice}}$ converge, because $\begin{cases} \bullet \text{ diagrams are connected} \\ \bullet\ G_0(\vec{r},\tau) \xrightarrow[r\to\infty \text{ (quickly)}]{} 0 \end{cases}$

Diagrams for the Zero-Range Model

We saw :

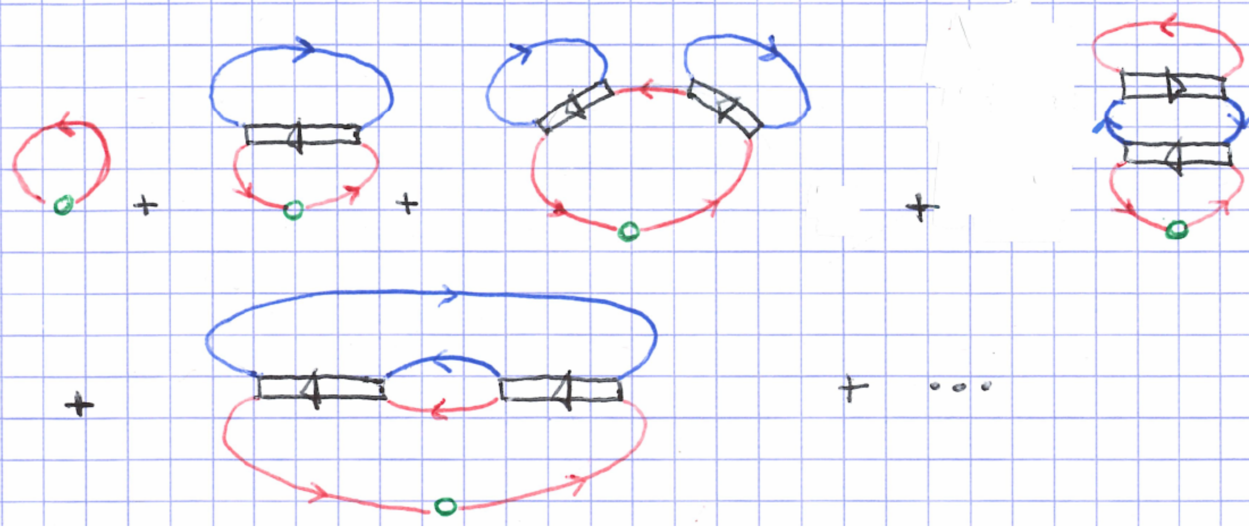
$n_\uparrow =$ 

Let us define :



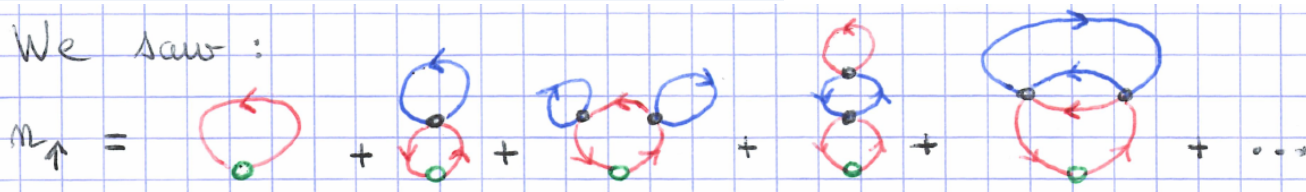
Γ_0

Then we express $n_\uparrow$ using the vertex  instead of •

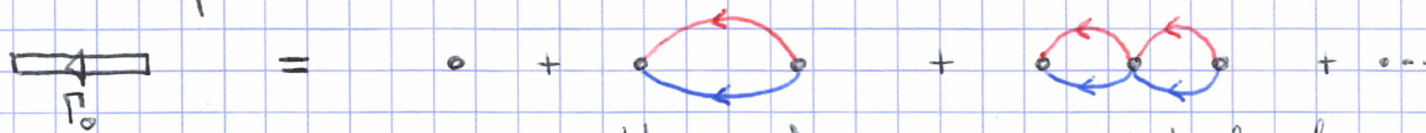
$n_\uparrow =$ 

Diagrams for the Zero-Range Model

We saw :



Let us define :



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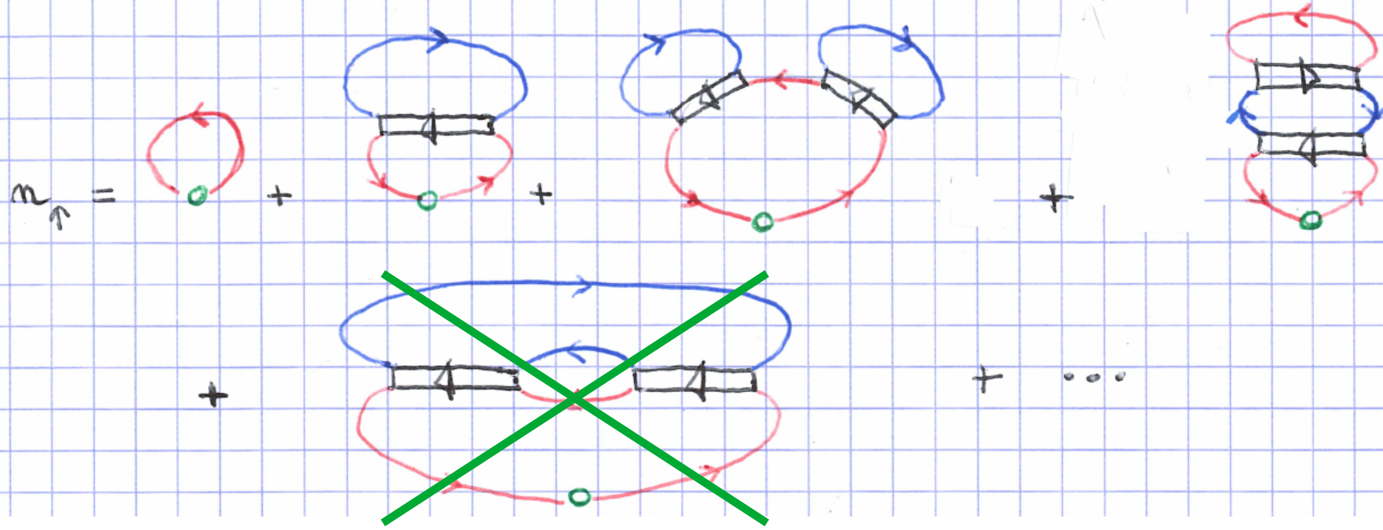
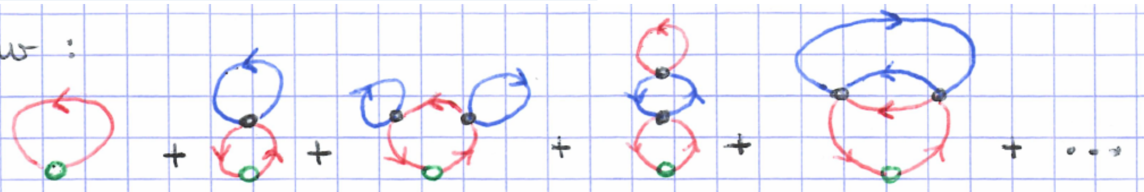


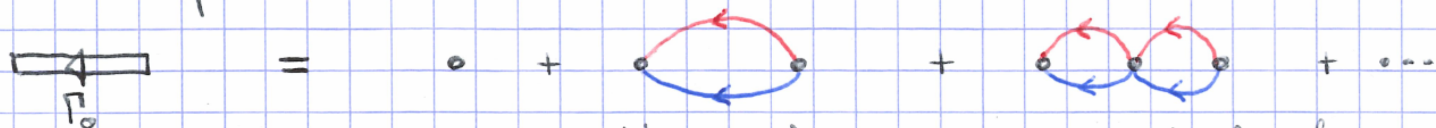
forbidden
(avoid double-counting)

Diagrams for the Zero-Range Model

We saw:

$n_\uparrow =$ 

Let us define:



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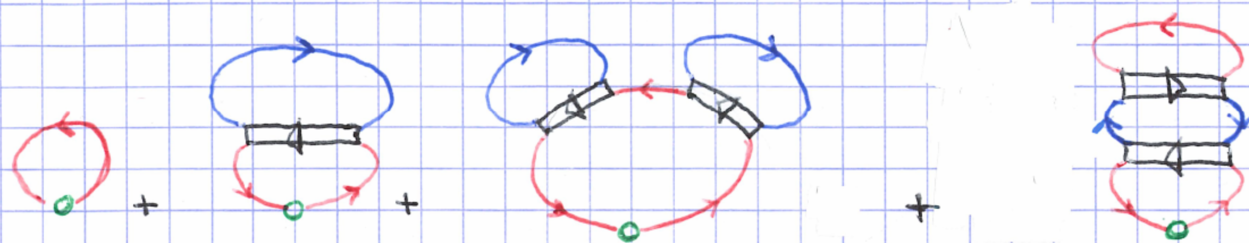
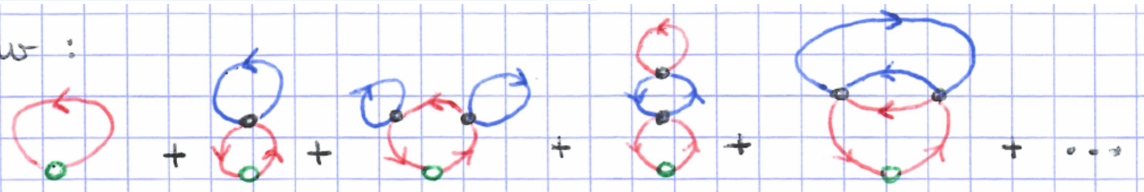
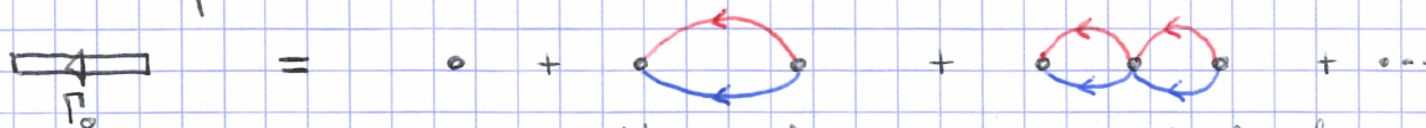
$n_\uparrow =$ 

