7 Algebraic Bethe Ansatz

It is possible to continue the formal developments of the previous chapter and obtain a completely algebraic derivation of the Bethe Ansatz equations (5.31) and the corresponding expression (5.24) for the transfer matrix eigenvalues. There are at least two reasons for following this route.

The first reason is that the elimination of unwanted terms is somewhat tricky, and it might be hard to see if one has taken into account all possible processes when generalising from $n = 1, 2, \ldots$ computations to the general case. The algebraic approach will add clarity to this step.

The second reason is that the algebraic Bethe Ansatz approach makes contact with a rich mathematical structure known as affine Hopf algebras. Within this structure, results about Lie algebras (and even Lie superalgebras) make possible the generalisation from the spin-$\frac{1}{2}$ six-vertex model to infinite classes of higher-spin integrable models. Exposing this in some detail requires to make contact also with the underlying quantum group, which for the six-vertex model is $U_q(sl(2))$. Since this is a more advanced subject, we refrain from treating it in this introduction.

The word "algebraic" should of course not conceal the fact that once the Bethe Ansatz equations for a given model have been found, there is usually a fair amount of analytical work to be done in order to derive the free energy in the thermodynamic limit and the critical exponents.

7.1 Monodromy matrix

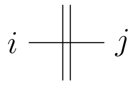
We define the monodromy matrix $T(u)$ as the same product over R-matrices as used in defining the transfer matrix (6.4), but without the trace over the auxiliary space:

$$T(u) = R_{\mathrm{a}L} R_{\mathrm{a}L-1} \cdots R_{\mathrm{a}2} R_{\mathrm{a}1} \,. \tag{7.1}$$

Thus $T(u)$ is an endomorphism on the auxiliary space V_{a} and we have

$$t(u) = \mathrm{Tr}_{\mathrm{a}}\, T(u) \,. \tag{7.2}$$

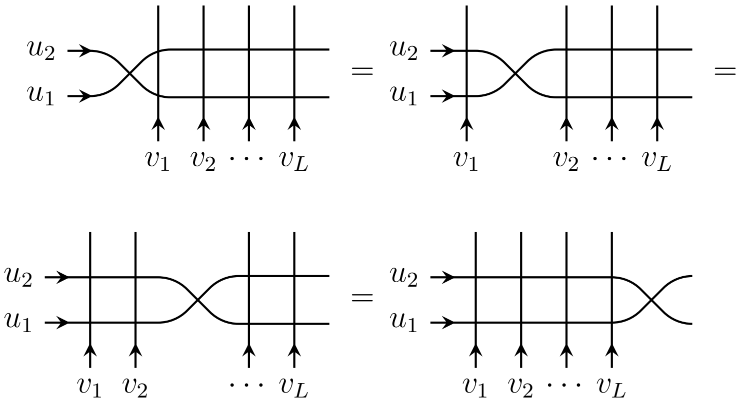
When several auxiliary spaces are involved we shall sometimes use the notation $T_{\mathrm{a}}(u)$ to make clear what space is involved. We shall also denote matrix elements of $T(u)$ using the same convention as for the R-matrix, and sometimes represent them graphically as

$$T_i^j(u) =$$ 

(7.3)

These matrix elements are operators acting in the quantum spaces, here shown symbolically as a double line.

We can repeat the reasoning of (6.11)

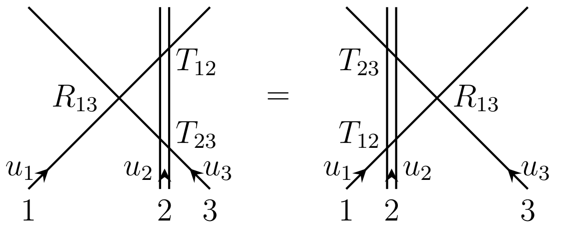


(7.4)

to establish that

$$R_{\rm ab}(u-v)\big(T_{\rm a}(u)T_{\rm b}(v)\big) = \big(T_{\rm b}(v)T_{\rm a}(u)\big)R_{\rm ab}(u-v)\,. \tag{7.5}$$

This identity is known popularly as the RTT relation. Using the double line convention of (7.3) it can also be written pictorially

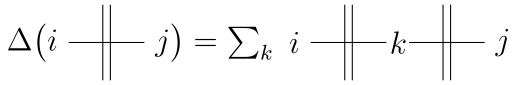


(7.6)

7.2 Co-product and Yang-Baxter algebra

A Yang-Baxter algebra $\mathcal{A}$ is a couple (R, T) satisfying the RTT relation (7.5). Its generators are the matrix elements $T_i^j(u)$. It is equipped with a product, obtained graphically by stacking two monodromy matrices along a common quantum space (represented as a double line).

