Algèbres, intégrabilité et Modèles Exactement Solubles
Written exam, 12 June 2014 from 1.30 pm to 4.30 pm

We consider the Q-state Potts model on a square lattice of width N columns and height M rows. The boundary conditions in the M-direction are periodic. In the N-direction there is no interaction between the spins in column 1 and N, so the boundary condition is non-periodic. In other words, we have an annulus of width N and periodic circumference M.

We impose a particular type of boundary condition on the left side of the annulus, by constraining the spins in column 1 to take only Q_1 distinct values, with $Q_1 \leq Q$. All other spins, including those on the right side of the annulus (column N) can take Q different values as usual. Apart from that, the Hamiltonian can be written as usual

$$H = -K \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j} \,, \tag{1}$$

and we suppose that the model is at the critical point, $\mathrm{e}^{K_c} - 1 = \sqrt{Q}$.

Question 1: Show how to write the partition function Z as a loop model. Let n denote the weight of loops not touching the left boundary, and n_1 the weight n_1 of loops touching at least once the left boundary. Write the relation between Q, Q_1 and n, n_1.

The algebraic framework for handling the loop model with weights n and n_1 is the one-boundary Temperley-Lieb algebra $\mathcal{B}(L, n, n_1)$ defined on $L = 2N$ strands. It is generated by $L - 1$ generators e_i (with $i = 1, 2, \ldots, L-1$) and one extra generator b called the 'blob'. The e_i are those of the usual TL algebra; they obey the usual algebraic relations

$$\begin{aligned} (e_i)^2 &= n e_i \,, && (2) \\ e_i e_{i \pm i} e_i &= e_i \,, && (3) \\ e_i e_j &= e_j e_i \text{ for } |i - j| > 1 \,; && (4) \end{aligned}$$

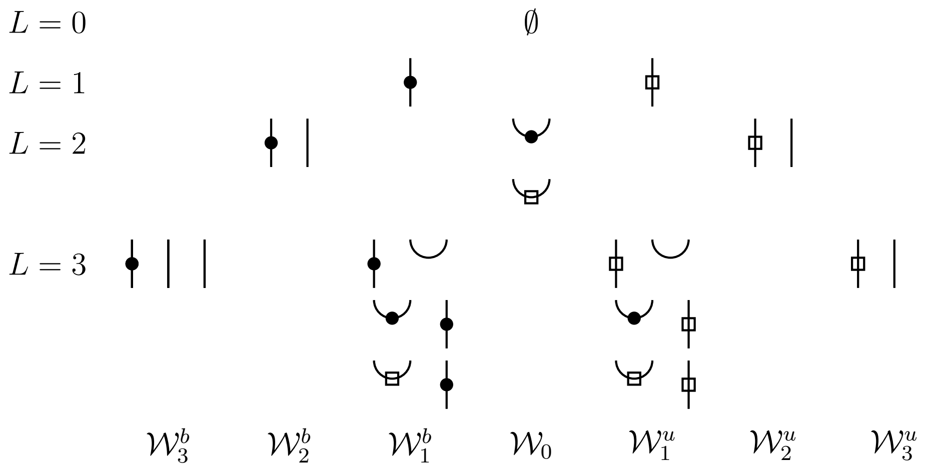
and their diagrammatic representation in terms of loop strands is the usual one. The extra relations involving b are

$$\begin{aligned} b^2 &= b \,, && (5) \\ e_1 b e_1 &= n_1 e_1 \,, && (6) \\ b e_i &= e_i b \text{ for } i > 1 \,. && (7) \end{aligned}$$

The diagrammatic representation of b is , i.e., it adds a special 'blob' symbol on the leftmost strand.

Question 2: Interpret the extra algebraic relations geometrically. Consider the word $w = e_3e_1e_3be_2e_3be_1$ within $\mathcal{B}(4, n, n_1)$. Reduce w to its shortest possible form using only the algebraic relations. Then draw w diagramatically and explain.

The set of reduced states describing the possible connectivities between the L loop strands in the uppermost row of a partially built lattice can be constructed as shown in the following figure:



We have here defined another generator $u = 1 - b$ called the 'antiblob', represented diagramatically as $\cdots$, i.e., it adds a special 'antiblob' symbol on the leftmost strand. As usual, the reduced states consist of arcs and strings (lines), representing respectively internal connections between points on the top of the system and connections back to the bottom of the system. From now on we adopt the convention that the leftmost string always carries either the b or u symbol.

The states for size $L+1$ can be constructed from those at size L as shown in the figure. In the left (resp. right) half of the diagram, a step down and to the left (resp. right) means adding a string on the right of the diagram; and a step down and to the right (resp. left) means bending the rightmost string to the extreme right so as to form an arc (possibly containing other arcs).

Example: From the state in row $L = 2$ and column $\mathcal{W}_2^b$ we can either take a step down and to the left to obtain the state in row $L = 3$ and column $\mathcal{W}_3^b$, or we can take a step down and to the right to obtain the state in row $L = 3$ and column $\mathcal{W}_1^b$.

The collection of states in a given column of the figure defines the standard modules $\mathcal{W}_j^b$, $\mathcal{W}_0$ and $\mathcal{W}_j^u$. For example, for $L = 3$ the standard module $\mathcal{W}_1^b$ consists of three states. The index j gives the number of strings in $\mathcal{W}_j^\alpha$, and $\alpha = b, u$ denotes whether the leftmost string is blobbed or antiblobbed.