Diagrams for the Zero-Range Model

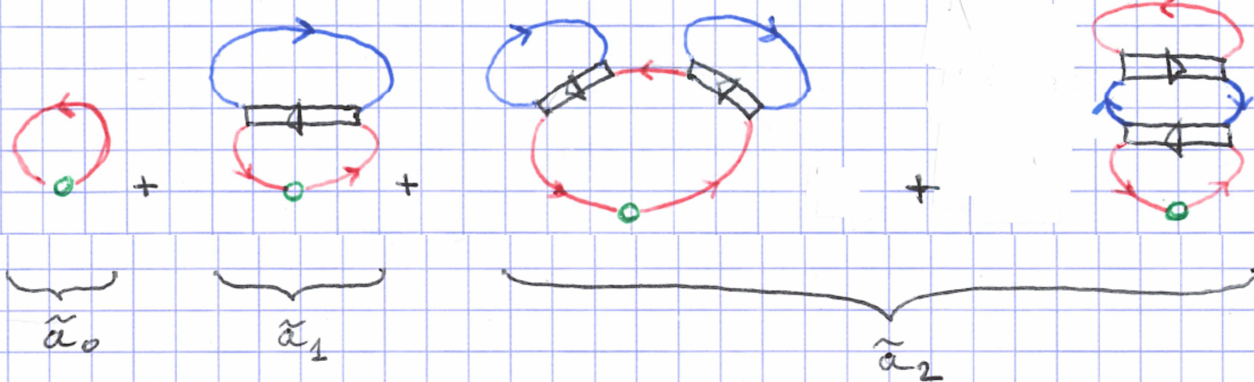
We saw :

$n_\uparrow =$ 

Let us define :



Then we express $n_\uparrow$ using the vertex  instead of •

$n_\uparrow =$ 

$$n_\uparrow = \sum_{N=0}^{\infty} \underbrace{\tilde{a}_N}_{\longrightarrow N\ \Gamma_0\text{- vertices}}$$

Continuum limit : $\Gamma_0 \longrightarrow$ Finite limit

$\tilde{a}_N \longrightarrow$ ——————

$n_\uparrow \simeq \tilde{a}_0 + \tilde{a}_1$ Nozières — Schmitt-Rink

Sum all $\tilde{a}_N$?

Diagrammatic MC : $\tilde{a}_N$, $N \leq N_m = 9$.

Problem : $\sum_{N=0}^{N_m} \tilde{a}_N$ diverges for $N_m \to \infty$.

$$|\tilde{a}_N| \sim (N!)^{1/5}$$

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$$|\tilde{a}_N| \sim (N!)^{1/5}$$

Solution : $\sum_{N=0}^{\infty} \tilde{a}_N$

we need to give a meaning to this sum

Sum all $\tilde{a}_N$?

Diagrammatic MC : $\tilde{a}_N$, $N \leq N_m = 9$.

Problem : $\displaystyle\sum_{N=0}^{N_m} \tilde{a}_N$ diverges for $N_m \to \infty$.

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Solution :

$$\sum_{N=0}^{\infty} \tilde{a}_N$$

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Problem : $\sum_{N=0}^{N_m} \tilde{a}_N$ diverges for $N_m \to \infty$.

$$|\tilde{a}_N| \sim (N!)^{1/5}$$

Solution :

$$f(z) \underset{z \to 0}{=} \sum_{N=0}^{\infty} \tilde{a}_N \, z^N$$

$\big\downarrow$ Unique.

$f(z)$ constructed so that $f(1) = n_\uparrow$

$\parallel$

$\langle n_\uparrow \rangle_{S(z)}$

$(T > T_c)$

Sum all $\tilde{a}_N$?

Diagrammatic MC : $\tilde{a}_N$, $N \leq N_m = 9$.

Problem : $\sum_{N=0}^{N_m} \tilde{a}_N$ diverges for $N_m \to \infty$.

$$|\tilde{a}_N| \sim (N!)^{1/5}$$

Solution :

$$f(z) \underset{z \to 0}{=} \sum_{N=0}^{\infty} \tilde{a}_N \, z^N$$

$\downarrow$ Unique. (conformal Borel transformation)

$f(z)$ constructed so that $f(1) = n_\uparrow$

$f(z) = \langle n_\uparrow \rangle_{S(z)}$

$(T > T_c)$

Sum all $\tilde{a}_N$?

Diagrammatic MC : $\tilde{a}_N$, $N \leq N_m = 9$.

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Solution :

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$\downarrow$ Unique. (conformal Borel transformation)

$f(z)$ constructed so that $f(1) = n_\uparrow$

$$f(z) = \langle n_\uparrow \rangle_{S(z)}$$

$(T > T_c)$

NB: For polaron problem:

- Very efficient algorithm $\Rightarrow$ can reach $N_m = 30$
- Series divergence is much slower, $\tilde{a}_N \underset{N\to\infty}{\sim} (-A)^N$ with $A \simeq 1.1$

$\Rightarrow$ resummable by simple conformal transformation

[Rossi *et al.*, PRB **101**, 045134 (2020)]

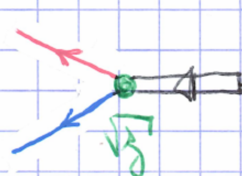
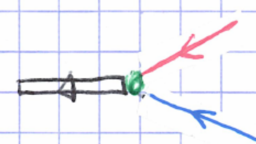
Appendix A : $\boxed{\text{Construction of } f(z)}$

Introduce a model defined by an action $S^{(z)}$

(z is a formal parameter) :

$$S^{(z)}[\underbrace{\varphi_\uparrow, \varphi_\downarrow}_{\substack{\text{fermionic}\\ \text{fields}\\ \text{(Grassmann)}}}, \underbrace{\eta}_{\substack{\text{bosonic}\\ \text{field}\\ \text{(complex)}}}]$$

- $S^{(0)}$ such that $\begin{cases} G_{0,\sigma}(\vec r,\tau) = -\langle \varphi_\sigma(\vec r,\tau)\,\bar\varphi_\sigma(\vec 0,0)\rangle_{S^{(0)}} \\ \Gamma_0(\vec r,\tau) = -\langle \eta(\vec r,\tau)\,\bar\eta\,(\vec 0,0)\rangle_{S^{(0)}} \end{cases}$

- interaction :  and 

- + term to avoid double-counting

$$S^{(z)}[\varphi_\uparrow,\varphi_\downarrow,\eta] = -\int d^3r \int_0^\beta d\tau \Big[\sum_{\sigma=\uparrow,\downarrow} \bar\varphi_\sigma\, G_{0,\sigma}^{-1}\,\varphi_\sigma + \bar\eta\,\Gamma_0^{-1}\,\eta$$
$$- z\,\bar\eta\,\Pi_0\,\eta + \sqrt{z}\,(\bar\eta\,\varphi_\downarrow\varphi_\uparrow + \bar\varphi_\uparrow\bar\varphi_\downarrow\,\eta)\Big](\vec r,\tau),$$

$\Pi_0 := G_{0\uparrow}\,G_{0\downarrow} =$ 

[Rossi et. al., PRB 93, 161102(R) (2016) ;
Rossi et al., PRL 121, 130405 (2018)]

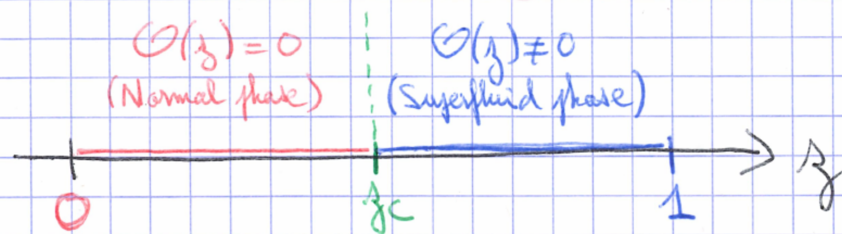
Appendix B : Extension to superfluid phase

Let $\mathcal{O}(z) := \langle \psi_\downarrow \psi_\uparrow \rangle_{S(z)}$.

$\mathcal{O}(1) \neq 0$. Thus :

$\longrightarrow$ if $S^{(0)}$ does not break $U(1)$ (i.e., conserves N)

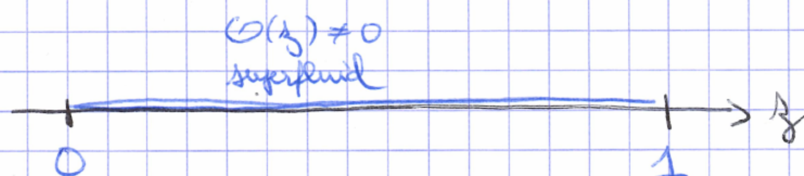
then : $\mathcal{O}(0) = 0$ (given that $S^{(0)}$ is quadratic).



$\Rightarrow$ a phase transition as a function of z, at $z_c < 1$

$\longrightarrow$ Solution : take $S^{(0)}$ which breaks $U(1)$,

e.g. $S^{(0)} = S_{BCS}$. Then, $\mathcal{O}(0) \neq 0$



The resulting diagrams contain "anomalous" propagators :

$$\begin{cases} \langle \psi_\uparrow(\vec{r},\tau)\, \psi_\downarrow(\vec{0},0) \rangle_{S^{(0)}} \\ \langle \psi^\dagger_\downarrow(\vec{r},\tau)\, \psi^\dagger_\uparrow(\vec{0},0) \rangle_{S^{(0)}} \end{cases}$$

Diagrammatic MC for the attractive Hubbard model in the Superfluid phase : [Spada et al., arXiv: 2103.12038]

Appendix C: $\boxed{\text{Determinant Diagrammatic Monte Carlo}}$

(a.k.a. Continuous-Time INTeraction expansion)

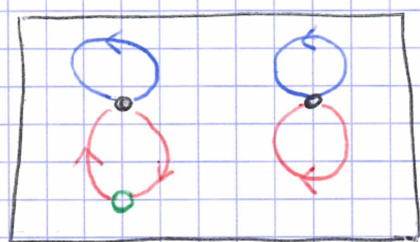
$$Q = \frac{A}{Z}$$ Separately expand in powers of g_0

(instead of expanding Q as above).

$$\Longrightarrow \quad A = \sum_{N=0}^{\infty} A_N \, g_0^{\,N} \, , \qquad Z = \sum_{N=0}^{\infty} Z_N \, g_0^{\,N} \, .$$

Now, disconnected diagrams are allowed.

E.g. A_2 contains the disconnected diagram:



$$\xrightarrow[\mathcal{V}\to\infty]{} \infty$$

$\Rightarrow$ need to work with finite system.

(This is actually a "conventional" QMC method, as discussed in Part 3)

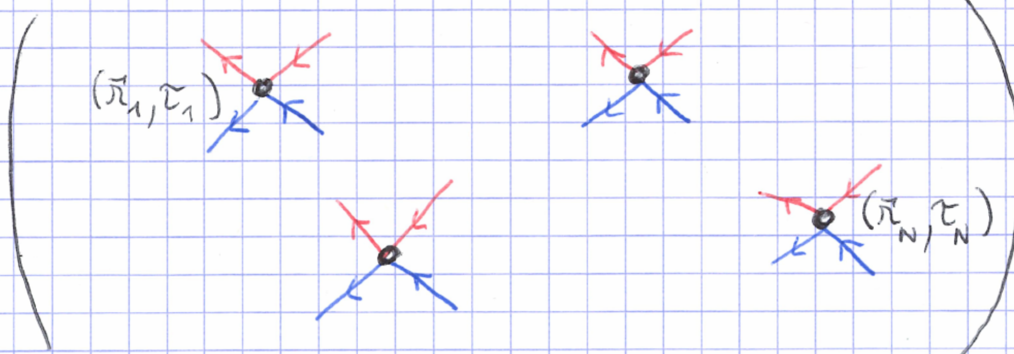
Also, each diagram $\longrightarrow 0$ in the continuum limit $b \to 0$.

