Algebra, Integrability and Exactly Solvable Models

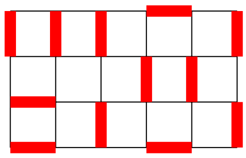
Written exam, 7 June 2012, 2.00-5.00 pm.

The AIMES lecture notes and any personal notes are allowed.
Results from the lecture notes can be freely referred to.
The two exercises are **almost** *independent.*

1 Dimer coverings in two dimensions

A dimer covering of a graph $G = (V, E)$ with vertex set V and edge set E is defined by assigning an occupation number $\alpha_e = 1$ (meaning that the edge is occupied) or $\alpha_e = 0$ (the edge is empty) to each edge $e \in E$, subject to the constraint that each vertex $v \in V$ is incident on exactly one occupied edge. Occupied edges are called dimers.

In the following G is taken as the two-dimensional square lattice. A possible dimer covering of a small piece of the square lattice is shown below.



We wish to study the model defined by the partition function

$$Z = \sum_{\mathcal{C}} \omega^{\#\text{horizontal dimers}}, \tag{1}$$

where $\mathcal{C}$ is the set of dimer coverings of G, and ω is some weight. The case $\omega = 1$ is referred to as the isotropic model.

On a strip of width L, we define the state of a row as the set of occupation numbers $\alpha = (\alpha_1, \ldots, \alpha_L)$ of the vertical edges. There are 2^L possible states of a row, so Z can be written as the trace of a 2^L-dimensional transfer matrix T. We impose periodic boundary conditions, so the index i is considered modulo L.

Given two row states α and β, the matrix element $T_{\beta\alpha}$ is the sum of the Boltzmann weights associated with the horizontal dimer configurations μ compatible with α and β:

$$T_{\beta\alpha} = \sum_{\mu|(\alpha,\beta)} \omega^{\mu_1+\ldots+\mu_L}. \tag{2}$$

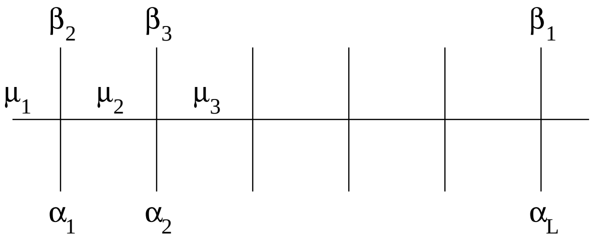


Figure 1: The row-to-row transfer matrix.

It is convenient to introduce a shift at each row in the numbering of columns, as shown in Fig. 1.

1.1

Express the compatibility criterion $\mu|(\alpha, \beta)$ formally in terms of the occupation numbers.

1.2

We call *particle* an empty even vertical edge *or* an occupied odd vertical edge. We suppose henceforth that L is even.

Show that T conserves the number of particles within a row. Deduce that T is block diagonal, and give the dimensions of each block. State the explicit rules for the dynamics of the particles (i.e., given the configuration of particles in row α, what are the possible configurations in the next row β?).

1.3

We consider first the case of $n = 1$ particle. Let $T^{(1)}$ be the corresponding block of the transfer matrix, and let $\Phi(x)$ denote the probability of finding the particle on edge x.

Write the action of $T^{(1)}$ on $\Phi(x)$.

1.4

Define the two-step cyclic permutation J of the sites by its action on the one-particle state Φ:

$$(J\Phi)(x) = \Phi(x+2) \tag{3}$$

Show that $[T^{(1)}, J] = 0$.

We want to take advantage of the translational invariance to diagonalize $T^{(1)}$. Supposing that z satisfies the condition $z^L = 1$, write the two-dimensional eigenspace $(\Phi_z, \overline{\Phi}_z)$ of J with eigenvalue z^2.

1.5

More generally, let z be a complex number of modulus unity. Write $T^{(1)}$ as a 2×2 matrix in the basis $(\Phi_z, \overline{\Phi}_z)$.

Let ψ_z, ψ'_z be the eigenvectors of this matrix. What are the corresponding eigenvalues $\Lambda(z), \Lambda'(z)$?

1.6

We henceforth use the parameterisation

$$\begin{aligned} z &= \exp(ik) & (4)\\ \Lambda(z) &= \exp[h + i(\phi+\theta)] & (5)\\ \Lambda'(z) &= \exp[-h + i(\phi-\theta)] & (6) \end{aligned}$$

where k, ϕ, θ are real and h is nonnegative. $\Lambda(z)$ is then the eigenvalue with greatest modulus. We refer to k as the *momentum*.

Show that one has

$$\cosh(2h) = 1 + 2\omega^2 \cos^2 k \tag{7}$$

when $|z| = 1$. Write an explicit expression for $h(k)$.

1.7

We next consider the case of $n = 2$ particles. Let $T^{(2)}$ be the corresponding block of T. Define the two-particle vector

$$\psi_{12}(x_1, x_2) = \psi_{z_1}(x_1)\psi_{z_2}(x_2)\ , \qquad x_1 < x_2 \tag{8}$$

Write the action of $T^{(2)}$ on $\psi_{12}(x_1, x_2)$. Deduce that the antisymmetric combination:

$$\psi(x_1, x_2) = \psi_{z_1}(x_1)\psi_{z_2}(x_2) - \psi_{z_2}(x_1)\psi_{z_1}(x_2) \tag{9}$$

is an eigenvector of $T^{(2)}$ with eigenvalue $\Lambda(z_1)\Lambda(z_2)$.

1.8

Give the eigenvectors and eigenvalues in the general n-particle sector (a detailed proof is *not* required).

1.9

Write the Bethe Ansatz equations for the quasi momenta z_j. Give the explicit solution of those equations.

1.10

Given L, what is the number of particles n that maximises the eigenvalue $\Lambda^{(n)}_{\max}$ of T?

1.11

Suppose now that L is a multiple of 4. Write the free energy per unit area

$$f^{(n)}_L = \frac{1}{L} \log \Lambda^{(n)}_{\max} \tag{10}$$

with n determined in question 1.10, in terms of the function $h(k)$ determined in question 1.6. Give the corresponding thermodynamic limit, $f_\infty = \lim_{L\to\infty} f^{(n)}_L$, in the form of an integral over k.

1.12

Find the dominant finite-L correction to f_∞ (hint: use the Euler-Maclaurin formula). Deduce the value of the central charge for the isotropic model of dimer coverings.

2 CFT treatment of the continuum limit

2.1

Show that a dimer configuration is equivalent to a height configuration, by defining an appropriate integer-valued height $H(\mathbf{x})$ on the faces of the square lattice G (i.e., on the vertices of the dual lattice G^*).

2.2

Argue that the continuum limit of H produces the theory of a free massless boson ϕ with Euclidean action

$$S = \frac{g}{2} \int \mathrm{d}^2 x \, \partial_\mu \phi \, \partial^\mu \phi \, . \tag{11}$$

2.3

Compute the propagator of $\partial\phi(z)$ in the usual complex coordinates $z = x^1 + ix^2$.

2.4

Admit without proof that the corresponding stress tensor in complex coordinates is

$$T(z) = -2\pi g : \partial\phi(z) \, \partial\phi(z) : \tag{12}$$

where $: \ldots :$ denotes normal ordering. Show that $\partial\phi(z)$ is a primary operator and compute its conformal weight (scaling dimension). (Hint: use the Wick theorem to compute the contractions with the normal order.)

2.5

Compute explicitly the operator product expansion (OPE) of $T(z)$ with itself. Deduce the value of the central charge, and compare with question 1.12.