In addition to this product, $\mathcal{A}$ is also equipped with a co-product Δ,[20] obtained graphically by glueing together two monodromy matrices along a common auxiliary space (represented as a single line). We have



(7.7)

The co-product thus serves to map the algebra $\mathcal{A}$ into the tensor product $\mathcal{A} \otimes \mathcal{A}$:

$$\begin{aligned} \Delta \quad &: \quad \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \\ T_i^j(u) \quad &\mapsto \quad \sum_k T_i^k(u) \otimes T_k^j(u) \end{aligned} \tag{7.8}$$

while preserving the algebraic relations of $\mathcal{A}$.

In particular, the co-product ΔT_i^j must again satisfy the RTT relation (7.5). It is a nice exercise to understand what this means and to prove it.

An algebra equipped with a product and a co-product is called a *bi-algebra*. To be precise, we need a little more structure (co-associativity, existence of a co-unit, ...). If in addition we have an antipode (and if various diagrams commute) one arrives at a Hopf algebra.

7.3 Six-vertex model

When the auxiliary space is $\mathbb{C}^2$, the matrix elements of the monodromy matrix are usually denoted as follows:

$$T_0^0(u) = A(u)\,, \quad T_1^0(u) = B(u)\,, \quad T_0^1(u) = C(u)\,, \quad T_1^1(u) = D(u)\,. \tag{7.9}$$

Recall that the structure constants of a Lie algebra provide a representation, known as the adjoint. In the same way, the R-matrix provides a representation of dimension 2 of the Yang-Baxter algebra. Indeed, in the

[20]This Δ should not be confused with the anisotropy parameter of the six-vertex model (XXZ spin chain).

special case where the double line is just a single line, the monodromy matrix reduces to the R-matrix:

$$\left(T_i^j(u)\right)_l^k = R_{il}^{jk}(u)\,. \tag{7.10}$$

The RTT relation is then nothing but the Yang-Baxter relation for the R-matrix.

The notation (7.9) just amounts to reading the R-matrix as a 2×2 matrix of blocks of size 2×2. As in (6.14) we have

$$R = \left[\begin{array}{cc|cc} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_3 & \omega_6 & 0 \\ \hline 0 & \omega_5 & \omega_4 & 0 \\ 0 & 0 & 0 & \omega_2 \end{array}\right] = \left[\begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array}\right]. \tag{7.11}$$

We recall the usual weights a, b, c (which now depend on the spectral parameter u), and we take the gauge $\eta = u$ in (6.15):[21]

$$a(u) = \sin(\gamma - u)\,, \quad b(u) = \sin u\,, \quad c(u) = \sin\gamma\,. \tag{7.12}$$

We have then in explicit notation, and in terms of Pauli matrices:

$$\begin{aligned} A(u) &= \left[\begin{array}{cc} a(u) & 0 \\ 0 & b(u) \end{array}\right] = \frac{a(u)+b(u)}{2} I + \frac{a(u)-b(u)}{2}\sigma^z\,, \\ B(u) &= \left[\begin{array}{cc} 0 & 0 \\ c(u) & 0 \end{array}\right] = \frac{c(u)}{2}(\sigma^x - i\sigma^y) = c(u)\sigma^-\,, \\ C(u) &= \left[\begin{array}{cc} 0 & c(u) \\ 0 & 0 \end{array}\right] = \frac{c(u)}{2}(\sigma^x + i\sigma^y) = c(u)\sigma^+\,, \\ D(u) &= \left[\begin{array}{cc} b(u) & 0 \\ 0 & a(u) \end{array}\right] = \frac{a(u)+b(u)}{2} I - \frac{a(u)-b(u)}{2}\sigma^z\,. \end{aligned} \tag{7.13}$$

Note that $B(u)$ (resp. $C(u)$) is acts as a creation (resp. annihilation) operator on the quantum space, with respect to the pseudo-vacuum in which all spins are up. We shall see below that this interpretation remains valid when taking co-products: $B(u)$ transforms n particle states into $n+1$ particle states (and vice versa for $C(u)$).