<u>But</u>:
- The series always converges
- For $\mu_\uparrow = \mu_\downarrow$: <u>efficient sign-free algorithm</u>

  $\Rightarrow$ one can reach very high orders (several 10^3)

Contribution to Z_N:

$$\sum_{\substack{\text{all} \\ \text{connections}}}$$



$$= \prod_{\sigma=\uparrow,\downarrow} \det \underbrace{\mathcal{G}^{(\sigma)}}_{\downarrow \; N^3 \text{ operations}} ,$$

$$\mathcal{G}^{(\sigma)}_{ij} := G_{0,\sigma}(\vec{r}_i - \vec{r}_j, \tau_i - \tau_j) \, .$$

[Burovski et al., PRL 96, 160402 (2006)] [also: Rubtsov, arXiv: 0302228; PRB 72, 035122]

Chapter 2

Many-body physics: some basic properties

Part 1: Unpolarized unitary gas

Equation of state

$T = 0$ $\qquad$ $\xi \simeq 0.37$

Experiments:

- MIT + correction(Heidelberg) [Ku *et al.*, Science **335**, 563 (2012); Zürn *et al.*, PRL **110**, 135301 (2013)]:

$$\xi = 0.370(5)_{\text{stat}}(8)_{\text{sys}}$$

- USTC [Li *et al.*, Science **375**, 528 (2022)]:

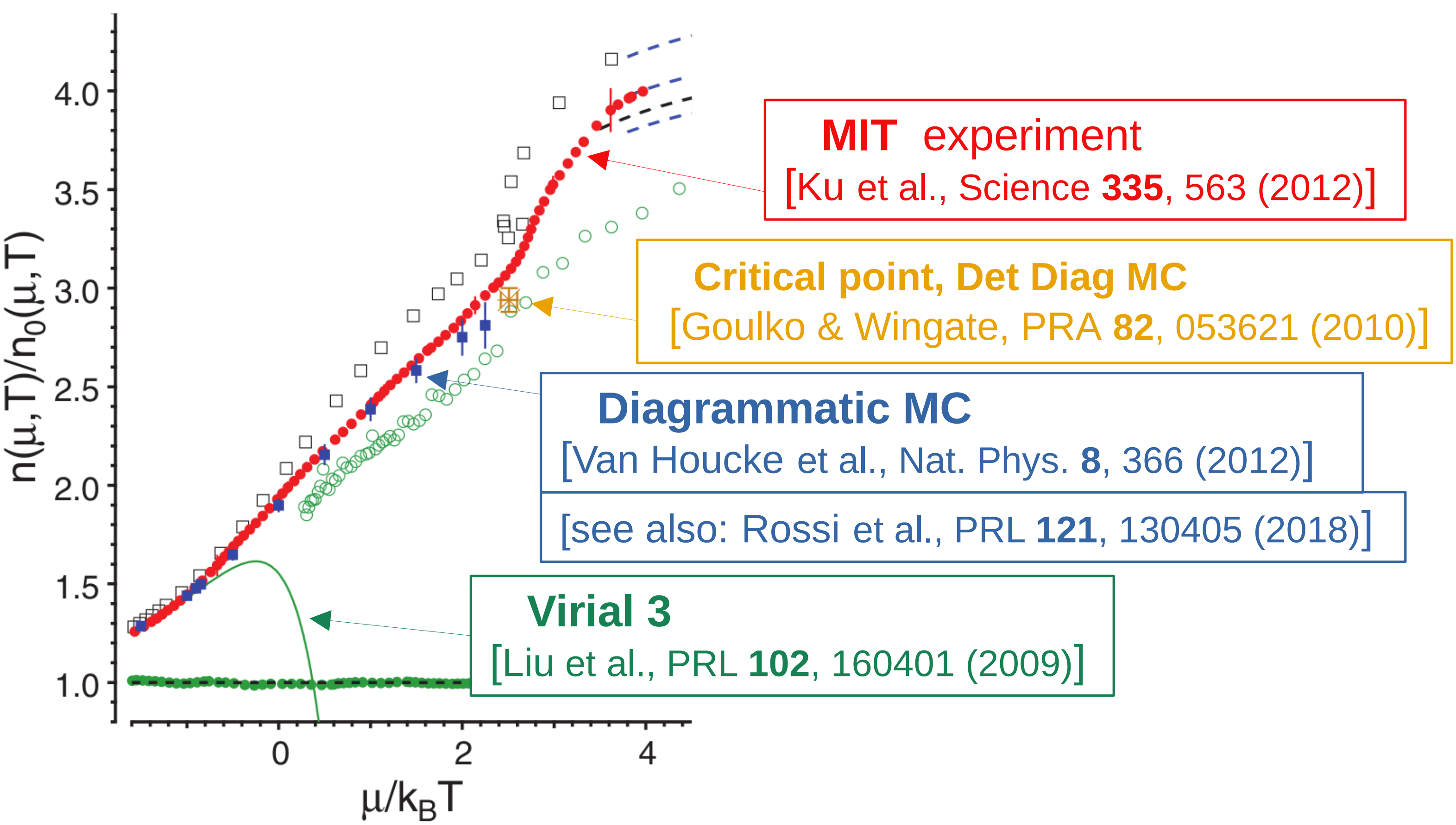
$$\xi = 0.367(9)$$

AFQMC [Carlson *et al.*, PRA **84**, 061602(R) (2011)]: $\qquad \xi = 0.372(5)$

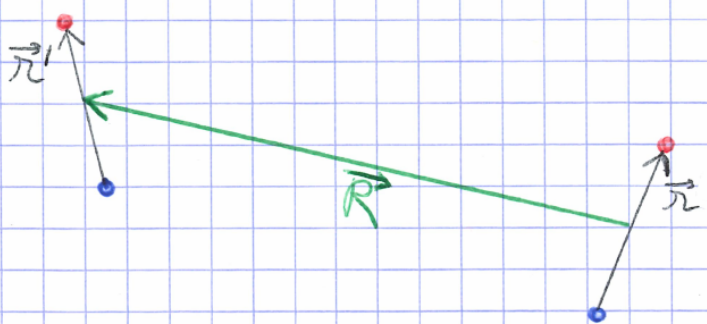
Variational calculations: BCS ansatz: $\xi < 0.59$

Fixed-node QMC: $\xi < 0.38$ [Forbes *et al.*, PRL **106**, 235303 (2011)]

Finite T



Long-range order



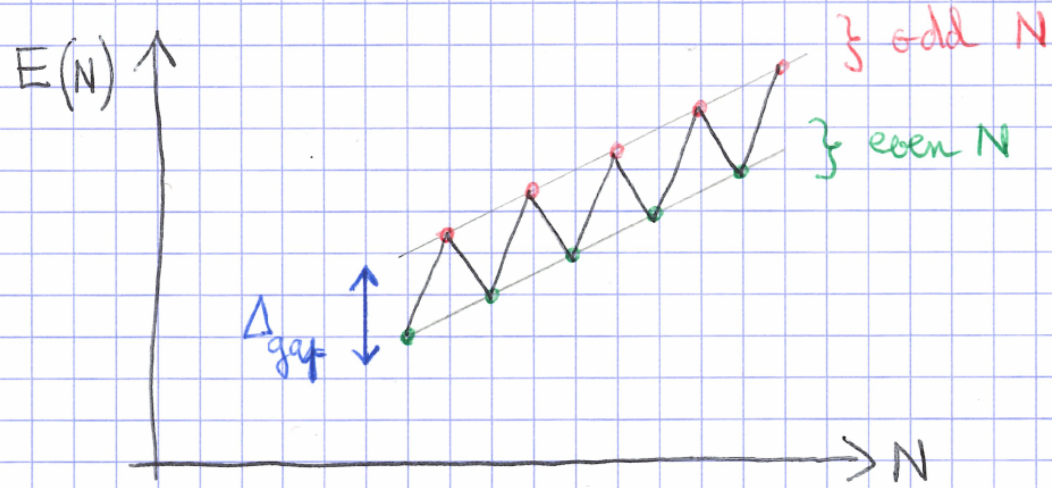
$$\left\langle \hat{\Psi}^{+}_{\uparrow}\left(\vec{R}+\frac{\vec{r}'}{2}\right)\hat{\Psi}^{+}_{\downarrow}\left(\vec{R}-\frac{\vec{r}'}{2}\right)\ \hat{\Psi}_{\downarrow}\left(-\frac{\vec{r}}{2}\right)\hat{\Psi}_{\uparrow}\left(\frac{\vec{r}}{2}\right)\right\rangle$$

$$\xrightarrow[R\to\infty]{} \left(\frac{m}{4\pi\hbar^2}\right)^2 |\Delta|^2 \ \phi_p(r')\ \underbrace{\phi_p(r)}_{\text{Pair wavefunction.}}$$

order parameter

$$\underset{r\to 0}{\sim} \frac{1}{r}$$

Pairing gap



$$\Delta_{gap} = \lim_{M\to\infty}\left[\frac{E(2M+1)+E(2M-1)}{2} - E(2M)\right]$$

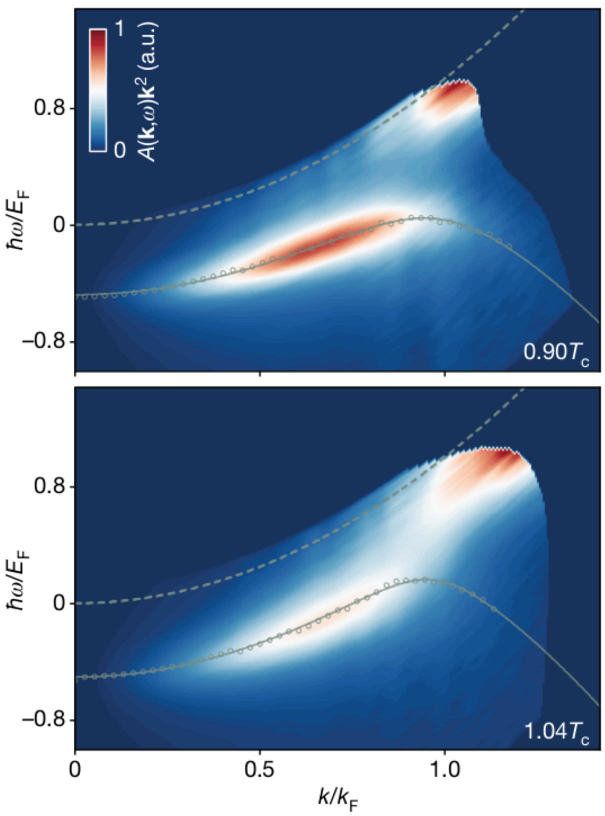
"cost for breaking a pair"

