

Superfluidity at Supersonic Speed?

Law *et al.* [1] recently reported that supersonic motion of an obstacle [a localized potential $U(x)$] could take place without any radiation in a one dimensional fluid described by the Gross-Pitaevski equation (GPE),

$$i\partial_t\Psi = -\frac{1}{2}\partial_{xx}\Psi - \Psi + |\Psi|^2\Psi + U(x-vt)\Psi. \quad (1)$$

(i) We first show that radiation emission is always nonzero for generic potentials. A dissipationless motion is possible when a stationary solution $\Psi(x-vt)$ of finite energy exists in the obstacle frame. The modulus r of Ψ obeys [1,2] (the fluid conservation equation serves to eliminate the phase of Ψ)

$$\frac{\partial_{xx}r}{2r} + F_v(r) = U(x), \quad (2)$$

$$F_v(r) = \frac{v^2}{2}\left(1 - \frac{1}{r^4}\right) + 1 - r^2.$$

Linearization of Eq. (2) for large $|x|$ (where the potential is negligible) as $r(x) = 1 + \eta(x)$ with $\eta \ll 1$ gives $\partial_{xx}\eta + 4(v^2 - 1)\eta = 0$. So, for a *supersonic* motion ($v > 1$), there are two oscillating modes at $x = -\infty$ and two at $x = +\infty$. Once $r \rightarrow 1$ is imposed at $x = -\infty$, the solution of the second order Eq. (2) is entirely determined and for a *generic* potential, one cannot avoid oscillations around $x = +\infty$. Therefore, there are *no* supersonic stationary solutions of finite energy. This is shown in Fig. 1 for a Gaussian potential $U(\sqrt{\beta}x) = U_0 \exp(-\beta x^2)$. Within a two-parameter potential family, oscillations can be absent for special velocity-dependent parameters. Such special examples were found in [1] but are, of course, not precisely physically realizable.

(ii) In a time-dependent context, the radiation emission rate is directly related to the amplitude A of the oscillatory tail [3]. A exponentially decreases with the ratio of the potential variation length to the GPE coherence length (see Fig. 1 inset in the Gaussian potential case) for *any* analytic potential that is weak enough for the flow to be locally supersonic everywhere [Eq. (3)]. This can be analytically shown as in a model water wave equation [3].

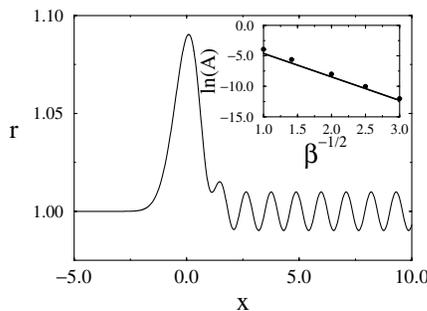


FIG. 1. Solution of Eq. (2) for $U_0 = 1$, $v = 3$, and $\beta = 1$. Inset: the oscillation amplitude for $x \gg 1$ when β is varied (circle). A line of slope $-I_c = -3.85$ [numerical evaluation of the integral in Eq. (5)] is drawn for comparison.

For small β , r is obtained in powers of β starting from the “adiabatic” approximation $r_0(x)$ which simply consists in dropping the derivative term in Eq. (2). This determines r_0 algebraically in function of U . The solution is real only when the maximum of $U(x)$ is smaller than the maximum $U_c = F_v(v^{1/3})$ of F_v attained for $r_c = v^{1/3}$. This gives (and explains) the condition

$$U_0 \leq U_c = (v^2 - 3v^{2/3} + 2)/2 \quad (3)$$

obtained in [1]. The function $r_0(x)$ as well as the higher orders in the expansion, being algebraic functions of $U(x)$, do not show oscillations at infinity. The oscillation amplitude is too small to be visible at any finite order in perturbation. It can be computed by resumming the expansion near one of its complex singularities which are either those of $U(z)$ or of the algebraic inversion giving r_0 from U . For the analytic Gaussian potential, the second possibility gives a square root singularity at $z_c = i\sqrt{\log(U_c/U_0)}$, where $U(z_c) = U_c$ and $r_0(z_c) = r_c$. Rescaling z and r in the neighborhood of $z = z_c$ as $z = z_c - i(\beta/r_c)^{2/5} \times (24|U'|)^{-1/5} \xi$, $r = r_c + \beta^{1/5}(432r_c)^{-1/5}|U'|^{2/5}Q$ shows that the leading order resummed expansion satisfies the (Painlevé) equation,

$$\partial_{\xi\xi}Q = -Q^2 + \xi. \quad (4)$$

Q (and therefore r) is not purely real on the imaginary z axis with $\text{Im}(Q) \sim \xi^{-1/8} \exp(-4\sqrt{2}\xi^{5/4}/5)$ for $\xi \rightarrow +\infty$. This signals the occurrence of oscillations in $r(x)$ for x real and large. Matching the asymptotic behavior of Q to the small- β expansion gives a WKB correction and the oscillation amplitude (A) asymptotics,

$$A \sim \beta \exp(-I_c/\sqrt{\beta}), \quad (5)$$

with $I_c = \int_{r_0^2(0)}^{r_c^2} \frac{d\rho}{F_v(\sqrt{\rho})} \sqrt{\log[F_v(\sqrt{\rho})/U_0]} (\frac{v^2}{\rho^3} - 1)^{3/2}$.

So, the absence of wave emission in the simulations of [1] is *unrelated* to the existence of special radiationless potentials. This explains the otherwise puzzling fact that it was also observed with a Gaussian potential. It is simply due to the choice of a slowly varying potential.

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