Question 3: Show that b and u are a complete set of projectors, and that the spaces onto which they project are complementary spaces (i.e., $\mathrm{Im}(b) = \mathrm{Ker}(u)$ and $\mathrm{Ker}(b) = \mathrm{Im}(u)$).

Question 4: Continue the figure by drawing all states corresponding to $L = 4$ at their proper position.

Question 5: Characterise the arcs and strings that can have a blob (b) or an antiblob (u) mark in this construction. Which arcs and strings have no mark at all? First deduce these features from the rules used to construct the figure above, then interpret them physically.

Question 6: Compute the dimensions d_j^α of the standard modules $\mathcal{W}_j^\alpha$ for any L, j, α. Hint: find a recurrence relation between $d_j^\alpha(L+1)$, $d_{j-1}^\alpha(L)$ and $d_{j+1}^\alpha(L)$.

We define also a variant algebra by

$$\mathcal{B}^b(2N, n, n_1) = b\mathcal{B}(2N, n, n_1)b\,. \tag{8}$$

It has the property that any string or arc on the leftmost site is blobbed.

Question 7: Suppose $n_1 \neq 0$. We set $1' = b$, $e_i' = be_{i+1}b$, and $b' = \frac{1}{n_1}be_1b$. Show that there is an isomorphism of algebras

$$\mathcal{B}^b(2N, n, n_1) \simeq \mathcal{B}(2N-1, n, n_1')\,. \tag{9}$$

by verifying that the primed generators $1'$, e_i' and b' satisfy the algebraic relations corresponding to the right-hand side of the isomorphism. Determine the parameter n_1'.

A symmetric bilinear form (scalar product) $\langle v_1, v_2\rangle$ is defined on $\mathcal{B}(L, n, n_1)$ as follows. Whenever v_1, v_2 belong to different standard modules, we set $\langle v_1, v_2\rangle = 0$. Otherwise, we reflect the reduced state corresponding to v_1 in a horizontal mirror, and glue it on top of v_2. If the resulting diagram is such that any string in v_1 gets glued to another string in v_1, or any string in v_2 gets glued to another string in v_2, we again set $\langle v_1, v_2\rangle = 0$. If this is not the case, the value of $\langle v_1, v_2\rangle$ is found by giving the proper weights (e.g., n or n_1) to each loop in the glued diagram. Note that the generators are supposed to be self-adjoint: $e_i^\dagger = e_i$ and $b^\dagger = b$.

Let $\mathcal{G}_j^\alpha$ be the Gram matrix of scalar products in $\mathcal{W}_j^\alpha$, i.e., the matrix elements are $(\mathcal{G}_j^\alpha)_{v_1,v_2} = \langle v_1|v_2\rangle$. We shall be interested in its determinant $\Delta_j^\alpha = \det \mathcal{G}_j^\alpha$.

Question 8: Compute Δ_2^b for size $L = 4$. Show that it factorises.

As in the study of the TL algebra we parameterise $n = q + q^{-1}$. We introduce q-deformed numbers by

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}\,. \tag{10}$$

In particular $n = [2]$. We also parameterise n_1 by a real number r so that

$$n_1 = \frac{[r+1]}{[r]}\,. \tag{11}$$

We define elements $P_k^b \in \mathcal{B}(k,n,n_1)$ recursively as follows:

$$\begin{aligned} P_1^b &= b\,, & (12)\\ P_{k+1}^b &= P_k^b - \gamma_k^b P_k^b e_k P_k^b \text{ for } k \geq 1\,. & (13) \end{aligned}$$

Question 9: Show that for a proper choice of the coefficients γ_k^b, the object P_k^b satisfies the properties

$$\begin{aligned} \forall i < k: \ e_i P_k^b &= P_k^b e_i = 0\,, & (14)\\ b P_k^b &= P_k^b b = P_k^b\,. & (15) \end{aligned}$$

We shall call P_k^b the Jones-Wenzl projector on $\mathcal{W}_k^b$.

Hints: γ_k^b has a simple expression in terms of q-deformed numbers. One may assume without proof that 'big projectors swallow smaller ones', namely $P_k^b P_\ell^b = P_\ell^b$ for $k \leq \ell$.

Question 10: We also define antiblob projectors P_k^u by replacing b by u in the definitions (12)–(13). Determine the coefficients γ_k^u so that P_k^u satisfy the Jones-Wenzl properties, namely (14)–(15) with b replaced by u.

The Markov trace Tr is defined as in the lecture notes by gluing the top and bottom of (full, not reduced) states. Non contractible unmarked (resp. blobbed) loops are given the weight n (resp. n_1), i.e., the same weights as contractible loops.

Question 11: Show that for any $M \in \mathcal{B}(L,n,n_1)$ the Markov trace decomposes as

$$\mathrm{Tr}\ M = \mathrm{tr}_{\mathcal{W}_0} M + \sum_{\alpha=b,u} \sum_{j=1}^{L} \mathrm{Tr}\{P_j^\alpha\}\ \mathrm{tr}_{\mathcal{W}_j^\alpha} M\,, \qquad (16)$$

where tr denotes the usual trace. Determine the quantum dimensions $D_j^\alpha = \mathrm{Tr}\{P_j^\alpha\}$.

Question 12: Express Δ_2^b (found in question 8) as a product of D_j^α. (For information: the same can be done for any Δ_j^α and leads to elegant formulae.)

Question 13: The representation theory is singular when some Δ_j^α, hence some D_j^α is zero. For which values of r can that happen? Discuss the corresponding boundary conditions (values of Q_1) for the case of the Ising model.