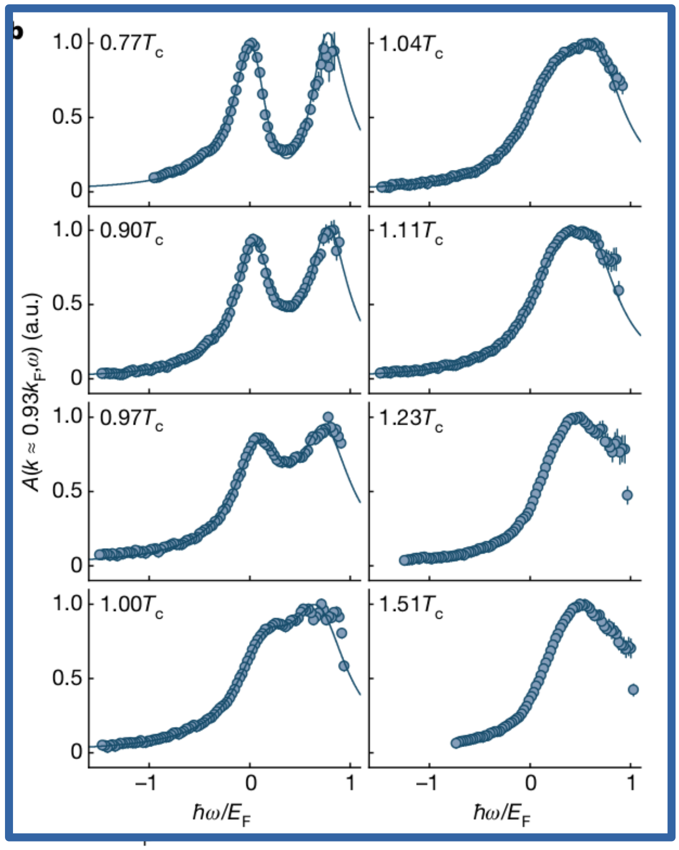
$\boxed{\Delta_{gap} \neq \Delta}$ (except for BCS theory, in regime $\mu > 0$)

	Fixed-node QMC [Carlson & Reddy, PRL 100, 150403 (2008)]	Experiments: MIT [Schirotzek et al, PRL 101, 140403 (2008)]	Experiments: Swinburne [Hoinka et al., Nat. Phys. 13, 943 (2017)]
$\frac{\Delta_{gap}}{\varepsilon_F}$	0.45(5)	0.44(3)	0.47(3)

Spectral function

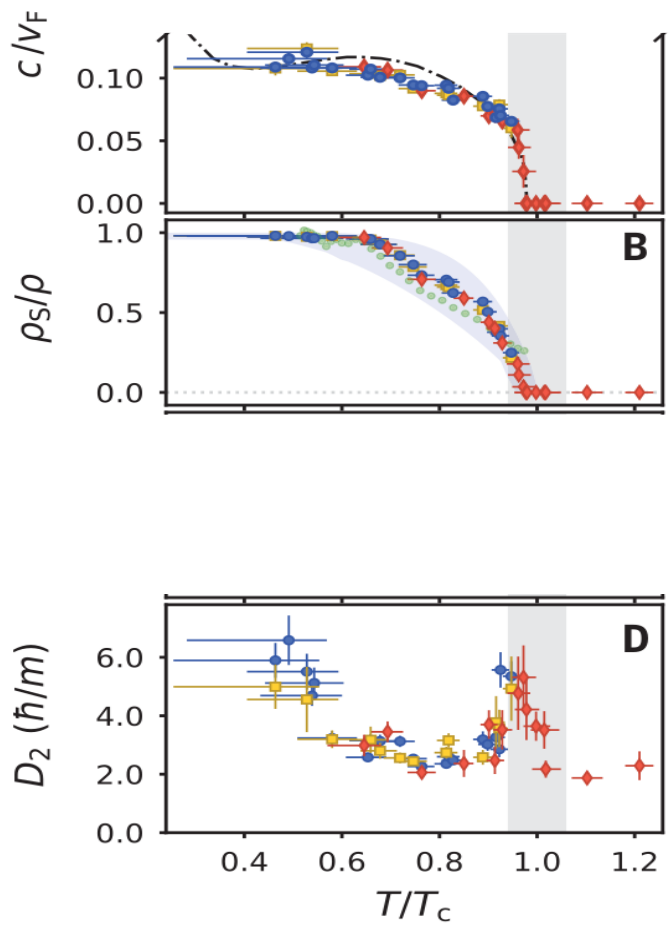
USTC [Li et al., Nature **626,** 288 (2024)]



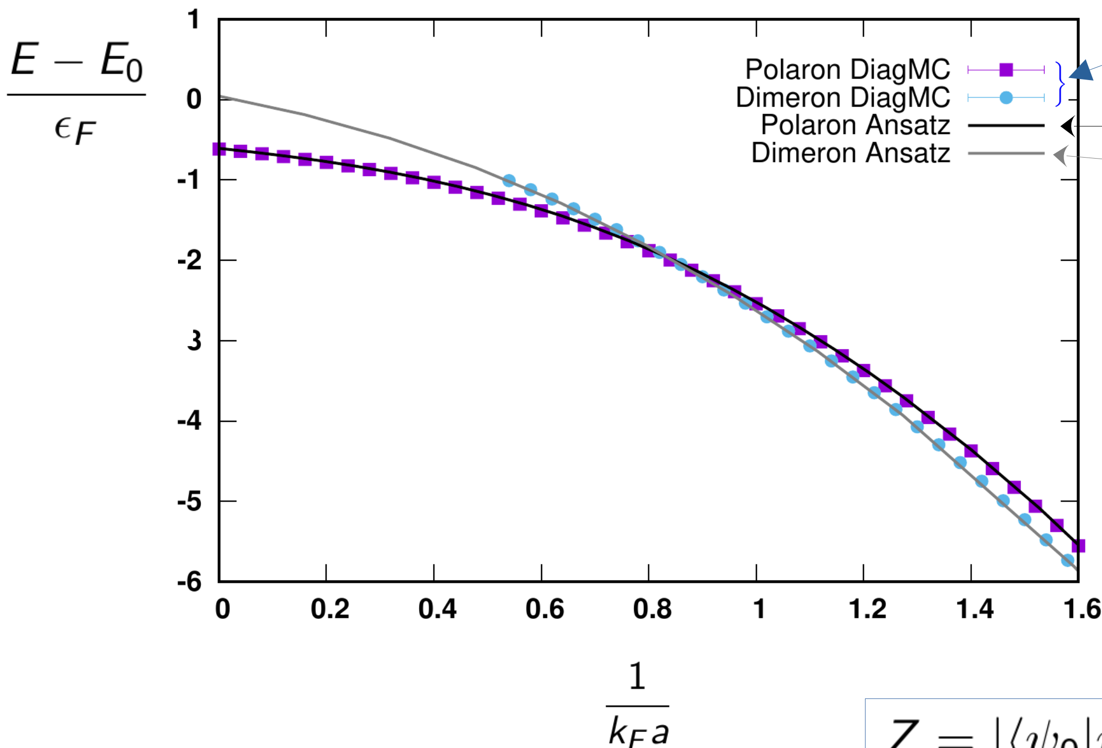


Second sound

MIT [Yan et al., Science **383**, 629 (2024)]



Part 2: Fermi polaron



[Vlietinck et al., PRB **87**, 115133 (2013) ; Prokof'ev & Svistunov, PRB **77**, 020408(R) (2008)]

[Chevy, PRA **74**, 063628 (2006)]

[Punk et al., PRA **80**, 053605 (2009)]

non-interacting: $\boxed{|\psi_0\rangle = c^\dagger_{\vec{0},\downarrow}\,|\mathrm{FS}_\uparrow\rangle}$ $\boxed{E_0}$

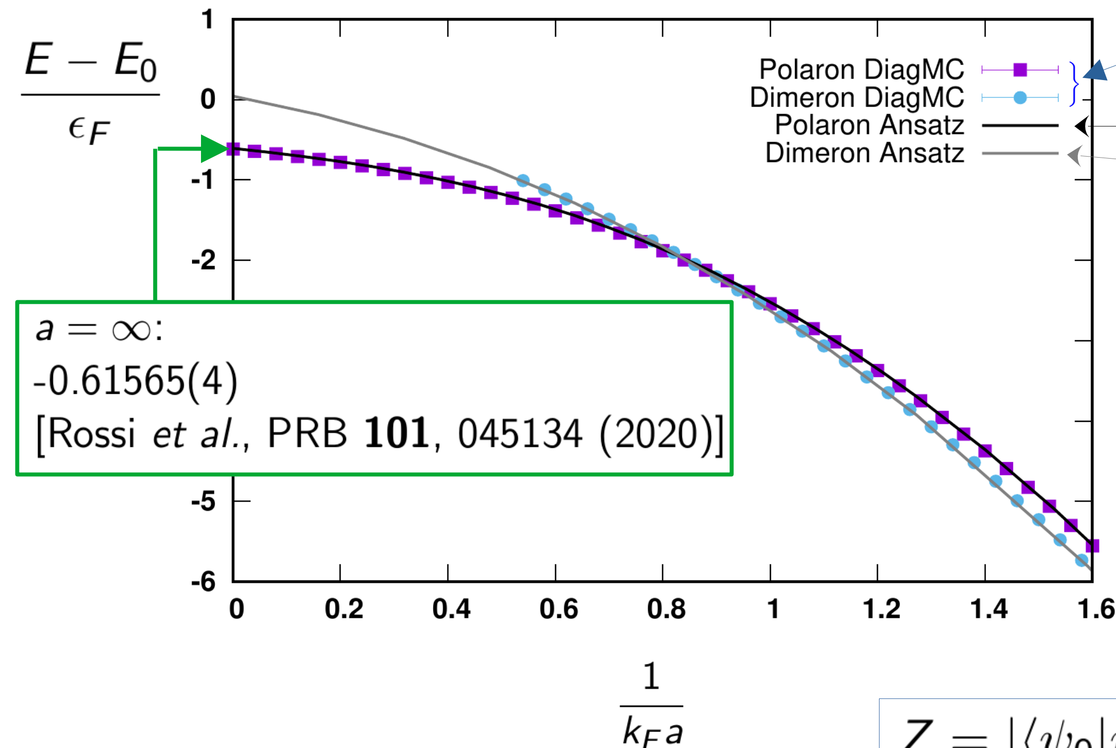
$$Z = |\langle\psi_0|\psi\rangle|^2 \neq 0\,, \quad \text{polaron}$$
$$= 0\,, \quad \text{dimeron}$$