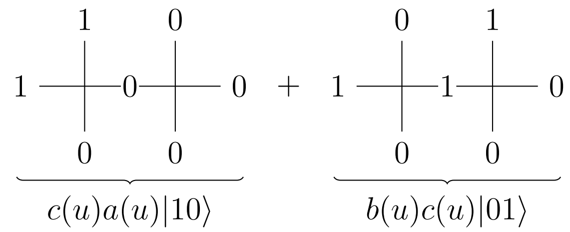
[21] To go from this convention to that of Gómez *et al*, take $u \to -iu$, $\gamma \to \pi - \gamma$, and divide all weights by $-i$.

7.3.1 Co-product

Establishing how the co-product acts on the operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$ will turn out to be an important ingredient in the sequel. In more formal terms, we wish to obtain a representation of the six-vertex Yang-Baxter algebra $\mathcal{A}$ on the space $V^{\otimes L}$.

Let us begin by examining the case $L = 2$ in details. Consider for instance the construction of $\Delta B(u)$. We have:

$\Delta B(u)|00\rangle =$



(7.14)

Here the left and right indices define $B(u) = T_1^0(u)$, and the co-multiplication implies a sum over the middle index. The bottom (resp. top) indices define the in-state (resp. out-state) of the quantum spaces, here denoted as kets.

Proceeding in the same way for the three other in-states, we find that $\Delta B(u)$ can be written in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ as

$$\begin{aligned}
\Delta B(u) &= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ b(u)c(u) & 0 & 0 & 0 \\ \hline c(u)a(u) & 0 & 0 & 0 \\ 0 & c(u)b(u) & a(u)c(u) & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline c(u)a(u) & 0 & 0 & 0 \\ 0 & c(u)b(u) & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ b(u)c(u) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & a(u)c(u) & 0 \end{array}\right] \\
&= B(u) \otimes A(u) + D(u) \otimes B(u)\,. \qquad (7.15)
\end{aligned}$$

Repeating the working for the three other operators, the complete co-multiplication table reads:

$$\begin{aligned}
\Delta A(u) &= A(u) \otimes A(u) + C(u) \otimes B(u)\,, \\
\Delta B(u) &= B(u) \otimes A(u) + D(u) \otimes B(u)\,, \\
\Delta C(u) &= C(u) \otimes D(u) + A(u) \otimes C(u)\,, \\
\Delta D(u) &= D(u) \otimes D(u) + B(u) \otimes C(u)\,. \qquad (7.16)
\end{aligned}$$

To generalise this construction from $L = 2$ to arbitrary L it suffices to use the associativity of the co-multiplication. Indeed for $L \geq 2$ we have

$$\begin{aligned} \Delta^{L-1} &: \mathcal{A} \to \mathcal{A}^{\otimes L} \\ \Delta^{L-1} &\mapsto (I^{\otimes L-2} \otimes \Delta)\Delta^{L-2} \,. \end{aligned} \tag{7.17}$$

It is a useful exercise to try this out for $L = 3$. On one hand, one can formally compute the co-products from (7.16)–(7.17). On the other hand, the results can be checked by a direct computation along the lines of (7.14).

In the following we shall simplify the notation and write, for example, $B(u)$ instead of $\Delta^{L-1}B(u)$. Thus $B(u)$ is an operator that acts on all L spaces in the tensor product $V^{\otimes L}$. Using (7.16)–(7.17) repeatedly it can be expanded in fully tensorised form, as an expression with 2^{L-1} terms. This expanded form is (7.16) for $L = 2$, and the expression for $L = 3$ is contained in the above exercice. The factors entering each term in the expanded form act on a single space V.

7.3.2 Commutation relations

The operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$ satisfy a set of commutation relations which follow as a direct consequence of the RTT relation (7.5).

To see in details how this works, we first write out the RTT relation in component form:

$$\sum_{j_1,j_2} R^{k_1k_2}_{j_1j_2}(u-v)T(u)^{j_1}_{i_1}T(v)^{j_2}_{i_2} = \sum_{j_1,j_2} T(v)^{k_2}_{j_2}T(u)^{k_1}_{j_1}R^{j_1j_2}_{i_1i_2}(u-v)\,. \tag{7.18}$$

This gives a relation for each choice of (k_1, k_2, i_1, i_2). Consider for instance the choice $(0, 0, 1, 0)$:

$$R^{00}_{00}(u-v)T(u)^0_1T(v)^0_0 = T(v)^0_1T(u)^0_0R^{01}_{10}(u-v) + T(v)^0_0T(u)^0_1R^{10}_{10}(u-v)\,. \tag{7.19}$$

