Algebra, Integrability and Exactly Solvable Models

Written exam, 13 June 2013, 1.30–4.30 pm.

The AIMES lecture notes and any personal notes are allowed.
Other documents and books as well as electronic devices are forbidden.
Results from the lecture notes can be freely referred to.
The exercises are independent.

1 Free Majorana fermion

We consider the free Majorana fermion $\Psi = (\psi, \bar{\psi})$ with two-dimensional Euclidian action

$$S[\Psi] = g \int \mathrm{d}^2x \, \left(\bar{\psi}\, \partial \bar{\psi} + \psi\, \bar{\partial} \psi\right) \, . \tag{1}$$

Average values are defined as usual:

$$\langle \ldots \rangle := \frac{1}{Z} \int \mathcal{D}\Psi \, \mathrm{e}^{-S[\Psi]} \, (\ldots) \quad \text{with} \quad Z := \int \mathcal{D}\Psi \, \mathrm{e}^{-S[\Psi]} \, . \tag{2}$$

Our first goal is to calculate the propagator $K_{ij}(\mathbf{x}, \mathbf{y}) := \langle \Psi_i(\mathbf{x}) \Psi_j(\mathbf{y}) \rangle$, where the spinor components $i, j = 1, 2$.

1.1

What are the classical equations of motion? Discuss their solutions.

1.2

Rewriting the action as

$$S[\Psi] = \frac{1}{2} \int \mathrm{d}^2x \mathrm{d}^2y \, \Psi_i(\mathbf{x}) A_{ij}(\mathbf{x}, \mathbf{y}) \Psi_j(\mathbf{y}) \, , \tag{3}$$

express the propagator $K_{ij}(\mathbf{x}, \mathbf{y})$ in terms of the kernel $A_{ij}(\mathbf{x}, \mathbf{y})$.

1.3

We recall that in complex coordinates $(z, \bar{z})$ the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{4}$$

Check that the Dirac delta function can be represented in complex coordinates as

$$\delta(\mathbf{x}) = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}}. \tag{5}$$

Hint: Compute $\int_M \mathrm{d}^2 x \, \delta(\mathbf{x}) f(z)$, where M is a domain containing the origin, and use the Gauss theorem to convert this into an integral along the boundary ∂M.

1.4

Use this representation to write down the differential equations satisfied by the components of $K_{ij}(\mathbf{x}, \mathbf{y})$. Show that the solutions are

$$\begin{aligned} \langle \psi(z, \bar{z}) \psi(w, \bar{w}) \rangle &= \frac{1}{2\pi g} \frac{1}{z - w}, \\ \langle \bar{\psi}(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle &= \frac{1}{2\pi g} \frac{1}{\bar{z} - \bar{w}}, \\ \langle \psi(z, \bar{z}) \bar{\psi}(w, \bar{w}) \rangle &= 0. \end{aligned} \tag{6}$$

Deduce the operator product expansion (OPE) of $\psi(z)$ with $\psi(w)$, in the limit $z \to w$.

1.5

We recall that the energy-momentum tensor (or stress tensor) $T^{\mu\nu}$ can be obtained from the Lagrangian $\mathcal{L}[\Psi]$ as

$$T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Psi)} \partial^\nu \Psi. \tag{7}$$

Write out the components T^{zz}, $T^{\bar{z}\bar{z}}$ and $T^{z\bar{z}}$. Is the stress tensor traceless and symmetric as required?

1.6

We recall that the holomorphic and antiholomorphic components of the stress tensor are

$$T(z) = -2\pi : T_{zz} : \quad \text{and} \quad \bar{T}(z) = -2\pi : T_{\bar{z}\bar{z}} : , \tag{8}$$

where $: \ldots :$ denotes the normal-ordered product. Compute the OPE of $T(z)$ with $\psi(w)$, in the limit $z \to w$. Deduce the conformal weight h_ψ of the fermion field. Is ψ primary / quasi-primary / descendent?