Part 2: Fermi polaron

[Vlietinck et al., PRB **87**, 115133 (2013) ; Prokof'ev & Svistunov, PRB **77**, 020408(R) (2008)]

[Chevy, PRA **74**, 063628 (2006)]

[Punk et al., PRA **80**, 053605 (2009)]



$a = \infty$:
-0.61565(4)
[Rossi *et al.*, PRB **101**, 045134 (2020)]

non-interacting: $|\psi_0\rangle = c^\dagger_{\vec{0},\downarrow}\, |\mathrm{FS}_\uparrow\rangle$ $\quad \boxed{E_0}$

$$Z = |\langle \psi_0 | \psi \rangle|^2 \neq 0\,, \quad \text{polaron}$$
$$= 0\,, \quad \text{dimeron}$$

Part 3: Polarized gas

SF – normal phase transition : 1st order at low T

exp: MIT [Shin et al., Nature **451**, 689 (2008)]
ENS [Nascimbène et al., Nature **463**, 1057 (2010) ;
Navon et al., Science **328**, 729 (2010)]

fixed-node QMC [Pilati & Giorgini, PRL **100**, 030401 (2008)]

unconventional SF phases:

p-wave

FFLO

Lecture 3

2-body and 3-body contacts

Lecture 3

2-body and 3-body contacts

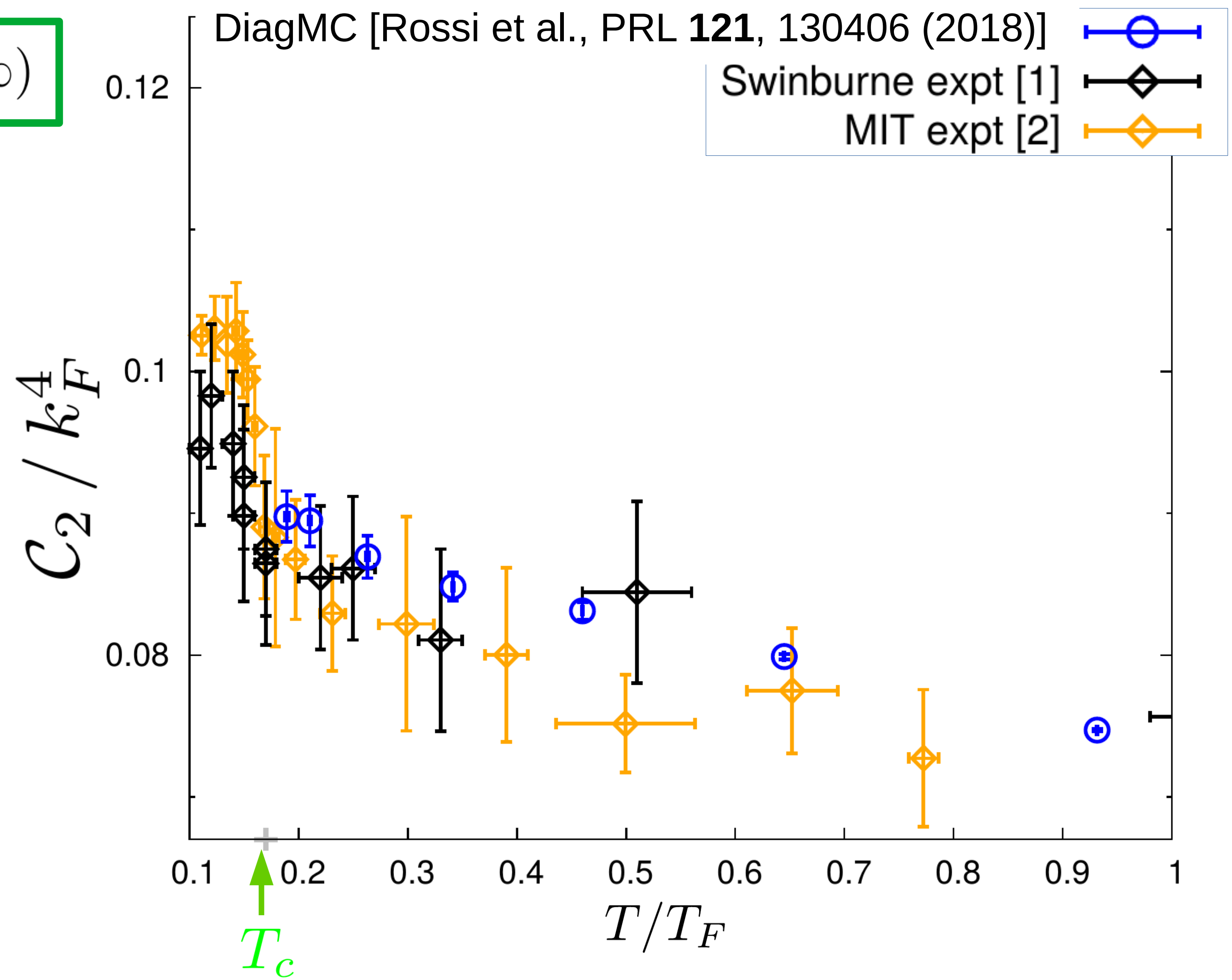
- S. Tan, Ann. Phys. **323**, 2952 (2008); Ann. Phys. **323**, 2971 (2008)
- with Y. Castin: Lect. Notes Phys. **836**, 127 (2012); PRA **86**, 013626 (2012)
- with X. Leyronas: C. R. Phys. **25**, 179 (2024)

Tan's two-body contact

$$\langle\, \hat{n}_\uparrow(\mathbf{r})\, \hat{n}_\downarrow(\mathbf{0})\,\rangle \underset{r\to 0}{\sim} \frac{\mathcal{C}_2}{(4\pi\, r)^2}$$

$$n_\sigma(k) \underset{k\to\infty}{\sim} \frac{\mathcal{C}_2}{k^4} \qquad (\sigma = \uparrow, \downarrow)$$

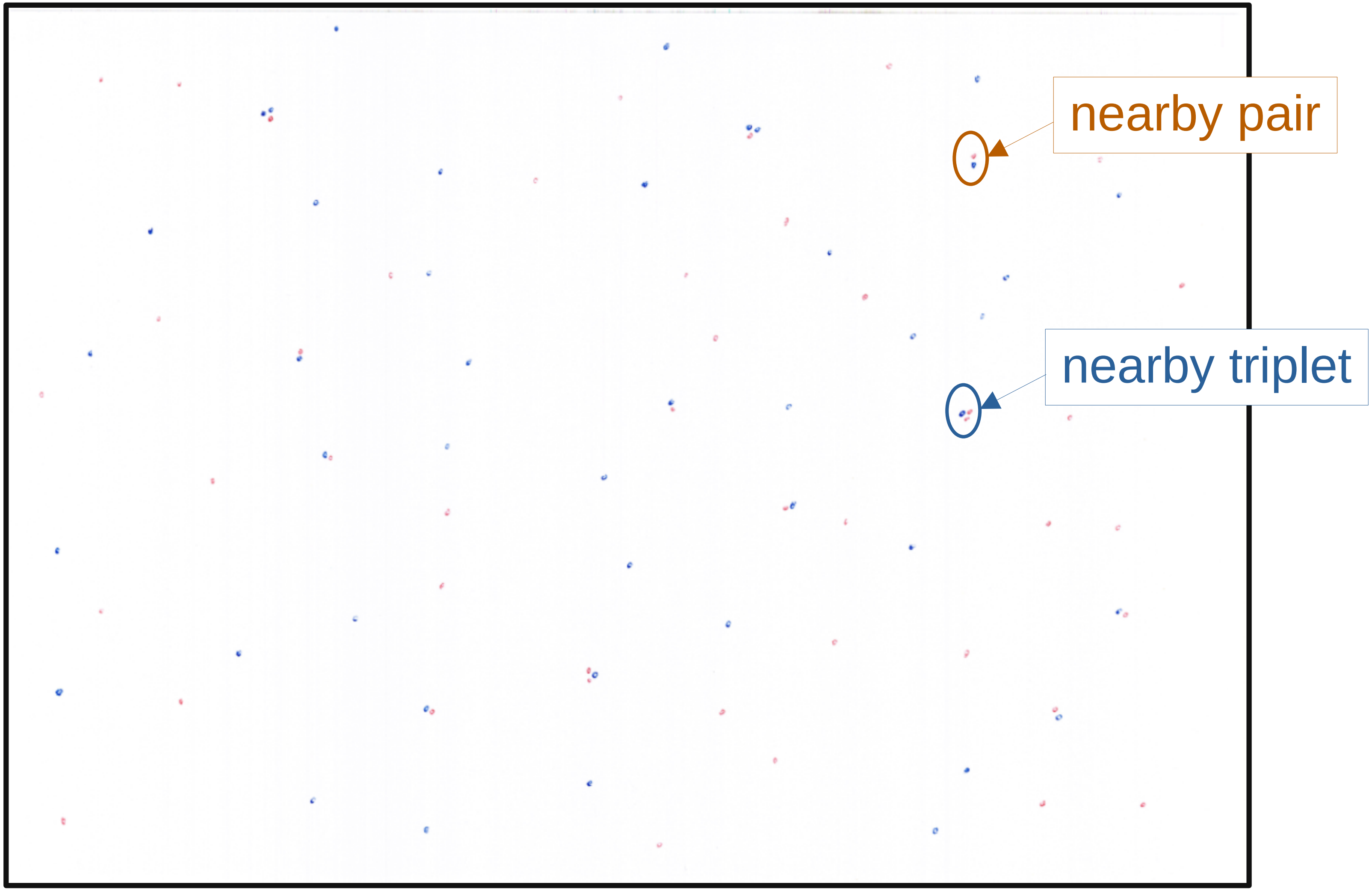
unitary Fermi gas $\quad (a_2 = \infty)$



[1] **Carcy, Hoinka, Lingham, Dyke, Kuhn, Hu, Vale, PRL 2019**
[2] **Mukherjee, Patel, Yan, Fletcher, Struck, Zwierlein, PRL 2019**

NUMBER OF NEARBY PAIRS & TRIPLETS

Gedankenexperiment: Measure positions of $|\uparrow\rangle$ and $|\downarrow\rangle$ atoms



NUMBER OF NEARBY PAIRS & TRIPLETS

$$\boxed{N_2(\epsilon) := \text{number of pairs separated by } r < \epsilon}$$

$$\left[\ \text{without interactions:}\quad N_2^{(0)}(\epsilon) \propto \epsilon^3\ \right]$$

$$\boxed{N_2(\epsilon) \underset{\epsilon\to 0}{\sim} C_2\ \frac{\epsilon}{4\pi}}$$

[Tan 2008]