Insert now the R-matrix elements from (7.11)–(7.12) and the monodromy matrix elements from (7.9), recalling that the former are just scalars, whereas the latter are (non-commuting) operators. This gives

$$a(u-v)B(u)A(v) = c(u-v)B(v)A(u) + b(u-v)A(v)B(u)\,. \tag{7.20}$$

Among all the possible commutation relations we shall actually only need a few. First, for two operators of the same type we have simply

$$\begin{aligned} A(u)A(v) = A(v)A(u)\,, \quad B(u)B(v) = B(v)B(u)\,, \\ C(u)C(v) = C(v)C(u)\,, \quad D(u)D(v) = D(v)D(u)\,. \end{aligned} \tag{7.21}$$

Second, to push an A or a D past a B we have

$$\begin{aligned} A(u)B(v) &= \frac{a(v-u)}{b(v-u)}B(v)A(u) - \frac{c(v-u)}{b(v-u)}B(u)A(v)\,, \\ D(u)B(v) &= \frac{a(u-v)}{b(u-v)}B(v)D(u) - \frac{c(u-v)}{b(u-v)}B(u)D(v)\,. \end{aligned} \tag{7.22}$$

The first of these relations follows from (7.20) after a relabelling $u \leftrightarrow v$ and some rearrangement. The second relation is obtained from a similar computation.

7.3.3 Algebraic Bethe Ansatz

We now have all necessary ingredients to treat the six-vertex model using the algebraic Bethe Ansatz.

As in the coordinate Bethe Ansatz, one starts from the pseudo-vacuum, or reference state, in which all spins point up and no particle world-lines are present. We denote this state as

$$| \Uparrow \rangle = | \uparrow\uparrow \cdots \uparrow \rangle = |00 \cdots 0\rangle\,. \tag{7.23}$$

Recall that $B(u)$ creates a particle (or equivalently, flips down one spin), whereas $C(u)$ annihilates a particle. Thus, an n-particle state (i.e., with n down spins) can be constructed as follows:

$$|\Psi_n\rangle = \prod_{i=1}^{n} B(u_i)| \Uparrow \rangle\,. \tag{7.24}$$

The states (7.24) are called algebraic Bethe Ansatz states.

Our goal is to diagonalise the transfer matrix

$$t(u) = \mathrm{Tr}_{\mathrm{a}}\, T(u) = A(u) + D(u)\,. \tag{7.25}$$

This means solving the eigenvalue equation

$$t(u)|\Psi_n\rangle = [A(u) + D(u)] \prod_{i=1}^{n} B(u_i)|\Uparrow\rangle = \Lambda_n(u;\{u_i\}) \prod_{i=1}^{n} B(u_i)|\Uparrow\rangle\,. \quad (7.26)$$

This can obviously only be done if the parameters $\{u_i\}$ satisfy certain conditions. These are precisely the Bethe Ansatz equations, and we shall rederive them now using the algebraic method.

This obviously implies that $\{u_i\}$ must somehow be related to the pseudo-momenta. The correct relation will be obtained from the algebraic method below, but we can obtain it already now by comparing out setup to that of the coordinate Bethe Ansatz.

We know from section 5.2.2 that a one-particle state reads $|\Psi_1\rangle = \sum_x g(x)|x\rangle$ with $g(x) = z^x = \mathrm{e}^{ikx}$. In the algebraic framework—and setting $L = 2$ for simplicity—this same state follows from (7.24) and (7.14):

$$|\Psi_1\rangle = \Delta B(u)|\Uparrow\rangle = c(u)a(u)|10\rangle + b(u)c(u)|01\rangle\,.$$

Comparing this with $|\Psi_1\rangle = \sum_x g(x)|x\rangle$ we identify[22]

$$z = \mathrm{e}^{ik} = \frac{a(u)}{b(u)}\,. \quad (7.27)$$

(To be quite honest, for $L = 2$ one cannot by comparing $|01\rangle$ and $|10\rangle$ say whether the particle has moved to the right or to the left, i.e., distinguish z and z^{-1}. But the above result will be rederived below by other means.)

To compute $[A(u) + D(u)] \prod_{i=1}^{n} B(u_i)|\Uparrow\rangle$ we use the commutation relations (7.22) to push $A(u)$ and $D(u)$ to the right, past the string of B's. When they have been pushed completely to the right, one applies the relations

$$A(u)|\Uparrow\rangle = a(u)^L|\Uparrow\rangle\,, \qquad D(u)|\Uparrow\rangle = b(u)^L|\Uparrow\rangle\,, \quad (7.28)$$

Note that (7.28) follows from the first and last lines of (7.16), generalised for $L = 2$ to arbitrary L. Consider for instance $\Delta A(u)$. It is easy to see that the right-hand side will contain a single term $A(u)^{\otimes L}$, and all remaining terms will contain at least one factor $C(u)$ in the tensor product. But this $C(u)$ will annihilate $|\Uparrow\rangle$, so the only contribution is $a(u)^L|\Uparrow\rangle$ indeed.