1.7

Compute finally the OPE of $T(z)$ with itself and deduce the value of the central charge c. Is T primary / quasi-primary / descendent?

2 XY spin chain

We consider a one-dimensional chain of N spin-1/2 particles interacting antiferromagnetically through the Hamiltonian

$$H = \sum_{i=1}^{N} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) . \tag{9}$$

The spin-1/2 operators S_i^α satisfy $[S_i^\alpha, S_j^\beta] = i\epsilon^{\alpha\beta\gamma}\delta_{ij}S_i^\gamma$, where $\epsilon^{\alpha\beta\gamma}$ is the completely antisymmetric tensor. They can be represented as Pauli matrices. We assume periodic boundary conditions, $\mathbf{S}_{N+1} = \mathbf{S}_1$, where $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$.

2.1

What is the relation between this model and the XXZ spin chain studied in the lectures?

2.2

We introduce raising operators $a_i^\dagger = S_i^x + iS_i^y$ and lowering operators $a_i = S_i^x - iS_i^y$. Write the algebraic relations satisfied by these operators. Express H in terms of them.

2.3

A proper fermionic description requires performing a Jordan-Wigner transformation:

$$c_i = \exp\left(i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) a_i\,, \qquad c_i^\dagger = a_i^\dagger \exp\left(-i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right)\,. \tag{10}$$

Invert this transformation, i.e., express a_i and $a_i^\dagger$ in terms of the c.

2.4

Show that c_i and $c_i^\dagger$ obey the canonical anticommutation relations:

$$\{c_i, c_j^\dagger\} = \delta_{ij}\,, \qquad \{c_i, c_j\} = 0\,, \qquad \{c_i^\dagger, c_j^\dagger\} = 0\,. \tag{11}$$

Focus on proving $\{c_i, c_j^\dagger\} = \delta_{ij}$, the other cases being similar.

Hints: Show first that

$$\exp\left(\pm i\pi \sum_{j=n}^{m} a_j^\dagger a_j\right) = \prod_{j=n}^{m} \exp\left(\pm i\pi a_j^\dagger a_j\right) \tag{12}$$

and that

$$\exp\left(\pm i\pi a_i^\dagger a_i\right) = 1 - 2a_i^\dagger a_i\,. \tag{13}$$

Compute next the commutator of $\exp\left(\pm i\pi \sum_{j=n}^{m} a_j^\dagger a_j\right)$ with a_i and $a_i^\dagger$ for $i \notin [n, m]$, and the anticommutator of the same operator with $a_i^\dagger$ for $i \in [n, m]$. Conclude.

2.5

Show that the Hamiltonian H can be expressed in terms of the c operators as

$$\begin{aligned} H &= \frac{1}{2}\sum_{i=1}^{N}\left(c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}\right) \\ &\quad -\frac{1}{2}\left(c_1^\dagger c_N + c_N^\dagger c_1\right)\left(\exp\left(i\pi \sum_{j=1}^{N} c_j^\dagger c_j\right) + 1\right)\,. \end{aligned} \tag{14}$$

Under what condition does the boundary term on the second line vanish?

2.6

We suppose this condition to be satisfied and focus on the cyclic chain

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(c_{i+1}^{\dagger}c_i + c_i^{\dagger}c_{i+1}\right) \tag{15}$$

with $c_{N+1} = c_1$ and $c_{N+1}^{\dagger} = c_1^{\dagger}$. What is the physical interpretation of this model? Diagonalise H by rewriting it in the form

$$H = \sum_k \epsilon_k \eta_k^{\dagger}\eta_k \,, \tag{16}$$

where η_k and $\eta_k^{\dagger}$ obey canonical anticommutation relations.

2.7

What is the physical meaning of the change of basis made in the preceding question? Describe the structue of the ground state of the spin chain. Determine the ground state energy per site.