NUMBER OF NEARBY PAIRS & TRIPLETS

$$\boxed{\text{(CC):} \quad \psi \propto 1/r \quad \text{for} \quad r \to 0}$$

$$\Downarrow$$

$$\boxed{N_2(\epsilon) := \text{number of pairs separated by } r < \epsilon}$$

$$\left[\text{ without interactions: } \quad N_2^{(0)}(\epsilon) \propto \epsilon^3 \right]$$

$$\boxed{N_2(\epsilon) \underset{\epsilon \to 0}{\sim} C_2 \, \frac{\epsilon}{4\pi}}$$

[Tan 2008]

NUMBER OF NEARBY PAIRS & TRIPLETS

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[Tan 2008]

NUMBER OF NEARBY PAIRS & TRIPLETS

$N_2(\epsilon) :=$ number of pairs separated by $r < \epsilon$

$$\boxed{N_2(\epsilon) \underset{\epsilon\to 0}{\sim} C_2 \, \frac{\epsilon}{4\pi}}$$

[Tan 2008]

$\left[\text{ without interactions: } \; N_2^{(0)}(\epsilon) \propto \epsilon^3 \right]$

$N_3(\epsilon) :=$ number of triplets of hyperradius $R < \epsilon$

$$R := \sqrt{\frac{2}{3}\left(r_{ij}^2 + r_{ik}^2 + r_{jk}^2\right)}$$

NUMBER OF NEARBY PAIRS & TRIPLETS

$N_2(\epsilon) :=$ number of pairs separated by $r < \epsilon$

$$\left[\text{ without interactions: } \; N_2^{(0)}(\epsilon) \propto \epsilon^3 \; \right]$$

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[Tan 2008]

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NUMBER OF NEARBY PAIRS & TRIPLETS

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[Tan 2008]

$\left[\text{ without interactions: } N_2^{(0)}(\epsilon) \propto \epsilon^3 \right]$

$N_3(\epsilon) :=$ number of triplets of hyperradius $R < \epsilon$

$$\boxed{N_3(\epsilon) \underset{\epsilon\to 0}{\sim} C_3\ \epsilon^{2s+2}}$$

$$C_3 = \underbrace{C_{2,1}}_{\uparrow\uparrow\downarrow} + \underbrace{C_{1,2}}_{\uparrow\downarrow\downarrow}$$

$s = s(2,1)$
$= 1.7727\ldots$

$\Uparrow$

short-range scaling law: $\psi \propto R^{s-2}$ for $R \to 0$

NUMBER OF NEARBY PAIRS & TRIPLETS

$N_2(\epsilon) :=$ number of pairs separated by $r < \epsilon$

$$\boxed{N_2(\epsilon) \underset{\epsilon \to 0}{\sim} C_2 \frac{\epsilon}{4\pi}}$$

[Tan 2008]

$$\left[\text{without interactions:} \quad N_2^{(0)}(\epsilon) \propto \epsilon^3 \right]$$

$N_3(\epsilon) :=$ number of triplets of hyperradius $R < \epsilon$

$$\boxed{N_3(\epsilon) \underset{\epsilon \to 0}{\sim} C_3 \, \epsilon^{2s+2}}$$

$$2s+2 = 5.54545\ldots$$

$$s = s(2,1) = 1.7727\ldots$$

$$C_3 = \underbrace{C_{2,1}}_{\uparrow\uparrow\downarrow} + \underbrace{C_{1,2}}_{\uparrow\downarrow\downarrow}$$

without interactions:

$$N^{(0)}_{3,\ \mathrm{fermions}}(\epsilon) \propto \epsilon^8$$

$$N^{(0)}_{3,\ \mathrm{distinguishable}}(\epsilon) \propto \epsilon^6$$

NUMBER OF NEARBY PAIRS & TRIPLETS

$N_2(\epsilon) :=$ number of pairs separated by $r < \epsilon$

$$N_2(\epsilon) \underset{\epsilon\to 0}{\sim} C_2 \, \frac{\epsilon}{4\pi}$$

[Tan 2008]

$\left[\text{ without interactions: } \quad N_2^{(0)}(\epsilon) \, \propto \, \epsilon^3 \right]$

$N_3(\epsilon) :=$ number of triplets of hyperradius $R < \epsilon$

$$N_3(\epsilon) \underset{\epsilon\to 0}{\sim} C_3 \, \epsilon^{\boxed{2s+2}}$$

$$\boxed{= 5.54545\ldots}$$

$s = s(2,1)$
$= 1.7727\ldots$

$$C_3 = \underbrace{C_{2,1}}_{\uparrow\uparrow\downarrow} + \underbrace{C_{1,2}}_{\uparrow\downarrow\downarrow}$$

without interactions:

$N^{(0)}_{3,\text{ fermions}}(\epsilon) \, \propto \, \epsilon^8$

$N^{(0)}_{3,\text{ distinguishable}}(\epsilon) \propto \epsilon^6$

BCS ansatz:
$N_{3,\,\text{BCS}}(\epsilon) \propto \epsilon^4$

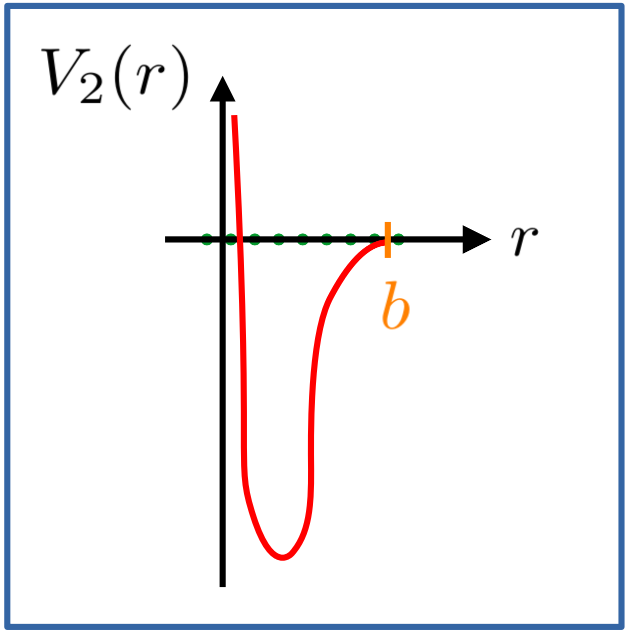
(wrong)

THREE-BODY LOSS RATE

a_3 defined by:

$$\Psi_{\mathsf{m}}(\mathbf{R}) \simeq \left(R^s - \frac{a_3}{R^s}\right)\frac{1}{R^2}\,\phi_{\mathsf{m}}(\boldsymbol{\Omega})$$

in the region $\{b \ll r_{ij} \ll |a_2|,\ \forall i < j\}$



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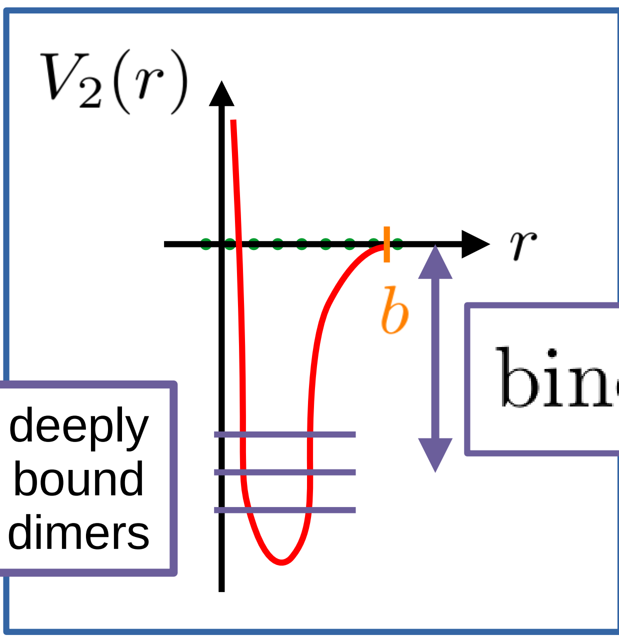


binding energy $\sim \dfrac{\hbar^2}{mb^2}$

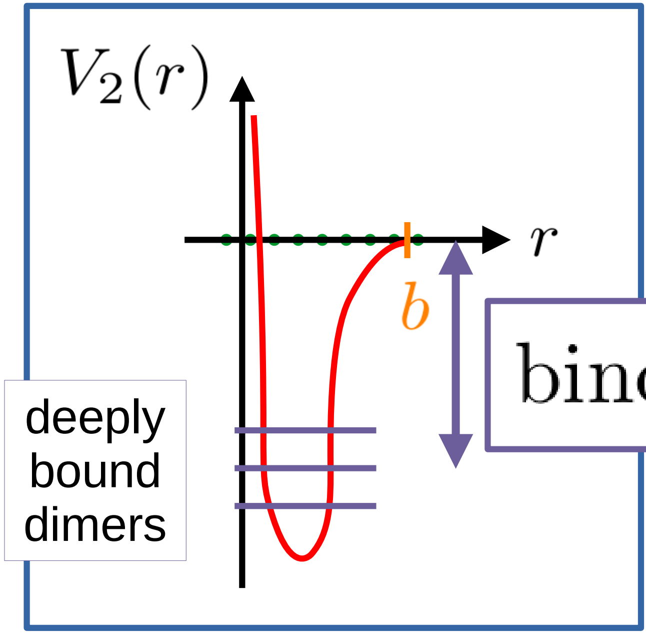
deeply bound dimers

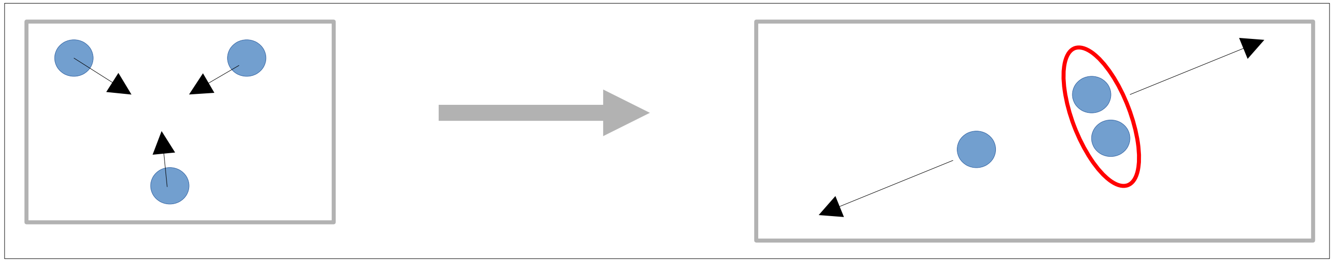
THREE-BODY LOSS RATE

a_3 defined by:

$$\Psi_{\mathsf{m}}(\mathbf{R}) \simeq \left(R^s - \frac{a_3}{R^s}\right)\frac{1}{R^2}\,\phi_{\mathsf{m}}(\mathbf{\Omega})$$

in the region $\{b \ll r_{ij} \ll |a_2|,\ \forall i<j\}$





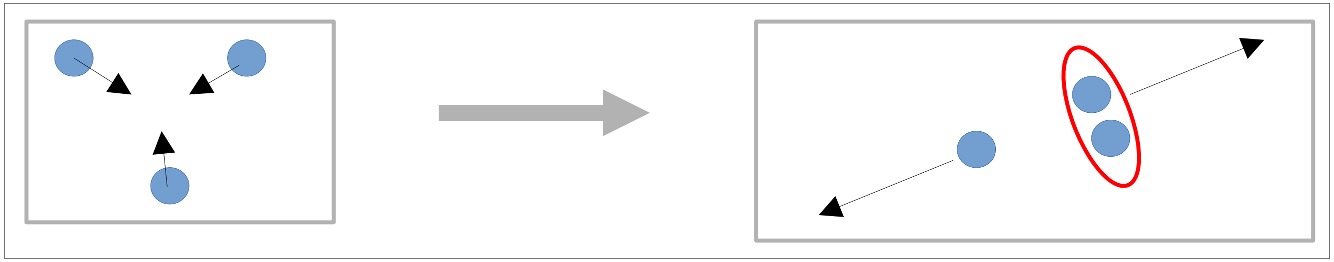
binding energy $\sim \dfrac{\hbar^2}{mb^2}$