Each time we push $A(u)$ one position towards the right, we obtain two contributions from the right-hand side of (7.22). The unique term obtained by always choosing the first contribution is a *wanted A-term*. In this term, the arguments of the $B(u_i)$ remain unchanged, and $A(u)$ simply "goes through". The remaining $2^n - 1$ terms are *unwanted A-terms*. In those terms, at least one of the arguments u_i of the B's has been changed into u, and so the state is not of the form (7.24). Similarly, there is one wanted and $2^n - 1$ unwanted D-terms.

The wanted A-term and the wanted D-term produces the expression for the eigenvalue of the transfer matrix

$$\Lambda_n(u;\{u_i\}) = a(u)^L \prod_{i=1}^{n} \frac{a(u_i - u)}{b(u_i - u)} + b(u)^L \prod_{i=1}^{n} \frac{a(u - u_i)}{b(u - u_i)} \,. \tag{7.29}$$

This should of course coincide with (5.24). This means that we should identify

$$L_i = L(z_i) = \frac{a(u_i - u)}{b(u_i - u)} \,. \tag{7.30}$$

On the other hand we have the definition (5.10) according to which

$$L_i = \frac{a(u)b(u) + \big(c(u)^2 - b(u)^2\big) z_i}{a(u)\big(a(u) - b(u) z_i\big)} \,, \tag{7.31}$$

where $z_i = \frac{a(u_i)}{b(u_i)}$. Inserting the parameterisations (7.12) and simplifying we find indeed

$$L_i = \frac{\sin(\gamma - (u_i - u))}{\sin(u_i - u)} = \frac{a(u_i - u)}{b(u_i - u)} \,. \tag{7.32}$$

The result $z_i = \frac{a(u_i)}{b(u_i)}$ was obtained above; alternatively one can find it by taking the $u \to 0$ limit of (7.31).

In the same way we can verify that

$$M_i = M(z_i) = \frac{a(u - u_i)}{b(u - u_i)} \tag{7.33}$$

is in agreement with the definition (5.11).

The condition that the unwanted A-terms cancel the unwanted D-terms is precisely the Bethe Ansatz equations

$$\left(\frac{a(u_i)}{b(u_i)}\right)^L = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{a(u_i - u_j)b(u_j - u_i)}{a(u_j - u_i)b(u_i - u_j)} \,. \tag{7.34}$$

Exercise: Prove this!

The form (7.34) is best compared with (5.33). We have therefore the identification of the scattering phases

$$\hat{S}_{ji} = \frac{a(u_j - u_i)b(u_i - u_j)}{a(u_i - u_j)b(u_j - u_i)} = -\frac{1 - 2\Delta z_i + z_i z_j}{1 - 2\Delta z_j + z_i z_j} \,. \tag{7.35}$$

Exercise: Verify this using the parameterisation (7.12).

Finally, we can compute the energy E of the Bethe Ansatz state (7.24). To this end we just need to recall the link (6.29) between the transfer matrix $t(u)$ and the Hamiltonian H. Taking expectation values with respect to the state (7.24) the operator H gets replaced by its expectation value E, and $t(u)$ gets replaced by the eigenvalue $\Lambda(u; \{u_i\})$. Therefore

$$E_n(\{u_i\}) = -\sin\gamma \,\frac{\partial}{\partial u} \log \Lambda_n(u; \{u_i\})\bigg|_{u\to 0} \,. \tag{7.36}$$

In (7.29) only the first term contributes in the $u \to 0$ limit:

$$\Lambda_n \simeq \sin^L(\gamma) \prod_{i=1}^{n} \frac{a(u_i - u)}{b(u_i - u)} \,. \tag{7.37}$$

Taking the derivative we arrive at

$$E_n(\{u_i\}) = -\sin\gamma \left(L \log(\sin\gamma) + \sum_{i=1}^{n} \frac{\sin\gamma}{\sin(u_i)\,\sin(\gamma - u_i)} \right) . \tag{7.38}$$

In the same way we can express the momentum of the Bethe Ansatz state:

$$-iP = \log\left(\frac{t(0)}{\sin^L(\gamma)}\right) = \sum_{i=1}^{n} \log\left(\frac{a(u_i)}{b(u_i)}\right) . \tag{7.39}$$