THREE-BODY LOSS RATE

a_3 defined by:

$$\Psi_{\mathsf{m}}(\mathbf{R}) \simeq \left(R^s - \frac{a_3}{R^s}\right)\frac{1}{R^2}\,\phi_{\mathsf{m}}(\boldsymbol{\Omega})$$

in the region $\{b \ll r_{ij} \ll |a_2|,\ \forall i < j\}$



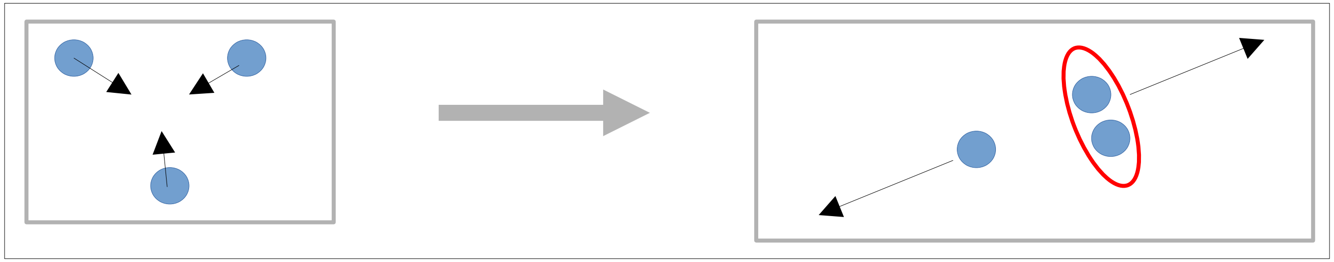
$$\boxed{\frac{dN}{dt} = -3\,\Gamma_3}$$

THREE-BODY LOSS RATE

a_3 defined by:

$$\Psi_{\mathsf{m}}(\mathbf{R}) \simeq \left(R^s - \frac{a_3}{R^s}\right) \frac{1}{R^2}\, \phi_{\mathsf{m}}(\mathbf{\Omega})$$

in the region $\{b \ll r_{ij} \ll |a_2|,\ \forall i < j\}$



$$\frac{dN}{dt} = -3\,\Gamma_3$$

$$\boxed{\Gamma_3 \simeq -\frac{\hbar}{m}\, 8s(s+1)\, C_3\, \mathrm{Im}\, a_3}$$

[FW & X. Leyronas, C. R. Phys. **25**, 179 (2024)]

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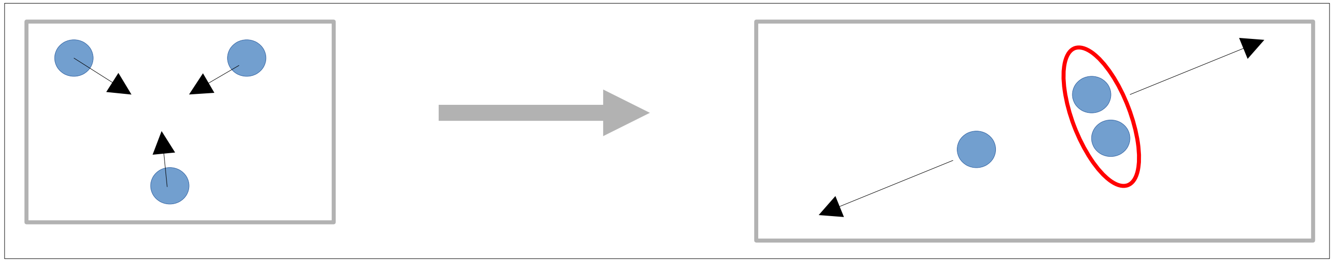

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order of magnitude:

$$a_3 \sim b^{2s}$$

$$\Rightarrow \frac{-\dot{N}/N}{\epsilon_F/\hbar} \sim (k_F b)^{2s} \ll 1$$

[Petrov *et al.*, PRL 93, 090404 (2004)]

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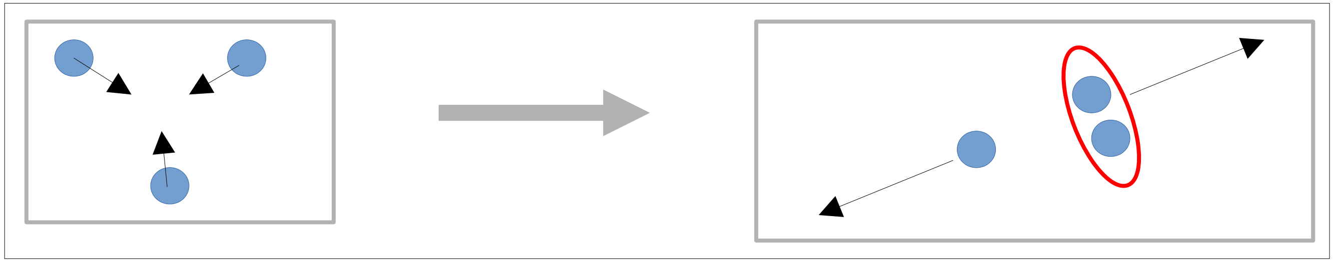

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$$C_3|_{\mathrm{finite-range}} \simeq C_3|_{\mathrm{zero-range}}$$

Derivation : $\mathbf{X} := (\mathbf{r}_1, \ldots, \mathbf{r}_N)$

$$\text{Gamov state:} \begin{cases} H\,\psi(\mathbf{X}) = E\,\psi(\mathbf{X}), \quad E \in \mathbb{C} \\ \psi(\mathbf{X}) \underset{\infty}{\sim} \text{outgoing (atom + deep-dimer) wave} \end{cases}$$

$$\Gamma_3 = \int_{\mathcal{S}_3} \mathbf{d^{3N-1}S} \cdot \frac{\hbar}{m}\,\mathrm{Im}\,(\psi^* \boldsymbol{\nabla}_{\mathbf{X}}\,\psi)$$

$$\psi(\mathbf{X}) \underset{\mathbf{X}\in\mathcal{S}_3}{\simeq} \left(R^s - \frac{a_3}{R^s}\right)\frac{1}{R^2} \times \sum_{\mathsf{m}=-1}^{+1} \phi_{\mathsf{m}}(\boldsymbol{\Omega})\; B_{\mathsf{m}}(\mathbf{C}; \mathbf{r}_4, \ldots, \mathbf{r}_N)$$

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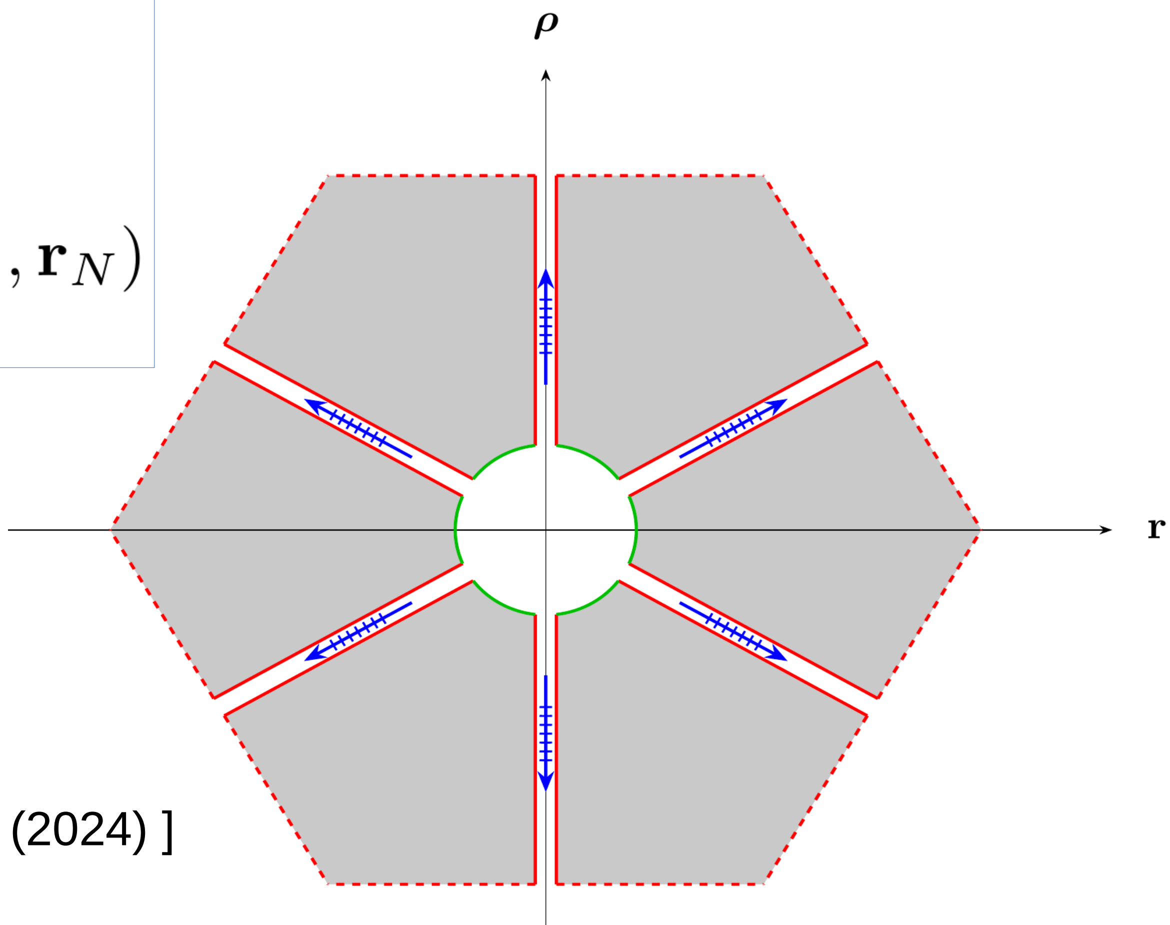

[FW & X. Leyronas, C. R. Phys. **25**, 179 (2024)]

C_3 in non-degenerate limit

[X. Leyronas & FW,
in preparation]

C_3 in non-degenerate limit

unitary Fermi gas $\quad (a_2 = \infty)$

$$\mathcal{C}_3 := \frac{C_3}{\text{Volume}}$$

Virial expansion
solution of 3-body problem

$$\mathcal{C}_3 \simeq n^3 \left(\frac{\hbar^2}{m k_B T}\right)^{2-s} \times 4.5552892\ldots$$

$$\mathcal{C}_3 = \zeta_3\left(\frac{T}{T_F}\right) n^{\frac{2s+5}{3}}$$

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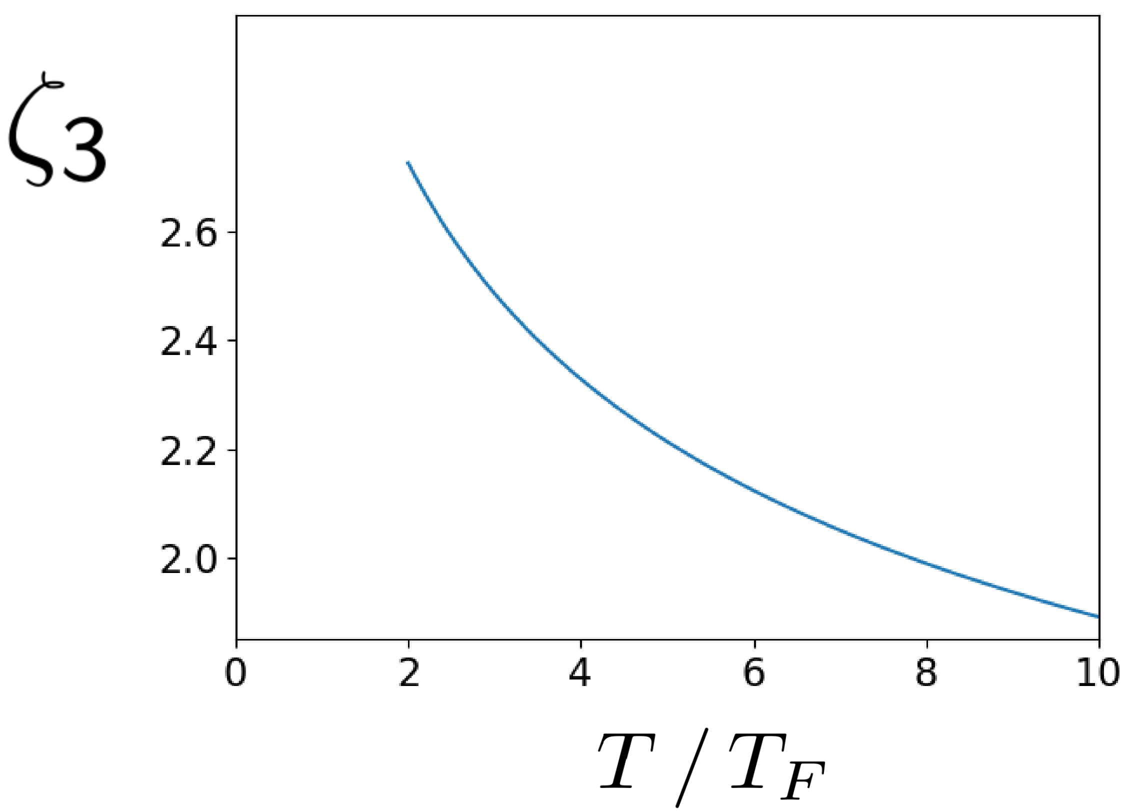

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$$\zeta_3 \propto \frac{1}{(T/T_F)^{2-s}}$$
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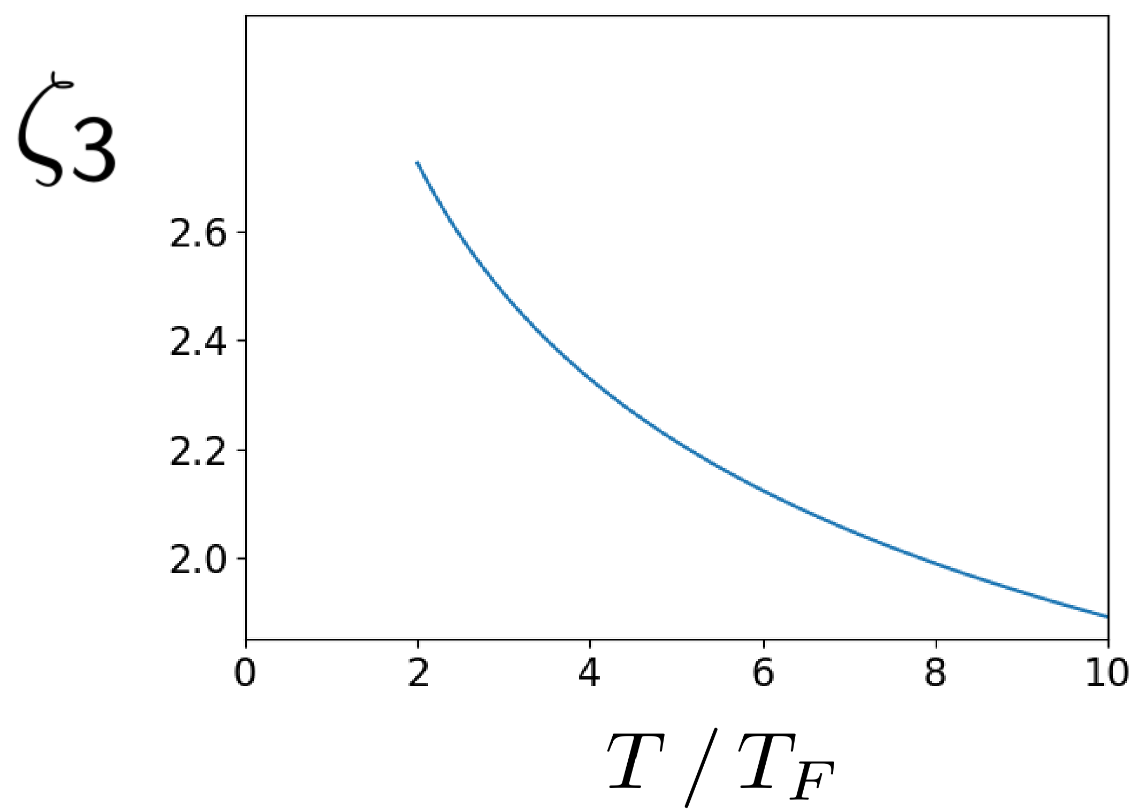

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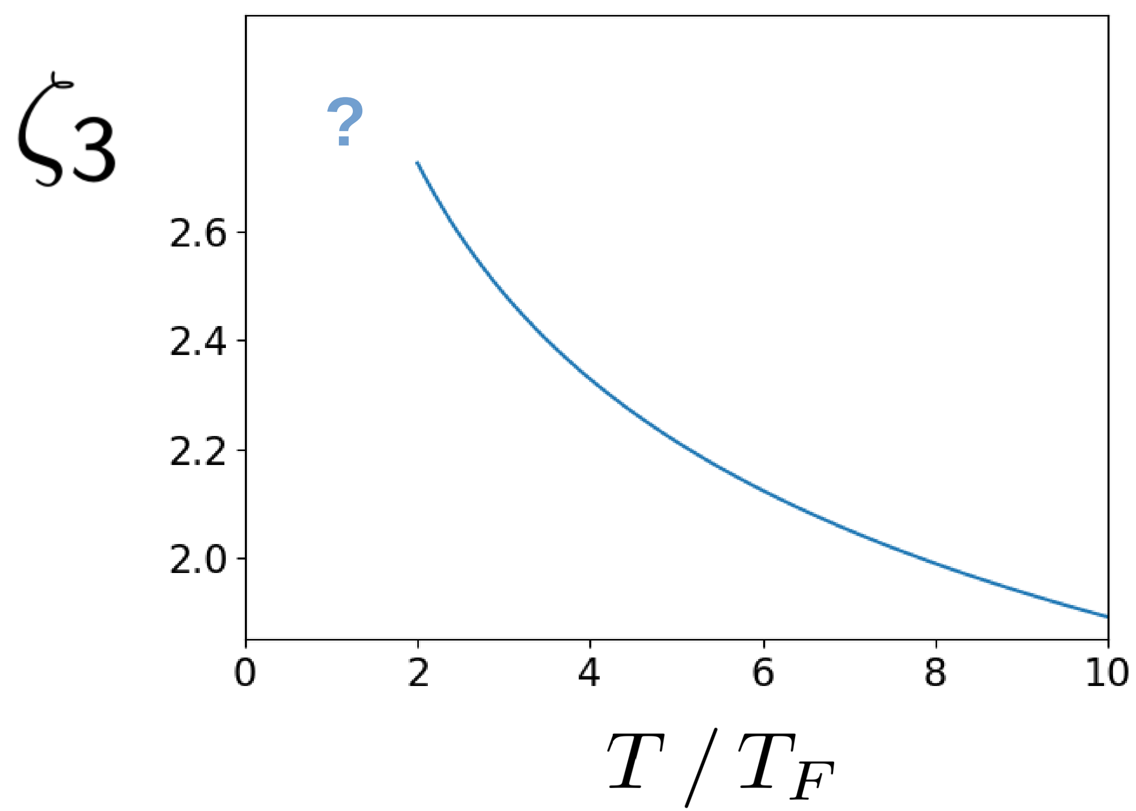

C_3 for the degenerate unitary gas

article in preparation

with

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Generalization

$$a_3^{(\uparrow\uparrow\downarrow)} \neq a_3^{(\uparrow\downarrow\downarrow)}$$

$$\Gamma_3 = \Gamma_3^{(\uparrow\uparrow\downarrow)} + \Gamma_3^{(\uparrow\downarrow\downarrow)}$$

$$\begin{cases} \Gamma_3^{(\uparrow\uparrow\downarrow)} = -\dfrac{\hbar}{m}\, 8s(s+1)\, C_3^{(\uparrow\uparrow\downarrow)}\, \mathrm{Im}\, a_3^{(\uparrow\uparrow\downarrow)} \\ \Gamma_3^{(\uparrow\downarrow\downarrow)} = -\dfrac{\hbar}{m}\, 8s(s+1)\, C_3^{(\uparrow\downarrow\downarrow)}\, \mathrm{Im}\, a_3^{(\uparrow\downarrow\downarrow)} \end{cases}$$

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